Part 1

Tilings and Packings

CHAPTER 1

Satterfield's Tomb

Imagine a stack of 20 cannonballs in the form of a regular tetrahedron. The layers of such a stack are shown in Figure 1.

The number of balls in each layer are the so-called *triangular numbers* $1, 3, 6, 10, \ldots$ having the form

$$1 + 2 + 3 + \ldots + L = L(L+1)/2 = \binom{L+1}{2}.$$

The number of balls in a tetrahedral stack of L layers is the sum of the first L triangular numbers, giving rise to the *tetrahedral numbers* $1, 4, 10, 20, \ldots$ for $L = 1, 2, 3, 4, \ldots$ respectively. In general, the tetrahedral number corresponding to a stack with L layers will have

$$\binom{2}{2} + \binom{3}{2} + \cdots + \binom{L+1}{2} = \frac{(L+2)(L+1)L}{6} = \binom{L+2}{3}$$

cannonballs in the stack. The stack of 20 balls has four layers, and indeed

$$\binom{4+2}{3} = \frac{6 \times 5 \times 4}{6} = 20.$$

Now imagine that the stack is fitted into a regular tetrahedron whose faces are tangent to the various balls on the outside of the stack. Also, we want to cut this tetrahedron into cells with each cell enclosing a ball. The sides of the cells will be the planes which are tangent to the balls and separate the balls from one another.

If were to separate a tetrahedral stack with five layers into cells in this way, the most regular cell would be the one enclosing the cannonball that is surrounded entirely by other balls. It turns out that this cell is the *rhombic dodecahedron* shown in Figure 2.

Rhombic dodecahedrons fill space just the way an infinitely large stack of cannonballs would fill space. We could cut our regular tetrahedron from this 3-dimensional lattice of dodecahedra, extend the walls of some of the cells to meet

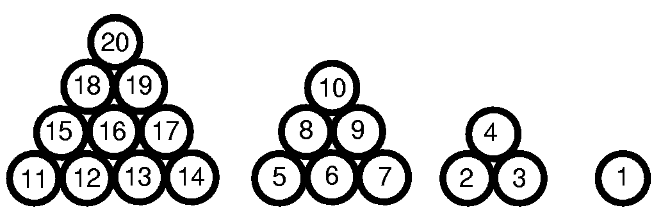

FIGURE 1. Twenty cannonballs in four layers

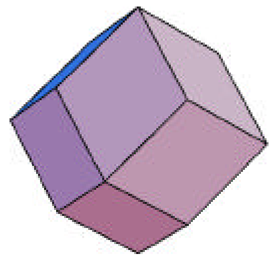

FIGURE 2. The Rhombic Dodecahedron

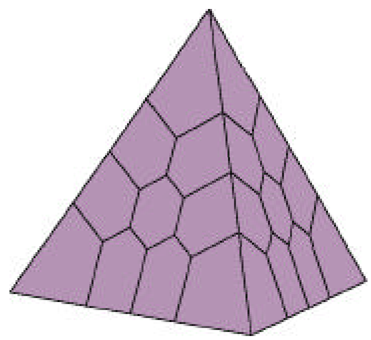

FIGURE 3. Satterfield's Tomb

the walls of tetrahedron, and delete some others (notably at the vertices of tetrahedron), and get the cellular decomposition of the tetrahedron shown in Figure 3.

The nice thing about the cells of the tetrahedron is that the same shapes of cells could be used to form tetrahedra with various sizes. We name the cells *vertex*, *edge*, *face*, and *interior*, as shown in Figures 4, 5, 6, and 2.

To build a tetrahedron with L layers we would need

$$\begin{array}{ll} 4\binom{L-2}{0} & \text{vertex,} \\ 6\binom{L-2}{1} & \text{edge,} \\ 4\binom{L-2}{2} & \text{face, and} \\ \binom{L-2}{3} & \text{interior cells.} \end{array}$$

When $L = 4$, these numbers[1] are 4 vertex, 12 edge, 4 face, and 0 interior cells.

[1] As usual, we define the notation $\binom{n}{k} = \frac{n!}{k!(n-k)!}$ when $0 \le k \le n$, and $\binom{n}{k} = 0$ otherwise.

