Lehmer's Conjecture on the Non-vanishing of Ramanujan's Tau Function

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Abstract

In this paper we prove Lehmer's conjecture on Ramanujan's tau function, namely $\tau(n) \neq 0$ for each $n \geq 1$ by investigating the additive group structure attached to $\tau(n)$ with the aid of unique factorization theorem.

[1] Let E_k $(k = 2, 4, \dots)$ be the normalized Eisenstein series $([4 : 108 - 122])$ given by

$$E_k = 1 - \frac{2k}{B_k} \sum_{n=1}^{\infty} \sigma_{k-1}(n) q^n \tag{1}$$

where $q := e^{i2\pi z}$ $(\Im(z) > 0)$, B_k the Bernoulli number defined by

$$\frac{x}{e^x - 1} = \sum_{k=0}^{\infty} B_k \frac{x^k}{k!}$$

and $\sigma_{k-1}(n)$ the divisor function:

$$\sigma_{k-1}(n) := \sum_{d|n} d^{k-1}.$$

For an elliptic curve given by

$$y^2 = 4x^3 - g_2(z)x - g_3(z) \tag{2}$$

where $g_2(z) = 120\zeta(4)E_4(z), g_3(z) = 280\zeta(6)E_6(z)$ and $E_k(z)$ given by equation (1) and $\zeta(k)$ is Riemann zeta function:

$$\zeta(k) := \sum_{n=1}^{\infty} \frac{1}{n^k}.$$

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A simple calculation $([1:14],[4:112])$ shows the discriminant $\Delta(z) := 4^4(x_1-x_2)^2(x_2-x_3)^2(x_3-x_1)^2$, where x_1, x_2 and x_3 are the roots the right side of equation (2), is given by

$$\Delta(z) = g_2(z)^3 - 27g_3(z)^2 = \frac{(2\pi)^{12}}{1728}(E_4(z)^3 - E_6(z)^2). \tag{3}$$

On the other hand Jacobi's theorem $([4:122])$ asserts that

$$(2\pi)^{-12}\Delta(z) = q\prod_{n=1}^{\infty}(1-q^n)^{24}. \tag{4}$$

From equation (4), Ramanujan has defined his tau function $\tau(n)$ $([1],[2],[3],[4:122],[5]-[7])$ by

$$q\prod_{n=1}^{\infty}(1-q^n)^{24} := \sum_{n=1}^{\infty}\tau(n)q^n. \tag{5}$$

Notice that each $\tau(n)$ $(n \geq 1)$ has an integer value. In a series of papers $([5]-[7])$, D.H. Lehmer investigated the properties of $\tau(n)$ for $n \leq 300$, proved that $\tau(n) \neq 0$ for $n < 3316799$, later for $n < 214928639999$ $([1:22])$. He also showed that if $\tau(n) = 0$ then n must be a prime. He then conjectured, what is nowadays known as Lehmer's conjecture ([6]) that

$$\tau(n) \neq 0 \ \text{ for } \quad \text{each } \ n \geq 1. \tag{6}$$

A simple calculation $([3:21-22],[4:122-123])$ shows

$$\tau(n) = \frac{65}{756}\sigma_{11}(n) + \frac{691}{756}\sigma_5(n) - \frac{691}{3}\sum_{j=1}^{n-1}\sigma_5(j)\sigma_5(n-j). \tag{7}$$

Since Lehmer's conjecture is equivalent to $3\tau(n) \neq 0$ for each $n \geq 1$, we write

$$A(n) := \frac{65}{252}\sigma_{11}(n) + \frac{691}{252}\sigma_5(n); \ B(n) := 691\sum_{j=1}^{n-1}\sigma_5(j)\sigma_5(n-j). \tag{8}$$

Then $3\tau(n) = A(n) - B(n)$. Observe that $A(n)$ takes on integer value for each $n \geq 1$ since both $\tau(n)$ and $B(n)$ do. Now Lehmer's conjecture is, in view of equations (7), (8) and the unique factorization theorem, equivalent to:

$$A(n) \neq B(n) \text{ for each } n \geq 1. \tag{9}$$

Recent calculation by Bosman confirms Lehmer's conjecture for $n \leq 22798241520242687999$. In this paper we prove equation (9) by showing that $\{\sum_{k=0}^{q-1} ka_{i,k} \bmod q\}_{k=0}^{q-1}$ forms an additive group of order q modulo q for $q \mid A(p),\ q > p,\ p \equiv -1 \bmod 691,\ [a_{i,k}]_{0 \leq i,k \leq q-1}$ $q \times q$-matrix, with the aid of the unique factorization theorem, the pigeonhole principle and the remainder theorem. We prove equation (9) first for prime p then for $p^\alpha, \alpha \geq 2$ and finally for any composite number n. Since $11 \nmid 690$ and since $(p+1) \mid (p^{11}+1)$, the following Lemma 1 evidently holds.

