ECO order theory I. Lattices of lattice paths*

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Abstract

We consider posets of lattice paths (endowed with a natural order) and begin the study of such structures. We give an algebraic condition to recognize which ones of these posets are lattices. Next we study the class of Dyck lattices (i.e., lattices of Dyck paths) and give a recursive construction for them. The last section is devoted to the presentation of a couple of open problems.

Keywords - ECO method, lattice paths, posets.

1 Introduction

When a class of objects is introduced in mathematics, the first problem that naturally arises is to count how many objects there are. Typically, one recognizes an interesting numerical parameter and tries to enumerate the objects according to it, so obtaining a sequence of nonnegative integers. This is the main problem of enumerative combinatorics.

The second step (after mere enumeration) is to look for some "mathematical" structure (like, e.g., operations) the class of objects naturally possesses. As a matter of fact, one of the simplest structure that can be found is an order relation, as Gian-Carlo Rota suggested in his masterful paper [12]. By the way, his theory of Möbius functions of posets leads to many deep enumerative results, so proving that a better knowledge of the mathematical structure of a set gives a better insight even on his combinatorial and enumerative properties.

In this work we deal with classes of lattice paths (like, e.g., Dyck paths, Motzkin paths, Schröder paths, Lukasiewicz paths, etc.) which can be ordered in a completely natural manner. Our goal is to begin a sistematic study of the posets of paths arising in this way. A similar point of view

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has been undertaken in [1], where the authors study the posets arising from Delannoy paths; however, the present work deals with essentially different classes of paths. The only general problem we tackle here is to determine in which cases a poset of paths is a lattice, and we propose a possible solution to this problem. Next, we focus on the study of a single type of paths, namely Dyck paths, and we provide an explicit (recursive) construction for the lattices of Dyck paths. It is then easy to see that a suitable, slight modification of such a construction can be successfully applied also to lattices of Schröder paths. We remark that in [11, 10] some similar problems are studied: the authors provide the construction of the lattice of partitions of a given integer n (with the dominance order) starting from the knowledge of the lattice of partitions of $n-1$. However, the methods used in the present work are completely different from those employed in [11, 10].

Quite surprisingly, our study sheds new light on a method of enumeration, usually called *ECO method*, which has proved fruitful in many problems of enumeration. A rough description of this method is given in section 3 (mainly with the help of the example of Dyck paths). What is important to point out here is that such a method provides (among other things) a partition of the objects of size n of a given class (when a suitable definition of size is provided); this fact is obviously crucial in trying to enumerate the objects of the class according to the size. What is typical of the ECO method when applied to the construction of a class of paths (Dyck, Motzkin, Schröder, and so on) is that the equivalence classes of the above mentioned partition have a nice order structure: they are *chains* with respect to the natural order of paths introduced in this paper. So the ECO method, if applicable, provides a chain partition of the lattice of paths under consideration.

In this paper we would like to begin a deep investigation on the order-theoretic aspects of the ECO method. We hope that the present work will be followed soon by analogous ones concerning several other classes of combinatorial objects naturally definable by means of some ECO-construction.

Throughout the whole paper $\mathbf{N}$ will denote the set of nonnegative integers. For $x, y \in \mathbf{Z}$, the expression $[x, y]$ will always indicate the interval of *integer* numbers between x and y.

Given a lattice $\mathbf{L}$, a *filter* of $\mathbf{L}$ is a subset F closed for meet and such that, if $x \in F$ and $x \leq y$, then $y \in F$. Dually, an *ideal* I of $\mathbf{L}$ is a subset closed for join and such that, if $x \in I$ and $y \leq x$, then $y \in I$. If $\mathbf{L}$ has minimum $\mathbf{0}$, a *rank function* for $\mathbf{L}$ is a map $r : \mathbf{L} \longrightarrow \mathbf{N}$ such that $r(\mathbf{0}) = 0$ and, if y covers x, then $r(y) = r(x) + 1$. For any other poset and lattice concept, we refer to the texts [6, 7]. In [13] a whole chapter is devoted to the study of posets, especially in connection with enumerative and algebraic combinatorics.

Our last remark concerns the use of the term "isomorphism" (and similar ones): in this paper it has to be considered a synonym of *lattice isomor-*

phism, i.e. a bijective map between two lattices which is both join-preserving and meet-preserving.

2 Definitions and a first characterization

Let Γ be a finite subset of $\mathbf{Z}$. We call Γ*-path* every element of the set:

$$\begin{aligned}\mathcal{C}^{\Gamma} &= \{f : \mathbf{N} \longrightarrow \mathbf{N} \mid \exists n \in \mathbf{N} : f(0) = f(n) = 0; \\ &\quad f(i) = 0, \forall i \geq n; f(k+1) - f(k) \in \Gamma, \forall k < n\}. \end{aligned} \tag{1}$$

The elements of the set Γ are called *steps*. It is clear that this definition is completely equivalent to the usual definition of a lattice path starting from the origin, ending on the x-axis and using steps of a prescribed type. Observe that this definition allows us to consider only lattice paths whose steps have length 1; so, for example, Schröder paths do not fall within our definition. The following, easy examples can be immediately checked by the reader:

- •) $\{-1, 1\}$-paths are the ordinary Dyck paths;
- •) $\{-1, 0, 1\}$-paths are the ordinary Motzkin paths.