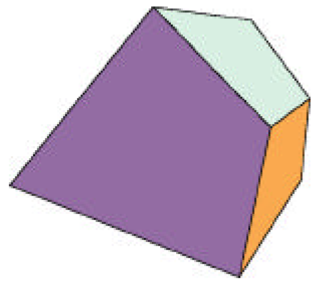

FIGURE 4. Vertex Cell

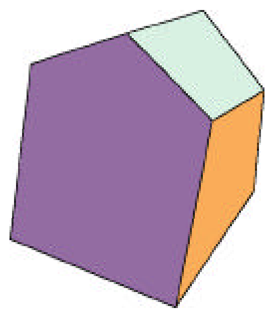

FIGURE 5. Edge Cell

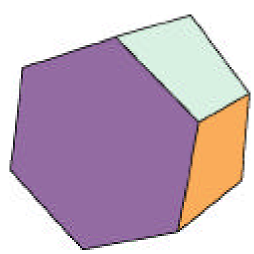

FIGURE 6. Face Cell

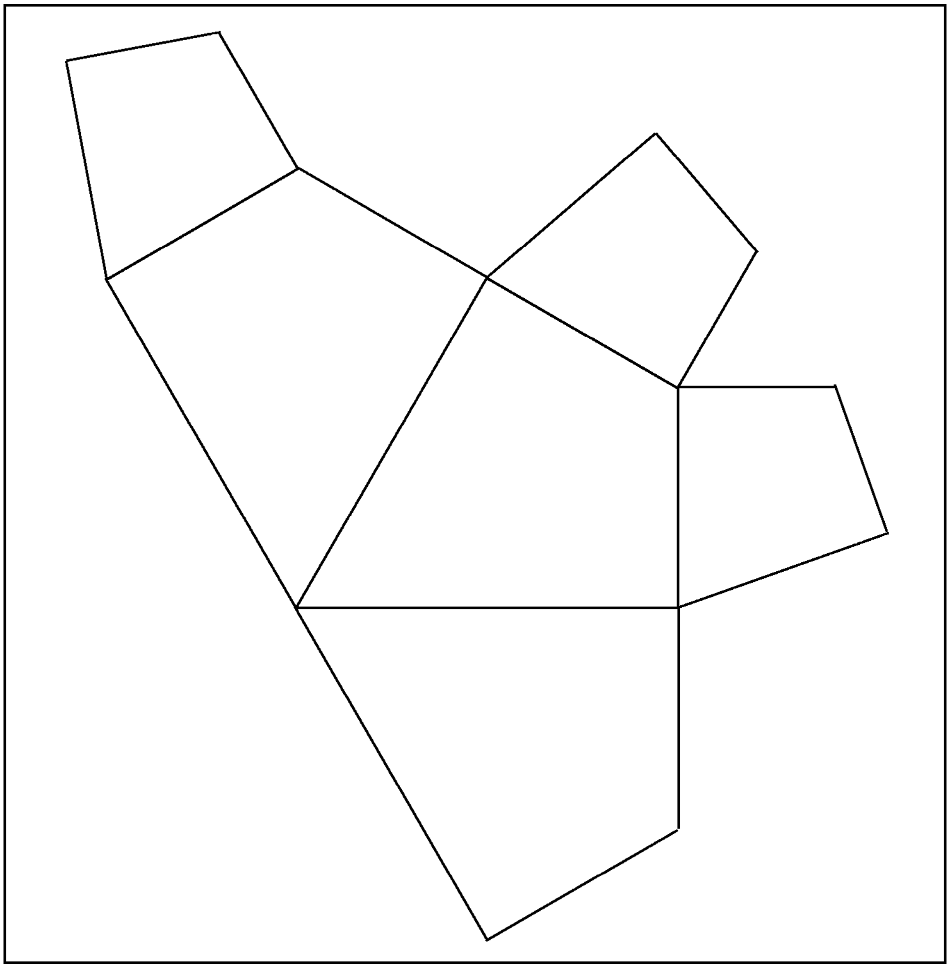

FIGURE 7. Vertex Cell Template

One of our fondest wishes is that some company would manufacture great numbers of these cells and make them available to us and interested readers. Until this wish is fulfilled, we must satisfy ourselves by making copies of the cells from tastefully colored stiff construction paper. To this end, we have furnished patterns for the various cells in Figures 7, 8, 9, and 10.

The reader might have these patterns copied with a larger magnification. Then tape a pattern temporarily to construction card and put pins through the vertex points of the pattern into the card. A utility knife can be used together with a steel rule to score the folds and cut out the shapes. Some skill will be required in taping the cells, and a real expert must conceal all tape inside the cell![2] Another method is to copy the patterns directly onto the construction paper.

[2]One could also add tabs, and use glue and tape.

