

AN ASYMPTOTIC PROBLEM IN DERANGEMENT THEORY*

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Abstract. N elements, divided into sets of respective cardinalities $\{n_1, n_2, \dots, n_a\}$, k sets of each size, where $N = k \sum_{i=1}^a n_i$, are given. The probability is considered that a random permutation of the N elements is a derangement, i.e., that it leaves none of the elements in the set to which it belonged initially. In particular, an asymptotic estimate of this probability as $k \rightarrow \infty$ is obtained.

It is known that the number of possible derangements can be represented by an integral involving products of Laguerre polynomials. The probability is obtained by asymptotic evaluation of a more general integral, involving the generalized Laguerre polynomials $L_n^{(\alpha)}(x)$, of which the integral required here is a special case.

Key words. derangement, Laguerre polynomials, asymptotic

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1. Introduction. Given finite sets, of respective cardinalities $\{n_1, n_2, \dots, n_a\}$, k sets of each size, we consider permutations of the entire set of $k \sum_{i=1}^a n_i$ elements. A permutation will be called a derangement if none of the elements is left in the set to which it initially belonged. Our main purpose is to estimate asymptotically, as $k \rightarrow \infty$, the probability that a random permutation of the elements is a derangement.

Let $D_k(n_1, n_2, \dots, n_a)$ denote the total number of possible derangements and let $P_k(n_1, \dots, n_a)$ denote the probability of a permutation being a derangement. It is clear that

$$(1.1) \quad P_k(n_1, \dots, n_a) = D_k(n_1, \dots, n_a) / \left(k \sum_{i=1}^a n_i \right) !.$$

Now it is known ([1, p. 135], [2, p. 4]) that

$$(1.2) \quad D_k(n_1, \dots, n_a) = \prod_{i=1}^a \{(-1)^{n_i} n_i !\}^k \int_0^\infty \left\{ \prod_{i=1}^a L_{n_i}(x) \right\}^k e^{-x} dx$$

where $L_n(x)$ denotes the Laguerre polynomial

$$(1.3) \quad L_n(x) = \sum_{m=0}^n (-1)^m \binom{n}{m} \frac{x^m}{m!}.$$

Since neither $D_k(n_1, \dots, n_a)$ nor $P_k(n_1, \dots, n_a)$ have closed-form expressions, integral representations such as (1.2) are useful for several reasons. First, we can use (1.2) and the recurrence relations for Laguerre polynomials to derive recurrence relations for $D_k(n_1, \dots, n_a)$. For a number of such recurrence relations, some of which seem difficult to prove from direct combinatorial considerations, see [2, p. 142]. Second, an integral representation such as (1.2) proves the positivity of the linearization of products of $\{(-1)^n L_n(x)\}$, and hence the existence of a discrete convolution structure associated with $\{(-1)^n L_n(x)\}$. Third, as we will see in this paper, (1.2) can be used to estimate the size of the D_k 's or P_k 's for large k . We write

$$(1.4) \quad s_r = \sum_{i=1}^a n_i^r \quad (r = 1, 2, \dots),$$

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and so

$$(1.5) \quad P_k(n_1, \dots, n_a) = (-1)^{ks_1} \left(\prod_{i=1}^a n_i! \right)^k \{(s_1 k)!\}^{-1} \int_0^\infty \left\{ \prod_{i=1}^a L_{n_i}(x) \right\}^k e^{-x} dx.$$

We will show that, as $k \rightarrow \infty$,

$$(1.6) \quad P_k(n_1, \dots, n_a) = \exp \left(-\frac{s_2}{s_1} \right) \left\{ 1 - \frac{s_1(2s_3 - s_2) - s_2^2}{2s_1^3 k} + O(k^{-2}) \right\}.$$

It follows, in particular, that in the special case $a = 1$,

$$(1.7) \quad \lim_{k \rightarrow \infty} P_k(n_1) = e^{-n_1},$$

a result previously obtained by Askey, Ismail, and Rashed [1, p. 5], though the method of proof there does not seem to extend to the case $a > 1$.

The limiting relation (1.7) is interesting because it shows that $P_k(n_1) \sim (P_k(1))^{n_1}$, i.e., the probability of having a derangement of type $n_1, n_1, \dots, n_1$ (k times), $k \rightarrow \infty$, is asymptotically equal to that of having n_1 independent copies of the classical derangement problem with $k \rightarrow \infty$. Now (1.6), which we will prove, is more interesting because it shows that

$$\{P_k(n_1, n_2, \dots, n_a)\}^{s_1} \sim \prod_{i=1}^a \{P_k(n_i)\}^{n_i},$$

a surprising result with an obvious combinatorial interpretation.

