UNIFORM(0,1) TWO-PERSON POKER MODELS

Chris Ferguson, TiltWare, Los Angeles
Tom Ferguson, Mathematics, UCLA
Céphas Gawargy, Mathematics, Université Paris 1

Section 1. Introduction and Summary.

The study of two-person zero-sum poker models with independent uniform(0,1) hands goes back to Borel and von Neumann. Borel discusses a model of poker in Chapter 5, "Le jeu de poker" of his 1938 book, *Applications aux Jeux des Hazard*. Von Neumann presents his analysis of a similar model of poker in the seminal book on game theory — *Theory of Games and Economic Behavior* by von Neumann and Morgenstern (1944). Most subsequent work on these models has been to extend the Borel model to allow several rounds of betting or more bet sizes. The von Neumann model, though more closely tied to actual play, is harder to treat mathematically. In this paper we solve several extensions to the von Neumann model. See Ferguson and Ferguson (2003) for a discussion and comparison of these two models.

1.1 The Model of von Neumann. In the von Neumann model, Players I and II both contribute an ante of 1 unit into the pot, so that the initial pot size is 2. Then they receive independent uniform(0,1) hands, x for Player I and y for Player II. Player I acts first either by checking (in which case, the hands are immediately compared and the higher hand wins the pot) or by betting a prescribed amount $B > 0$ (putting that amount into the pot). If Player I bets, then Player II acts by either folding (and conceding the pot to Player I) or calling (and putting B into the pot). If Player II calls the bet of Player I, the hands are compared and the player with the higher hand wins the entire pot. That is, if $x > y$ then Player I wins the pot; if $x < y$ then Player II wins the pot. We do not have to consider the case $x = y$ since this occurs with probability 0.

The solution may be described as follows. Player I has a unique optimal strategy of the form for some numbers a and b with $a < b$: bet if $x < a$ or if $x > b$, and check otherwise. Although there are many optimal strategies for Player II, and von Neumann finds all of them, there is a unique admissible one. (A strategy is admissible if no other strategy gives a better expected payoff against one strategy of the opponent without giving a worse expected payoff against another strategy of the opponent.) It has the simple form for some number c: call if $y > c$, and fold if $y < c$. The optimal values of a, b and c in terms of B are

$$a = \frac{B}{(1+B)(4+B)} \qquad b = \frac{2+4B+B^2}{(1+B)(4+B)} \qquad c = \frac{B(3+B)}{(1+B)(4+B)}. \tag{1.1}$$

The value to Player I is

$$V_I(B) = \frac{B}{(1+B)(4+B)}. \tag{1.2}$$

Since this is positive, the game favors Player I. When $B = 2$, the bet is the size of the pot (the maximum bet in pot-limit poker), and we have $a = 1/9$, $b = 7/9$, and $c = 5/9$, and the value is $v(2) = 1/9$.

The region $x < a$ is the region in which Player I bets as a bluff. It is noteworthy that Player I must bluff with his worst hands, and not with his moderate hands. It is interesting to note that there is an optimal bet size for Player I. It may be found by setting the derivative of $V_I(B)$ to zero and solving the resulting equation for B. It is $B = 2$. In other words, the optimal bet size is the size of the pot.

1.2. Summary. In this paper, we treat three simple extensions of the von Neumann model. In Section 2, we allow Player II to bet if Player I checks. This game favors Player II. Both players bluff with their worst hands. In Section 3, we allow Player II to raise if Player I bets. Player I retains the advantage. An interesting feature of the solution is that the optimal size of the raise by Player II is the size of the pot. In Section 4, we allow either player to initiate the betting, followed by a raise of the other player. For simplicity, we restrict attention to pot-limit bet sizes. This allows Player I the possibility of the check-raise, sometimes called "sandbagging"; this is the tactic of checking with a very strong hand in the hopes that Player II will bet so that Player I can then raise. The possibility of bluff sandbagging also appears. We discuss briefly the changes in the optimal strategies needed for the limit poker case.

There are two important extensions of the von Neumann model in the literature. One, by Newman (1959), allows Player I to choose the bet size B as an arbitrary nonnegative number depending on x. Betting 0 is equivalent to checking. Newman's result may be summarized as follows. An optimal strategy for Player I is to check if $1/7 < x < 4/7$, to bet B if $x = 1-(12/7)(2+B)^{-2}$ for $x > 4/7$, and to bet B if $x = (4/7)(2+3B)(2+B)^{-3}$ for $x < 1/7$. An optimal strategy for Player II is to call a bet of B if $y > 1-(12/7)(2+B)^{-1}$. The value is $1/7$. Cutler (1976) solves the problem when the bets are restricted to an interval [a,b] on the positive real line, as in "limit poker".

In this model, Player I's optimal strategy gives away a lot of information about his hand; a bet of B tells Player II that Player I's hand is either $1 - \frac{7}{12}(2 + B)^2$ or $\frac{7}{36}(2 + 3B)(2 + B)^2$. Such a strategy would be poor in the models treated in this paper where Player II is allowed to raise the bet of Player I.

The other important extension, by Cutler (1975), is to allow an unlimited number of raises under pot-limit rules, when each bet or raise must be the size of the pot. Cutler treats two cases. The first is when Player I is forced to bet on the first round. Thereafter the players in turn may fold, call or raise indefinitely until someone folds or calls. This is solved by recursion. In the other case, Player I is allowed to check on the first round, but he is forbidden to raise if he checks and Player II bets. This game is also solved completely. He says, "However, solving the problem with check raises appears to be quite difficult . . . ". One of the objects of the present paper to solve a problem that allows check-raises with two rounds of betting.

Section 2. One Round of Betting.

In the first extension of the von Neumann model, we allow Player II to bet if Player I checks. As before, Player I receives "hand" x and Player II receives "hand" y. It is assumed that x and y are chosen from a uniform distribution on the interval (0,1) and that x and y are independent. Throughout the play, both players know the value of their own hand, but not that of the opponent. Players ante 1 unit each, so the initial pot size is 2 units.

Player I acts first either by betting (adding a fixed amount $B > 0$ to the pot), or by checking. If Player I bets, Player II may fold, in which case Player I wins the pot, or she may call by putting the amount B into the pot, in which case the hands are compared and the player with the higher hand wins the pot. However, if Player I checks, Player II may check, in which case the hands are compared, or she may bet the same amount B, in which case Player I may fold, conceding the pot to Player II, or he may call. The betting tree is in Figure 2.1.

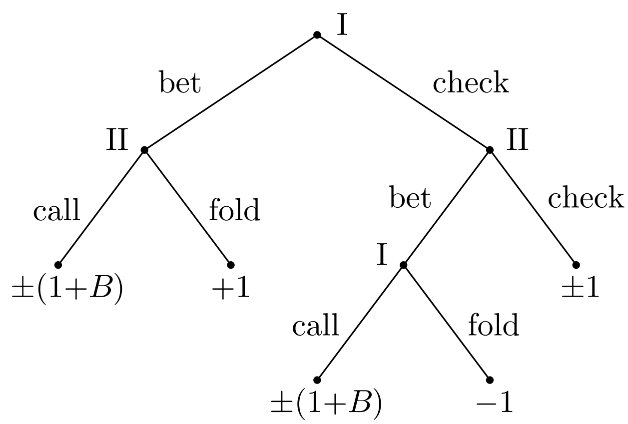


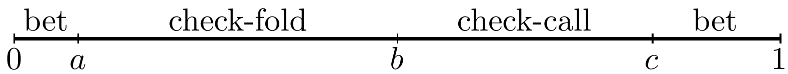
Fig. 2.1. The Betting Tree.