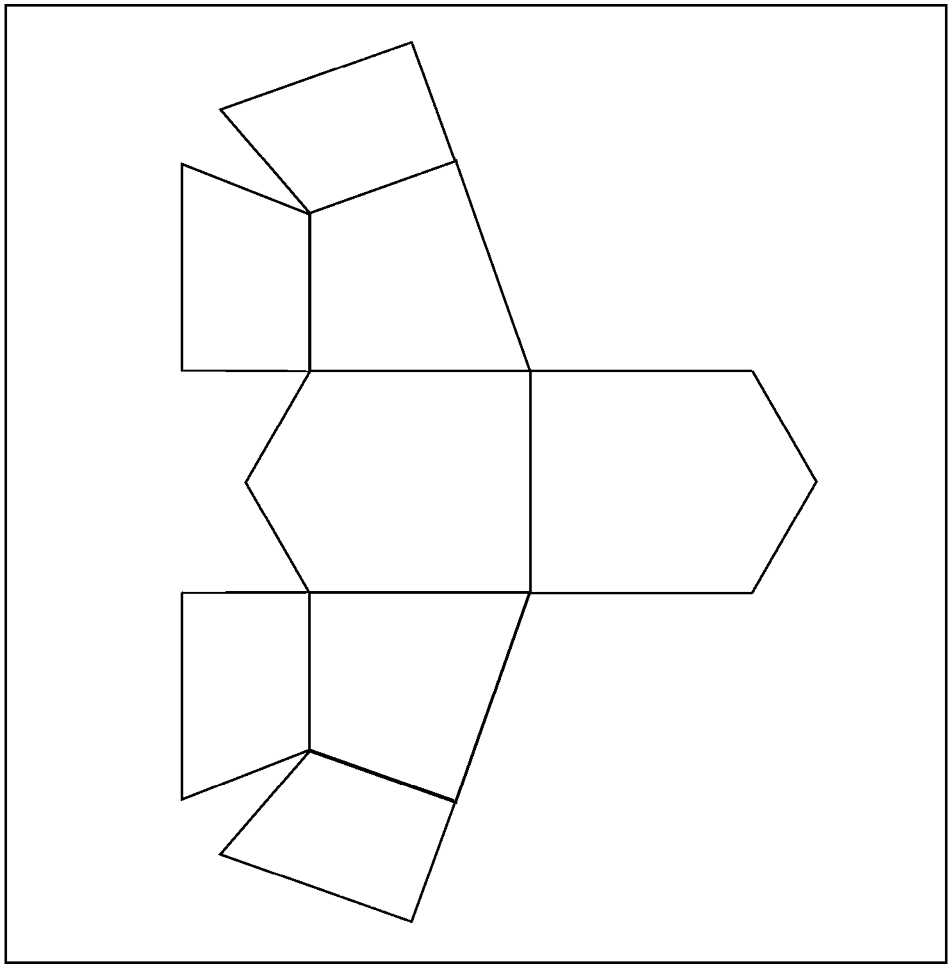

FIGURE 8. Edge Cell Template

The Story of the Four Little Bears

There is a puzzle[3] on the market which involves assembling various pieces made out of balls into a six layer tetrahedral stack of balls. The pieces are made of marbles glued together at points. At least four of the pieces of the puzzle are shaped like little bears as suggested in Figure 11.

It is a remarkable fact that these four little bears can be assembled in two quite different ways to form a four layer tetrahedral stack. These two assemblages are indicated in Figures 12 and 13.

We wondered what the little bears would look like if they were made to fill the tetrahedron without leaving holes between the marble segments of the bears. Somehow we imagined the marbles enclosed in a tetrahedron and then we let the marbles expand like soap bubbles unitl they pressed against each other and the constraining walls of the tetrahedral box enclosing them. What should would the

[3] I've been unable to determine exact the puzzle being referred to here. *Editor*