Equation (1.6) will follow from the asymptotic estimation, as $k \rightarrow \infty$, of the integral in (1.2). However, we will begin by examining a more general integral, namely,

$$(1.8) \quad I(\alpha) = \int_0^\infty \left\{ \prod_{i=1}^a (-1)^{n_i} L_{n_i}^{(\alpha)}(x) \right\}^k x^\alpha e^{-x} dx$$

where the $L_n^{(\alpha)}$ are generalized Laguerre polynomials defined (for $\alpha > -1$) by

$$(1.9) \quad L_n^{(\alpha)}(x) = \sum_{m=0}^n (-1)^m \binom{n+\alpha}{n-m} \frac{x^m}{m!}.$$

It is not known whether the integrals $I(\alpha)$ of (1.8) are nonnegative for $\alpha > -1$. When $\alpha = 0$ the nonnegativity of $I(\alpha)$ follows from the combinatorial interpretation of [2]. The nonnegativity of $I(\alpha)$ for $\alpha \neq 0$ is difficult to prove because each Laguerre polynomial $L_n^\alpha(x)$ in the integrand has n simple zeros in the range of integration. It will be shown that, for large k ,

$$(1.10) \quad I(\alpha) = (2\pi)^{1/2} (ks_1)^{ks_1 + \alpha + 1/2} \left\{ \prod_{i=1}^a n_i! \right\}^{-k} \exp \left\{ -ks_1 - \frac{s_2 + \alpha s_1}{s_1} \right\} \left\{ 1 + \frac{h}{k} + O(k^{-2}) \right\}$$

where

$$h = \{(1 + 12\alpha + 18\alpha^2)s_1^2 - 12s_1s_3 + 6s_2^2 + 6(\alpha + 1)s_1s_2\} / 12s_1^3.$$

In particular,

$$(1.11) \quad \begin{aligned} I(0) = (2\pi)^{1/2} (ks_1)^{ks_1 + 1/2} \left\{ \prod_{i=1}^a n_i! \right\}^{-k} \exp \left\{ -ks_1 - \frac{s_2}{s_1} \right\} \\ \cdot \left\{ 1 - \frac{1}{12ks_1} - \frac{s_1(2s_3 - s_2) - s_2^2}{2ks_1^3} + O(k^{-2}) \right\}. \end{aligned}$$

Now,

$$\begin{aligned}
 P_k(n_1, \dots, n_a) &= \left\{ \prod_{i=1}^a n_i! \right\}^k \{(ks_1)!\}^{-1} I(0) \quad (\text{by (1.5)}) \\
 &= (2\pi)^{1/2} (ks_1)^{ks_1+1/2} \{(ks_1)!\}^{-1} \\
 (1.12) \quad &\cdot \left\{ 1 - \frac{1}{12ks_1} - \frac{s_1(2s_3-s_2)-s_2^2}{2ks_1^3} + O(k^{-2}) \right\} \\
 &\cdot \exp \left\{ -ks_1 - \frac{s_2}{s_1} \right\},
 \end{aligned}$$

while, by Stirling's Formula,

$$(1.13) \quad (ks_1)! = (2\pi)^{1/2} (ks_1)^{ks_1+1/2} \exp(-ks_1) \left\{ 1 + \frac{1}{12ks_1} + O(k^{-2}) \right\}.$$

Estimate (1.6) now follows immediately from (1.12) and (1.13). It remains to establish (1.10).

2. Proof of (1.10). We will use the Laplace approach (cf. [4, p. 81]), writing

$$(2.1) \quad I(\alpha) = \int_0^\infty e^{f(x)} dx$$

where

$$(2.2) \quad f(x) = k \sum_{i=1}^a \ln \{(-1)^{n_i} L_{n_i}^{(\alpha)}(x)\} + \alpha \ln x - x.$$

It will turn out that interest centres entirely on large values of x , where $(-1)^{n_i} L_{n_i}^{(\alpha)}(x) > 0$, and we may therefore ignore questions about logarithms of negative quantities.