Lemma 1 *Let $A(p)$ be given by equation* (8). *Then the following two conditions* (10) *and* (11) *are equivalent:*

$$A(p) \equiv 0 \mod 691. \tag{10}$$

$$p \equiv -1 \mod 691. \tag{11}$$

If $691 \nmid A(p)$ or equivalently p does not satisfy equation (11) then we trivially have $A(p) \neq B(p)$ by equation (8). It suffices therefore to prove Lehmer's conjecture for prime p satisfying equation (10) or (11). In what follows, prime p satisfies either equation (10) or (11). We first prove:

Lemma 2 *Let p satisfy equation* (10) *or* (11). *Then $A(p)$ has at least one prime factor q greater than p.*

Proof. Write $A(p)$ from equation (8) as

$$\begin{array}{ll} A(p) & = \frac{65}{252}(1+p^{11}) + \frac{691}{252}(1+p^5) \\ & = 3 + K_5 p^5 \end{array} \tag{12}$$

where $K_5 := \frac{691}{252} + \frac{65}{252}p^6$ is an integer with $p^5 < K_5 < p^6$ since $A(p)$ is an integer with $p^{10} < A(p) < p^{11}$. Suppose $A(p)$ has no prime factor greater than p. $A(p)$ then is written via the unique factorization theorem as

$$A(p) = 2^{e_0} q_1^{e_1} \dots q_m^{e_m},\ q_i < p,\ e_i \geq 1\ (1 \leq i \leq m). \tag{13}$$

Notice that A(p) has an even factor 2^{e_0} by substituting equation (11) into equation (8). Write $x = [x] + \{x\}$ where $[x]$ represents the integral part of x and $\{x\}$ the nonintegral part of x. Since $p^{10} < A(p) < p^{11}$, the representation for $A(p)$ in the base p is uniquely given from equation (13) by

$$A(p) = \sum_{i=0}^{10} b_i p^i,\ b_i := [p\{\frac{A(p)}{p^{i+1}}\}]\ (0 \leq i \leq 10). \tag{14}$$

We show that each $b_i \neq 0$ $(0 \leq i \leq 10)$. Indeed we prove

$$b_i \geq 5\ (1 \leq i \leq 10),\ b_0 \neq 0. \tag{15}$$

The fact that $b_0 \neq 0$ follows from $p \nmid A(p)$. Likewise $b_{10} \geq 5$ follows from $p \nmid A(p)$ and $b_{10} \geq [\frac{65}{252}p] \geq [\frac{p}{252}] \geq 5$ in view of equations (14) and (17), where $p \geq 1381$. ¿From equation (8) or (12), we readily have

$$\frac{65}{252}p^{11} < A(p) < (\frac{65}{252} + 3p^{-6})p^{11}. \tag{16}$$

Equation (16) is equivalent to

$$\frac{65}{252}p^{10-i} < \frac{A(p)}{p^{i+1}} < (\frac{65}{252} + 3p^{-6})p^{10-i}\ (1 \leq i \leq 9). \tag{17}$$

Write

$$65p^{10-i} = 252Q_i + R_i,\ Q_i \geq 1,\ 1 \leq R_i \leq 251\ (1 \leq i \leq 9). \tag{18}$$

Substitution of equation (18) into equation (17) reveals

$$Q_i + \frac{R_i}{252} < \frac{A(p)}{p^{i+1}} < (Q_i + \frac{R_i}{252} + 3p^{4-i})\ (1 \leq i \leq 9). \tag{19}$$

Inequality (19) implies $\{\frac{A(p)}{p^{i+1}}\} \geq \frac{R_i}{252} \geq \frac{1}{252}$ $(1 \leq i \leq 9)$ and hence we have since $p \geq 1381$

$$b_i = [p\{\frac{A(p)}{p^{i+1}}\}] \geq [\frac{p}{252}] \geq 5\ (1 \leq i \leq 9). \tag{20}$$

This establishes inequality (15). Rewrite equation (14) as

$$A(p) = L_5 p^5 + \sum_{i=0}^{4} b_i p^i \tag{21}$$

where $L_5 := \sum_{i=0}^{5} b_{5+i} p^i$. Subtraction of equation (21) from equation (12) with rearrangement of terms leads us to

$$\begin{aligned} p^5 &\leq (K_5 - L_5)p^5 \\ &= (b_0 - 3) + \textstyle\sum_{i=1}^{4} b_i p^i \\ &< (p-1)\textstyle\sum_{i=0}^{4} p^i \\ &= p^5 - 1. \end{aligned} \tag{22}$$

Since each $b_i \geq 1$ $(0 \leq i \leq 10)$ from equation (15), $K_5 - L_5 \geq 1$ follows from the the second line equality of equation (22) regardless of the value of $b_0 \neq 0$. Since $1 \leq b_i \leq p-1$ $(0 \leq i \leq 10)$ from equation (15) and since $b_0 - 3 < p - 1$, the third line inequality follows. Inequality (22) is absurd. Consequently the assumption that $A(p)$ has no prime factor $> p$ is false. This establishes Lemma 2.

It is easy to check our proof for Lemma 2 works for all primes $p > 252$ with inequality (15) replaced by $1 \leq b_i$ $(0 \leq i \leq 10)$. A simple computation reveals Lemma 2 also holds for primes $p \leq 252$. Consequently Lemma 2 holds for all primes p. Thus the assumption that the prime p generated by equation (11) in Lemma 2 is redundant.