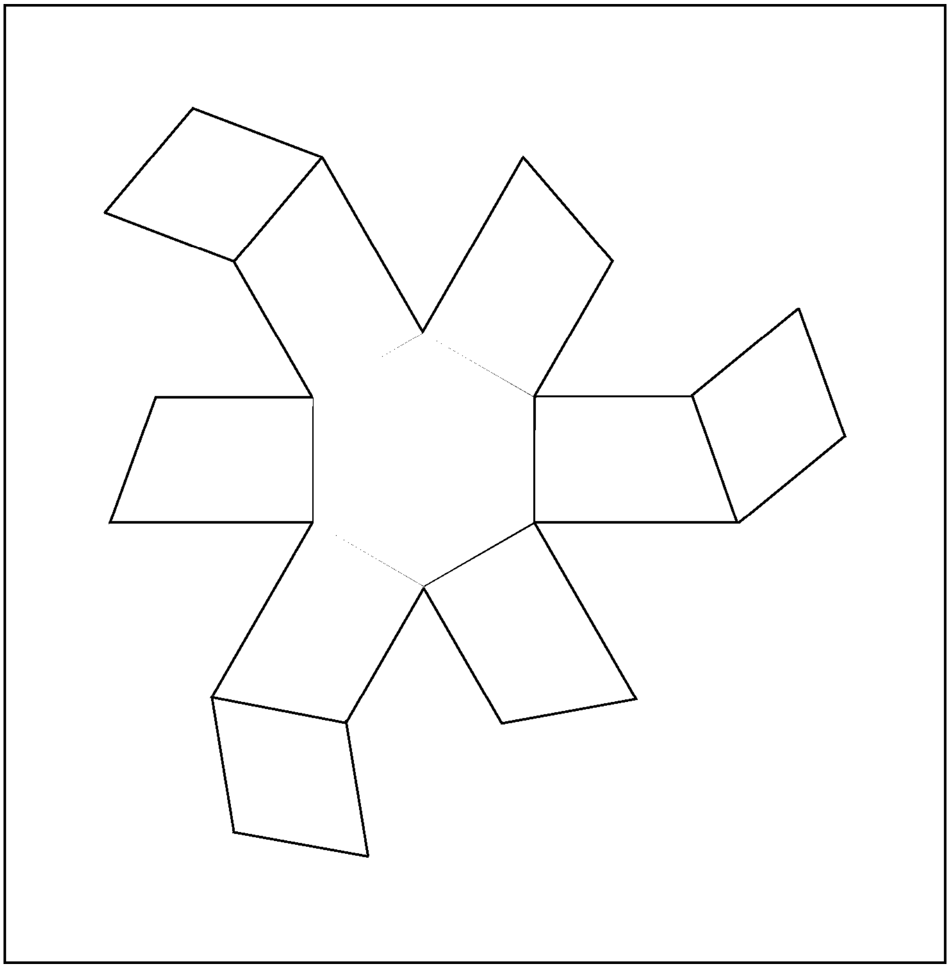

FIGURE 9. Face Cell Template

cells be? And after the cells of a little bear were glued together, what would the shape of a bear be like? The bears interlock in a complicated way; would they pull apart when they were made of cells instead of marbles? What other shapes analogous to the bears could fill the four-layer tetrahedron? All these questions concerning the four little bears, and our inability to imagine in detail what they would look like assembled into a tetrahedron was the inspiration for this article.

One of us (Satterfield), used a computer to determine the various shapes of the cells, and all the measurements of the the line segments, the angles between them, the dihedral angles between planar faces, and so on. These detailed measurements enabled us to build and take photos of a tinkertoy-like model[4].

Also, the computer was used to compute and print the patterns shown in Figures 7, 8, 9, and 10. Because of the near-death state induced by the effort involved

[4]Because these models, computer programs, patterns, and photos were unavailable, we constructed versions them using Mathematica 4.0. *Editor.*

