A TEXT ON TRIGONOMETRY BY LEVI BEN GERSON (1288—1344)

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divided into months of either 29.5 or 30 days each.[12]

Therefore, based on the previous discussion, Nordenskiöld concluded that the quipus used by the Incas and examined by him contained: (1) the number 7 combined in various forms and often represented, (2) numbers that expressed days, and (3) astronomical numbers.

In conclusion, one should fully be aware of the fact that the secrets of the Peruvian quipus have not yet been completely unlocked. This is clearly demonstrated when we note that Nordenskiöld's study occurred in 1925, and little progress has been made since then. In summation, we do know that the Incas counted in groups of ten, used a decimal placement system, and figured on the abacus. The results of these computations were then transferred to the quipu by the Quipucamayoc, who was a specialist in his field. The purposes of this mnemonic device were varied: to record historical events, to keep statistical records, to aid in the confessions of Incan converts, to record magic numbers, and perhaps even to record astronomical data.

Much additional research needs to be undertaken. The possibilities of further discovery are still present for those who are intrigued by the mysteries that are tied into the quipu.

[12] Nordenskiöld, *Calculations with Years and Months in the Peruvian Quipus* (Sweden: Comparative Ethnographical Studies, 1925), II, Part 2, 5–34.

ADDITIONAL REFERENCES

BUSHNELL, G. H. S. *Peru.* New York: Frederick A. Praeger, 1960, pp. 125–28.

CIEZA DE LEON, PEDRO DE. *The Incas.* Norman, Okla.: University of Oklahoma Press, 1959, pp. 39–40, 105, 163, 172–75, 177, 187.

PRESCOTT, WILLIAM H. *History of the Conquest of Peru.* Philadelphia: J. B. Lippincott Co., 1847, pp. 122–26.

A TEXT ON TRIGONOMETRY BY LEVI BEN GERSON (1288–1344)

By Pamela H. Espenshade, Princeton Day School, Princeton, New Jersey

LEVI ben Gerson (1288–1344), a prominent Hebrew writer who lived in Provence, is most noted for his religious and philosophical writings. Nonetheless, his astronomical works, although largely unstudied, are quite imposing and voluminous. His chief astronomical work, contained in *Wars of the Lord,* Book V, Part 1 (136 chapters), was translated into Latin by the Augustinian hermit Peter of Alexandria in 1342 and dedicated to Pope Clement VI. Since, however, Clement VI was elected May 7, 1342, it is probable that the work was undertaken for Benedict XIII (1334–1342).[1] The Latin version of the table of contents was published by Renan in 1843.[2]

A section of these astronomical writings was circulated separately under the title of *De Sinibus, Chordis et Arcubus, item*

[1] L. Thorndike, *A History of Magic and Experimental Science,* pp. 309–11.

[2] E. Renan, "Les écrivains juifs français du XIV siècle," pp. 633 ff.

Instrumento Revelatore Secretorum. Curtze discussed this text and presented portions of it in 1898 and 1900,[3] and Braunmühl commented upon Curtze's presentation.[4] *De Sinibus, Chordis et Arcubus, item Instrumento Revelatore Secretorum* includes an introduction to trigonometry and a description of the Jacob staff as an astronomical instrument.[5] Levi ben Gerson introduces the sine and versine functions, but does not mention the tangent function. After defining his terms he develops the fundamental theorems for deriving a sine table arranged at intervals of 15′. His derivation of the sin $\frac{1}{4}°$ follows closely that of Ptolemy's for chord 1°,[6] and his result is

$$\sin \tfrac{1}{4}° = 0;0,15,42,28,32,7.^{7}$$

The text which follows is a translation of Chapter 2, Sections 1–4, of *De Sinibus, Chordis et Arcubus, item Instrumento Revelatore Secretorum* as excerpted by Curtze.[8] This portion corresponds to *Wars of the Lord*, Book V, Part 1, Chapter 4, Sections 1–4.[9] The fifth section containing the sine table, as described at the end of the fourth section, appears in MS Vatican Latin, 3098. Where Curtze's text presented difficulties, I consulted two Latin manuscripts of *Wars of the Lord* (MSS Vatican Latin, 3098 and 3380). Moreover, Dr. Bernard R. Goldstein of Yale University kindly translated for me parallel passages from MS Paris Hebrew, 724. Following the translation, I present some comments on the mathematical content of the treatise.

Translation of the Latin Text

(Second Chapter)

Section 1

Astronomers are accustomed to divide each sphere or circle into 360 parts, which they name degrees. Moreover, they divide any degree into 60 parts, and they call these minutes. Also, they divide a minute into 60 seconds, each second into 60 thirds, each third into 60 fourths, and so on ad infinitum. They also divide the zodiac into 12 parts, which they name signs. Each sign is divided into 30 parts, which they call degrees, which are mentioned above, and by this measurement the twelve signs also embrace 360°. The diameter of a sphere is divided into 120 degrees, though a single degree on the diameter is not equal to a single degree on the circumference.

