Linear difference operators with sequence coefficients having infinite-dimentional solution spaces

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In memory of Marko Petkovšek

Abstract

The notion of lacunary infinite numerical sequence is introduced. It is shown that for an arbitrary linear difference operator L with coefficients belonging to the set R of infinite numerical sequences, a criterion (i.e., a necessary and sufficient condition) for the infinite-dimensionality of its space V_L of solutions belonging to R is the presence of a lacunary sequence in V_L.

Key words: Lacunary sequence, linear difference operator with sequence coefficients, solution space dimension

1 Introduction

Finding sequences satisfying linear difference equations with constant coefficients (called C-finite sequences) is a classical and well-studied topic: the solution set forms a vector space and its dimension is equal to the order of the operator. If one passes from linear to nonlinear equations, questions about the structure of the set of solutions in sequences become more complicated [2] or even undecidable [3]. However, one can still define and study an interesting notion of the dimension of the solution set as was shown in [4]. In this paper, we take a different route by keeping the equations linear but substantially relaxing the restrictions on the coefficients: following [1], we allow any sequences as coefficients. In other words, we consider equations in an unknown sequence $\{x(n)\}_{n\in\mathbb{Z}}$:

$$a_r(n)x(n+r) + \ldots + a_1(n)x(n+1) + a_0(n)x(n) = 0 \quad \text{for every } n \in \mathbb{Z}, \tag{1}$$

where $a_1, \ldots, a_n$ arbitrary sequences acting as coefficients. One can see that the solutions of (1) for a vector space. The main question we study in this paper is *under which conditions the solution space of* (1) *has infinite dimension.* In [1, Sec. 3], a specific sequence was exhibited such that, if it is a solution of the equation, then the solution space has infinite dimension. We develop this approach to a complete criterion: our main result (Theorem 1) is that this happens if and only if (1) has a solution of certain type which we call a “lacunary sequences” (by analogy with lacunary power series), that is, sequences containing arbitrary long finite zero intervals.

2 Example of infinite dimension

The following example shows that indeed, if the coefficients of (1) are arbitrary sequences, the dimension of the solution space can be infinite.

Example 1. Let $L = \Sigma_{k=0}^{r} c_k(n)\sigma^k$, where

$$(r+1) \nmid (n+k) \Rightarrow c_k(n) = 0, \tag{2}$$

and a sequence $a(n)$ be such that

$$(r+1) \mid n \Rightarrow a(n) = 0, \tag{3}$$

then by (2) we have $c_k(n)a(n+k) = 0$ for all such n, that $(r+1) \nmid (n+k)$. In turn, (3) gives $c_k(n)a(n+k) = 0$ for all n such that $(r+1) \mid (n+k)$. Thus

$$L(a(n)) = \Sigma_{k=0}^{r} c_k(n)a(n+k) = 0$$

for all $n \in \mathbb{Z}$, i.e. V_L contains all numeric sequences satisfying (3). As the values of $a(n)$ for n such that $(r+1) \nmid n$ can be chosen arbitrary, we conclude that $\dim V_L = \infty$.

3 Lacunary sequences

We consider a linear difference equation in an unknown sequence $\{x(n)\}_{n\in\mathbb{Z}}$:

$$a_r(n)x(n+r) + \ldots + a_1(n)x(n+1) + a_0(n)x(n) = 0 \tag{4}$$

where $\{a_0(n)\}_{n\in\mathbb{Z}}, \ldots, \{a_r(n)\}_{n\in\mathbb{Z}}$ are arbitrary sequences.

We will call the number r the *order* of (4). For a sequence $\{a(n)\}_{n\in\mathbb{Z}}$, its *support* is defined as

$$\operatorname{supp}(\{a(n)\}) := \{i \in \mathbb{Z} \mid a(i) \neq 0\}.$$

A sequence $\{a(n)\}_{n\in\mathbb{Z}}$ will be called *lacunary* if the difference between the consequent elements of the support (that is, i and j in $\operatorname{supp}\{a(n)\}$ such that $i < j$ and there is no $k \in \operatorname{supp}\{a(n)\}$ with $i < k < j$) can be arbitrary large.

Theorem 1. *The following statements are equivalent:*

1. *the dimension of the solution space of* (4) *is infinite;*
2. (4) *has a lacunary solution.*

The proof of the theorem will be based upon the following lemmas.