Let q be an odd prime prime factor of $A(p)$ greater than p. Existence of such a prime q is guaranteed by Lemma 2. Construct matrix $[a_{i,k}]_{0\le i,k\le q-1}$ as follows:

$$a_{i,k} := \sum_{\substack{j=1 \\ i691\sigma_5(j)\sigma_5(p-j) \equiv k \bmod q}}^{p-1} 1. \tag{23}$$

Since $\sigma_5(j)\sigma_5(p-j) = \sigma_5(p-j)\sigma_5(p-(p-j))$, we have from equation (23)

$$a_{i,k} = 2 \sum_{\substack{j=1 \\ i691\sigma_5(j)\sigma_5(p-j) \equiv k \bmod q}}^{(p-1)/2} 1. \tag{24}$$

Then the matrix $[a_{i,k}]_{0\le i,\, k\le q-1}$ has the following properties:

$$a_{i,k} \equiv 0 \mod 2 \qquad (0 \le i, k \le q-1). \tag{25}$$

$$a_{0,0} = p-1,\ a_{0,k} = 0 \qquad (1 \le k \le q-1). \tag{26}$$

$$a_{i,0} = a_{j,0} \qquad (1 \le i \ne j \le q-1). \tag{27}$$

$$a_{i,k} = a_{q-i,q-k} \qquad (1 \le i, k \le q-1). \tag{28}$$

$$i691\sum_{j=1}^{p-1}\sigma_5(j)\sigma_5(p-j) \equiv \sum_{k=1}^{q-1} k a_{i,k} \mod q \qquad (1 \le i \le q-1). \tag{29}$$

$$\sum_{k=1}^{q-1} k a_{i,k} \equiv i\sum_{k=1}^{q-1} k a_{1,k} \mod q \qquad (1 \le i \le q-1). \tag{30}$$

Notice that given $a_{1,k}$ $(1 \le k \le q-1)$, $a_{i,k}$ $(2 \le i \le q-1, 1 \le k \le q-1)$ are reshuffles of $a_{1,k}(1 \le k \le q-1)$ and vice versa determined by

$$a_{i,k} = a_{1,i^{-1}k \bmod q} \Longleftrightarrow a_{1,k} = a_{i,ik \bmod q} \ \ (1 \le i, k \le q-1). \tag{31}$$

For each $i = 1, 2, \ldots, q-1$, write $f_{j_i} := i691\sigma_5(j)\sigma_5(p-j) \bmod q$. Then $f_{j_i} = f_{p-j_i}$ $(1 \leq i \leq q-1)$ from equation (23). Define for each $i = 1, 2, \ldots, q-1$:

$$\begin{array}{lcl} S_{i,l} & := & \{k : a_{i,k} = 2l\} \ (1 \leq k \leq q, 1 \leq l \leq q_0) \\ \Longleftrightarrow & = & \{(j_{i_1}, j_{i_2}, \ldots, j_{i_l}) : \ f_{j_{i_1}} = f_{j_{i_2}} = \cdots = f_{j_{i_l}} = k\} \ (1 \leq j_{i_1} < j_{i_2} < \cdots < j_{i_l} \leq \frac{(p-1)}{2}) \\ S_{i,l} & = & \varnothing \ (1 \leq l \leq q_0) \text{ for } l > q_0. \end{array} \tag{32}$$

Since $q > p$ and since $a_{i,k}$ $(0 \leq i, k \leq q-1)$ cannot be too large even number from equations (23) and (24), a positive integer $q_0 < q-1$ exists, depending on p and q, satisfying the last line of equation (32). It is clear from equation (32) with the aid of equation (31) that for each $l = 1, 2, \ldots, q_0$:

$$S_{i,l} = S_{j,l} \ (1 \leq i < j \leq q-1). \tag{33}$$

For each $q \mid A(p)$ with $q > p$, we then have from equations (31) – (33) that

$$\textstyle\sum_{l=0}^{q_0} \mid S_{i,l} \mid \ = q \ (1 \leq i \leq q-1). \tag{34}$$

$$\textstyle\sum_{k=1}^{q-1} a_{i,k} = \displaystyle\sum_{l=1}^{q_0} 2l \mid S_{i,l} \mid = p-1 \ (1 \leq i \leq q-1). \tag{35}$$

Equation (35) reads when $q \mid A(p)$ with $q < p$ that:

$$\sum_{k=0}^{q-1} a_{i,k} = p-1 \ (1 \leq i \leq q-1). \tag{36}$$

Equations (25) – (31) readily follow from equations (23) and (24). Equation (29) is a restatement of the remainder theorem in view of equations (23), (32) and (35). Equations (34), (35) and (36) follow from the pigeonhole principle. Since we exclusively use $S_{1,l}$ $(1 \leq l \leq q_0)$ in what follows, we show the following inequality:

$$\mid S_{1,l-1} \mid > \ l \mid S_{1,l} \mid \ \ (2 \leq l \leq q_0). \tag{37}$$