A part of the circumference of a circle is called an arc. A straight line which subtends an arc is called a chord. The mean of a chord of an arc doubled is called the sine of the arc. The mean of the chord is the mean of the arc which the chord subtends. Also, the straight line passing from the midpoint of the chord to the midpoint of the arc is named the versine (versed sine). This is called the versine of the mean of the arc.

The motion which Ptolemy calls epicyclic motion we call motion of anomaly. The place which Ptolemy names apogee (*augem*) we call perigee [?] (*inaugem*).[10]

Section 2

It is well known that the chord of any arc in the circumference of a circle is also the chord of the remainder. Likewise, it is known that the sine of one arc is the sine of the arc remaining in an arc of 180°. Moreover Euclid instructs that the ver-

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[3] M. Curtze, "Die Abhandlung des Levi ben Gerson über Trigonometrie und den Jacobstab," *Bibliotheca Mathematica*, Neue Folge, XII, 97–112, and "Urkunden zur Geschichte der Trigonometrie im christlichen Mittelalter," *Bibliotheca Mathematica*, Dritte Folge, I, 321–416.

[4] A. Von Braunmühl, *Vorlesungen über Geschichte der Trigonometrie*, Erster Teil, pp. 103–7.

[5] E. G. R. Taylor, "Cartography, Survey, and Navigation, 1400–1750," *A History of Technology*, Singer *et al.*, III, 530–57.

[6] Cf. A. Aaboe, *Episodes from the Early History of Mathematics*, pp. 112–21.

[7] I follow here the notation introduced by O. Neugebauer in *The Exact Sciences in Antiquity*, p. 13, note 1.

[8] Curtze, "Urkunden zur Geschichte der Trigonometrie im christlichen Mittelalter," *Bibliotheca Mathematica*, Dritte Folge. I, 372–80.

[9] Renan, *op. cit.*

[10] MS P. Hebrew, 724, fol. 8a reads: "The place which Ptolemy names apogee we shall also call apogee, though the word does not have the same meaning, for we do not set the apogee further from the earth than the other points on the sphere."

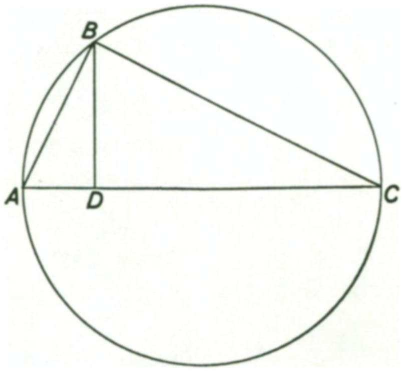

FIGURE 1

sine always forms a right angle with the sine. And if the versine is extended to the circumference, it passes through the center.

With these propositions I wish now to demonstrate the following:

First Theorem: Each chord of a minor arc in a semicircle is equal to the square root of the product of the versine of that arc and the entire diameter.

Let ABC be a semicircle above the diameter AC (see Fig. 1). In this semicircle AB is a minor arc whose chord is AB. I say that the square of the line AB is equal to the product of line AD and line AC. For our demonstration of this let the straight line BC be drawn. I say that angle ABC and angle ADB are equal because both are right angles. Angle A is held in common by triangle ABC and triangle ADB. Therefore angle C in the larger triangle is equal to angle B in the smaller. Thus there exists a proportion of line AD, subtending angle ABD, to line AB, subtending the right angle ADB, and such also is the proportion of line AB, subtending angle C, to the line AC, subtending the right angle ABC. Thus the product of the first part with the fourth is equal to the product of the second and third. Therefore the product of AB and itself is equal to the product of AD and AC, which is what I wished to prove.

From this it is apparent that if the versine AD is known, the chord AB is known. Or, if the chord AB is known, the versine AD is known. The first statement is evident because if the versine AD is multiplied with the diameter AC, the square root of the result is the chord AB. The second statement is evident because if the square of the chord AB is divided by diameter AC, the quotient will be versine AD. It is clear from this that if the chord or versine of any arc is known, the sine of that arc is known.

From a knowledge of the versine comes the knowledge of the chord, and from a knowledge of the chord comes a knowledge of the versine, as has been said. From the versine and chord the sine is immediately known because the chord is related to the two squares, that of the versine and that of the sine, as is apparent from the preceding figure and discussion of sines. From this it follows that the square of the chord AB is equal to the sum of the two squares, clearly sine BD and versine AD. Therefore, if the square of the known versine AD is subtracted from the square of the known chord AB, the root of the remainder will be the sine BD.