2.1. Maximum of the integral. Let $Q(x) = \prod_{i=1}^a \{(-1)^{n_i} L_{n_i}^{(\alpha)}(x)\}$. Then

$$f(x) = k \ln Q(x) + \alpha \ln x - x$$

and so the extremals of $f(x)$ are to be sought among the roots of the equation

$$(2.3) \quad (x - \alpha)Q(x) = kxQ'(x).$$

This is a polynomial equation of degree $s_1 + 1$. It is clear that, for large k , there will be a root close to $x = 0$, and $s_1 - 1$ roots close to those of $Q'(x)$, all bounded independently of k . On the other hand, if

$$Q(x) = a_0 x^{s_1} + a_1 x^{s_1-1} + \dots,$$

then (2.3) becomes

$$kx(s_1 a_0 x^{s_1-1} + \dots) = (x - \alpha)(a_0 x^{s_1} + a_1 x^{s_1-1} + \dots),$$

i.e.,

$$a_0 x^{s_1+1} - \{(ks_1 + \alpha)a_0 - a_1\}x^{s_1} + \dots = 0$$

so that the sum of all $(s_1 + 1)$ roots is $ks_1 + \alpha - a_1/a_0$. It follows that the remaining root will be

$$(2.4) \quad x_0 = ks_1 + O(1).$$

For large k this will clearly be the largest root and it is easily verified that $f(x)$ will attain its global maximum there. It remains to estimate x_0 more precisely.

For any fixed $\alpha > -1$, write

$$(2.5) \quad v_n(x) = L_n^{(\alpha)'}(x) / L_n^{(\alpha)}(x);$$

then (2.3) becomes

$$(2.6) \quad k \sum_{i=1}^a v_{n_i}(x) = 1 - \frac{\alpha}{x}.$$

But $y = L_n^{(\alpha)}(x)$ satisfies the differential equation ([3, p. 781])

$$(2.7) \quad xy'' + (\alpha + 1 - x)y' + ny = 0$$

and hence v_n satisfies

$$(2.8) \quad x(v_n' + v_n^2) + (\alpha + 1 - x)v_n + n = 0.$$

However, $L_n^{(\alpha)}(x)$ is a polynomial of degree n , and so v_n must be of the form

$$\frac{n}{x} + O\left(\frac{1}{x^2}\right)$$

for large x . We deduce, by successive approximation in (2.8), that

$$(2.9) \quad v_n(x) = \frac{n}{x} + \frac{n(n+\alpha)}{x^2} + \frac{n(n+\alpha)(2n+\alpha-1)}{x^3} + O(x^{-4}).$$

It follows that

$$(2.10) \quad V(x) = \sum_{i=1}^a v_{n_i}(x) = \frac{s_1}{x} + \frac{s_2 + \alpha s_1}{x^2} + \frac{c}{x^3} + O(x^{-4})$$

where

$$(2.11) \quad c = 2s_3 + (3\alpha - 1)s_2 + \alpha(\alpha - 1)s_1.$$

Equation (2.6) now becomes

$$(2.12) \quad 1 - \frac{\alpha}{x} = k \left(\frac{s_1}{x} + \frac{s_2 + \alpha s_1}{x^2} + \frac{c}{x^3} + \dots \right).$$

Starting from (2.4) we obtain, by successive approximation in (2.12),

$$(2.13) \quad x_0 = ks_1 + \frac{s_2 + 2\alpha s_1}{s_1} + \frac{t}{ks_1^3} + O(k^{-2})$$

where

$$(2.14) \quad t = 2s_1s_3 - s_2^2 - s_1s_2 - \alpha(\alpha + 1)s_1^2.$$

2.2. Taylor expansion of $f(x)$ about $x = x_0$. Since

$$(2.15) \quad f'(x) = kV(x) + \frac{\alpha}{x} - 1$$

it follows that, for $r \geq 2$,

$$(2.16) \quad f^{(r)}(x_0) = kV^{(r-1)}(x_0) + (-1)^{r-1}(r-1)! \alpha x_0^{-r}.$$

Substituting in (2.16) from (2.10) and (2.13) we get, after some manipulation,

$$(2.17) \quad f(x) = f(x_0) - A(x - x_0)^2 + B(x - x_0)^3 - C(x - x_0)^4 \cdots$$

where

$$(2.18) \quad \begin{aligned} A &= -\frac{1}{2}f''(x_0) = \frac{1}{2ks_1} \left\{ 1 - \frac{\alpha}{ks_1} + \frac{t}{k^2s_1^4} + O(k^{-3}) \right\}, \\ B &= \frac{1}{6}f'''(x_0) = \frac{1}{3k^2s_1^2} \left\{ 1 - \frac{2\alpha}{ks_1} + \frac{3t}{k^2s_1^4} + O(k^{-3}) \right\}, \\ C &= -\frac{1}{24}f^{(iv)}(x_0) = \frac{1}{4k^3s_1^3} \left\{ 1 - \frac{3\alpha}{ks_1} + \frac{6t}{k^2s_1^4} + O(k^{-3}) \right\}. \end{aligned}$$