To prove inequality (37) we use the second line of equation (32) for the definition of $S_{1,l}$. Consider the map $\beta : S_{1,l} \mapsto S_{1,l-1} \times S_{1,l-1} \times \cdots \times S_{1,l-1}$ given by

$$\begin{aligned} \beta(j_1, j_2, \ldots, j_l) := \quad & ((\beta_1(j_1), \beta_2(j_1), \ldots, \beta_{l-1}(j_1)), (\beta_1(j_2), \beta_2(j_2), \ldots, \beta_{l-1}(j_2)), \ldots, \\ & (\beta_1(j_l), \beta_2(j_l), \ldots, \beta_{l-1}(j_l))) \end{aligned} \tag{38}$$

such that for each $i = 1, 2, \ldots, l$:

$$\begin{aligned} \beta_1(j_i) \quad & := \min_{j_{i_k}} \{ \mid j_i - j_{i_k} \mid : a_{1, j_{i_k}} = 2(l-1) \\ f_{\beta_1(j_i)} \quad & = f_{\beta_{2(j_i)}} = \cdots = f_{\beta_{l-1}(j_i)}. \end{aligned} \tag{39}$$

In the second line of equation (39), $\beta_k(j_i)$ $(2 \leq k \leq l-1, 1 \leq i \leq l)$ are uniquely determined once $\beta_1(j_i)$ $(1 \leq i \leq l)$ is determined by the first line of equation (39). Observe that $(\beta_1(j_i), \beta_2(j_i), \ldots, \beta_{l-1}(j_i)) \in S_{1,l-1}$ $(1 \leq i \leq l)$ are distinct from equations (32) and (39). To show that the map β given by equation (38) maps $S_{1,l}$ into a proper subset of $S_{1,l-1}$, write

$$\begin{aligned} \beta_1(j_1') \quad & := \min_{j_{1_k}} \{ \mid j_1 - j_{1_k} \mid : a_{1, j_{1_k}} = 2(l-1), \ \beta_1(j_1) \neq \beta_1(j_1') \\ f_{\beta_1(j_1')} \quad & = f_{\beta_2(j_1')} = \cdots = f_{\beta_{l-1}(j_1')}. \end{aligned} \tag{40}$$

Observe that $(\beta_1(j_1'), \beta_2(j_1'), \ldots, \beta_{l-1}(j_1')) \in S_{1,l-1}$ and distinct from $(\beta_1(j_i), \beta_2(j_i), \ldots, \beta_{l-1}(j_i))$ $(1 \leq i \leq l)$ from equations (32), (39) and (40). Equations (39) and (40) imply that the map $\beta : S_{1,l} \mapsto S_{1,l-1} \times S_{1,l-1} \times \cdots \times S_{1,l-1}$ given by equation (38) maps $S_{1,l}$ into a proper subset of $S_{1,l-1}$ in a fashion of 1 to l. This establishes inequality (37). See Table 1 for examples of primes p with $q \mid A(p)$, $q > p$, satisfying inequality (37), where $q_0 \leq 4$.

Lehmer's conjecture therefore is equivalent via equation (29) for $i = 1$ to:

$$\sum_{k=0}^{q-1} k a_{1,k} \not\equiv 0 \mod q. \tag{41}$$

Since both $A(p)$ and $B(p)$ are even and divisible by 691, we have $(A(p), B(p)) \geq 1382$. Suppose q divides both $A(p)$ and $B(p)$. Then by equation (29), we have:

$$\sum_{k=0}^{q-1} k a_{i,k} \equiv 0 \mod q \ (0 \leq i \leq q-1). \tag{42}$$

Clearly equation (42) is equivalent by equation (30) to:

$$\sum_{k=0}^{q-1} k a_{1,k} \equiv 0 \mod q. \tag{43}$$

Since $\sum_{k=0}^{q-1} k a_{0,k} = 0 \equiv 0 \bmod q$ by equation (26), it follows that $\{\sum_{k=0}^{q-1} k a_{i,k} \bmod q\}_{i=0}^{q-1} = \{0\}$, the trivial additive group modulo q. Conversely, equation (42) or (43) implies both $q \mid A(p)$ and $q \mid B(p)$ by equation (29). On the other hand, since nonzero $a_{i,k}$ $(0 \leq i \leq q-1)$ is even and ≥ 2 from equation (25), with the aid of the unique factorization theorem, equation (42) or (43) is equivalent to:

$$\min_{1 \leq i < j \leq q-1} \left(\sum_{k=0}^{q-1} k a_{i,k}, \ \sum_{k=0}^{q-1} k a_{j,k}\right) = 2q. \tag{44}$$

Consequently equation (42), (43) or (44) completely characterizes common odd prime factors of both $A(p)$ and $B(p)$. We thus have:

Lemma 3 *The following conditions are equivalent:*

(i) q *divides both* $A(p)$ *and* $B(p)$.

(ii) $\sum_{k=0}^{q-1} k a_{i,k} \equiv 0 \mod q \ (0 \leq i \leq q-1)$.