This is proved in another way because line BD is the middle position of the proportion between the lines AD and DC, as Euclid demonstrates. From this it follows that the product of line AD and line DC is equal to the product of BD with itself; and because the product of line AD and line DC is known, since DC is the remainder of the diameter, it follows that line BD is known, because it is the square root of the product of lines AD and DC; and this is what I wished to prove. From this it is known that if the chord of one arc is known, the chord of that arc doubled is known, because from the chord of the first is found its sine, and that sine doubled is the chord of the arc doubled.

It is also known that if a versine is known, the chord of the arc remaining in 180° is known. That is to say, line AD and line DC are known, for if line AD, the known versine, is subtracted from diameter AC, line DC remains, which is the versine of the remaining arc. But if line

DC, the versine, is known, the chord BC is also known by the first corollary.

Second Theorem: The versine plus the sine of the arc remaining in 90° is equal to the radius.

Let ABC be a fourth of a circumference about a center E (see Fig. 2). Let line CE

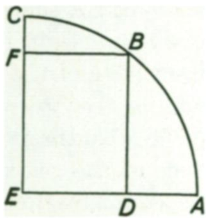

FIGURE 2

be drawn, and let line AD be the versine of arc AB, and let BF be the sine of the arc remaining in 90°, clearly arc BC. Let line BD be drawn. I say that line AD, the versine, added to line BF, the sine of the arc remaining in 90°, is equal to the radius AE. Certainly it is clear that angle E is right, so also that angle D is right, and that angle F is right. Consequently, angle B will be right and line BD is equidistant and equal to line ED. If line AD is added to each, the augmented sums will be equal; that is, line AD plus line BF equals line ADE.

From this it is apparent that if the versine is known, the sine of the arc remaining in 90° is known; and if the sine is known, the versine of the arc remaining in 90° will be known.

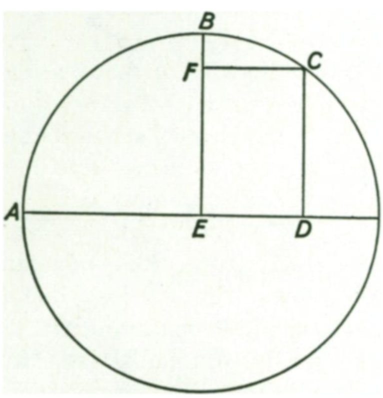

FIGURE 3

Third Theorem: The versine of an arc greater than 90° equals the radius plus the sine of the arc superfluous of 90°.

Let arc ABC be an arc greater than 90°, and BC be the arc superfluous of 90° (see Fig. 3). Let the center be point E, the versine of the major arc greater than 90° be line AED, and line CF be the sine of arc BC. Let there be drawn line CD, which is the sine of arc ABC, and line BFE. We have lines FC and ED equidistant and equal as above. From this it follows that line AE plus line FC is equal to line AED, which is what I wished to prove.

Fourth Theorem: If the two sines and versines of two different arcs are known, the chords of the sum and of the difference of said arcs are known.

For example, let AB and BC be two different arcs, let arc AB be the greater of

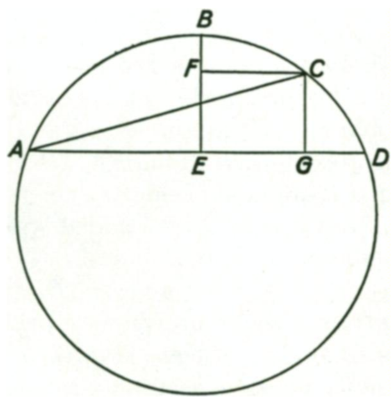

FIGURE 4

the two (see Fig. 4). Let the sine of arc AB be the line AE, its versine be line BE. And let the sine of arc BC be line CF, and its versine be line BF. It is known that line BF necessarily falls upon line BE because each line comes from point B through the center to the circumference. Let arc BD be equal to arc BA, and let the straight line AE be drawn to the point D. It is likewise known that said line comes directly to point D. Because the chord of the arc ABD is divided by the point E into two equal parts, and line AE is the sine of arc AB, it is known that line

ED is equal to line AE. Let there be drawn from point C a line perpendicular to line AD, which is line CG. Line CG in the first figure falls within the circumference and in the second figure, outside (see Fig. 5).