Lemma 1. *Assume that the dimension of the solution space of* (4) *is infinite. Then there is a ray $\mathcal{S}$ in $\mathbb{Z}$, that is, a set of the form $\mathbb{Z}_{\geqslant i_0} := \{i \in \mathbb{Z} \mid i \geqslant i_0\}$ or $\mathbb{Z}_{\leqslant i_0} := \{i \in \mathbb{Z} \mid i \leqslant i_0\}$, such that the subspace of the solutions of* (4) *with the support contained in $\mathcal{S}$ is also infinite-dimensional.*

Proof. Consider the subspace V of the solution space of (4) consisting of the sequences $\{x(n)\}_{n\in\mathbb{Z}}$ satisfying $x(0) = x(1) = \ldots = x(r) = 0$. This is a subspace of a finite codimension in the whole solution space (that is it is defined by finitely many constraints), so it must be also infinite-dimensional. Consider a solution $\{x(n)\}$ of (4) belonging to V. Since every next term in a solution of (4) depends only on the previous r, this solution can be represented as a sum of

$$x^+(n) = \begin{cases} x(n), & \text{if } n > r, \\ 0, & \text{otherwise} \end{cases} \qquad \text{and} \quad x^-(n) = \begin{cases} x(n), & \text{if } n < 0, \\ 0, & \text{otherwise} \end{cases}$$

Thus, V is a direct sum of subspaces of solutions of (4) with the support contained in $\mathbb{Z}_{>r}$ and $\mathbb{Z}_{<0}$, respectively. Therefore, at least one of these subspaces has infinite dimension. □

Lemma 2. *Assume that the dimension of the space W of solutions of (4) with the support contained in $\mathbb{Z}_{\geqslant 0}$ is infinite. There exists an index $J > 0$ such that there exists a nonzero solution of (4) with the support contained in $[1, \ldots, J]$.*

Proof. For every $i > 0$, we denote by W_i the projection of the space W onto the coordinates $[1, \ldots, i]$. Since the dimension of W is infinite, the dimension of W_i can be arbitrarily large. Consider J such that $\dim W_J \geqslant r + 2$. We will prove that such J satisfies the conditions of the lemma. Consider a subspace of W_J consisting of sequences satisfying

$$x(J - r) = x(J - r + 1) = \ldots = x(J) = 0$$

It has a codimension at most $r + 1$, so its dimension is at least one. Therefore, there exists a nonzero element in W of the form

$$x_0(n) = \begin{cases} 0, \text{ if } n \leqslant 0, \\ x(n), \text{ if } 0 < n < J - r, \\ 0, \text{ if } J - r \leqslant n \leqslant J, \\ x(n), \text{ if } J < n. \end{cases}$$

Since the possibility of extending a finite solution to an infinite one depends only on the last r terms, W also must contain the following solution

$$x_1(n) = \begin{cases} 0, \text{ if } n \leqslant 0, \\ x(n), \text{ if } 0 < n < J - r, \\ 0, \text{ if } J - r \leqslant n. \end{cases}$$

Since $\operatorname{supp}(x_1) \subset [1, \ldots, J]$, the lemma is proved. □

Proof of Theorem 1. Implication 1 $\implies$ 2. Again, let W be the space of solutions of (4) with the suppport belonging to $\mathbb{Z}_{\geqslant 0}$. Lemma 1 implies that, after shifting and reversing indices if necessary, we can further assume that the dimension of W is infinite. We will inductively construct a sequence of indices $J_1, J_2, \ldots$ and solutions $A_1, A_2, \ldots$ of (4) as follows. We apply Lemma 2 and obtain index J_1 and the corresponding solution A_1. Now we assume that the index J_ℓ and solution A_ℓ are already constructed. Then we apply Lemma 2 to the solutions of (4) with the support contained in $\mathbb{Z}_{> J_\ell + \ell}$. We thus will obtain index $J_{\ell+1}$ and solution $A_{\ell+1}$. The constructed solutions $A_1, A_2, \ldots$ have final supports, and the difference $\min \operatorname{supp}(A_{i+1}) - \max \operatorname{supp}(A_i)$ is at least $i + 1$ by construction. Consider an infinite sum $A_1 + A_2 + A_3 + \ldots$. It is well-defined since the supports of the summands do not intersect. This sum is the desired lacunary solution.