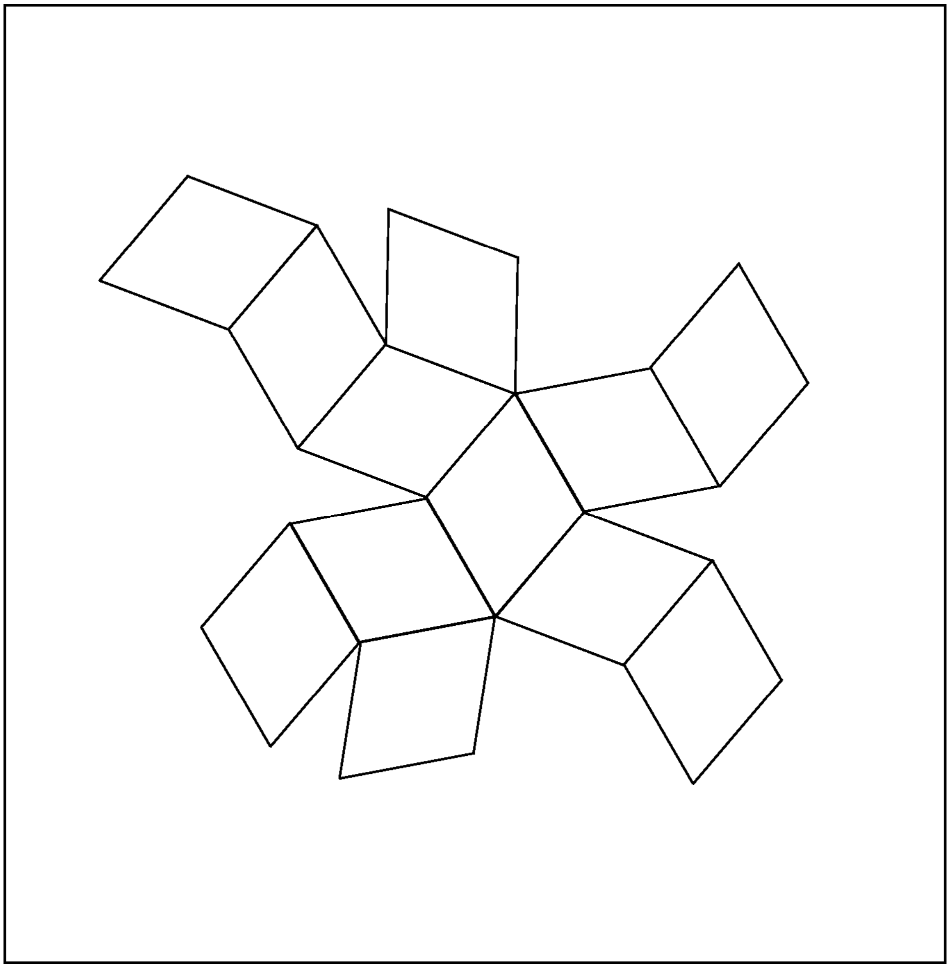

FIGURE 10. Interior Cell Template

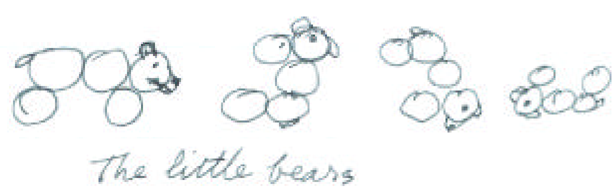

FIGURE 11. The Little Bears

in writing these computer programs, we refer to the structure in Figure 3 as *Satterfield's Tomb*.

We urge the reader to make a set of colored cells to use while reading the rest of this paper. Make the cells for four bears colored red, blue, green, and purple, say. A set of colored cells will consist of one vertex cell, three edge cells, and one

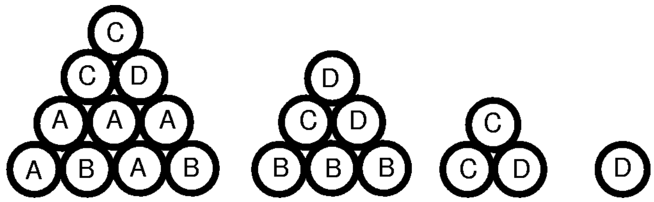

FIGURE 12. The Little Bears form a four layer pyramid

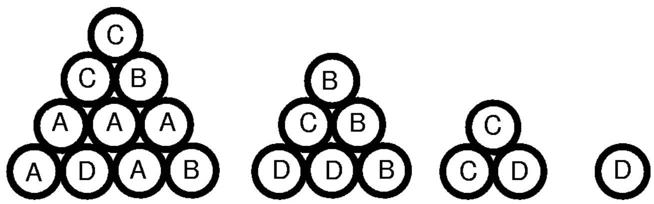

FIGURE 13. The Little Bears form a four layer pyramid a different way.

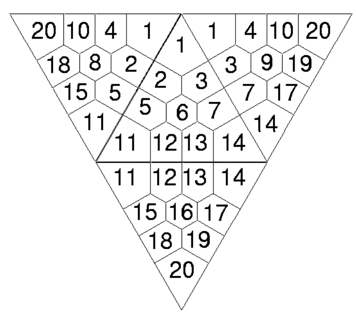

FIGURE 14. Cell numbering for Satterfield's Tomb

face cell. Our first question is how many different animals can be made with these five cells? To answer this question, we need some notation provided by Figure 14. The figure shows a numbering of the cells as seen on the surface of the tetrahedron, cut along some of its edges and flattened. This is the numbering we will use for the cells in the tetrahedron.

Another useful device we will need is the permutations of the cells induced by rotating and reflecting the tetrahedron. There are 24 symmetries of the tetrahedron corresponding to rotations and reflections. Such symmetries will permute the four vertex cells among themselves, the twelve edge cells among themselves, and the four face cells among themselves. These 24 permutations are listed in Table 1.