2.1 Conjectured Optimal Strategies. For each $x \in (0,1)$, Player I has three choices; he may bet, check-fold (i.e. check and fold if Player II bets), or check-call (i.e. check and call if Player II bets). We will show there is an optimal strategy for Player I that divides the interval (0,1) into four regions. In the lowest region, say $(0,a)$, Player I will bet as a bluff. In the next region, say (a,b), Player I will check-fold. In the third region, say (b,c), Player I will check-call, and in the last region, $(c,1)$, Player I will bet for value. We have $0 < a < b < c < 1$.

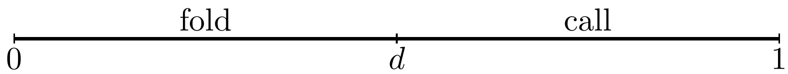
If Player I bets, Player II has two possible actions for each $y \in (0,1)$; she may call or fold. We will show there is an optimal strategy for II that folds if $y \in (0,d)$ and calls if $y \in (d,1)$ for some number d. If Player I checks, Player II may either bet or check. We will show that in this case, there is an optimal strategy for Player II that bets as a bluff if $y \in (0,e)$, checks if $y \in (e,f)$ and bets for value if $y \in (f,1)$. We have $0 < e < f < 1$.

We assume $a < e < b < f < c$ and $a < d < c$ and look for relations among these quantities that satisfy the principle of indifference at the boundaries. This principle states that if Player I receives hand $x = a$, for example, then against an optimal strategy of Player

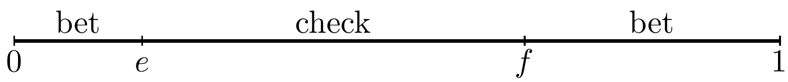
Player I:



Player II if Player I bets:



Player II if Player I checks:



II, he will be indifferent between betting and check-folding. In other words, at $x = a$ his expected return will be the same if he bets or if he check-folds. The same principle holds for all six boundary points. The equations obtained are called the indifference equations.

2.2 The Indifference Equations. In the analysis, we assume that the pot belongs to neither player. This makes the game constant-sum with sum 2. But it also makes the derivation of the equations somewhat simpler. The bet size is denoted by $B > 0$.

1. For Player I to be indifferent at a: If I bets at a, he wins 2 with probability d and loses B with probability $1 - d$. If I check-folds at a, he wins 0. Player I is indifferent if these two are equal, namely if $2d - B(1 - d) = 0$, we find $d = B/(2 + B)$. (This requires $a < d$ and $a < e$.)

2. For Player I to be indifferent at b: If I check-folds at b, he wins 2 with probability $b - e$ and nothing otherwise. If I check-calls, he wins $2 + B$ with probability e, 2 with probability $b-e$ and loses B with probability $1-f$. Equating expectations gives $2(b-e) = (2 + B)e + 2(b - e) - B(1 - f)$, or $(2 + B)e + Bf = B$. (This requires $e < b$ and $b < f$.)

3. For Player I to be indifferent at c: If I check-calls at c, he wins $2+B$ with probability $e + (c - f)$, 2 with probability $f - e$ and loses B with probability $1 - c$. If I bets, he wins 2 with probability d, $2 + B$ with probability $c - d$ and loses B with probability $1 - c$. Equating expectations gives $d + e - f = 0$. (This requires $f < c$ and $d < c$.)

4. For Player II to be indifferent at d: If II folds a bet at d, she wins nothing. If II calls a bet at d, she wins $2 + B$ with probability $a/(a + 1 - c)$ and loses B with probability $(1 - c)/(a + 1 - c)$. Equating the expectation to zero gives $(2 + B)a + Bc = B$. (This requires $a < d < c$.)

5. For Player II to be indifferent at e: If II bets at e, she wins 2 with probability $(b - a)/(c - a)$ and loses B with probability $(c - b)/(c - a)$. If II checks at e, she wins 2 with probability $(e - a)/(c - a)$. Equating expectations gives $(2 + B)b - Bc - 2e = 0$. (This requires $a < e < b$.)

6. For Player II to be indifferent at f: If II checks at f, she wins 2 with probability $(f - a)/(c - a)$. If II bets at f, she wins 2 with probability $(b - a)/(c - a)$, $2 + B$ with

probability $(f-b)/(c-a)$ and loses B with probability $(c-f)/(c-a)$. Equating expectations gives $b+c-2f=0$. (Requires $b<f<c$.)

Solving these six equations in six unknowns gives:

$$
\begin{aligned}
a &= \frac{2B}{(2+B)^2(1+B)} \qquad e = \frac{B}{(1+B)(2+B)} \\
b = d &= \frac{B}{2+B} \qquad f = \frac{B}{1+B} \qquad c = \frac{B(3+B)}{(2+B)(1+B)}
\end{aligned}
\tag{2.1}
$$

In the important special case of pot-limit poker, we have $B=2$, so that $a=1/12$, $e=1/6$, $b=d=1/2$, $f=2/3$ and $c=5/6$. Player I checks three-fourth of the time and bets one-fourth of the time. One-third of his bets are bluffs. If Player I bets, Player II calls one-half of the time. If Player I checks, Player II checks half the time and bets half the time.

2.3 Proof of Optimality. Starting with a reasonable guess at the form of an optimal strategy, we used the principle of indifference to solve for the critical points and found that the critical points satisfied the assumed form. This is strong confirmation that our guess is correct but it does not yet prove that the derived strategies are optimal. To prove optimality, we must show that the strategies are in equilibrium; that is, that each is a best response to the other.

We start by finding I's best response to II's strategy. Suppose I receives the hand x. If I bets, his expected return is

$$
v_{\mathrm{b}}(x) = \begin{cases} 0 & \text{if } 0<x<d \\ 2(1+B)(x-d) & \text{if } d<x<1. \end{cases}
\tag{2.2}
$$

If I check-folds, his expected return is

$$
v_{\mathrm{cf}}(x) = \begin{cases} 0 & \text{if } 0<x<e \\ 2(x-e) & \text{if } e<x<f \\ 2(f-e) & \text{if } f<x<1. \end{cases}
\tag{2.3}
$$

If I check-calls, his expected return is

$$
v_{\mathrm{cc}}(x) = \begin{cases} 2(1+B)(x-e) & \text{if } 0<x<e \\ 2(x-e) & \text{if } e<x<f \\ 2(1+B)(x-d) & \text{if } f<x<1. \end{cases}
\tag{2.4}
$$

It is easy to find these formulas knowing only the slopes of the lines in the intervals and that the functions are continuous. For example in (2.3), since $v_{\mathrm{cf}}(x)=0$ in the interval $0<x<e$, and has slope 2 in the interval $e<x<f$, we must have $v_{\mathrm{cf}}(x)=2(x-e)$ there, and since it is constant in the interval $f<x<1$, it must have value $2(f-e)$ there.