The positive integer n in the definition of $\mathcal{C}^{\Gamma}$ depends on the single path f, and it is called the *length* of f. We will denote $\mathcal{C}_n^{\Gamma}$ the set of Γ-paths of length n.

It is possible to introduce a natural order between the elements of each set $\mathcal{C}_n^{\Gamma}$, by defining $f \leq g$ in $\mathcal{C}_n^{\Gamma}$ whenever $f(i) \leq g(i)$, for any $i \leq n$. The fact that $[\mathcal{C}_n^{\Gamma}; \leq]$ is a poset is immediate. Instead, it could be interesting to wonder whether such an order is a lattice order, that is: for which $\Gamma \subseteq \mathbf{Z}$ the poset $[\mathcal{C}_n^{\Gamma}; \leq]$ is a lattice? Obviously, the meet and join operations induced on $\mathcal{C}_n^{\Gamma}$ by the order introduced above are defined pointwise, i.e.:

$$(f \vee g)(i) = f(i) \vee g(i),$$
$$(f \wedge g)(i) = f(i) \wedge g(i),$$

where the join and meet in the r.h.s. are computed in $\mathbf{N}$.

We will give an answer to the above problem in the next theorem. To state and prove our result we need to introduce a few notation. We set $\gamma_+ = \max \Gamma$, $\gamma_- = \min \Gamma$; it is clear that $\gamma_+ \geq 0$ and $\gamma_- \leq 0$ (that is, Γ must contain both up and down steps, otherwise the set of Γ-paths would be empty). We call *diameter* of Γ the difference between the maximum and the minimum of Γ, that is $\overline{\gamma} = \operatorname{diam} \Gamma = \gamma_+ - \gamma_-$. Finally, we define the set $\Delta_n^{\Gamma} \subseteq \mathbf{Z}$ as follows (for any $n \in \mathbf{N}$):

$$\Delta_n^{\Gamma} = \left\{ \sum_{i=1}^{n} (x_i - y_i) \mid x_i, y_i \in \Gamma, x_i \neq y_j, \forall i, j \right\}. \tag{2}$$

Now we are ready to state our main result concerning the algebraic characterization of lattices of lattice paths.

Theorem 2.1 *$\mathcal{C}_n^\Gamma$ is a (finite) distributive lattice if and only if, for every $h \in \mathbf{N}$:*

$$(\Delta_h^\Gamma + \Gamma) \cap [\gamma_-, \gamma_+] \subseteq \Gamma. \tag{3}$$

Proof. It is clear that all the axioms of distributive lattices are trivially satisfied by $\mathcal{C}_n^\Gamma$. The only difficult thing to check in order to prove that $\mathcal{C}_n^\Gamma$ is a lattice is that it is closed with respect to the natural join and meet operations described above, i.e., if $f, g \in \mathcal{C}_n^\Gamma$, then $f \wedge g, f \vee g \in \mathcal{C}_n^\Gamma$. So take $f, g \in \mathcal{C}_n^\Gamma$, and suppose that, for a given $k \in \mathbf{N}$, $f(k) < g(k)$ and $f(k+1) > g(k+1)$ (this is clearly the only difficult case to study).

For suitable $\gamma_1, \gamma_2 \in \Gamma$, we have

$$f(k+1) = f(k) + \gamma_1,$$
$$g(k+1) = g(k) + \gamma_2.$$

In order that the path $f \vee g$ belongs to Γ, we must impose that the step $f(k+1) - g(k)$ is contained in Γ. Now $f(k+1) - g(k) = (f(k) - g(k)) + \gamma_1$. Obviously both $f(k)$ and $g(k)$ can be expressed as a sum of k elements of Γ; more precisely, we can set:

$$f(k) = \sum_{i=1}^{k} \tilde{x}_i, \qquad g(k) = \sum_{i=1}^{k} \tilde{y}_i,$$

where $\tilde{x}_i, \tilde{y}_i \in \Gamma$. Thus we can express the difference $f(k) - g(k)$ as a sum of differences between elements of Γ; therefore, possibly deleting pairs of equal steps appearing both in the expressions of $f(k)$ and $g(k)$, for a suitable $h \in \mathbf{N}$, we have:

$$f(k) - g(k) = \sum_{i=1}^{h} (x_i - y_i),$$

where $x_i \neq y_j$, for every $i, j \leq h$ ($x_i, y_j \in \Gamma$).