(iii) $\sum_{k=0}^{q-1} k a_{1,k} \equiv 0 \mod q$.

(iv) $\{\sum_{k=0}^{q-1} k a_{i,k} \bmod q\}_{i=0}^{q-1} = \{0\}$, *the trivial additive group modulo* q.

(v) $\min_{1\leq i<j\leq q-1}(\sum_{k=0}^{q-1} ka_{i,k},\ \sum_{k=0}^{q-1} ka_{j,k}) = 2q.$

Lemma 4 *(Main Lemma) Let* p *satisfy equation* (10) *or* (11) *and let* $q \mid A(p)$ *with* $q > p$*. Then* $\{\sum_{k=0}^{q-1} ka_{i,k} \bmod q\}_{i=0}^{q-1}$ *forms an additive group of order* q *modulo* q*.*

Proof. Let $a_{i,k}$ $(0 \leq i, k \leq q-1)$ be defined by equation (23). We have for each $i = 1, 2, \ldots, q-1$:

$$\begin{array}{rllll} & \sum_{k=0}^{q-1} ka_{i,k} & + & \sum_{k=0}^{q-1} ka_{q-i,k} & \\ = & \sum_{k=1}^{q-1} ka_{i,k} & + & \sum_{k=1}^{q-1} ka_{i,q-k} & \text{by } (28) \\ = & \sum_{k=1}^{q-1} ka_{i,k} & + & \sum_{k=1}^{q-1}(q-k)a_{i,k} & \\ = & q\sum_{k=1}^{q-1} a_{i,k} & & & \\ = & q\sum_{l=1}^{q_0} 2l \mid S_{1,l} \mid & \text{by} & (35) & \\ = & q(p-1) & \text{by} & (35) & \end{array} \tag{45}$$

Notice that equation (45) holds regardless of $\{\sum_{k=0}^{q-1} ka_{i,k} \bmod q\}_{i=0}^{q-1}$ being trivial or not. We claim that $\{\sum_{k=0}^{q-1} ka_{i,k}\}_{i=0}^{q-1}$ are all distinct. To show the claim observe that $\{S_{1,l}\}_{l=0}^{q_0}$ are disjoint from equation (32). Since $a_{i,k} = a_{1,i^{-1}k \bmod q}$ from equation (31), we have for each $l = 1, 2, \ldots, q_0$:

$$\begin{array}{lcll} \sum\limits_{k\in S_{1,l}} ka_{i,k} & = & \sum\limits_{k\in S_{1,l}} ka_{1,i^{-1}k \bmod q} & = \\ \sum\limits_{k\in S_{1,l}} ik \pmod q\ a_{1,k} & = & 2l\sum\limits_{k\in S_{1,l}} ik \pmod q. & \end{array} \tag{46}$$

It is evident for each $1 \leq i \neq j \leq q-1$ and each l $(1 \leq l \leq q_0)$ that:

$$\sum_{k\in S_{1,l}} ik \pmod q \neq \sum_{k\in S_{1,l}} jk \pmod q. \tag{47}$$

For each $1 \leq i \neq j \leq q-1$, conjunction of equations (46) and (47) leads us to

$$\begin{array}{rll} & \sum_{k=0}^{q-1} ka_{i,k} & \\ = & \sum_{l=1}^{q_0} 2l \sum\limits_{k\in S_{1,l}} ik \pmod q & \text{by } (46) \\ \neq & \sum_{l=1}^{q_0} 2l \sum\limits_{k\ inS_{1,l}} jk \pmod q & \text{by } (37)\ \&\ (47) \\ = & \sum_{k=0}^{q-1} ka_{j,k} & \text{by } (46). \end{array} \tag{48}$$

Equation (48) establishes the claim. Since $\sum_{k=0}^{q-1} ka_{1,k} \bmod q$ is a generator for the additive group $\{\sum_{k=0}^{q-1} ka_{i,k} \bmod q\}_{i=0}^{q-1}$ from equation (30) if it is nontrivial, it suffices therefore to show that

$$\sum_{k=0}^{q-1} ka_{1,k} \not\equiv 0 \bmod q. \tag{49}$$

Write

$$C_i := \sum_{k=0}^{q-1} ka_{i,k} \ (1 \leq i \leq q-1). \tag{50}$$

Notice that $\{C_i\}_{i=1}^{q-1}$ are distinct from equation (48). Rename C_i $(1 \leq i \leq q-1)$ again as C_i $(1 \leq i \leq q-1)$ in ascending order as follows:

$$C_1 < C_2 < \cdots < C_{q-1}. \tag{51}$$

We claim that there is at least one pair $\{C_j, C_{j+1}\}$ $(1 \leq j \leq q-2)$ from equation (51) such that

$$C_{j+1} - C_j < q-1 \text{ for some } j \ (1 \leq j \leq q-2). \tag{52}$$

Assume equation (52) is false. We then have:

$$\begin{array}{ll} & C_{q-1} \\ := & \max\limits_{1\leq i\leq q-1} \sum_{k=1}^{q-1} ka_{i,k} \ \text{ by (51)} \\ := & \sum_{k=1}^{q-1} ka_{i_0,k} \ \text{ for some } \ i_0 \ (1 \leq i_0 \leq q-1) \\ = & C_1 + \sum_{k=1}^{q-2}(C_{k+1} - C_k) \\ \geq & C_1 + \sum_{k=1}^{q-2}(q-1) \ \text{ by assumption} \\ = & C_1 + (q-2)(q-1) \\ > & (q-2)(q-1). \end{array} \tag{53}$$