2.3. Estimation of $e^{f(x_0)}$. By (2.2),

$$(2.19) \quad e^{f(x_0)} = x_0^\alpha e^{-x_0} \left\{ \prod_{i=1}^a (-1)^{n_i} L_{n_i}^{(\alpha)}(x_0) \right\}^k.$$

We write

$$(2.20) \quad (-1)^n L_n^{(\alpha)}(x) = \frac{x^n}{n!} G_n(x)$$

where

$$G_n(x) = 1 - \frac{n(n+\alpha)}{x} + \frac{n(n-1)(n+\alpha)(n+\alpha-1)}{2x^2} + O(x^{-3})$$

so that

$$(2.21) \quad \ln G_n(x_0) = -\frac{n(n+\alpha)}{x_0} - \frac{n(n+\alpha)(2n+\alpha-1)}{2x_0^2} + O(x_0^{-3}),$$

and hence

$$(2.22) \quad \begin{aligned} \prod_{i=1}^a (-1)^{n_i} L_{n_i}^{(\alpha)} &= x_0^\alpha \left\{ \prod_{i=1}^a n_i! \right\}^{-1} \\ &\quad \cdot \exp \left\{ -\frac{s_2 + \alpha s_1}{x_0} - \frac{2s_3 + (3\alpha - 1)s_2 + \alpha(\alpha - 1)s_1}{2x_0^2} + O(x_0^{-3}) \right\}, \\ e^{f(x_0)} &= x_0^\alpha e^{-x_0} \left\{ \prod_{i=1}^a (-1)^{n_i} L_{n_i}^{(\alpha)}(x) \right\}^k \\ &= x_0^{ks_1 + \alpha} \left\{ \prod_{i=1}^a n_i! \right\}^{-k} \\ &\quad \cdot \exp \left\{ -kx_0 - \frac{k(s_2 + \alpha s_1)}{x_0} - \frac{k[2s_3 + (3\alpha - 1)s_2 + \alpha(\alpha - 1)s_1]}{2x_0^2} + O(kx_0^{-3}) \right\}. \end{aligned}$$

Substituting from (2.13) we get, after some manipulation,

$$(2.23) \quad e^{f(x_0)} = x_0^{ks_1 + \alpha} \left\{ \prod_{i=1}^a n_i! \right\}^{-k} \exp \left\{ -ks_1 - \frac{2s_2 + 3\alpha s_1}{s_1} - \frac{g}{2ks_1^3} + O(k^{-2}) \right\}$$

where

$$g = 6s_1s_3 - 4s_2^2 - 3(\alpha + 1)s_1s_2 - \alpha(5\alpha + 3)s_1^2.$$

But

$$x_0 = ks_1 \left\{ 1 + \frac{s_2 + 2\alpha s_1}{ks_1^2} + \frac{t}{k^2 s_1^4} + O(k^{-3}) \right\}$$

and so

$$(2.24) \quad \ln \left(\frac{x_0}{ks_1} \right) = \frac{s_2 + 2\alpha s_1}{ks_1^2} + \frac{2t - (s_2 + 2\alpha s_1)^2}{2k^2 s_1^4} + O(k^{-3})$$

whence

$$(2.25) \quad (ks_1 + \alpha) \ln \left(\frac{x_0}{ks_1} \right) = \frac{s_2 + 2\alpha s_1}{s_1} + \frac{p}{2ks_1^3} + O(k^{-2})$$

where

$$(2.26) \quad p = 4s_1 s_3 - 3s_2^2 - 2(\alpha + 1)s_1 s_2 - 2\alpha(\alpha + 1)s_1^2,$$

$$(2.27) \quad \left(\frac{x_0}{ks_1} \right)^{ks_1 + \alpha} = \exp \left\{ \frac{s_2 + 2\alpha s_1}{s_1} + \frac{p}{2ks_1^3} + O(k^{-2}) \right\}.$$

From (2.23) and (2.27) we get

$$(2.28) \quad e^{f(x_0)} = (ks_1)^{ks_1 + \alpha} \left\{ \prod_{i=1}^a n_i! \right\}^{-k} \exp \left\{ -ks_1 - \frac{s_2 + \alpha s_1}{s_1} - \frac{q}{2ks_1^3} + O(k^{-2}) \right\}$$

where

$$(2.29) \quad q = 2s_1 s_3 - s_2^2 - (\alpha + 1)s_1 s_2 - \alpha(3\alpha + 1)s_1^2.$$