Now we are very close to complete our proof. Imposing that $f(k+1) - g(k) = (f(k) - g(k)) + \gamma_1 \in \Gamma$ leads in general to the condition

$$((\Delta_h^\Gamma \cap \;] - \overline{\gamma}, 0]) + \Gamma) \cap [\gamma_-, \gamma_+] \subseteq \Gamma, \qquad \forall h \in \mathbf{N}, \tag{4}$$

(recall the definition of the sets Δ_h^Γ given above). The reasons for which we have considered the intersection $\Delta_h^\Gamma \cap \;] - \overline{\gamma}, 0]$ are the following:

- •) $f(k) - g(k)$ must be negative, since we have supposed that $f(k) < g(k)$;

•) the paths f and g must cross, so the difference between $f(k)$ and $g(k)$ has to allow the crossing: this is why such difference must be greater than $-\overline{\gamma}$.

Also the fact that we consider the intersection with $[\gamma_-, \gamma_+]$ needs an explanation. Since we are supposing there is a crossing, necessarily we have $f(k+1) \geq g(k+1)$ and $f(k) \leq g(k)$, whence:

$$\gamma_- \leq g(k+1) - g(k) \leq f(k+1) - g(k) \leq f(k+1) - f(k) \leq \gamma_+.$$

Thus, condition (4) is necessary for the existence of the sup $f \vee g$ in $\mathcal{C}_n^\Gamma$. Analogously, it can be shown that a similar condition must be verified for the existence of the inf $f \wedge g$, and precisely:

$$((\Delta_h^\Gamma \cap [0, \overline{\gamma}[\ ) + \Gamma) \cap [\gamma_-, \gamma_+] \subseteq \Gamma, \qquad \forall h \in \mathbf{N}. \tag{5}$$

Putting things together, we have proved that, if $\mathcal{C}_n^\Gamma$ is a lattice, then:

$$(\Delta_h^\Gamma + \Gamma) \cap [\gamma_-, \gamma_+] \subseteq \Gamma, \qquad \forall h \in \mathbf{N}. \tag{6}$$

Conversely, suppose that condition (3) is satisfied. Then, if the Γ-paths f and g cross between k and $k+1$, the fact that $f(k+1) - g(k)$ and $g(k+1) - f(k)$ both belong to Γ is ensured by the hypothesis, and we are done. ■

Corollary 2.1 *If Γ is a set of consecutive integers (i.e. $\Gamma = [\gamma_1, \gamma_2]$), then $\mathcal{C}_n^\Gamma$ is a lattice.*

Proof. Indeed, in this case $\Gamma = [\gamma_-, \gamma_+]$, and so we get immediately $(\Delta_h^\Gamma + \Gamma) \cap \Gamma \subseteq \Gamma$. ■

Corollary 2.2 *If $\Gamma = \{-b, a\}$ $(a, b \in \mathbf{N})$, then $\mathcal{C}_n^\Gamma$ is a lattice.*

Proof. First observe that, in the assumed hypothesis, we have:

$$\Delta_h^\Gamma = \{(a+b)z \mid z \in \mathbf{Z}\},$$

and so

$$\Delta_h^\Gamma + \Gamma = \{(a+b)z + a, (a+b)z - b \mid z \in \mathbf{Z}\}.$$

Now consider the set $(\Delta_h^\Gamma + \Gamma) \cap [-b, a]$, i.e. the elements of $\Delta_h^\Gamma + \Gamma$ contained in the integer interval $[-b, a]$. If c is such an element then two cases are possible.

i) $c = (a+b)z + a$: then z must necessarily be of the form

$$z = \frac{c-a}{a+b} \in \mathbf{Z};$$

the r. h. s. is an increasing function of c, moreover $c = -b$ implies that $z = -1$ and $c = a$ implies that $z = 0$. Since $z \in \mathbf{Z}$, this means that no value of c in $[-b, a]$ is allowed other than $c = -b$ and $c = a$, and so $c \in \{-b, a\} = \Gamma$.

ii) $c = (a+b)z - b$: a similar argument leads to the same conclusion, i.e. $c \in \{-b, a\} = \Gamma$. ■

Remark. Each of the previous results holds also if Γ is infinite (on the right, on the left, on both sides): to obtain the right stataments it is enough to set $\gamma_- = -\infty$ and/or $\gamma_+ = +\infty$.

Examples.

1. Motzkin paths of a given length n have a lattice structure. Indeed, as we have seen before, Motzkin paths correspond to Γ-paths for $\Gamma = \{-1, 0, 1\}$, so in this case Γ is an integer interval and we can apply corollary 2.1.

2. Dyck paths are precisely $\{-1, 1\}$-paths, so, thanks to corollary 2.2, they constitute a lattice for every even length $2n$. Analogously, also $\{2, -1\}$-paths of any given length (necessarily of the form $3n$) constitute a lattice.