On the other hand, we estimate C_{q-1} from equations (23) and (46). Since each nonzero $a_{i_0,k}$ $(0 \leq i_0 \leq q-1)$ is even ≥ 2 from equation (25), there are at most $(p-1)/2$ -numbers of nonzero $a_{i_0,k} \geq 2$ $(0 \leq k \leq q-1)$. Notice that each nonzero $a_{i_0,k}$ is a small even number due to equations (32) and (35) with $2 \leq a_{i_0,k} \leq 2q_0$ $(0 \leq k \leq q-1)$. It follows that there are at least $(q-(p-1)/2)$ -numbers of $a_{i_0,k}=0$ $(0\leq k\leq q-1)$. We then have:

$$\begin{array}{ll} & C_{q-1} \\ = & \sum_{k=0}^{q-1} k a_{i_0,k} \\ = & \sum_{k=0}^{q-1} i_0 k \pmod q\ a_{1,k} \quad \text{by (31)} \\ = & \sum_{l=1}^{q_0} 2l(\sum\limits_{k\in S_{1,l}} i_0 k \pmod q) \text{ by (46)} \\ < & (\sum_{l=1}^{q_0} 2l \mid S_{1,l} \mid)(q-1) \\ = & (p-1)(q-1) \quad \text{by (35)} \\ < & (q-2)(q-1). \end{array} \tag{54}$$

In the last line of inequality (54), we use the assumption $p+1<q$ and hence $p-1<q-2$. The last line of inequality (54) contradicts inequality (53). This establishes inequality (52). For j chosen from inequality (52), since each nonzero $a_{i,k}\geq 2$ $(1\leq i\leq q-1,\ 0\leq k\leq q-1)$, we then have:

$$2\leq (C_j,\ C_{j+1})=(C_j,\ C_{j+1}-C_j)<q-1. \tag{55}$$

Equation (55) implies $C_j:=\sum_{k=0}^{q-1} k a_{u,k}$ and $C_{j+1}:=\sum_{k=0}^{q-1} k a_{v,k}$ for some u,v $(1\leq u,\ v\leq q-1)$, have no common factor q, which leads to $q\nmid \sum_{k=0}^{q-1} k a_{1,k}$ in view of equation (30), thereby proving equation (49). Consequently, each $\sum_{k=0}^{q-1} k a_{i,k}$ $(1\leq i\leq q-1)$ has no factor q from equations (30) and (49). We thus have:

$$\sum_{k=0}^{q-1} k a_{i,k} \not\equiv 0 \bmod q,\ 1\leq i\leq q-1. \tag{56}$$

Equation (56) is equivalent that the map:

$$\{\sum_{k=0}^{q-1} k a_{i,k} \bmod q\}_{i=0}^{q-1} \longmapsto \mathbb{Z}/q\mathbb{Z}$$

is an isomorphism. Furthermore equations (45) and (56) reveal the structure of the additive group $\{\sum_{k=0}^{q-1} k a_{i,k} \bmod q\}_{i=0}^{q-1}$ which is nontrivial, namely

$$\sum_{k=0}^{q-1} k a_{i,k}+\sum_{k=0}^{q-1} k a_{q-i,k}\equiv\ 0 \bmod q,\ 1\leq i\leq q-1. \tag{57}$$

Equations (56) and (57) show $\sum_{k=0}^{q-1} ka_{i,k}$ mod q and $\sum_{k=0}^{q-1} ka_{q-i,k}$ mod q are additive inverse to each other modulo q for each $i = 1, 2, \ldots q-1$. Clearly $\sum_{k=0}^{q-1} ka_{0,k} = 0 \equiv 0 \pmod q$ is the additive identity modulo q from equation (26). This completes the proof of Lemma 4.

Since $p \nmid A(p)$ from equation (12), conjunction of Lemma 3 and Lemma 4 leads us to:

Corollary 5 *Let* $691 \mid A(p)$. *An odd prime* q *divides both* $A(p)$ *and* $B(p)$ *only if* $q < p$.

¿From Lemma 4, we have in particular for $i = 1$:

$$B(p) = 691 \sum_{j=1}^{p-1} \sigma_5(j)\sigma_5(p-j) \equiv \sum_{k=0}^{q-1} ka_{1,k} \not\equiv 0 \bmod q \text{ by (29) \& (56).} \tag{58}$$

Equation (58) implies $q \nmid B(p)$ and hence $A(p) \neq B(p)$ and $\tau(p) = (A(p) - B(p))/3 \neq 0$ via the unique factorization theorem if $691 \mid A(p)$. If $691 \nmid A(p)$, then since $691 \mid B(p)$ from equation (8), we trivially have $A(p) \neq B(p)$ and $\tau(p) = (A(p) - B(p))/3 \neq 0$ via the unique factorization theorem in this case too. We thus have:

Theorem 6 $\tau(p) \neq 0$ *for each prime* p.