2.4. Estimation of $\int_0^\infty e^{f(x)-f(x_0)} dx$. It follows from (2.17) that the integral can be estimated by

$$(2.30) \quad \int_{-x_0}^\infty e^{-Au^2 + Bu^3 - Cu^4} du = \left(\int_{-\infty}^\infty - \int_{-\infty}^{-x_0} \right) e^{-Au^2 + Bu^3 - Cu^4} du \\ = X - Y \quad (\text{say}).$$

We begin by estimating Y :

$$(2.31) \quad 0 < Y = \int_{-\infty}^{-x_0} e^{-Au^2 + Bu^3 - Cu^4} du \\ < \int_{-\infty}^{-x_0} e^{-Au^2} du \quad \text{since } A, B, C, \text{ are all positive} \\ = \int_{x_0}^\infty e^{-Au^2} du.$$

But for large k , $x_0 > ks_1$, by (2.13), and hence

$$(2.32) \quad 0 < Y < \int_{ks_1}^\infty e^{-Au^2} du \\ = A^{-1/2} \operatorname{Erfc}(ks_1 A^{1/2}) \\ \sim (2ks_1)^{1/2} \operatorname{Erfc}(\sqrt{ks_1/2}) \quad \text{by (2.18)} \\ \sim 2\pi^{-1/2} \exp(-ks_1/2)$$

(cf. [3, p. 298].

To estimate X we write $v^2 = Au^2 - Bu^3 + Cu^4$ leading to

$$(2.33) \quad u = av + \beta v^2 + \gamma v^3 \dots$$

where

$$(2.34) \quad \alpha = A^{-1/2}, \quad \gamma = \frac{1}{8}A^{-7/2}(5B^2 - 4AC).$$

We then get

$$\begin{aligned} (2.35) \quad X &\sim \int_{-\infty}^{\infty} e^{-v^2} du \\ &= \int_{-\infty}^{\infty} e^{-v^2} \frac{du}{dv} dv \\ &= \int_{-\infty}^{\infty} e^{-v^2} (\alpha + 3\gamma v^2 + \dots) dv \\ &\sim \sqrt{\pi}(\alpha + 3\gamma/2) \\ &= \frac{1}{16} \sqrt{\pi} A^{-7/2} (16A^3 + 15B^2 - 12AC). \end{aligned}$$

Substituting again from (2.18), we obtain

$$(2.36) \quad \int_0^{\infty} \exp \{f(x) - f(x_0)\} dx = \sqrt{2\pi k s_1} \left\{ 1 + \frac{1+6\alpha}{12k s_1} + O(k^{-2}) \right\}.$$

2.5. Proof of (1.10) and (1.6). It follows from (2.1) that

$$\begin{aligned} I(\alpha) &= e^{f(x_0)} \int_0^{\infty} e^{f(x) - f(x_0)} dx \\ &= (k s_1)^{k s_1 + \alpha} \left\{ \prod_{i=1}^a n_i ! \right\}^{-k} \exp \left\{ -k s_1 - \frac{s_2 + \alpha s_1}{s_1} - \frac{q}{2k s_1^3} + O(k^{-2}) \right\} \\ &\quad \cdot \sqrt{2\pi k s_1} \left\{ 1 + \frac{1+6\alpha}{12k s_1} + O(k^{-2}) \right\} \quad \text{by (2.28) and (2.36)} \\ &= (2\pi)^{1/2} (k s_1)^{k s_1 + \alpha + 1/2} \left\{ \prod_{i=1}^a n_i ! \right\}^{-k} \exp \left\{ -k s_1 - \frac{s_2 + \alpha s_1}{s_1} \right\} \\ &\quad \cdot \left\{ 1 + \frac{(1+6\alpha)s_1^2 - 6q}{12k s_1^3} + O(k^{-2}) \right\} \\ &= (2\pi)^{1/2} (k s_1)^{k s_1 + \alpha + 1/2} \left\{ \prod_{i=1}^a n_i ! \right\}^{-k} \exp \left\{ -k s_1 - \frac{s_2 + \alpha s_1}{s_1} \right\} \left\{ 1 + \frac{h}{k} + O(k^{-2}) \right\}, \end{aligned}$$

i.e., (1.10).

Equation (1.6) now follows as shown in § 1, (1.12), and (1.13).

3. Some numerical results. Table 1 may give some idea as to the accuracy of approximation (1.6) for even quite small values of k . All results are correct to the fourth decimal place.

TABLE 1

$(n_1, \dots, n_a)$	k	Direct evaluation by (1.5)	Asymptotic approximation
2	4	0.1179	0.1184
2	6	0.1243	0.1241
3	5	0.0398	0.0398
1, 2	3	0.1703	0.1726
1, 2, 3	2	0.0779	0.0799

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