3. Recalling the preceding remark, we can consider also paths with an infinite set of steps as, for example, the so-called Lukasiewicz paths. According to our vocabulary, they are Γ-paths for $\Gamma = \{-1\} \cup \mathbf{N}$. By translating corollary 2.1 into its right version for the infinite set Γ, it is immediate to conclude that Lukasiewicz paths of any given length form a lattice.

Remark. As we have said at the beginning, in this paper we will be concerned with paths using exclusively single steps (i.e., of length 1). Anyway, any class of paths of the same length can be endowed with the natural order we have defined, therefore it makes sense to ask whether such order is a lattice order. For example, it is natural to wonder whether Schröder paths of a given length possess a lattice structure. In this case, it is not difficult to see that the answer is positive, even if we prefer not to go into details. The reason for which everything works is that, if f, g are Schröder paths (of the same length) and $g(k) - f(k) = 1$, it means that either f or g possesses a double horizontal step having its middle point in k, so f and g can not cross at $k + 1$.

3 Dyck lattices

In the previous section we have shown that a lot of classes of paths often used in combinatorics can be endowed with a lattice structure in a natural way. In this section we try to examine in details the lattice structure of a very well-known class of paths, namely Dyck paths. For a very detailed survey on Dyck paths and their enumeration, see [8].

We have just proved that, for any given $n \in \mathbf{N}$, Dyck paths of length $2n$ constitute a finite distributive lattice: we will denote it by $\mathcal{D}_n$. Throughout this whole section we will always denote the minimum and the maximum of $\mathcal{D}_n$ by $\mathbf{0}$ and $\mathbf{1}$, respectively (independently from n). In fig. 1 we have drawn the Hasse diagrams of $\mathcal{D}_n$ for small values of n.

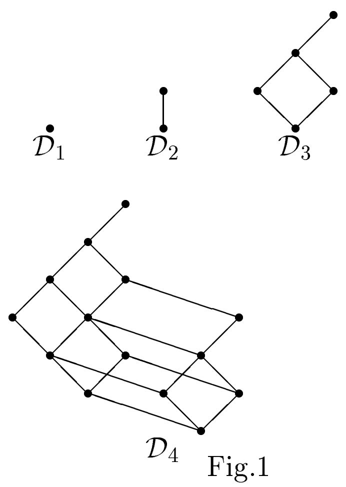

Fig.1

Our goal is to give a description of the shape of the lattices $\mathcal{D}_n$ and then to provide an efficient way for constructing $\mathcal{D}_n$ starting from the knowledge of $\mathcal{D}_{n-1}$, in the same spirit of [11], where the same problem is solved (in a completely different way) for lattices of integer partitions.

The first thing we observe is that $\mathcal{D}_n$ has a *rank function*: this follows immediately from the fact that it is a distributive lattice (see [7]). The determination of the rank of an element of $\mathcal{D}_n$ is not particularly difficult.

Proposition 3.1 *If we denote by r_n the rank function of $\mathcal{D}_n$, then, given $f \in \mathcal{D}_n$, we have:*

$$r_n(f) = \frac{\alpha(f) - n}{2}, \tag{7}$$

where $\alpha(f)$ denotes the area *of f, that is, by definition, the sum of the values of f:*

$$\alpha(f) = \sum_{k=1}^{n} f(k).$$

Proof. The minimum $\mathbf{0} \in \mathcal{D}_n$ is clearly the function defined as follows (for $k \leq 2n$):

$$\mathbf{0}(k) = \begin{cases} 0, & \text{if } k \text{ is even;} \\ 1, & \text{if } k \text{ is odd.} \end{cases}$$

Thus we have immediately $r_n(\mathbf{0}) = 0 = \frac{n-n}{2} = \frac{\alpha(\mathbf{0})-n}{2}$. Now, by induction, suppose that $r_n(f) = \rho = \frac{\alpha(f)-n}{2}$ for a given $f \in \mathcal{D}_n$, and take $g \in \mathcal{D}_n$ such that $f \prec g$ (this is a standard notation to mean that f is covered by g). Then, it is easy to see that there exists precisely one k such that $2 \leq k \leq 2n-2$, $f(h) = g(h)$ for every $h \neq k$ and $g(k) = f(k)+2$. Therefore, by the definition of area given above, $\alpha(g) = \alpha(f)+2$ and, by the definition of rank function,

$$r_n(g) = \rho + 1 = \frac{\alpha(f)-n}{2} + 1 = \frac{\alpha(f)+2-n}{2} = \frac{\alpha(g)-n}{2}. \quad \blacksquare$$

Remark. Thanks to the above proposition, we can assert that two Dyck paths of the same length have the same rank if and only if they have the same area.

Now we come to describe the construction of $\mathcal{D}_{n+1}$ starting from $\mathcal{D}_n$. The first thing to observe is that it is possible to determine a very natural chain partition of any Dyck lattice, suggested by a particular method of enumeration, called the *ECO method*, which in this context reveals some unexpected peculiarities in exploring order-theoretic properties of a combinatorial structure.