Armed with a notation for the cells and the permutations of the cells induced by rotating or reflecting (possibly neither, either, or both), we now argue that there are exactly 19 five-celled animals consisting of one vertex cell, three edge cells, and

TABLE 1. Permutations of the twenty cells induced by symmetries of the tetrahedron.

a_3	$(2,4)(5,10)(6,9)(11,20)(12,19)(13,17)(15,18)$
a_4	$(2,3)(5,7)(8,9)(11,14)(12,13)(15,17)(18,19)$
a_5	$(2,3,4)(5,7,10)(6,9,8)(11,14,20)(12,17,18)(13,19,15)$
a_6	$(2,4,3)(5,10,7)(6,8,9)(11,20,14)(12,18,17)(13,15,19)$
$b_1 a_1 = b_1$	$(1,11)(2,5)(3,12)(4,15)(7,13)(9,16)(10,18)$
$b_1 a_2 = b_2$	$(1,11)(2,5)(3,15)(4,12)(6,8)(7,18)(9,16)(10,13)(14,20)(17,19)$
$b_1 a_3 = b_3$	$(1,20,11)(2,10,15)(3,19,12)(4,18,5)(6,9,16)(7,17,13)$
$b_1 a_4 = b_4$	$(1,14,11)(2,7,12)(3,13,5)(4,17,15)(8,9,16)(10,19,18)$
$b_1 a_5 = b_5$	$(1,14,20,11)(2,7,19,15)(3,17,18,5)(4,13,10,12)(6,9,16,8)$
$b_1 a_6 = b_6$	$(1,20,14,11)(2,10,17,12)(3,18,7,15)(4,19,13,5)(6,8,9,16)$
$c_1 a_1 = c_1$	$(1,14)(2,13)(3,7)(4,17)(5,12)(8,16)(10,19)$
$c_1 a_2 = c_2$	$(1,20,14)(2,18,13)(3,10,17)(4,19,7)(5,15,12)(6,8,16)$
$c_1 a_3 = c_3$	$(1,14)(2,17)(3,7)(4,13)(5,19)(6,9)(8,16)(10,12)(11,20)(15,18)$
$c_1 a_4 = c_4$	$(1,11,14)(2,12,7)(3,5,13)(4,15,17)(8,16,9)(10,18,19)$
$c_1 a_5 = c_5$	$(1,20,11,14)(2,19,5,17)(3,10,15,13)(4,18,12,7)(6,9,8,16)$
$c_1 a_6 = c_6$	$(1,11,20,14)(2,15,19,7)(3,5,18,17)(4,12,10,13)(6,8,16,9)$
$d_1 a_1 = d_1$	$(1,20)(2,18)(3,19)(4,10)(5,15)(6,16)(7,17)$
$d_1 a_2 = d_2$	$(1,14,20)(2,13,18)(3,17,10)(4,7,19)(5,12,15)(6,16,8)$
$d_1 a_3 = d_3$	$(1,11,20)(2,15,10)(3,12,19)(4,5,18)(6,16,9)(7,13,17)$
$d_1 a_4 = d_4$	$(1,20)(2,19)(3,18)(4,10)(5,17)(6,16)(7,15)(8,9)(11,14)(12,13)$
$d_1 a_5 = d_5$	$(1,11,14,20)(2,12,17,10)(3,15,7,18)(4,5,13,19)(6,16,9,8)$
$d_1 a_6 = d_6$	$(1,14,11,20)(2,17,5,19)(3,13,15,10)(4,7,12,18)(6,16,8,9)$

one face cell. From now on, an "animal" means a connected set of cells of these three types in these numbers.

We can assume without loss of generality that the animal has vertex cell 1. (If the vertex cell is 11, 14, or 20, we use inverses of the permutations b, c, or d, respectively to change the vertex cell into 1). If 1 is the vertex cell, the face cell is either 6, 8, 9, or 16. First, consider face cell 16. There are exactly six animals with cells 1 and 16, namely, $\{1,2,5,12,16\}$, $\{1,2,5,15,16\}$, $\{1,4,10,19,16\}$, $\{1,3,7,13,16\}$, $\{1,3,7,17,16\}$, and $\{1,4,10,18,16\}$. It is easy to check that these cells are

$$\{a_i \cdot 1, a_i \cdot 2, a_i \cdot 5, a_i \cdot 12, a_i \cdot 16\}$$

for $i = 1,2,3,4,5,6$, respectively, so they are really all equivalent. We take the lexically smallest of these as representative. So far, we have one animal, and all animals having the vertex cell opposite its face cell are congruent to this one: $\{1,2,5,12,16\}$.

Now we consider animals with vertex cell 1, and face cell either 6, 8, or 9. First note that there is no loss in generality to assume that if the face cell is one of these three, it can be assumed to be 6. If the face cell is 8, apply a_2 to change it to 6. If the face cell is 9, apply a_3 to change it to 6. Both a_2 and a_3 do not change cell 1, so we have not violated the earlier assumption that the vertex cell is 1. Given that the vertex cell is 1 and the face cell is 6, it is fairly easy to see that without any loss of generality, one of the three edge cells can be assumed to be cell 2. Since cell

1 has to be connected to the other cells in the animal we must select a non-empty subset of the edge cells $\{2,3,4\}$. Some of these subsets do contain cell 2 in the first place: $\{2\}$, $\{2,3\}$, $\{2,4\}$, $\{2,3,4\}$. Two subsets which do not contain 2 are $\{3\}$ and $\{3,4\}$, but in these cases we can apply a_4, which does not change cell 1 or cell 6, but does change cell three to cell 2. Only the subset $\{4\}$ remains, but there is no connected animal that includes cells 1, 4 and 6 but excludes cells 2 and 3.