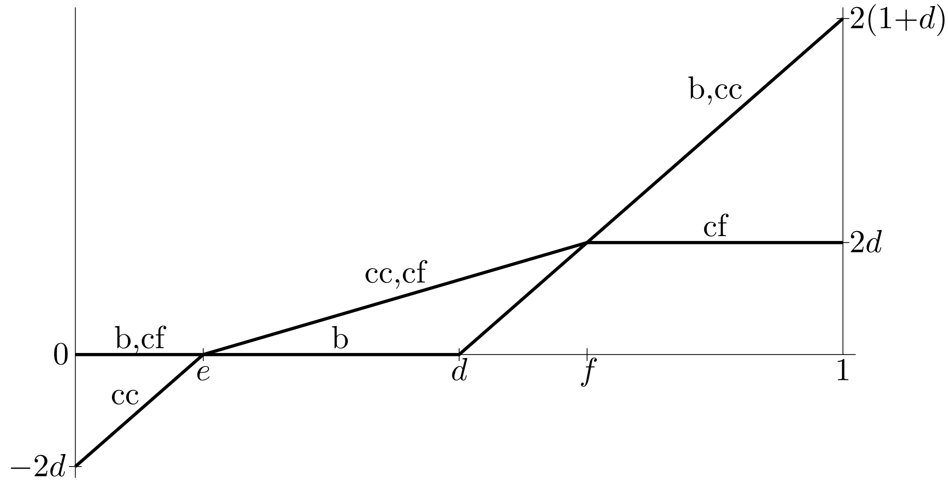


Fig. 2.2. I's payoffs against II's strategy.

For a given x, any response that yields the maximum of these three quantities is a best response to II's given strategy. (See Figure 2.2.)

We see that any strategy for Player I is a best response provided it bets or check-folds when $x < e$, check-folds or check-calls when $e < x < f$, and bets or check-calls when $x > f$. I's given strategy satisfies this and so is a best response to II's strategy.

Now we find II's best response to I's given strategy. This is done is two steps. Suppose II receives the hand y and I bets. If II folds, she receives the payoff of $w_{\mathrm{f}}(y) = 0$ for all y. If II calls, her expected return is

$$w_{\mathrm{ca}}(y) = \begin{cases} 2(1+B)(y-a) & \text{if } 0 < y < a \\ 0 & \text{if } a < y < c \\ 2(1+B)(y-c) & \text{if } c < y < 1. \end{cases} \tag{2.5}$$

Thus any strategy of II is a best response to I's given strategy if it folds hands $y < a$ and calls hands $y > c$. II's given strategy, in the case that I bets, satisfies this and so is a best response.

Suppose then that II receives the hand y and I checks. If II bets, her expected payoff is

$$w_{\mathrm{b}}(y) = \begin{cases} 2(e-a) & \text{if } 0 < y < b \\ 2(e-a) + 2(1+B)(y-b) & \text{if } b < y < c \\ 2(e-a) + 2(1+B)(c-b) & \text{if } c < y < 1. \end{cases} \tag{2.6}$$

If II checks, her expected payoff is

$$w_{\mathrm{ch}}(y) = \begin{cases} 0 & \text{if } 0 < y < a \\ 2(y-a) & \text{if } a < y < c \\ 2(c-a) & \text{if } c < y < 1. \end{cases} \tag{2.7}$$

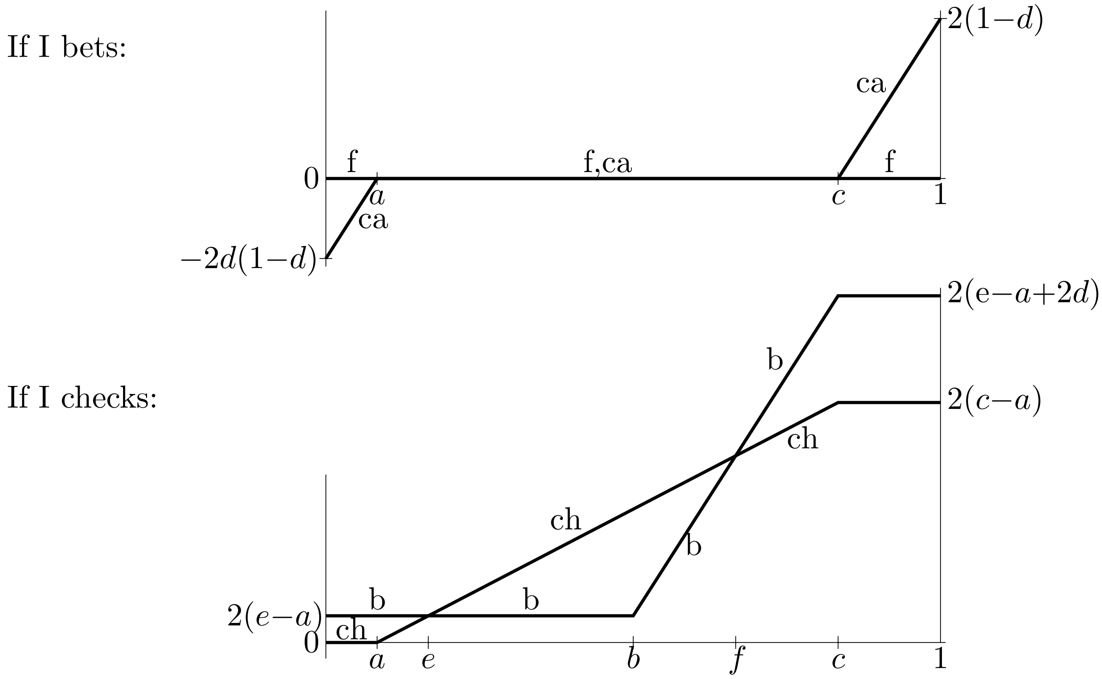


Fig. 2.3. II's payoffs against I's strategy.

II's optimal response here is essentially unique. She must bet for $y < e$, check for $e < y < f$, and bet for $y > f$. This is II's given strategy in the case that I checks. (Figure 2.3.)

2.4 The Value of the Game. The value to Player I of this game may be found as the expectation of the maximum of (2.2), (2.3) and (2.4) when x has the uniform(0,1) distribution, minus the original ante. This is the sum of the areas of two triangles in Figure 2.2 minus the original ante, which is 1; namely,

$$V_I = \frac{1}{2}2d(d-e) + \frac{1}{2}(1-d)(2+2d) - 1 = -de = \frac{-B^2}{(1+B)(2+B)^2}. \tag{2.8}$$

For the pot-limit game, $V_I = -1/12$. For all $B > 0$, the game favors Player II. As a function of B on $[0,\infty)$, this is zero at $B = 0$, decreases to a minimum at $B = 1+\sqrt{5}$, and increases to zero as $B \to \infty$.

2.5 Mistakes. We say that a pure strategy for Player I is a mistake if there exists a minimax strategy for Player II that gives Player I an expected return strictly less than the value of the game. We define similarly a mistake for Player II. From Figures 2.2 and 2.3, we can point out some mistakes a player can make.

Player I makes a mistake if he check-calls with an $x < e$; or if he bets with $e < x < f$; or if he check-folds with $f < x < 1$.

When Player I bets, Player II makes a mistake if she calls with $y < a$; or if she folds with $y > c$.

When Player I checks, Player II makes a mistake if she checks with $y < e$; or if she bets with $e < y < f$; or if she checks with $y > f$.

2.6 Another Optimal Strategy for Player I. Player I's optimal strategy is not unique. Others may be found by moving the region of betting with high hands down a bit. One such new strategy is: bet if $0 < x < a$, check-fold if $a < x < b$, check-call if $b < x < f$, bet if $f < x < c$ and check-call if $c < x < 1$.