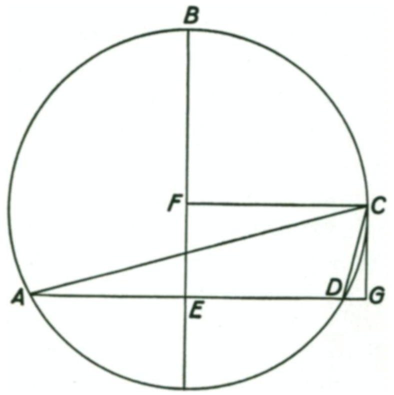

FIGURE 5

Draw lines AC, CD, of which line AC is the chord of the sum of the two arcs mentioned before, evidently AB, BC. Line CD is the chord of the difference of said arcs because it is assumed that arc BCD is equal to arc AB. I say that if AE and FC, sines of arcs AB, BC, and BE, BF, versines of the same, are known, line AC as the chord of arcs AB, BC added will be known. I also say that if AE and FC, sines of the two arcs AB, BC, and BE, BF, versines of the same arcs, are known, then the straight line CD as the chord of their difference will be known. It is proved because line AC is related to the sum of the squares of the two lines AE, FC added, and the square of line FE, the difference of the two versines, whose square is known because their lines are known. The chord of the arc CD is related to the square of the difference of the two sines plus the square of the difference of their two versines. This is apparent because the angles F, E, G are right; and so it follows that lines CF, EG are parallel, as are lines EF, CG. Opposite parallel lines are equal, evidently here lines EF, CG and lines CF, EG. Because line CG is perpendicular to line AG forming a right angle, it is known that line AC is related to the squares of the two lines CG, GA. But line GA is equal to the two sines AE, FC, and line CG is equal to line EF, which is the difference of the two versines. Therefore, it is known that the square of the chord of the two arcs added is equal to the sum of the square of the two sines added and the square of the difference of their versines. For the same reason it is known that the square of the chord of the difference of the two arcs mentioned above, which is the straight line CD, is equal to the two squares of the two lines CG, GD. The first, CG, is equal to the difference of the two versines; the second, GD, is equal to the difference of the two sines. These are what I wished to prove.

Fifth Theorem: When a sine is known, its versine is known.

Either the sine belongs to an arc of 90° —then the sine and versine are mutually equal—or belongs to an arc less than 90°, or belongs to an arc greater than 90°. If the sine belongs to an arc of 90°, then the sine and versine are the same because, as is the sine, the versine is a radius. If the arc is major or minor, I say that the distance between the sine and the center is known, and consequently the versine, because the square of a distance remains when the square of the sine is subtracted from the square of the radius. The root of this squared distance subtracted from the radius produces the versine of the arc less than 90°. Or if that same root is added, the versine of an arc greater than 90° is

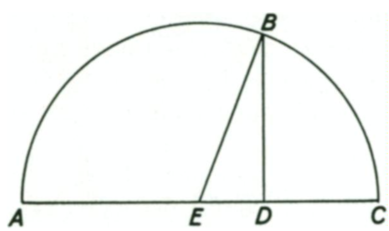

FIGURE 6

produced. Let ABC be a semicircle above center E (see Fig. 6). Line BD is the known sine of the two arcs, of which one is greater than 90° and the other less than. Let the major arc be AB, the minor arc be BC. Then I say that if the sine BD is known, so also is DE, the distance from the center E, and consequently DC, the versine of the minor arc, and AD, versine of the major arc. Let there be drawn line BE, a radius, whose square is equal to the square of lines BD, DE, because angle BDE is right. Subtract the known square of BD from the known square of BE; the square of DE remains, whose root added to the radius produces the versine of the arc greater than 90°. And that same root subtracted from the radius produces DE the versine of the arc less than 90°, evidently BC. These are what I wished to prove.

Sixth Theorem: If the chord of the double arc is known, the chord of half the double arc is also known.

Because the chord of a double arc is known, one may find the mean of said chord, which is the sine of half the double arc. When a sine is known, its versine follows, and from that the chord of half the double arc is known, which is what I wished to prove.

Section 3

If the sines and versines are determined at intervals of $\frac{1}{4}$°, from $\frac{1}{4}$° up to 45°, to a certain number of sexagesimal places, we know sufficiently the remaining sines and versines.

Indeed, if the sine and versine of $\frac{1}{4}$° is known, we know the sine and versine of $89\frac{3}{4}$°. Because if the quantity of the sine $\frac{1}{4}$° is known, we know that the remainder of the radius left by the quantity mentioned is the versine of the arc remaining in 90°. This is apparent by the second theorem of the section above. Likewise, if the sine and versine of $\frac{1}{4}$° is known, the sine and versine of $179\frac{3}{4}$° is known. It is evident because the same is the sine of both, as was said in the beginning of the second theorem. In subtraction of one versine from the entire diameter remains the versine of the other, as is apparent from the latter part of the first section.