So now we want to find all inequivalent animals which include cells 1, 2, 6 and include two more cells from the remaining eleven edge cells 3, 4, 5, 7, 10, 12, 13, 15, 17, 18, 19. We assume in turn that the smallest of these cells selected is $3, 4, 5, \ldots$ in turn and see if there is a second cell larger than it which connects the animal. It is easy to check under these assumptions we can only get equivalent animals when cell 3 is selected. Then we get equivalent pairs $\{1,2,6,3,5\}$ and $\{1,2,6,3,7\}$ (apply a_4 to the first of these to get the second), and $\{1,2,6,3,12\}$ and $\{1,2,6,3,13\}$, again apply a_4 to see that they are congruent.

The complete list of animals is given in Table 2 and Figure 15.

TABLE 2. Representatives of the 19 congruence class of animals

$(A_1)\{1,2,3,4,6\}$	$(A_7)\{1,2,4,6,12\}$	$(A_{14})\{1,2,6,7,12\}$
$(A_2)\{1,2,3,5,6\}$	$(A_8)\{1,2,4,6,13\}$	$(A_{15})\{1,2,6,7,13\}$
$(A_3)\{1,2,3,6,12\}$	$(A_9)\{1,2,5,6,7\}$	$(A_{16})\{1,2,6,7,17\}$
$(A_4)\{1,2,4,5,6\}$	$(A_{10})\{1,2,5,6,12\}$	$(A_{17})\{1,2,6,12,13\}$
$(A_5)\{1,2,4,6,7\}$	$(A_{11})\{1,2,5,6,13\}$	$(A_{18})\{1,2,3,12,15\}$
$(A_6)\{1,2,4,6,10\}$	$(A_{12})\{1,2,5,6,15\}$	$(A_{19})\{1,2,6,13,17\}$
	$(A_{13})\{1,2,5,12,16\}$	

Now we turn to the problem of determining which of these nineteen animals can be used to fill the tetrahedron with four congruent copies of themselves. It can be assumed without loss of generality that one of the copies is a representative A_i listed in Table 2. Then we have to apply one of the permutations $b_1, b_2, b_3, b_4, b_5, b_6$ to A_i to fill vertex 11, then apply one of the permutations $c_1, c_2, c_3, c_4, c_5, c_6$ to A_i to fill vertex 14, and finally apply one of the permutations $d_1, d_2, d_3, d_4, d_5, d_6$ to fill vertex 20. Such a solution might be recorded $(A_i : b_u, c_s, d_t)$; not all of these are distinct. It might happen that $A_i = a_4 A_i$ in which case $(A_i : a_4 b_u, a_4 c_s, a_4 d_t)$ is a solution equivalent to $(A_i : b_u, c_s, d_t)$.

The reader may want to have fun looking for these solutions by making four sets of colored cells and filling the tetrahedron with colored animals. Because the angles of these cells are not the usual right angles, it is an intriguing pastime to build with these little cells! We close with our list of results for each animal[5].