For $691 \mid A(p)$ and $q \mid A(p)$ with $q > p$, since $\{\sum_{k=0}^{q-1} ka_{i,k}\}_{i=0}^{q-1}$ are distinct from equation (48) and since $q \nmid \sum_{k=0}^{q-1} ka_{i,k}$ $(1 \leq i \leq q-1)$ from Lemma 4, we have $\sum_{k=0}^{q-1} ka_{i,k} = 2^s t$ $(s \geq 1,\ t = \text{odd},\ 1 \leq i \leq q-1)$, where $q \nmid t$ from Lemma 4. Since $q > p$, and since each nonzero $a_{i,k} \geq 2$ from equation (25), there is at least one i $(1 \leq i \leq q-1)$ such that $\sum_{k=0}^{q-1} ka_{i,k} = 2t,\ q \nmid t$. We thus have from Lemma 1 (statement (v))with the aid of unique factorization theorem:

Corollary 7 *Suppose p satisfies equation (10) or (11). Let $q \mid A(p)$ with $q > p$. Then*

$$\min_{1\leq i<j\leq q-1}(\sum_{k=0}^{q-1} ka_{i,k}, \sum_{k=0}^{q-1} ka_{j,k}) = 2.$$

Now let $\alpha \geq 2$. Then equations (10) and (11) are no longer equivalent. As in the case of $\alpha = 1$, since $A(p^\alpha) \equiv 3 \bmod p^5$ and $p^{11\alpha-1} < A(p^\alpha) < p^{11\alpha}$ from equation (8), an almost identical proof of Lemma 2 works for $\alpha \geq 2$, where in equation (14), the upper limit for the sum is replaced by $11\alpha - 1$. We thus have:

Lemma 8 *Let $691 \mid A(p^\alpha)$ for $\alpha \geq 2$. There is at least one prime $q \mid A(p^\alpha)$ with $q > p^\alpha$.*

For $q \mid A(p^\alpha)$, construct matrix $[a_{i,k}]_{0\leq i,\, k\leq q-1}$ exactly the same way as in equation (23). Then properties (25) – (31), (33) – (37) hold with p replaced by p^α. Likewise almost identical proof of Lemma 4 works for $\alpha \geq 2$. We thus have:

Lemma 9 *Let $691 \mid A(p^\alpha)$ for $\alpha \geq 2$. Let $q \mid A(p^\alpha)$ with $q > p^\alpha$. Then $\{\sum_{k=0}^{q-1} ka_{i,k} \bmod q\}_{i=0}^{q-1}$ forms an additive group of order q modulo q.*

In particular for $i = 1$ from Lemma 9 and equation (29), we have for $\alpha \geq 2$

$$B(p^\alpha) = 691 \sum_{j=1}^{p^\alpha-1} \sigma_5(j)\sigma_5(p^\alpha - j) \equiv \sum_{k=0}^{q-1} ka_{1,k} \not\equiv 0 \mod q. \tag{59}$$

Equation (59) implies $q \nmid B(p^\alpha)$ and hence $A(p^\alpha) \neq B(p^\alpha)$ and $\tau(p^\alpha) = (A(p^\alpha) - B(p^\alpha))/3 \neq 0$ by the unique factorization theorem. If $691 \nmid A(p^\alpha)$, since $691 \mid B(p^\alpha)$ from equation (8), we then trivially have $A(p^\alpha) \neq B(p^\alpha)$ and $\tau(p^\alpha) = (A(p^\alpha) - B(p^\alpha))/3 \neq 0$ via the unique factorization theorem in this case too. We thus have:

Theorem 10 *$\tau(p^\alpha) \neq 0$ for each $\alpha \geq 2$.*

Finally we show that $\tau(n) \neq 0$ for any positive integer n.

Theorem 11 *(Lehmer's Conjecture)* $\tau(n) \neq 0$ *for each* $n \geq 1$.

Proof. Since $\tau(1) = 1$, it suffices to prove the theorem when n is composite from Theorem 6 and Theorem 10. Write

$$n = p_0^{s_0} p_1^{s_1} \ldots p_u^{s_u},\ p_0 := 2,\ s_0 \geq 0,\ s_j \geq 1,\ 1 \leq j \leq u.$$

Since $\tau(n)$ is multiplicative $([\ 1:92-93\ ], [\ 2:52-53\ ], [\ 4:122\ ], [\ 5\ ], [\ 6\ ])$, Theorem 11 readily follows from Theorem 6 or Theorem 10 , namely

$$\begin{array}{rcl} \tau(n) & = & \prod_{j=0}^{u} \tau(p_j^{s_j}) \\ & \neq & 0. \end{array} \tag{60}$$

This completes the proof.