When the sine and versine of $89\frac{3}{4}$° is known, that the sine and versine of $90\frac{1}{4}$° is known is evident in the same manner. And by this process the same is known about other arcs through 45°. Henceforth, I assume three sines to be known. First, the sine of 90° is equal to its versine because both the sine and versine equal the radius. Second, the sine of an arc of 30° is known, for Euclid demonstrates that the chord of an arc of 60° is 60. The versine of 30° is known to be 8;2,18,30,48. The third known sine is 18°, because, as Euclid instructs, the side of a decagon plus one fourth of the diameter is related to the square of one fourth of the diameter. But the root of these two squares is 71;4,55,20,30,[11] because the two said squares added are 4500. From this root let there be subtracted one fourth of the diameter, evidently 30, and 31;4,55,20,30[12] remains. By these divisions the sine of arc of 18° remains equal, whose sine is 15;32,27,40,15.[13] From this information it is apparent that its versine is 29;56,11, 47,58.[14]

With these suppositions and the knowledge of the sine and versine of 90°, we know the sine and versine of an arc 45°, and through that we know the sine and versine of $22\frac{1}{2}$° and $11\frac{1}{4}$°. Further, from the sine and versine of 30° we have a notion of the sine and versine of 15°, $7\frac{1}{2}$°, $3\frac{3}{4}$°. From the values of the sine and versine of 18°, we learn the sine and versine of 9°, $4\frac{1}{2}$°, $2\frac{1}{4}$°, and also 36°. Since we know the sine and versine of arcs of 30° and 18°, we can learn the sine and versine of arc of 24°, which is the mean of the two arcs added. Similarly, we may find the sines and versines of the arcs 12°, 6°, 3°, $1\frac{1}{2}$°, $\frac{3}{4}$°. Now we are able to find all the sines and versines of all arcs that are multiples of $\frac{3}{4}$°. So, for example, from the

[11] With Vat. Latin, 3380, fol. 81a, and Vat. Latin, 3098, fol. 4b, col. 2, read: 67;4,55,20,30.

[12] With Vat. Latin, 3380, fol. 81a and Vat. Latin, 3098, fol. 4b, col. 2, read: 37;4,55,20,30.

[13] With Vat. Latin, 3380, fol. 81a, and Vat. Latin, 3098, fol. 4b, col. 2, read: 18;32,27,40,15.

[14] With Vat. Latin, 3380, fol. 81a, and Vat. Latin, 3098, fol. 4b, col. 2, read: 2;56,11,35,58.

sines and versines of $8\frac{1}{4}°$ and $1\frac{1}{2}°$ we may learn the sine and versine of arc $9\frac{3}{4}°$.

We will now find the sine of $\frac{1}{4}°$, using the preceding information. For from the sine of arc $8\frac{1}{4}°$ the sine of arc $4\frac{1}{8}°$ is found, and so by continuing this procedure the sine of $\frac{1}{4}°$ plus $\frac{1}{128}°$ is known. And through this approach from the sine of arc of 4° less $\frac{1}{4}°$, we will have the sine of $\frac{1}{4}°$ less $\frac{1}{64}°$. By another procedure I find that the ratio of the sine of an arc of $\frac{1}{4}°$ plus $\frac{1}{128}°$ to the sine of an arc of $\frac{1}{4}°$ less $\frac{1}{64}°$ is equal to the ratio of the first arc to the second arc, so far that a difference in fourths of minutes of a fraction of the ratio is not apparent. It is allowable that a little may appear in the fifth place. So, by this method it is concluded that the ratio of the sine of an arc of $\frac{1}{4}°$ plus $\frac{1}{128}°$ to the sine of an arc of $\frac{1}{4}°$ is the same as the ratio of the first arc to the second arc.

From the sines discovered by this procedure and the ratio of their arcs, one may assert that the sine of an arc of $\frac{1}{4}°$ is 0;15,42,28,32,27.[15] From this the sine of arcs of $\frac{1}{4}°$ and the sines of arcs of $\frac{1}{2}°$, 2°, 4°, 8°, 20°, and 40°. In the same way we will easily know the sines and versines of all degrees from $\frac{1}{4}°$ to the complement of 45°, with intervals of 15′. Consequently, we will know all remaining sines and versines, as was said above.

I have given myself the task of discovering sines and versines from $\frac{1}{4}°$ every 15′, because I have found in the tables which proceed from one degree by a single degree, a defect of 15 minutes, or thereabouts, may come about in certain places in the circle. If, indeed, one seeks an arc from a known sine, and especially if the arc is near 90° in measure, the defect mentioned will be found. For example, in my tables the sine of arc 89°30′ is 59;59, 57;[16] and in the tables proceeding degree by degree, if the arc is minor in the semicircle, the sine above is the sine of arc 89;45,21;[17] and if the arc is greater than 90°, the above sine is the sine of arc 90;14,33; and this demonstrates that the table by degrees is incorrect by about 15′. Accordingly, the versine of arc 0;14,33 would be 0;0,8. If the versine is 0;0,8, the square of the sine corresponding to that versine would be 0;15,59,58,56, because it is the product of 0;0,8 and the complement of the diameter. From this it follows that the sine of arc 0;14,33° is 0;30,59,0,53. But that is incorrect, because that sine is 0;29,35 approximately, and therefore it is known that the arc of a sine positioned is greater than, or less than, 0;29,35, approximately. From this follows, in another approach, an error of 15′, and therefore I have arranged the tables from $\frac{1}{4}°$ every 15′, because in this there does not follow in working with sines proportionately the error notable in the work discussed before.