The ECO method was introduced in a series of articles, due to Pinzani et al., in which they fruitfully apply a particular tool, namely *succession rules* (deeply studied by West, see e.g. [14]), to give a purely combinatorial construction performing a local recursive expansion of a given class of objects. Such a construction often allows to determine a functional equation satisfied by the generating function of the combinatorial structure under consideration. This, in turn, can be solved (in many cases) to give an explicit expression for the desired generating function. For a detailed survey concerning the ECO method, we suggest [2].

In this work, we are mainly interested in the ECO construction of Dyck paths and in its properties with respect to the order introduced on Dyck paths. So let us briefly recall how such construction works.

Consider a Dyck path P of length $2n$, and suppose that the length of its last descent (i.e., of its last sequence of descending steps) is k. Then we construct $k+1$ Dyck paths of length $2n+2$ starting from P (they will be called the *sons* of P) simply by inserting a peak in every point of its last descent. In fig. 2 it is shown in a concrete example how this construction works.

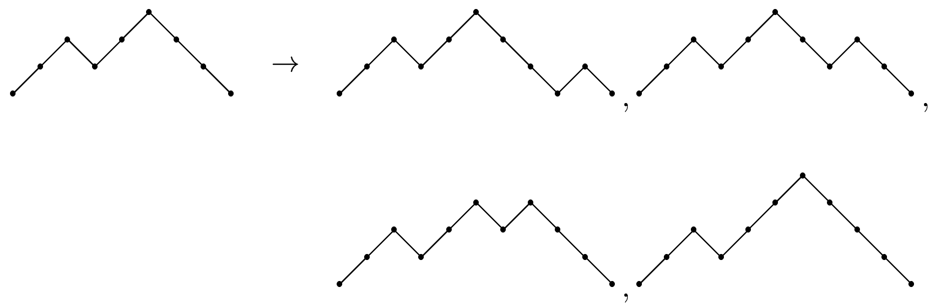

Fig.2

It is not difficult to see that, by performing this construction on all Dyck paths of length $2n$, one obtains every Dyck path of length $2n+2$ exactly once. Moreover, such a construction possesses strong recursive properties, which can be encoded by the following succession rule:

$$\Omega : \begin{cases} (2) \\ (k) \rightsquigarrow (2)(3)\dots(k)(k+1). \end{cases} \tag{8}$$

In the above expression it has to be intended that every Dyck path whose last descent has length $k-1$ (encoded by the label (k)) produces k sons (through the above ECO construction) having last descents of length $1, 2, \dots, k-1, k$ respectively. A possible way of visualizing this succession rule is to draw its *generating tree*, that is the infinite rooted labelled tree in which the root is labelled (2) and every node labelled (k) has (k) sons labelled $(2), (3), \dots, (k), (k+1)$ respectively. Thus the number of Dyck paths of length $2n$ is precisely the number of the nodes at level n of the generating tree.

The above ECO construction of Dyck paths provides a partition of the sets of Dyck paths of any fixed length. So one can consider the equivalence classes of those Dyck paths having the same father. The following result is stated without proof, since it is completely trivial; nevertheless, it contains the main order-theoretic feature of the ECO construction of Dyck paths.

Theorem 3.1 *The set of the sons of a Dyck path of length $2n-2$ forms a saturated chain in the lattice $\mathcal{D}_n$ (i.e. a chain such that for any two elements x and y in it, $x < y$ implies that x is covered by y). Therefore, the ECO method provides a partition into saturated chains of every Dyck lattice, which we will refer to as the* ECO-partition *of $\mathcal{D}_n$.*

In fig. 3 the ECO-partition of $\mathcal{D}_4$ is drawn.

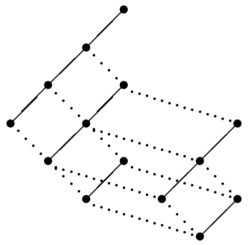

Fig.3

Observe that the rule in (8) is not the only possible rule defining Catalan numbers; in [3] many different rules for Catalan numbers are derived, and the same thing can be done for any other numerical sequence. However, it turns out that this is the "best" one from an order-theoretical point of view, meaning that the nice order structure of the equivalence classes gives a considerable help in the description of Dyck lattices.

To understand the structure of Dyck lattices it remains now to investigate how the saturated chains of the ECO-partition of $\mathcal{D}_n$ are linked together. This is precisely what we are going to do in the next pages.

First of all we describe another partition of $\mathcal{D}_n$ into sublattices (not necessarily chains) which follows in a very natural way from some elementary geometric properties of Dyck paths. More precisely, the sublattices of such a partition are certain filters of $\mathcal{D}_{n-1}$.