Player I:

bet check-fold check-call bet check-call

0 a b f c 1

From Figure 2.2, this strategy is a best response to Player II's optimal strategy. To check that it is indeed optimal, it suffices to check that II's optimal strategy is a best response to it. We must recompute the formulas that led to Figure 2.3. Against this conjectured optimal strategy of Player I, we have

$$w_{\mathrm{f}}(y) = 0 \qquad \text{for all } y$$

$$w_{\mathrm{ca}}(y) = \begin{cases} 2(1+B)(y-a) & \text{if } 0 < y < a \\ 0 & \text{if } a < y < f \\ 2(1+B)(y-f) & \text{if } f < y < c \\ 2(1+B)(c-f) & \text{if } c < y < 1 \end{cases}$$

$$w_{\mathrm{b}}(y) = \begin{cases} 2(e-a) & \text{if } 0 < y < b \\ 2(e-a) + 2(1+B)(y-b) & \text{if } b < y < f \\ 2(e-a) + 2(1+B)(f-b) & \text{if } f < y < c \\ 2(e-a) + 2(1+B)(y-f) & \text{if } c < y < 1 \end{cases}$$

$$w_{\mathrm{ch}}(y) = \begin{cases} 0 & \text{if } 0 < y < a \\ 2(y-a) & \text{if } a < y < f \\ 2(f-a) & \text{if } f < y < c \\ 2(f-a+y-c) & \text{if } c < y < 1. \end{cases}$$

Only slight changes are required in Figure 2.3. These are produced in Figure 2.4.

Player II's optimal strategy achieves the largest of these returns and so Player I's new strategy is optimal as claimed. Should Player I have a secondary preference for one of these two strategies? Admissibility doesn't help us answer this question since both strategies are admissible. The new strategy of Player I is not as good as the original strategy at taking advantage of the mistake of Player II may make by checking I's check with $y > f$, but it is better at taking advantage of the mistake Player II may make by folding Player I's bet with $y > f$. However, the former mistake is more likely than the latter since Player II's optimal strategy has f as the cutoff point between checking and betting. In other words, Player II has to be only a little too timid to make the first error, while she must be much too timid, if not downright pessimistic, to make the second type of error. Thus, the original optimal strategy for Player I has a secondary advantage in a practical sense.

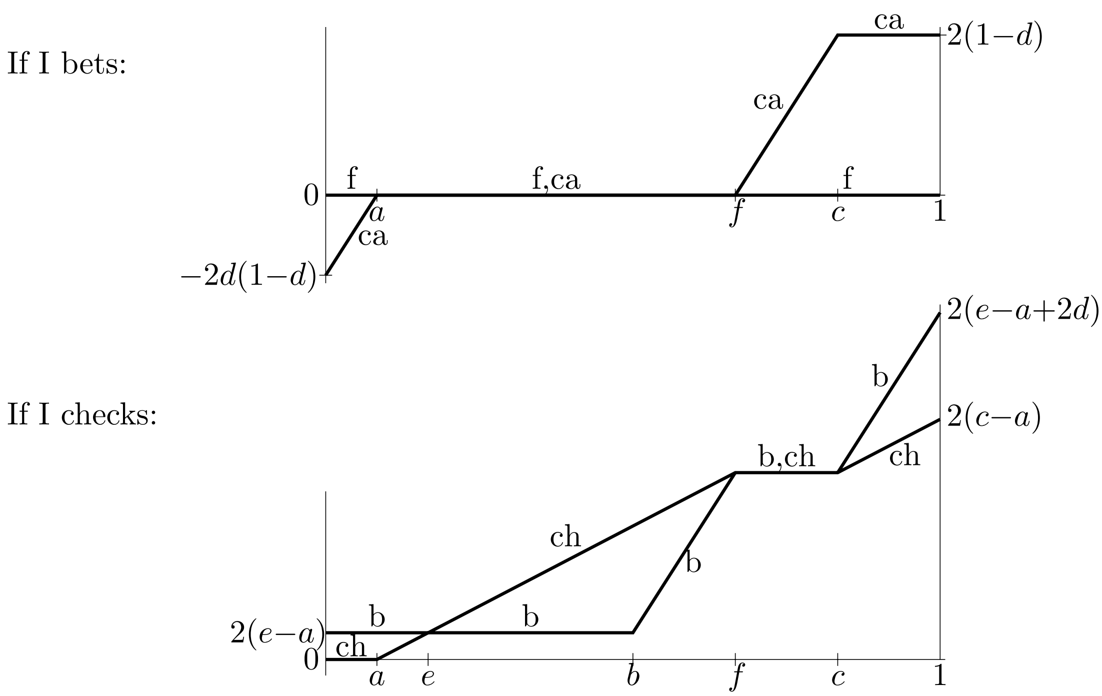


Fig. 2.4. II's payoffs against I's modified strategy.

Further insight into the difference between these two optimal strategies may be obtained by expanding the model to give the players different sizes of bets. Let B_1 denote the size of Player I's bet, and B_2 the size of Player II's bet. It may be shown by methods used above that if $B_2 < B_1$ then Player I's optimal strategy is of the form found in Section 2.2, while if $B_1 < B_2$, then Player I's optimal strategy is of the form found in this section.

Section 3. The von Neumann model allowing Player II to raise.

As in the von Neumann model, the two players each ante 1 unit into the pot, and receive independent uniform(0,1) hands. Player I acts first either checking or by betting $B > 0$. If Player I checks, the hands are immediately compared and the higher hand wins the pot. However, if Player I bets, Player II now has three choices. Player II may fold, in which case Player I wins the pot. Player II may call by putting the amount B into the pot, after which the hands are compared and the higher hand wins the pot. Or Player II may raise by putting the amount $B + R$ into the pot, where $R > 0$ represents the size of the raise. If Player II raises, Player I must decide whether to call or fold. Figure 3.1 shows the betting tree for this model.

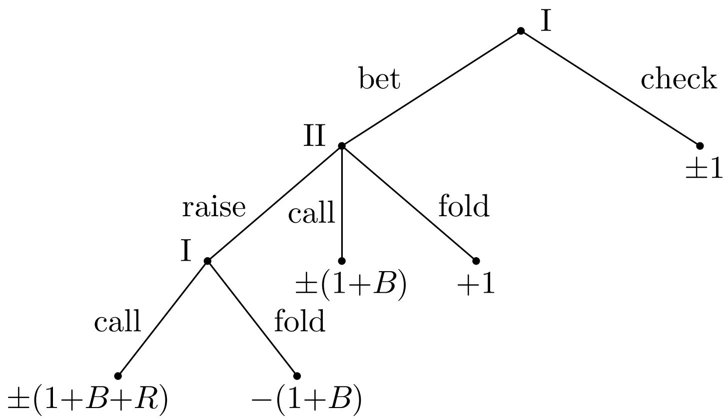
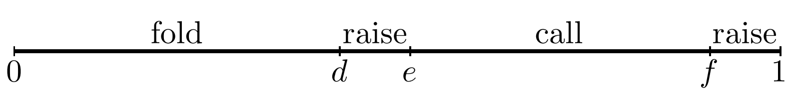


Fig. 3.1. The Betting Tree.

3.1 Conjectured Optimal Strategies. For each $x \in (0, 1)$, Player I has three choices; he may check, he may bet-fold (i.e. bet and fold a raise by Player II), or he may bet-call (i.e. bet and call a raise by Player II). We claim there is an optimal strategy for Player I that divides the interval (0,1) into four regions. If x is in the lowest region, say $(0, a)$, Player I will bet-fold. In the next region, say (a, b), Player I will check. In the third region, say (b, c), Player I will bet-fold, and in the last region, $(c, 1)$, Player I will bet-call. We have $0 < a < b < c < 1$.