[5]MS ends here, without the list of results. *Editor*

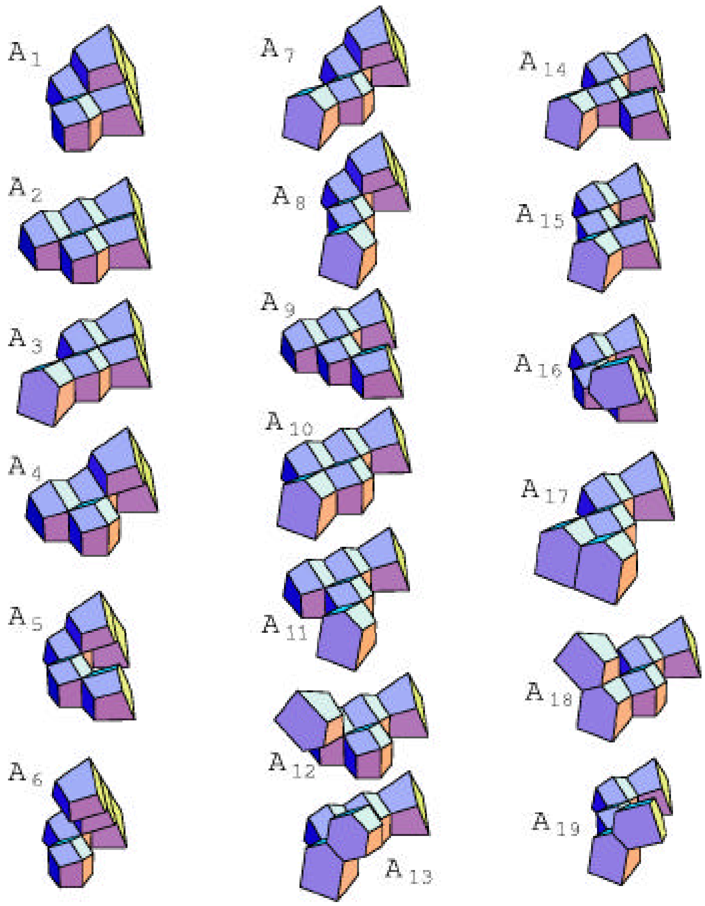

FIGURE 15. The nineteen animals. The little bear is A_{15}.

Results

Animal	# *Non-isomorphic packings, and representatives*	
A_1	2	$\{1,2,3,4,6\}\{7,9,13,14,17\}\{8,10,18,19,20\}\{5,11,12,15,16\}$ $\{1,2,3,4,6\}\{7,9,13,14,17\}\{5,8,11,12,15\}\{10,16,18,19,20\}$
A_2	1	$\{1,2,3,5,6\}\{7,9,14,17,19\}\{4,8,10,18,20\}\{11,12,13,15,16\}$
A_3	3	$\{1,2,3,6,12\}\{4,5,8,11,15\}\{13,16,18,19,20\}\{7,9,10,14,17\}$ $\{1,2,3,6,12\}\{4,5,8,11,15\}\{7,9,10,19,20\}\{13,14,16,17,18\}$ $\{1,2,3,6,12\}\{13,16,18,19,20\}\{5,8,10,11,15\}\{4,7,9,14,17\}$
A_4	1	$\{1,2,4,5,6\}\{10,16,17,19,20\}\{8,11,12,15,18\}\{3,7,9,13,14\}$
A_5	2	$\{1,2,4,6,7\}\{3,9,13,14,17\}\{5,8,11,12,18\}\{10,15,16,19,20\}$ $\{1,2,4,6,7\}\{3,9,13,14,17\}\{8,10,15,19,20\}\{5,11,12,16,18\}$
A_6	1	$\{1,2,4,6,10\}\{5,8,11,12,13\}\{9,15,18,19,20\}\{3,7,14,16,17\}$
A_7	0	
A_8	1	$\{1,2,4,6,13\}\{5,8,10,11,12\}\{3,9,18,19,20\}\{7,14,15,16,17\}$
A_9	1	$\{1,2,5,6,7\}\{3,9,14,17,19\}\{4,8,10,15,20\}\{11,12,13,16,18\}$
A_{10}	1	$\{1,2,5,6,12\}\{13,16,17,19,20\}\{8,10,11,15,18\}\{3,4,7,9,14\}$
A_{11}	1	$\{1,2,5,6,13\}\{12,16,17,19,20\}\{4,8,11,15,18\}\{3,7,9,10,14\}$
A_{12}	1	$\{1,2,5,6,15\}\{9,14,17,18,19\}\{3,4,8,10,20\}\{7,11,12,13,16\}$
A_{13}	1	$\{1,2,5,12,16\}\{6,13,17,19,20\}\{9,10,11,15,18\}\{3,4,7,8,14\}$
A_{14}	2	$\{1,2,6,7,12\}\{3,9,10,14,17\}\{4,5,8,18,20\}\{11,13,15,16,19\}$ $\{1,2,6,7,12\}\{3,9,10,14,17\}\{4,5,8,11,18\}\{13,15,16,19,20\}$
A_{15}	3	$\{1,2,6,7,13\}\{5,8,10,11,18\}\{12,15,16,19,20\}\{3,4,9,14,17\}$ $\{1,2,6,7,13\}\{5,8,10,11,18\}\{3,4,9,19,20\}\{12,14,15,16,17\}$ $\{1,2,6,7,13\}\{5,8,10,15,20\}\{3,4,9,14,17\}\{11,12,16,18,19\}$
A_{16}	0	
A_{17}	1	$\{1,2,6,12,13\}\{4,5,8,10,11\}\{3,7,9,19,20\}\{14,15,16,17,18\}$
A_{18}	0	
A_{19}	1	$\{1,2,6,13,17\}\{5,12,16,19,20\}\{3,4,8,11,15\}\{7,9,10,14,18\}$