Denote by $\mathcal{D}_{nk}$ the set of all Dyck paths starting with precisely k rise steps (so that the $(k+1)$-st step is a fall step). In symbols:

$$\mathcal{D}_{nk} = \{f \in \mathcal{D}_n \mid \forall i \leq k, f(i) = i, f(k+1) = k-1\}. \tag{9}$$

It is clear that this definition is meaningful only when $k \leq n$. In particular, $\mathcal{D}_{n0} = \emptyset$ and $\mathcal{D}_{nn} = \{\mathbf{1}\}$. In fig. 4 it is shown how a path in $\mathcal{D}_{n5}$ starts.

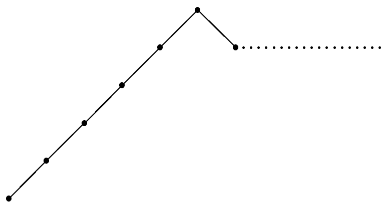

Fig.4

Proposition 3.2 *For every $k \leq n$, $\mathcal{D}_{nk}$ is a sublattice of $\mathcal{D}_n$. Moreover, the family of sublattices $\{\mathcal{D}_{nk} \mid 0 < k \leq n\}$ constitutes a partition of $\mathcal{D}_n$.*

Proof. It is trivial to observe that, if two paths starts with the same sequence of steps, then this happens for their join and meet as well: hence the first part of the proposition immmediately follows. The second part is obvious, since the starting sequence of rise steps is uniquely determined for every Dyck path. ■

In particular, for $\mathcal{D}_{n1}$ we have the following, nice result.

Proposition 3.3 $\mathcal{D}_{n1} \simeq \mathcal{D}_{n-1}$.

Proof. Consider the function mapping each element $f \in \mathcal{D}_{n1}$ into the element of $\mathcal{D}_{n-1}$ obtained by removing the first two steps of f. It is clear that this is a bijection which preserves joins and meets. ■

The above, very simple proposition states that, for every n, $\mathcal{D}_n$ contains a copy of $\mathcal{D}_{n-1}$, which is precisely the ideal of the elements of $\mathcal{D}_n$ starting with a hill (i.e. a peak at height 1).

Now consider the sublattice $\mathcal{D}_{nk}$ of $\mathcal{D}_n$, for $k > 1$. The next proposition asserts that this lattice "lives" also inside $\mathcal{D}_{n-1}$ as a particular filter.

Proposition 3.4 *Denote by $F_{(n-1)(k-1)}$ the subset of $\mathcal{D}_{n-1}$ of all Dyck paths starting with $k-1$ rise steps (without any further hypothesis on the kth step). Then $F_{(n-1)(k-1)}$ is a filter, and is isomorphic to $\mathcal{D}_{nk}$.*

Proof. The fact that $F_{(n-1)(k-1)}$ is a filter of $\mathcal{D}_{n-1}$ is immediate (the argument is the same as that of proposition 3.2). Now consider any path in $\mathcal{D}_{nk}$ and delete its first peak (which, by definition, occurs precisely after the first $k-1$ rise steps). In this way we obtain a path of $\mathcal{D}_{n-1}$ starting with $k-1$ rise steps, that is an element of $F_{(n-1)(k-1)}$. It is clear that this correspondence is a meet- and join-preserving bijection between $\mathcal{D}_{nk}$ and $F_{(n-1)(k-1)}$, so the proof is complete. ■

Remark. Notice that, as a filter of $\mathcal{D}_{n-1}$, $F_{(n-1)(k-1)}$ is the principal filter generated by the least Dyck path starting with a hill of height $k-1$ (that is a sequence of $k-1$ rise steps followed by a sequence of $k-1$ fall steps).

Next we show how the sublattices $\mathcal{D}_{nk}$ are linked together to form the whole $\mathcal{D}_n$.

Proposition 3.5 *Consider the function* $\varphi_{nk} : \mathcal{D}_{n(k+1)} \longrightarrow \mathcal{D}_{nk}$ *defined as follows. Let* $f \in \mathcal{D}_{n(k+1)}$*, so that* $f(k+1) = k+1$*,* $f(k+2) = k$*; we set by definition* $\varphi_{nk}(f) = g \in \mathcal{D}_{nk}$*, where* $f(i) = g(i), \forall i \neq k+1$*, and* $g(k+1) = k-1$*. Then* φ_{nk} *is a lattice monomorphism. Moreover, if* $g = \varphi_{nk}(f)$*, then* $f \prec g$ *in* $\mathcal{D}_n$*.*

