In the inner sum, write $r = k + (i - j)$. Since $i < j$, we have $r < k$. Reindexing,

$$S = \sum_{r<k\le n} \binom{n}{k}^2 \sum_j \binom{k}{n+k-r-j}\binom{n-k}{j-k}.$$

Applying (1) once more, this time to the inner sum, gives

$$S = \sum_{r<k\le n} \binom{n}{k}^2 \binom{n}{n-r} = \sum_{r<k\le n} \binom{n}{r}\binom{n}{k}^2,$$

as desired.

Editorial comment. As noted by several readers, the second identity is equivalent to

$$\sum_k \binom{n}{k}^2 \binom{2k}{n} = \sum_k \binom{n}{k}^3,$$

which is equation (29) of V. Strehl, Binomial identities – combinatorial and algebraic aspects, *Discrete Mathematics* **136** (1994) 309–346. Two copies of the requested identity plus Strehl's identity yield

$$\sum_i \sum_j \binom{n}{i}\binom{n}{j}\binom{i+j}{n} = \sum_i \sum_j \binom{n}{i}\binom{n}{j}^2,$$

which can be proved by applying the Vandermonde convolution on both sides.

Strehl explored his identity in the context of hypergeometric techniques. It can also be proved by using the Vandermonde convolution twice along with various other identities, or by showing that both sides count the ways to start with n black cards and n white cards, designate an equal number of cards of each color as bad, and discard n bad cards.

Also solved by M. Apadogu, R. Chapman (U. K.), P. P. Dályay (Hungary), R. Dutta, N. Ghosh, Y. J. Ionin, B. Karaivanov (U. S. A.) & T. S. Vassilev (Canada), O. Kouba (Syria), P. Lalonde (Canada), J. Nieto (Venezuela), M. Prasad, J. H. Smith, A. Stadler (Switzerland), R. Stong, R. Tauraso (Italy), M. Tiso, M. Wildon (U. K.), Y. Zhao, GCHQ Problem Solving Group (U. K.), NSA Problems Group, and the proposer.

Balanced Tilings of a Rectangle with Three Rows

11929 [2016, 831]. *Proposed by Donald Knuth, Stanford University, Stanford, CA.* Let a_n be the number of ways in which a rectangular box that contains $6n$ square tiles in three rows of length $2n$ can be split into two connected pieces of size $3n$ without cutting any tiles. Thus $a_1 = 3$, $a_2 = 19$, and one of the 85 ways for $n = 3$ is shown.

Taking $a_0 = 1$, find a closed form for the generating function $A(z) = \sum_{n=0}^{\infty} a_n z^n$. What is the asymptotic nature of a_n as $n \to \infty$?

Solution by the editors. The generating function is

$$A(z) = \frac{1+\sqrt{1-4z}}{(\sqrt{1-4z}+z)^2}\frac{1}{\sqrt{1-4z}} - \frac{1-z^2+2z^3}{(1-z)^3}.$$

The coefficients a_n are asymptotic to $4^{n+2}/\sqrt{\pi n}$.

In a splitting of a 3-by-m board into two connected pieces, call the piece containing more of the three cells in the first column *black* and the other piece *white*. Let $f_m(b, w)$ be the number of splittings having b black and w white cells and let $F(X, Y, Z) = \sum_{m=0}^{\infty} \sum_{b+w=3m} f_m(b, w) X^m Y^b Z^w$. We derive an explicit expression for $F(X, Y, Z)$ by relating f_m to paths in a directed multigraph G. The vertices of G represent cases for a column of the 3-by-m board using connectivity information from the cells to the left. We process columns of a tiling from left to right, using 11 states:

1. Start,
2. BBB, with no white cells anywhere to the left,
3. BBB, with some white cells to the left,
4. BBW or WBB,
5. BWB, with the two black cells connected via cells to the left,
6. BWB, with the two black cells not connected via cells to the left,
7. BWW or WWB,
8. WBW, with the two white cells connected via cells to the left,
9. WBW, with the two white cells not connected via cells to the left,
10. WWW,
11. End.

Due to the black-majority convention, *Start* leads next only to vertices 2, 4 (in two ways), 6, or 11. The possible transitions are encoded in the matrix M below. The entry in position (i, j) encodes a step from state i to state j as one column is added. When the transition is possible, it augments the power of X by 1 (for length) and the sum of the powers of Y and Z by 3 (for the three tiles). The coefficient is 2 when there are two ways to make the transition. The requirement that both pieces are connected is encoded by the impossibility of various transitions.

$$\begin{bmatrix}
0 & XY^3 & 0 & 2XY^2Z & 0 & XY^2Z & 0 & 0 & 0 & 0 & 1\\
0 & XY^3 & 0 & 2XY^2Z & XY^2Z & 0 & 2XYZ^2 & 0 & XYZ^2 & XZ^3 & 1\\
0 & 0 & XY^3 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1\\
0 & 0 & XY^3 & XY^2Z & 0 & 0 & XYZ^2 & 0 & XYZ^2 & XZ^3 & 1\\
0 & 0 & XY^3 & 0 & XY^2Z & 0 & 2XYZ^2 & 0 & 0 & XZ^3 & 1\\
0 & 0 & XY^3 & 0 & 0 & XY^2Z & 0 & 0 & 0 & 0 & 0\\
0 & 0 & XY^3 & XY^2Z & 0 & XY^2Z & XYZ^2 & 0 & 0 & XZ^3 & 1\\
0 & 0 & XY^3 & 2XY^2Z & 0 & 0 & 0 & XYZ^2 & 0 & XZ^3 & 1\\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & XYZ^2 & XZ^3 & 0\\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & XZ^3 & 1\\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0
\end{bmatrix}$$

For the example given, the state path is $\langle 1, 2, 5, 7, 4, 9, 10, 11\rangle$. Splittings with m columns correspond to paths from state 1 to state 11 using $m + 1$ transitions; we seek the coefficient of $X^{2n}Y^{3n}Z^{3n}$ in position $(1, 11)$ of M^{2n+1}. Thus, $F(X, Y, Z) = (I - M)^{-1}_{1,11}$.