Section 4

I have omitted tables of versines and chords, but I tabulated the sines for every 15′, because from a known sine the chord of that arc doubled is found, and from that one can learn the versines. If the arc is less than 90°, the sine of the arc remaining in 90° is sought in the tables. The sine, once found, may be subtracted from the radius, and that which remains is the versine that was sought.

The arc may be found from the versine because if the versine is less than 60, it may be subtracted from that, and the result sought in the table will be the sine of the arc of the remainder, which arc found may be subtracted from 90°; and that which remains is the arc of the versine which was being sought.

If, indeed, the arc is greater than 90°, the sine of the difference is sought; when the sine is found it may be added to the radius, and the versine is obtained. If you wish to know the arc from a versine greater than 60, it is that arc which is the sine of the difference, versine superfluous to 60,

[15] With Vat. Latin, 3380, fol. 81b and Vat. Latin, 3098, fol. 4b, col. 2, read: 0;15,42,28,32,7.

[16] With Vat. Latin, 3380, fol. 81b and Vat. Latin, 3098, fol. 4b, col. 2, read: 59;59,52.

[17] With Vat. Latin, 3380, fol. 81b and Vat. Latin, 3098, fol. 4b, col. 2, read: 89;45,27.

which when the arc is found is added to 90° and the arc of the versine which was being sought is found. That is known from the second section of that chapter.

Thus, when Your Holiness[18] wishes to find the sines of known arcs, he may look up the known arc in the tables and therein find directly the sine. If he does not find the exact arc in the tables, he should seek the one closest to it, and their difference should be noted. The ratio of the difference of the two arcs located approximate to it ought to be recognized as a difference in the sine, too. In similar manner, an arc may be found from its sine. From this proportion an error occurs near 90°. Therefore, one should consider the arc remaining in 90°, in order to find the arc near 90°, whose sine is known, and this may be done according to the rules we have mentioned.

I have divided these tables into three columns. In the first column I have put the arcs from $\frac{1}{4}$° to 90° at intervals of 15′. (In the second column I have put those arcs supplementary in 180° to the arcs in the first column. In the third are the sines corresponding to the arcs in the first two columns.)[19]

Comments

On Section 2

Levi ben Gerson first establishes the basic trigonometric identities. He proves the six theorems stated below by employing the Pythagorean theorem and similarity of triangles. We introduce the following functions, where $R = 60$:

1. $\text{Sin}\,\alpha = R \sin \alpha.$
2. $\text{Vers}\,\alpha = R\,\text{vers}\,\alpha.$
3. $\text{Crd}\,\alpha = 2\,\text{Sin}\,\dfrac{\alpha}{2}.$

Levi discusses the theorems in terms of line segments. Here they are stated as expressions of the functions just introduced. For Theorems 1, 2, 3, 5, and 6, see Figure C-1, and for Theorem 4, see Figures C-2 and C-3.

First Theorem: $(\text{Crd}\,\alpha)^2 = d\,\text{Vers}\,\alpha.$

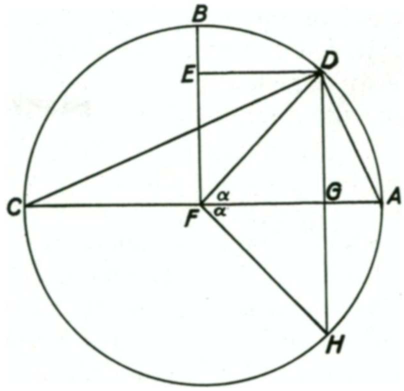

Figure C-1

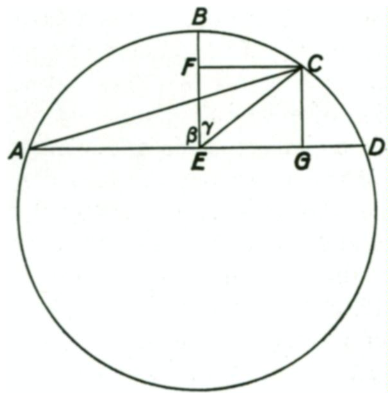

Figure C-2

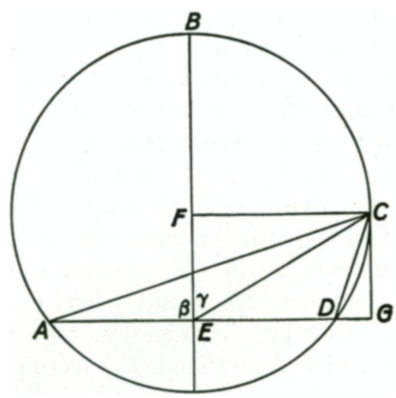

Figure C-3

[18] Your Holiness: not found in the Hebrew text (cf. MS P. Hebrew, 724, fol. 11a).