Proof. First observe that φ_{nk} is well-defined, since $\varphi_{nk}(f)$ is clearly an element of $\mathcal{D}_{nk}$. If $\varphi_{nk}(f_1) = \varphi_{nk}(f_2)$, then $f_1(i) = f_2(i)$ at least for every $i \neq k+1$; moreover $f_1(k+1) = k+1 = f_2(k+1)$ (since $f_1, f_2 \in \mathcal{D}_{n(k+1)}$), so that $f_1 \equiv f_2$. Now consider $f, f' \in \mathcal{D}_{n(k+1)}$, so that f and f' coincide for $i \leq k+2$. Then it is clear that also $f \vee f'$ coincides with both f and f' for $i \leq k+2$. Therefore, for $i \neq k+1$, we have $\varphi_{nk}(f \vee f')(i) = (f \vee f')(i) = (\varphi_{nk}(f) \vee \varphi_{nk}(f'))(i)$, whereas $\varphi_{nk}(f \vee f')(k+1) = k-1 = (\varphi_{nk}(f) \vee \varphi_{nk}(f'))(k+1)$, and so $\varphi_{nk}(f \vee f') = \varphi_{nk}(f) \vee \varphi_{nk}(f')$. The same argument can be used also for meet. Finally, it is clear that $f \prec \varphi_{nk}(f)$, since $\alpha(f) + 2 = \alpha(\varphi_{nk}(f))$ and $f < g$ (recall that $\alpha(f)$ denotes the area of f, as defined in proposition 3.1). ■

Fig. 5 illustrates how the map φ_{n3} works.

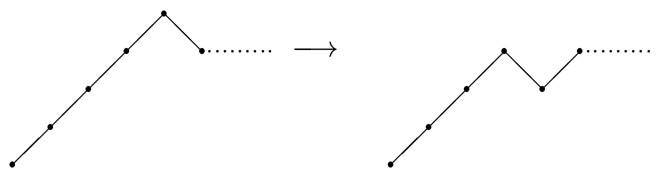

Fig.5

Remark. Observe that $\varphi_{nk}(\mathcal{D}_{n(k+1)})$ is a filter of $\mathcal{D}_{nk}$. More precisely, it is the filter of all the paths of $\mathcal{D}_{nk}$ whose $(k+2)$-nd step is a rise step (the reader can easily check that this is actually a filter of $\mathcal{D}_{nk}$).

Let us summarize what we have done until now. We have discovered a special partition of $\mathcal{D}_n$ into sublattices (the $\mathcal{D}_{nk}$'s). We have shown that a certain filter of $\mathcal{D}_{n-1}$ is isomorphic to $\mathcal{D}_{nk}$ [1], which, in turn, is isomorphic to a filter of $\mathcal{D}_{n(k-1)}$. As a consequence of this fact, we have that $\mathcal{D}_n$ can be constructed starting from $\mathcal{D}_{n-1}$ or, which is the same, from $\mathcal{D}_{n1}$ by glueing together some of its filters. Notice that this is all we need to draw the Hasse diagram of $\mathcal{D}_n$, since $x \prec y$ in $\mathcal{D}_n$ if and only if either $x \prec y$ in $\mathcal{D}_{nk}$, for some k, or $x \in \mathcal{D}_{nk}$, $y \in \mathcal{D}_{n(k+1)}$ and $y = \varphi_{nk}(x)$.

Now, to give a complete formalization of our construction we need to define a particular operation on lattices.

[1] In particular, $\mathcal{D}_{n1}$ is isomorphic to the whole $\mathcal{D}_{n-1}$.

Given a lattice $\mathbf{L}$, let F be a filter of $\mathbf{L}$. We define the *F-filtered doubling* of $\mathbf{L}$ to be the lattice

$$\mathbf{L} \times_F \mathbf{2} = \{(x,n) \in L \times \{0,1\} \mid n = 1 \Rightarrow x \in F\} \tag{10}$$

endowed with coordinatewise meet and join.

Remark. We wish to point out that this construction is not new at all. In fact, this is merely a particular instance of a *subdirect product* of lattices. Recall that a subdirect product of two lattices $\mathbf{L}$ and $\mathbf{M}$ is, by definition, any sublattice $\mathbf{N}$ of the direct product $\mathbf{L} \times \mathbf{M}$ such that both the projections of $\mathbf{N}$ onto $\mathbf{L}$ and $\mathbf{M}$ are surjective homomorphisms (see, for example, [6]). So, an F-filtered doubling of $\mathbf{L}$ is nothing more than a particular subdirect product of the lattices $\mathbf{L}$ and $\mathbf{2} = [\{0,1\}; \vee, \wedge]$. By the way, this also proves that $\mathbf{L} \times_F \mathbf{2}$ is actually a lattice. Observe that, if $F = L$, then $\mathbf{L} \times_F \mathbf{2} = \mathbf{L} \times \mathbf{2}$.