Player I:

Player II if Player I bets: (3.1)



If Player I bets, Player II has three possible actions for each $y \in (0, 1)$; she may fold, call or raise. We claim there is an optimal strategy for II that divides the interval (0,1) into four regions. In the lowest region, say $(0, d)$, Player II will fold. In the next region, say (d, e), Player II will raise as a bluff. In the third region, say (e, f), Player II will call, and in the last region, $(f, 1)$, Player II will raise for value. We have $0 < d < e < f < 1$.

3.2 The Indifference Equations. In deriving the indifference equations, we assume

$$0 < a < e < b < c < f < 1 \qquad \text{and} \qquad 0 < d < e. \tag{3.2}$$

(We allow d to be larger or smaller than a.)

In the analysis, we assume that the pot belongs to neither player. This makes the game constant-sum with sum 2. But it also makes the derivation of the equations somewhat simpler. We denote the bet size by $B > 0$ and the raise size by $R > 0$.

As in Section 2.2, we may find the indifference equations as follows. For Player I to be indifferent at a: If I checks at a, he wins 2 with probability a. If I bet-folds at a, he wins 2 with probability d and loses B with probability $(1-d)$. Equating these two expectations, we find $2a = 2d - B(1-d)$. We may find all six equations in a similar manner:

1. For Player I to be indifferent at a: $-2a - (2+B)d = B$.
2. For Player I to be indifferent at b: $2Bb + (2+B)d - 2(1+B)e = B$.
3. For Player I to be indifferent at c: $-(2+2B+R)d + (2+2B+R)e + Rf = R$.
4. For Player II to be indifferent at d: $(2+B)a - (2+B)b + (2+2B+R)c = B+R$.
5. For Player II to be indifferent at e: $-2(1+B)b + (2+2B+R)c = R$.
6. For Player II to be indifferent at f: $f = (1+c)/2$.

These six linear equations in six unknowns have a unique solution, namely

$$\begin{aligned}
a &= \frac{B^2(2+2B+R)^2}{(1+B)\Delta} \\
b &= 1 - \frac{(2+B)}{B}a \\
c &= 1 - \frac{2B(2+B)(2+2B+R)}{\Delta} \\
d &= \frac{B}{2+B} + \frac{2}{2+B}a \\
e &= \frac{B}{1+B} - a \\
f &= 1 - \frac{B(2+B)(2+2B+R)}{\Delta}
\end{aligned} \tag{3.3}$$

where

$$\Delta = B(4+B)(2+2B+R)^2 + (1+B)(2+B)^2 R. \tag{3.4}$$

It is easy to check that the presumed inequalities (3.2) hold for these values.

For the pot limit case, we have $B = 2$ and $R = 6$. This leads to

$$\begin{aligned} a &= \;\; 2/21 = .095\ldots \\ b &= 17/21 = .810\ldots \\ c &= 19/21 = .905\ldots \\ d &= 23/42 = .547\ldots \\ e &= 12/21 = .571\ldots \\ f &= 20/21 = .952\ldots \end{aligned}$$

It is interesting to note that Player I checks about 70% of the time. Of Player I's bets, one-third are bluffs (those below a). Of Player II's raises, one-third are bluffs (those between d and e).

3.3 Proof of Optimality. First suppose Player II uses the conjectured optimal strategy, and suppose that Player I receives hand x. Then, if Player I checks, his expected return is

$$v_{\rm c}(x) = 2x \qquad \text{for all } 0 < x < 1.$$

If I bet-folds, his expected return is

$$v_{\rm bf}(x) = \begin{cases} 2a & \text{if } 0 < x < e \\ 2a + 2(1+B)(x-e) & \text{if } e < x < f \\ 2a + 2(1+B)(f-e) & \text{if } f < x < 1. \end{cases}$$

If I bet-calls, his expected return is

$$v_{\rm bc}(x) = \begin{cases} 2a - 2(1+B+R)(e-d) & \text{if } 0 < x < d \\ 2a - 2(1+B+R)(e-x) & \text{if } d < x < e \\ 2a + 2(1+B)(x-e) & \text{if } e < x < f \\ 2a + 2(1+B)(f-e) + 2(1+B+R)(x-f) & \text{if } f < x < 1. \end{cases}$$

To see these computations, first look at $v_{\rm bf}(x)$ for $0 < x < e$. The actual return is $2d - B(1-d) = (2+B)d - B$ which is equal to $2a$ from the formula for d in (3.3). Similarly, $v_{\rm bc}(x)$ at the point $x = e$ may be computed using $2d + (2+B+R)(e-d) - B(f-e) - (B+R)(1-f) = -(B+R)d + (2+2B+R)e + Rf - B - R$, which from the third indifference equation is equal to $-(B+R) + (2+2B+R)d - B = (2+B)d - B = 2a$. The rest is computed by noting that the functions are continuous piecewise linear and finding the slopes.

Suppose now that Player I uses his conjectured optimal strategy, and suppose that Player II receives hand y. Then, if Player II folds (i.e. uses the fold strategy), her expected return is

$$w_{\rm f}(y) = \begin{cases} 0 & \text{if } 0 < y < a \\ 2(y-a) & \text{if } a < y < b \\ 2(b-a) & \text{if } b < y < 1. \end{cases}$$

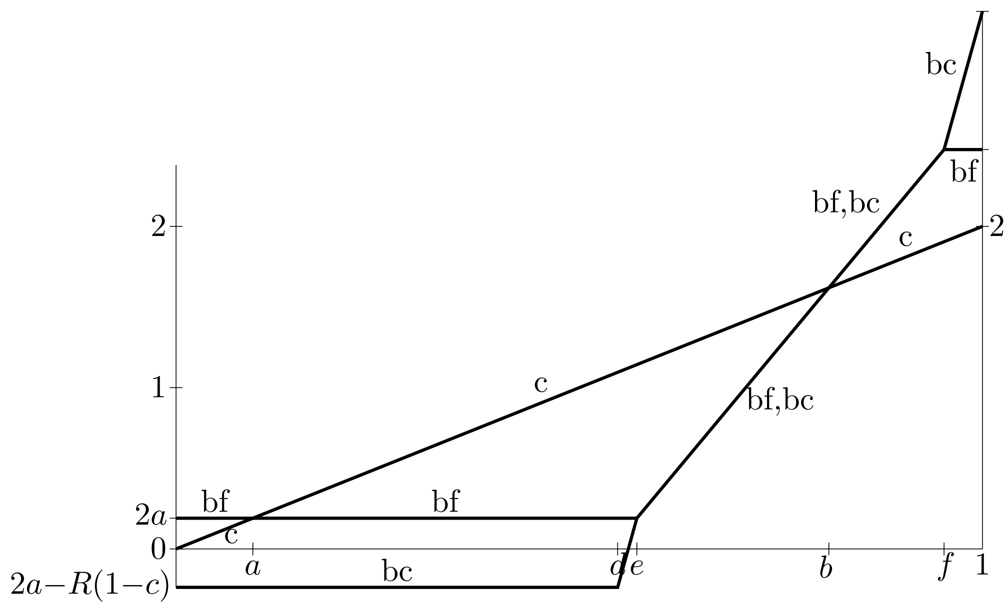


Fig. 3.2. I's payoff against II's strategy.