Suppose for each $\alpha \geq 1$,

$$A(p^{\alpha}) \equiv 0 \mod 691. \tag{61}$$

Equation (61) is equivalent to:

$$p^{(\alpha+1)} \equiv 1 \mod 691 \text{ and } (p-1,\ 691) = 1. \tag{62}$$

Equation (62) implies the following periodicity theorem modulo 691:

Theorem 12 *(periodicity modulo* 691*) Suppose* $691 \mid A(p^{\alpha})$ *for* $\alpha \geq 1$. *Then we have:*

$$A(p^{\alpha+k(\alpha+1)}) \equiv 0 \mod 691,\ k = 0, 1, 2, \ldots$$

The values of α satisfying the periodicity of $A(p^\alpha) \equiv 0 \bmod 691$ for each $\alpha \geq 1$ have gaps in view of equation (62) and Fermat's little theorem, namely $A(p^\alpha) \not\equiv 0 \bmod 691$ if and only if the factors of $\alpha + 1$ do not divide $690 = 2.3.5.23$. Thus $A(p^\alpha) \not\equiv 0 \bmod 691$ for α in the following set S of numbers:

$$S := \{6, 10, 12, 16, 18, 28, 30, 36, 40, 42, 46, 48, 52, 58, \dots\}$$

Needless to say $A(p^\alpha) \neq B(p^\alpha)$ and hence $\tau(p^\alpha) \neq 0$ for each $\alpha \in S$ by equation (8) with the aid of the unique factorization theorem.

Remark 13 *If an odd prime* $q \mid A(p^\alpha),\ \alpha \geq 1$ *with* $q < p^\alpha$*, as long as* $\{\sum_{k=0}^{q-1} k a_{i,k} \bmod q\}_{i=0}^{q-1}$ *forms an additive group of order* q *modulo* q*, then* $q \nmid B(p^\alpha)$ *by Lemma* 4 *or Lemma* 9*. It follows that* $A(p^\alpha) \neq B(p^\alpha)$ *and hence* $\tau(p^\alpha) = (A(p^\alpha) - B(p^\alpha))/3 \neq 0$ *in this case too. For* $691 \mid A(p)$*, computer simulation reveals* $A(p)$ *has at least one odd prime factor* $q \neq 691,\ q \mid A(p)$ *with* $q < p$ *for which* $q \nmid B(p)$ *for each prime* $p \leq 1100000$ *except* $p = 186569, 290219, 464351, 671651$*. Let* $691 \mid A(p)$ *and let* $A_1(p)$ *be the product of prime divisors* $q \mid A(p)$ *for which* $q < p$ *with their respective powers and* $A_2(p)$ *the product of prime divisors* $q \mid A(p)$ *for which* $q > p$ *with their respective powers. Computer simulation shows* $C_1 p^2 < A_1(p) < C_2 p^5$ *and* $C_3 p^6 < A_2(p) < C_4 p^{10}$ *with absolute constants* $C_1, C_2, C_3, C_4 < 1$ *for primes* $p \leq 1100000$*.*

In Table 1, we list primes p such that both 691 and q divide $A(p)$ with $q > p$ and the cardinality $\mid S_{1,l} \mid \ (1 \leq l \leq 5)$, thereby confirming inequality (37) with $q_0 \leq 4$. Notice that in Table 1, each prime p with the associated prime $q \mid A(p)$ with $q > p$, satisfies equations (34) and (35). Computer simulation reveals that the majority of respective relatively large odd prime factors less than p of both $A(p)$ and $B(p)$ are distinct. Likewise an overwhelming majority of common odd prime factors of both $A(p)$ and $B(p)$ for which $691 \mid A(p)$ are relatively small apart from 691, thereby confirming Corollary 5. In Table 2, we list primes $p \leq 3000000$ such that $691 \mid A(p)$ and the odd prime factors of $(A(p),\ B(p))$ are ≥ 11.

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Table 1

p	q	$\mid S_{1,0} \mid$	$\mid S_{1,1} \mid$	$\mid S_{1,2} \mid$	$\mid S_{1,3} \mid$	$\mid S_{1,4} \mid$	$\mid S_{1,5} \mid$
8291	216113	212008	4065	40	0	0	0
29021	1357091	1342657	14358	76	0	0	0
30403	1283839	1268731	15015	93	0	0	0
34549	789673	772578	16918	175	2	0	0
51133	112919	89995	20474	2267	174	9	0
53897	371549	345582	25014	925	28	0	0
96739	392957	347376	42917	2543	118	3	0

Table 2

p	$(A(p),\ B(p))$
547271	2.3.11.691
610843	2.3.17.691
988129	2.3.5.13.691
1112509	2.3.5.23.691
1336393	2.3.101.691
1405493	2.3.113.691
1716463	$2.3^2.23.691$
1875373	2.23.691
1940327	$2^2.3^2.13.691$
2126897	$2.3^3.19.691$
2128279	$2^2.5.11.691$

p	$(A(p),\ B(p))$
2161447	$2^2.23.691$
2198761	$2.43.691$
2447521	$2.23.691$
2479307	$2.23.691$
2538733	$2.11.691$
2542879	$2^4.3.5.23.691$
2956097	$2.23.691$

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