The resulting expression for F is the fraction with numerator

$$\begin{aligned}
&1 - X^5Y^6Z^4(Y + Z)\left(Y^4 + 3Y^3Z + 2Y^2Z^2 - 2Z^4\right)\\
&\quad + X^4Y^4Z^2\left(Y^6 + 4Y^5Z + 7Y^4Z^2 + 7Y^3Z^3 + 6Y^2Z^4 - Z^6\right)\\
&\quad - X^3Y^2Z\left(Y^6 + 3Y^5Z + 8Y^4Z^2 + 9Y^3Z^3 + 5Y^2Z^4 + YZ^5 + Z^6\right)\\
&\quad + X^2YZ\left(4Y^4 + 3Y^3Z + 3Y^2Z^2 + 3YZ^3 + 2Z^4\right)\\
&\quad - X\left(Y^3 + 2YZ^2 + Z^3\right)
\end{aligned}$$

and denominator

$$\left(XY^3-1\right)^2\left(XZ^3-1\right)\left(XY^2Z-1\right)\left(XYZ^2-1\right)\left(XY^2Z+XYZ^2-1\right).$$

The coefficients of $X^{2n}Y^{3n}Z^{3n}$ for the first few values of n are 1, 3, 19, 85, 355, and 1435. The problem was first investigated in 2009, with these counts appearing in R. H. Hardin, number of ways to partition a $2n\times 3$ grid into 2 connected equal-area regions, oeis.org/A167242.

To extract the generating function $A(z)$, consider $H=F(X,Y,1/Y)$. To have equal count in black and white, we seek the coefficient of Y^0 in H. Viewing H as a Laurent series in Y, we seek the constant term h_0 (an expression in X). The Cauchy coefficient formula applies to H (see P. Flajolet and R. Sedgewick, *Analytic Combinatorics*, Cambridge University Press, New York, 2009). We obtain $h_0=\frac{1}{2\pi i}\oint_C \frac{1}{Y}H\,dY$, where C is a small counterclockwise circle around the origin. Now h_0 is the sum of the residues with respect to the Y-poles. Since the denominator of H is $(XY^3-1)^2(XY^{-3}-1)(XY-1)(XY^{-1}-1)(XY+XY^{-1}-1)$, the poles are at $0, X$, the three cube roots of X, and $(1-\sqrt{1-4\,X^2})/(2X)$. There are eight additional poles, but they lie outside C when X is small. With the help of *Mathematica*, we find an exact expression for h_0, and then changing variables from X to $\sqrt{z}$ gives $A(z)$ as stated earlier.

The asymptotic behavior of a_n is governed by the singularity of $A(z)$ at $z=1/4$. Write $A(z)=B(z)+16/\theta-Q$ with $\theta=\sqrt{1-4z}$. We have $16/\theta=\sum_{n=1}^{\infty}16\binom{2n}{n}z^n$, with coefficients asymptotic to $4^{n+2}/\sqrt{\pi n}$ by Stirling's formula. Setting Q equal to $3086/27$ means that $B(z)/\theta$ is bounded in a disk of radius larger than $1/4$. Hence, a "transfer theorem" applies: Use Theorem VI.4 of the book of Flajolet and Sedgewick cited above to deduce that $a_n=4^{n+2}/\sqrt{\pi n}+O(4^n/n^{3/2})$. In that theorem, use $\zeta=1/4, a=-1/2, \sigma(z)=0$, and $\tau(z)=1-t$.

With more work, one can obtain a formula for a_n, with F_m denoting the mth Fibonacci number:

$$a_n=-3+n-n^2-\frac{1}{5}\big((n-5)F_{3n+1}+(2n-1)F_{3n}\big)$$
$$+\frac{1}{5}\sum_{m=0}^{n}\binom{2(n-m)}{n-m}\big((3m+5)F_{3m+1}-(4m+3)F_{3m}\big).$$

Also solved by J. Semonsen, R. Tauraso (Italy), GCHQ Problem Solving Group, and the proposer.

A Telescoping Series with Inverse Hyperbolic Sine

11930 [2016, 831]. *Proposed by Cornel Ioan Vălean, Timiş, Romania.* Find

$$\sum_{n=1}^{\infty}\sinh^{-1}\left(\frac{1}{\sqrt{2^{n+2}+2}+\sqrt{2^{n+1}+2}}\right).$$

Solution by Ángel Plaza, Universidad de Las Palmas de Gran Canaria, Spain. Writing

$$\frac{1}{\sqrt{2^{n+2}+2}+\sqrt{2^{n+1}+2}}=\frac{\sqrt{2^{n+2}+2}-\sqrt{2^{n+1}+2}}{2^{n+1}}$$
$$=\sqrt{\frac{1}{2^n}}\cdot\sqrt{1+\frac{1}{2^{n+1}}}-\sqrt{\frac{1}{2^{n+1}}}\cdot\sqrt{1+\frac{1}{2^n}},$$

we see that

$$\sinh^{-1}\left(\frac{1}{\sqrt{2^{n+2}+2}+\sqrt{2^{n+1}+2}}\right)=\sinh^{-1}\left(\sqrt{\frac{1}{2^n}}\right)-\sinh^{-1}\left(\sqrt{\frac{1}{2^{n+1}}}\right).$$