If II calls, her expected return is

$$w_{\rm c}(x)=\begin{cases}-2(1+B)(a-y) & \text{if } 0<y<a\\ 2(y-a) & \text{if } a<y<b\\ 2(b-a)+2(1+B)(y-b) & \text{if } b<y<1.\end{cases}$$

If II raises, her expected return is

$$w_{\rm r}(y)=\begin{cases}0 & \text{if } 0<y<a\\ 2(y-a) & \text{if } a<y<b\\ 2(b-a) & \text{if } b<y<c\\ 2(b-a)+2(1+B+R)(y-c) & \text{if } c<y<1.\end{cases}$$

In these computations, indifference equation 4 is used to show that $w_{\rm r}(y)=0$ for $0<y<a$, and the formula for b in (3.3) is used to show $w_{\rm c}(a)=0$.

3.4 The Value of the Game. As in Section 2.4, we may compute the value of the game as the area under the upper envelope of the payoffs in Figure 3.2, minus the original ante which is 1. Though the computation is somewhat more difficult, the answer is surprisingly simple,

$$V_I(B,R)=a=\frac{B^2(2+2B+R)^2}{(1+B)\Delta}.$$

This is always positive so the game is in favor of Player I.

We may also find the optimal values of the bet and the raise for this model, assuming that Player I first chooses the size of the bet, B, and Player II then chooses the size of the raise, R.

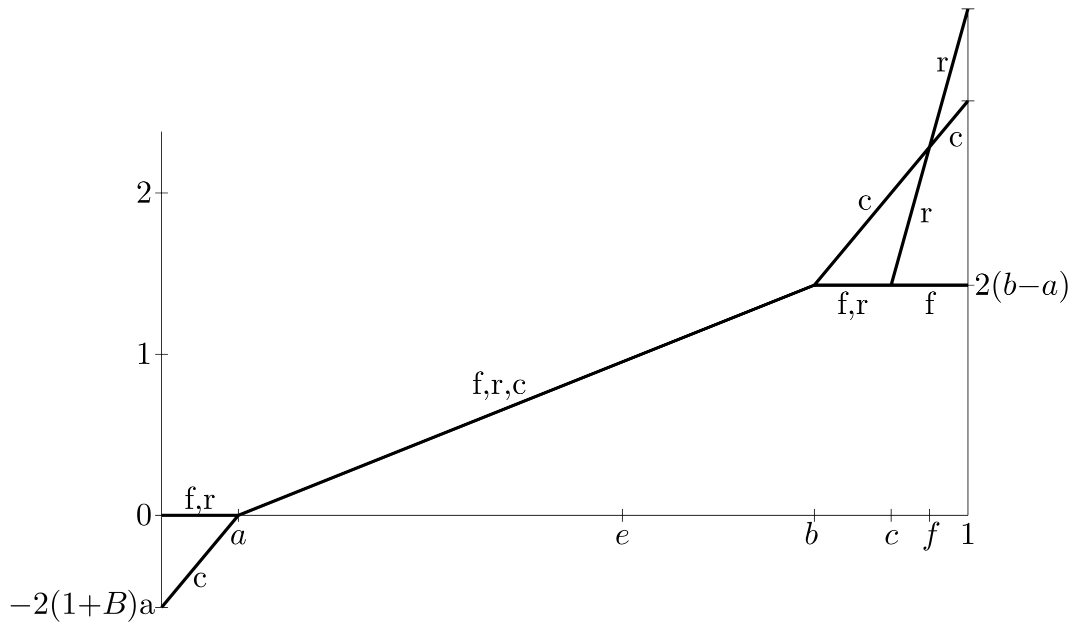


Fig. 3.3. II's payoff against I's strategy.

Player II will choose $R \geq 0$ to minimize $V_I(B,R)$, where Δ is given by (3.4). Upon dividing numerator and denominator by $(2+2B+R)^2$, we see this is equivalent to choosing R to minimize $(2+2B+R)^2/R$. Setting the derivative to zero and solving give the unique inflection point $R = 2+2B$, which is easily seen to be a minimum. This means that the optimal size of the raise is always the size of the pot!

Now Player I will choose B to maximize

$$V_I(B, 2+2B) = \frac{8B^2}{(1+B)(4+36B+9B^2)}.$$

Setting the deravitive of this to zero gives a unique inflection point on the positive axis at the root of $9B^2 - 18B - 4 = 0$. This occurs at $B = 1 + \sqrt{13/9} = 2.20185\cdots$, and is clearly a maximum. Thus the optimal bet size is slightly bigger than the size of the pot.

3.5 Other Optimal Strategies for Player II. The optimal strategy for Player I is unique. However, Player II has many optimal strategies that differ only in what she does with hands $y \in (0,e)$. In this region, she may fold or raise arbitrarily provided the total probability of raising there is $e-d$. For example, it is optimal to raise with hands $y \in (0, e-d)$ and fold with hands in $(e-d, e)$. It is also optimal to raise with probability $(e-d)/e$ and to fold with probability d/e for all hands $y \in (0,e)$.

Though all of these strategies are optimal, only one takes maximal advantage of a mistake Player I may make by bet-calling for some $x \in (0,e)$ or bet-folding for some $x \in (a,e)$. That is the strategy suggested above in (3.1). In other words, the only admissible optimal strategy for Player II is the strategy given in (3.1).

Section 4. Two Rounds of Betting With Sandbagging.

In this section, we allow either player to initiate betting, followed by a possible raise by the other player. For simplicity, we restrict attention to pot limit poker. Since the analysis follows the same lines as in the case of one round of betting found in Section 2, we omit many of the details. The betting tree is as follows.

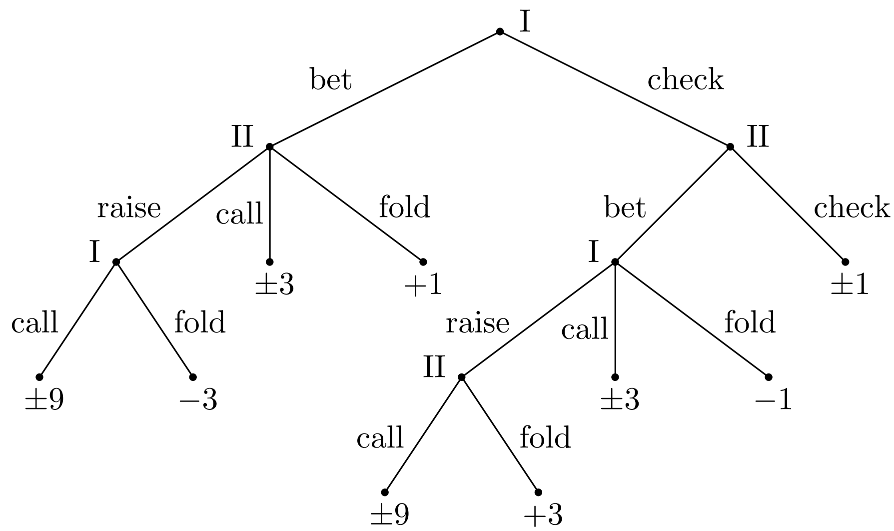


Fig. 4.1. The Betting Tree.

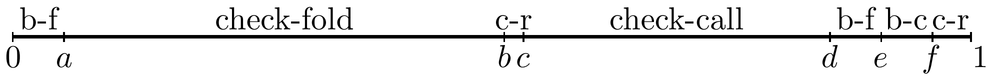
4.1 The Strategies. Player I now has five choices for each x: bet-fold, bet-call, check-fold, check-call and check-raise. We claim there there are now seven regions in an optimal policy for Player I. The lowest region, $(0, a)$ is a bluffing region, bet-fold. In the next region, (a, b), Player I will check-fold. In the next region, (b, c), he will check-raise as a sandbagging bluff. In the next region, (c, d), he will check-call. In the next region, (d, e), he will bet-fold. In the next region, (e, f), he will bet-call. In the last region, $(f, 1)$, he will check-raise as a sandbag. We assume $0 < a < b < c < d < e < f < 1$.