[19] Added from Vat. Latin, 3098, fol. 5a, col. 1. Vat. Latin, 3380 ends somewhat differently, but the sense agrees with the passage above.

Second Theorem: $\text{Vers}\,\alpha + \text{Sin}\,(90 - \alpha) = R$.

Third Theorem: $\text{Vers}\,(180 - \alpha) = R + \text{Sin}\,(90 - \alpha)$.

Fourth Theorem: $[\text{Crd}\,(\beta \pm \gamma)]^2 = (\text{Vers}\,\beta - \text{Vers}\,\gamma)^2 + (\text{Sin}\,\beta \pm \text{Sin}\,\gamma)^2$.

Fifth Theorem: $(\text{Vers}\,\alpha)^2 = (\text{Crd}\,\alpha)^2 - (\text{Sin}\,\alpha)^2$.

Sixth Theorem: $(\text{Crd}\,\alpha)^2 = (\frac{1}{2}\,\text{Crd}\,2\alpha)^2 + (\text{Vers}\,\alpha)^2$.

In Figure C-1, $\text{Crd}\,\alpha = AD$, $\text{Vers}\,\alpha = AG$, $\text{Sin}\,\alpha = \text{Sin}\,(180 - \alpha) = DG$, $R = 60$, $d = 2R = CA$, $\text{Vers}\,(180 - \alpha) = CG$, $\text{Crd}\,(180 - \alpha) = CD$, $\text{Sin}\,(90 - \alpha) = DE$, $\text{Crd}\,2\alpha = DH$. In Figures C-2 and C-3, $\text{Crd}\,(\beta + \gamma) = AC$, $\text{Crd}\,(\beta - \gamma) = CD$, $\text{Sin}\,\beta = EA = ED$, $\text{Sin}\,\gamma = FC$, $\text{Vers}\,\beta = BE$, $\text{Vers}\,\gamma = BF$.

On Section 3

Here Levi ben Gerson pursues the derivation of his sine table. He points out that the sine and versine of some arcs may be obtained from complementary and supplementary arcs or from the summation and difference of other arcs, as discussed in the previous section. He declares that the sines of 90°, 30°, and 18° are known. The latter may be obtained from the side of a decagon, which is the chord of 36°.[20] Through application of the Pythagorean theorem, he is able to find the sine of half of a given angle. The first step is to determine the sine and versine of that angle. Then

$$(\text{Crd}\,\alpha)^2 = (\text{Sin}\,\alpha)^2 + (\text{Vers}\,\alpha)^2$$

and

$$\text{Sin}\,\frac{\alpha}{2} = \frac{1}{2}\,\text{Crd}\,\alpha.$$

Finally, he addressed himself to the task of obtaining $\text{Sin}\,\frac{1}{4}°$. His method is as follows:

From the sine of an angle, one may find the sine of half that angle. Thus, from the sine of $8\frac{1}{4}°$ and the sine of $3\frac{3}{4}°$, one obtains $\text{Sin}\,(\frac{1}{4} + \frac{1}{128})°$ and $\text{Sin}\,(\frac{1}{4} - \frac{1}{64})°$. Levi states that

$$\frac{\text{Sin}\left(\frac{1}{4} + \frac{1}{128}\right)}{\text{Sin}\left(\frac{1}{4} - \frac{1}{64}\right)} = \frac{\frac{1}{4} + \frac{1}{128}}{\frac{1}{4} - \frac{1}{64}}$$

with no error apparent in the first four sexagesimal places. Therefore, by linear interpolation

$$\frac{\text{Sin}\left(\frac{1}{4} + \frac{1}{128}\right)}{\text{Sin}\,\frac{1}{4}} = \frac{\frac{1}{4} + \frac{1}{128}}{\frac{1}{4}}.$$

Thus, he finds $\text{Sin}\,\frac{1}{4}° = 0;15,42,28,32,7$, whereas the true value is $0;15,42,28,29,18$.