Now consider, for every positive integer $k < n$, all the filters $F_{(n-1)k}$ of $\mathcal{D}_{n-1}$ as defined above. So, for example, $F_{(n-1)1}$ is the whole lattice $\mathcal{D}_{n-1}$ and $F_{(n-1)(n-1)}$ is the singleton of the maximum of $\mathcal{D}_{n-1}$. We introduce the following recursive notation:

$$\begin{aligned} \beta_1(\mathcal{D}_{n-1}) &= \mathcal{D}_{n-1} \times_{F_{(n-1)1}} \mathbf{2} = \mathcal{D}_{n-1} \times \mathbf{2}, \\ \beta_k(\mathcal{D}_{n-1}) &= \beta_{k-1}(\mathcal{D}_{n-1}) \times_{F_{(n-1)k}} \mathbf{2}, \qquad \text{for } k < n. \end{aligned}$$

Observe that the operations performed make sense since, in general, $F_{(n-1)k}$ is a filter of $\beta_{k-1}(\mathcal{D}_{n-1})$ (this is a consequence of propositions 3.4 and 3.5).

The following theorem condenses in a single formula the whole construction of $\mathcal{D}_n$ from $\mathcal{D}_{n-1}$ described in this section, making use of the notations introduced above.

Theorem 3.2 $\mathcal{D}_n = \beta_{n-1}(\mathcal{D}_{n-1})$.

3.1 Schröder lattices

As we have said in section 2, Schröder paths do not fall within the class of paths captured by our definition. Nevertheless, they can be naturally ordered in the same way, and it is easy to prove that they constitute a lattice. It turns out that the study of Schröder lattices follows essentially the same lines as those for Dyck lattices. Also in this case, there is a privileged rule among all those defining Schröder numbers, which is precisely

$$\Omega : \begin{cases} (2) \\ (2k) \rightsquigarrow (2)(4)^2(6)^2 \ldots (2k-2)^2(2k)^2(2k+2). \end{cases} \tag{11}$$

It is immediate to see that the ECO-construction of Schröder paths encoded by the above rule provides a partition of any Schröder lattice $\mathcal{S}_n$

into saturated chain. Besides, the construction of $\mathcal{S}_n$ starting from $\mathcal{S}_{n-1}$ can be carried out in an analogous way to Dyck lattices: the interested reader will find no difficulties in reproducing it. In fig. 6 we give the Hasse diagram of $\mathcal{S}_3$ (lattice of Schröder paths of length 6).

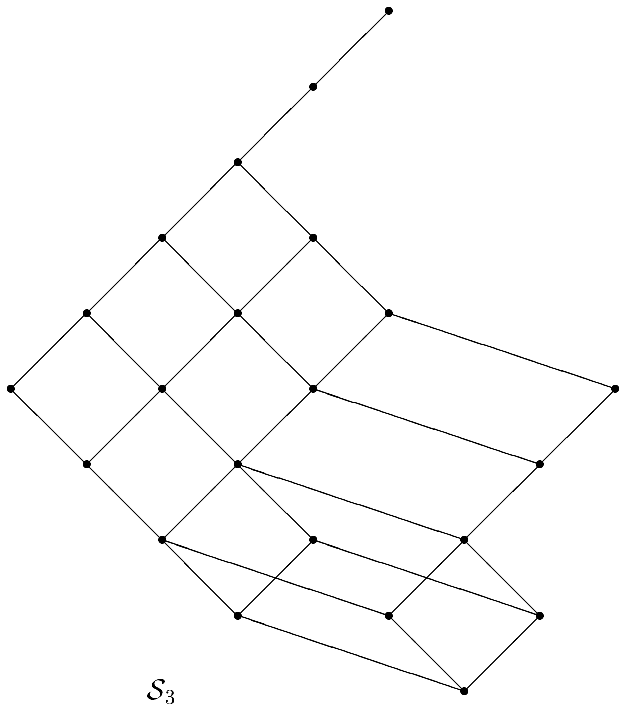

Fig.6

4 Open problems

1. Recall that a finite sequence of numbers $a_0, \ldots a_n$ is said to be *unimodal* whenever there exists an index k such that $a_0 \leq a_1 \leq \ldots \leq a_k$ and $a_k \geq a_{k+1} \geq \ldots \geq a_n$. It seems natural to think that Dyck lattices are rank-unimodal (i.e. the sequence given by the elements of the same rank is unimodal), even if we have not succeeded in proving it yet. This problem was also mentioned in [4], both for Dyck and Schröder lattices, where it is presented as an "intriguing open question". Obviously the same question can be asked for any other lattice of paths. In [5] the author studies problems related with the area and the moments of Dyck paths, and in [9], exercise 5.2.12 gives formulas for the area under various paths: these references could be of some help in tackling this problem.

2. The approach used in the study of Dyck and Schröder lattices does

not work for Motzkin lattices. In particular, the ECO decomposition for Motzkin lattices does not provide a good description, as it happens for Dyck and Schröder lattices. In fact, it is still true that the sons of a Motzkin path constitute a chain of the Motzkin lattice they belong to, but such a chain is not saturated (see [2] for a precise description of the usual ECO construction of Motzkin paths). Therefore we are not helped by the ECO method in drawing the Hasse diagrams of Motzkin lattices. So the study of such lattices (as well as of other lattices of paths of some interest in combinatorics) should be done, maybe using different techniques from those developed in the present work.

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