If Player I checks, Player II has three strategies for each y: check, bet-fold and bet-call. There are four regions in an optimal strategy. In the lowest region, $(0, g)$, Player II will bet-fold as a bluff. In the next region, (g, h), she will check. In the next region, (h, j), she will bet-fold. In the last region, $(j, 1)$, she will bet-call. It is assumed that $0 < g < h < j < 1$.

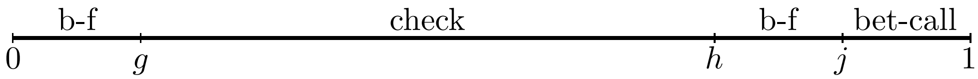
If Player I bets, Player II has three strategies, and there are four regions in an optimal strategy. In the first region, $(0, k)$, she will fold. In the next region, (k, m), she will raise as a bluff. In the next region, (m, n), she will call. In the last region, $(n, 1)$, she will raise. It is assumed that $0 < k < m < n < 1$.

We also assume $a < g < b < c < h < d < j < f$ and $a < k < m < d < e < n < f$ to find the indifference equations.

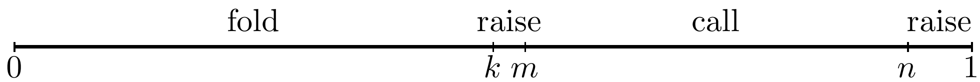
Player I:



Player II if Player I checks:



Player II if Player I bets:



(4.1)

4.2 The Indifference Equations. We take the initial pot size to be 2. Since the game is pot limit, the bet size is 2 and the raise size is 6. There are now 12 unknowns.

Player II makes Player I indifferent to:
1. bet-fold and check-fold at a: $2k = 1$.
2. check-fold and check-raise at b: $g - h + 3j = 2$.
3. check-raise and check-call at c: $-h + 2j = 1$.
4. check-call and bet-fold at d: $g - h - 2k + 3m = 0$.
5. bet-fold and bet-call at e: $-2k + 2m + n = 1$.
6. bet-call and check-raise at f: $g - h - 3j + 4k - 3m + 3n = 0$.

Player I makes Player II indifferent to:

7. bet-fold and check at g after a check: $2b - d + f - g = 1$.
8. check and bet-fold at h after a check: $2b - 3c - d + f + 2h = 1$.
9. bet-fold and bet-call at j after a check: $-2b + 2c + f = 1$.
10. fold and raise at k after a bet: $a - d + 3e - 2f = 0$.
11. raise and call at m after a bet: $d - 2e + f = 0$.
12. call and raise at n after a bet: $e + f - 2n = 0$.

The solution of these 12 linear equations in 12 unknowns is as follows. All numbers are to be divided by 150:

$$a = 8, \quad g = 20, \quad b = 77, \quad c = 80, \quad h = 110, \quad d = 128, \quad j = 130, \quad f = 144$$
$$k = 75, \quad m = 80, \quad e = 136, \quad n = 140.$$

Note that the assumed inequalities are satisfied for these values. The placement of these values on the lines in (4.1) is taken according to these numbers so that one can see what the optimal strategies are like.

4.3 Proof of Optimality. The following are the expected payoffs to Player I for each of his five pure strategies, when he has x and Player II uses the given strategy.

If I check-folds or check-calls,

$$V_{\text{cf}}(x) = \begin{cases} 0 & \text{if } 0 < x < g \\ 2(x-g) & \text{if } g < x < h \\ (6/5) & \text{if } h < x < 1 \end{cases} \qquad V_{\text{cc}}(x) = \begin{cases} 6x - (4/5) & \text{if } 0 < x < g \\ 2(x-g) & \text{if } g < x < h \\ 6(x-h) + (6/5) & \text{if } h < x < 1 \end{cases}$$

If I check-raises or bet-folds,

$$V_{\mathrm{cr}}(x) = \begin{cases} 0 & \text{if } 0 < x < g \\ 2(x-g) & \text{if } g < x < h \\ 6/5 & \text{if } h < x < j \\ 18(x-j) + (6/5) & \text{if } j < x < 1 \end{cases} \qquad V_{\mathrm{bf}}(x) = \begin{cases} 0 & \text{if } 0 < x < m \\ 6(x-m) & \text{if } m < x < n \\ 12/5 & \text{if } n < x < 1 \end{cases}$$

If I bet-calls,

$$V_{\mathrm{bc}}(x) = \begin{cases} -3/5 & \text{if } 0 < x < k \\ 18(x-k) - (3/5) & \text{if } k < x < m \\ 6(x-m) & \text{if } m < x < n \\ 18(x-n) + (12/5) & \text{if } n < x < 1 \end{cases}$$

Player I's given strategy achieves the maximum of the five functions at each x. (See Figure 4.2.)

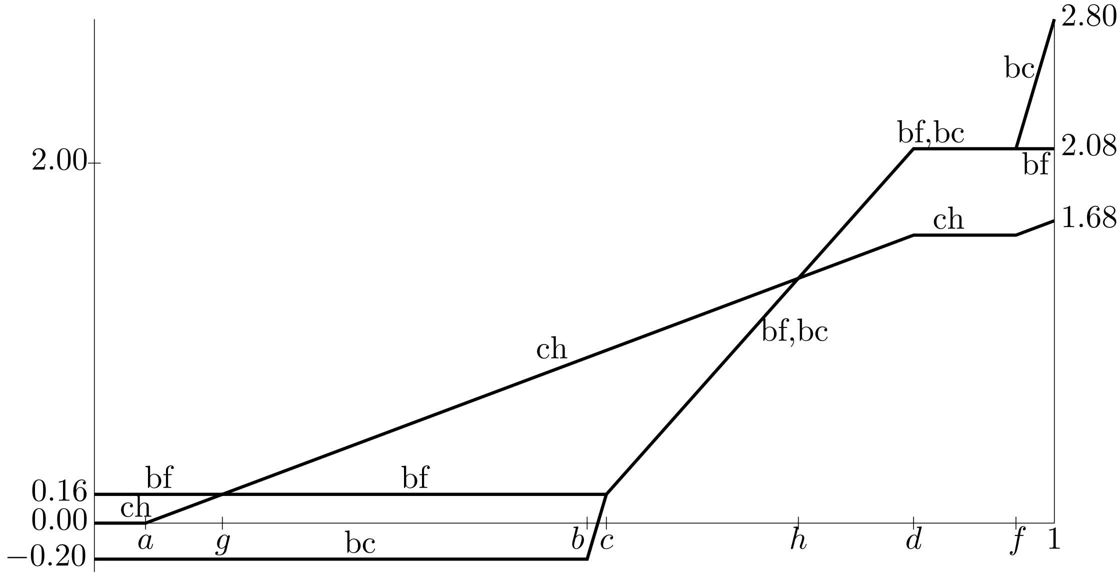


Fig. 4.2. I's payoffs against II's strategy.

The expected payoffs to Player II for each of her three pure strategies when she has y and Player I checks using the given strategy are as follows. The payoffs are piecewise linear.