He concludes this section with a proof that his tables are more accurate than a table with intervals of 1°. Levi states that his table reads $\text{Sin}\,89;30° = 59;59,52$. But by linear interpolation in a table with intervals of 1°, the arc sine of 59;59,52 would be either 89;45,27° or 90;14,33°,[21] for $\text{Sin}\,89° = 59;59,27$ and $\text{Sin}\,90° = 60;0,0$. Thus we see the error of 15′, mentioned by Levi, in the arc sine near 90° which results from linear interpolation. In addition, Levi points out that if one accepts $\text{Sin}\,90;14,33° = 59;59,52$, it follows that $\text{Vers}\,0;14,33° = 0;0,8$. Then $\text{Sin}\,0;14,33° = 0;30,59,0,53$, which may be derived from

$$(\text{Sin}\,\alpha)^2 = (\text{Vers}\,\alpha)(d - \text{Vers}\,\alpha);$$

but, according to Levi, $\text{Sin}\,0;14,33° = 0;29,35$. For these reasons, he feels that a table with intervals of 15′ is more accurate than one with a larger interval.

On Section 4

Here Levi gives the instructions for using his tables. His final paragraph remains somewhat unclear. He may refer to the fact that linear interpolation is less accurate when the angle under consideration is near 90° than when it is close to 0°. The translation is based on this under-

[20] Cf. Aaboe, *Episodes from the Early History of Mathematics*, pp. 112–13.

[21] An example of a sine table with intervals of 1° is the Latin text Table 58a, *Die Astronomischen Tafeln des Muhammed Ibn Musa Al-Khwarizmi*, H. Suter (Copenhagen, 1914).

standing, but other interpretations are possible. The section concludes with another description of linear interpolation to find the sine of any angle.

BIBLIOGRAPHY

AABOE, A. *Episodes from the Early History of Mathematics*. New York: Random House, 1964.

VON BRAUNMÜHL, A. *Vorlesungen über Geschichte der Trigonometrie*, Erster Teil. Leipzig: Teubner, 1900.

CURTZE, M. "Die Abhandlung des Levi ben Gerson über Trigonometrie und den Jacobstab," *Bibliotheca Mathematica*, Neue Folge. XII (1898), 97–112.

CURTZE, M. "Urkunden zur Geschichte der Trigonometrie im christlichen Mittelalter," *Bibliotheca Mathematica*, Dritte Folge. I (1900), 321–416.

NEUGEBAUER, O. *The Exact Sciences in Antiquity*. New York: Harper & Brothers, 1962.

RENAN, E. "Les écrivains juifs français du XIV siècle," *Histoire litteraire de la France* XXXI (1843), 351–830.

SUTER, H. *Die Astronomischen Tafeln des Muhammed Ibn Musa Al-Khwarizmi*. Copenhagen, 1914.

TAYLOR, E. G. R. "Cartography, Survey, and Navigation, 1400–1750," *A History of Technology*, Singer *et al.*, Oxford: 1957. III, 530–57.

THORNDIKE, L. *A History of Magic and Experimental Science*. New York: Columbia University Press, 1934.

Letters to the Editor

DEAR EDITOR:

I read with a great deal of interest both Melvin Hausner's article in the April 1966 issue of THE MATHEMATICS TEACHER, and Louis Kassel's letter commenting upon it in the February 1967 issue of the same magazine.

Mr. Kassel's points are all well taken, but I should like to point out that his objections can easily be satisfied within the framework of Mr. Hausner's original approach. The following incomplete program is the FORTRAN version of a program originally written in Easycoder for a Honeywell 200 Computer.

```
   DIMENSION I(169)
   I(1)=3
   I(2)=5
   L=2
   DO 1 NMBR=7,999999, 2
   N=Ø
   7 N=N+1
   IF (NMBR-NMBR/I(N)*I(N))3, 1, 6
 6 IF (I(N+1)*I(N+1).LE.NMBR) GO TO 7
   IF (169.LE.L) GO TO 1
   L=L+1
   I(L)=NMBR
 1 CONTINUE
 3 STOP
```

It should be noted that—

1. The program tests only odd numbers.
2. It uses only primes as test divisors.
3. It makes only as many test divisions as are necessary to assure primality.

While I hesitate to comment upon the relative pedagogic value of either method, I think the fact that this modified version of Mr. Hausner's approach is much less of a "brute force" method than the sieve approach makes it at least esthetically the more desirable of the two to serve as the students' sole contact with computer methods.

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DEAR EDITOR:

The listing on page 239 of the March issue specifies *incorrectly* that "How to Lie with Statistics" is free.

On page 240, Item 10 should be revised to read "The Low Achiever in Mathematics," U.S. Department of Health, Education, Office of Education, Superintendent of Documents, Catalog No. FS 5.229:29061. U.S. Government Printing Office, Washington: 1965. Price 35 cents.

I regret the appearance of errors and resulting inconvenience to all concerned.

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