If II checks,

$$W_{\mathrm{ch}}(y) = \frac{1}{25} \begin{cases} 0 & \text{if } 0 < y < a \\ 0 \text{ to } 40 & \text{if } a < y < d \\ 40 & \text{if } d < y < f \\ 40 \text{ to } 42 & \text{if } f < y < 1 \end{cases}$$

If II bet-folds,

$$W_{\mathrm{bf}}(y) = \frac{1}{25} \begin{cases} 4 & \text{if } 0 < y < c \\ 4 \text{ to } 52 & \text{if } c < y < d \\ 52 & \text{if } d < y < 1 \end{cases}$$

If II bet-calls,

$$W_{\mathrm{bc}}(y) = \frac{1}{25}\begin{cases} -5 & \text{if } 0 < y < b \\ -5 \text{ to } 4 & \text{if } b < y < c \\ 4 \text{ to } 52 & \text{if } c < y < d \\ 52 & \text{if } d < y < f \\ 52 \text{ to } 70 & \text{if } f < y < 1 \end{cases}$$

Player II's given strategy, when I checks, achieves the maximum of the three functions at each y. (See Figure 4.3.)

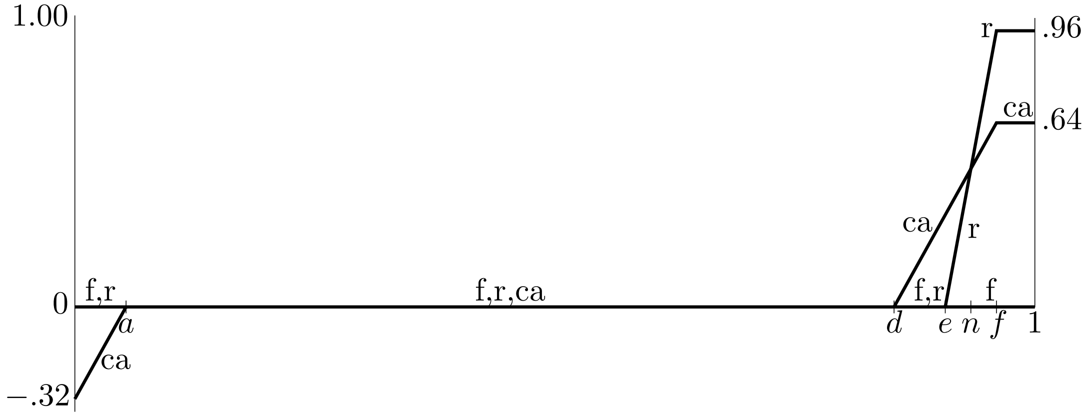


Fig. 4.3. II's payoffs against I's strategy when I checks.

The expected payoffs to Player II for each of his three pure strategies when she has y and Player I bets using the given strategy are as follows.

If II folds,

$$W_{\mathrm{f}}(y) = 0 \qquad \text{for all } y.$$

If II calls,

$$W_{\mathrm{ca}} = \frac{1}{25}\begin{cases} -8 \text{ to } 0 & \text{if } 0 < y < a \\ 0 & \text{if } a < y < d \\ 0 \text{ to } 16 & \text{if } d < y < f \\ 16 & \text{if } f < y < 1 \end{cases}$$

If II raises,

$$W_{\mathrm{r}} = \frac{1}{25}\begin{cases} 0 & \text{if } 0 < y < e \\ 0 \text{ to } 24 & \text{if } e < y < f \\ 24 & \text{if } f < y < 1 \end{cases}$$

Player II's given strategy, when I bets, achieves the maximum of the three functions at each y. (See Figure 4.4.) It is interesting to note that when Player I bets while using his optimal strategy, then Player II never needs to fold, it being at least as good to raise.

4.4 The Value of the Game. The value to Player I of this game may be found as the area under the upper envelope of the payoffs in Figure 4.2, minus 1. This is $V_I = -2/25$.

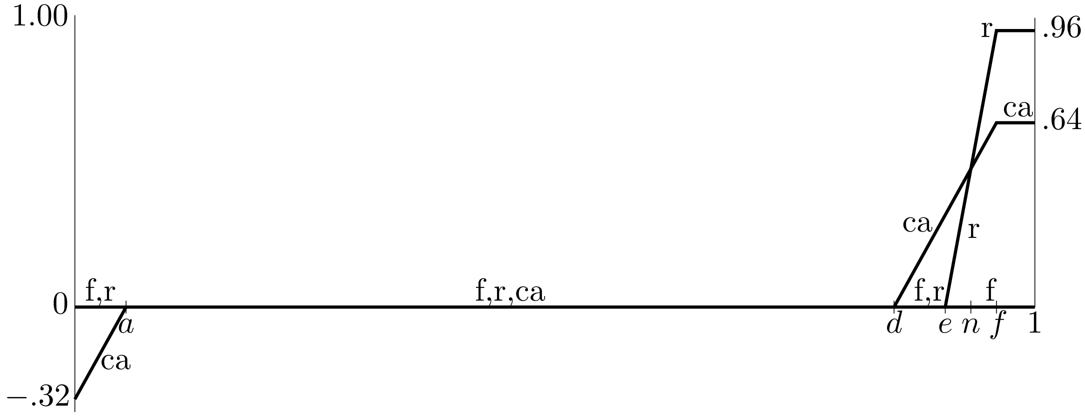


Fig. 4.4. II's payoffs against I's strategy when I bets.

The value for Player II is the negative of this. We see that this game is in favor of Player II.

4.5 Limit Games. The methods of this section can also be used to treat Limit Games. In such games, the size of the bet or raise is held fix throughout a round of betting. Often in the last round of betting, the pot size is large compared to the bet size. It is of interest to see how the optimal strategies change in our model if $R = B$ and if B is small compared to the initial pot size, which above has been taken to be 2.

We consider three cases: the bet size is equal to the initial pot size, the bet size is 1/3 the initial pot size, and the bet size is 1/6 the initial pot size. It turns out that the form of the optimal strategies is the same as given in (4.1); only the values of the twelve parameters change. These numerical values are given in the following table. The values for the pot-limit case are given for comparison.

	Initial pot size			Pot
	B	$3B$	$6B$	limit
a	.059	.069	.051	.053
b	.511	.269	.160	.513
c	.526	.283	.170	.533
d	.821	.644	.555	.853
e	.850	.690	.595	.906
f	.939	.919	.910	.960
g	.140	.117	.080	.133
h	.719	.530	.443	.733
j	.790	.609	.505	.867
k	.500	.250	.143	.500
m	.526	.283	.170	.533
n	.895	.804	.752	.933

One sees, as expected, that in limit poker where the pot size is relatively large compared to the bet size, one should call more often and fold less often. It is interesting to note that the optimal strategy for limit poker with bet equal to the initial pot size is not far from the optimal strategy for pot-limit poker, the biggest difference being the change in the value of j. The bet-call region for Player II is slightly bigger at the expense of the bet-fold region below it.

However, when the bet size is 1/6 the initial pot size, the optimal strategies become strikingly different. Player I bets about 40% of the time, whereas in pot limit Player I bets only 16% of the time. Moreover, the check-fold region shrinks to about 11% from the original 46% for pot limit poker.

When the bet size is 1/6 the initial pot size and Player I checks, Player II checks only about 36% of the time, compared to 60% of the time for pot limit poker, and bet-calls about 50% of the time compared to 13% of the time for pot limit. In the same case when Player I bets, Player II folds only 14% of the time, compared to 50% for pot limit, and raises-for-value 25% of the time compared to 7% for pot limit.

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