Implication 2 $\implies$ 1. Let $\{x(n)\}_{n \in \mathbb{Z}}$ be a lacunary solution of (4). Consider finite intervals of zeroes in $\{x(n)\}$ of length greater than r, that is, two indices $i < j$ such that

$$x(i) = x(i+1) = \ldots = x(i+r) = x(j) = x(j+1) = \ldots = x(j+r) = 0 \quad \text{and} \quad \exists i < k < j \colon x(k) \neq 0.$$

Since the order of the equation is r, the sequence

$$x_0 = \begin{cases} x(n), \text{ if } i + r < n < j, \\ 0, \text{ otherwise} \end{cases}$$

is a nonzero solution of (4) with finite support. Repeating this operation for other pairs of zero intervals of length greater than r which do not overlap with each other (this is always possible because there are infinitely many of them due to lacunarity), we obtain nonzero solutions $\{x_1(n)\}, \{x_2(n)\}, \ldots$ of (4) with finite non-intersecting supports. They are linearly independent, so the dimension of the whole solution space is infinite as well. □

Example 2. Go back to Example 1. The equation $L(y) = 0$ has lacunary solutions, e.g., the sequence

$$l(n) = \begin{cases} 1, & \text{if } n = 2^m(r+1)+1 \text{ for some } m \in \mathbb{Z}, \\ 0, & \text{otherwise.} \end{cases}$$

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Counting clean words according to the number of their clean neighbors

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In fond memory of Marko Petkovšek (1955-2023), a great summer and enumerator.

Preface

Our good friend and collaborator, Marko Petkovšek ([PWZ]), passed away on March 23, 2023, and we already wrote a eulogy [Z], and donated to the Online Encyclopedia of Integer Sequences in his memory (see `https://oeisf.org/donate` and search for Petkovsek). However we believe that we can do more than that to commemorate Marko. We looked through his list of publications, and found the delightful article [KMP] by Marko, joint with Sandi Klavžar and Michel Mollard, and realized that the beautiful methodology that they used to solve one very *specific* enumeration problem is applicable to a wide class of enumeration problems of the same flavor. More important, since Marko was such an authority in *symbolic* computation, we decided to **implement** the method, and wrote a Maple package

`https://sites.math.rutgers.edu/~zeilberg/tokhniot/Marko.txt` ,

that can very fast answer these kind of questions. In particular as we will soon see, Theorem 1.1 of [KMP] can be gotten (in its equivalent form in terms of generating functions stated as $f(x, y)$ on top of p. 1321) by typing

`WtEs( {0,1 },{[1,1]},y,x,3);` .

Our Maple package, `Marko.txt`, gives, in 0.057 seconds, the answer

$$-\frac{x^2y^2 - x^2y - xy - 1}{x^3y^2 - x^3y - x^2y - xy + 1} \quad .$$

The problem treated so nicely by Klavžar, Mollard, and Petkovšek

There are 2^n vertices in the n-dimensional unit cube $\{0, 1\}^n$ and every such vertex has **exactly** n neighbors (i.e. vertices with Hamming distance 1 from it). The **Fibonacci lattice** consists of those vertices whose 01 vector **avoids** two consecutive 1s, in other words of words in the alphabet $\{0, 1\}$ avoiding as a **consecutive subword** the two-letter word 11. Such words are called **Fibonacci words**, and there are, not surprisingly, F_{n+2} of them (why?).

Each such word has n neighbors, but some of them are not Fibonacci words. The question answered so elegantly in [KMP] was:

For any given n and k, how many Fibonacci words of length n are there that have exactly k Fibonacci neighbors? Calling this number $f_{n,k}$, [KMP] derived an explicit expression for it, that is equivalent to the generating function (that they also derived)

$$f(x,y) \, = \, \sum_{n,k \geq 0} f_{n,k} \, x^n \, y^k \, = \, -\frac{x^2y^2 - x^2y - xy - 1}{x^3y^2 - x^3y - x^2y - xy + 1} \quad .$$

They also considered the analogous problem for **Lucas words** that consists of Fibonacci words where the first and last letter can't both be 1. This problem is also amenable to far-reaching generalization, but will not be handled here.

The general problem

Input:

- A finite alphabet A (In the [KMP] case $A = \{0, 1\}$).
- A finite set of words M, (of the same length) in the alphabet A. (In the [KMP] case M is the singleton set $\{11\}$).

Definition: A word in the alphabet A is called **clean** if it does not have, as *consecutive substring*, any of the members of M.

In other words writing $w = w_1 \dots w_n$, a word is **dirty** if there exists an i such that $w_i w_{i+1} \dots w_{i+k-1} \in M$. For example if $A = \{1, 2, 3\}$ and $M = \{123, 213\}$, then 12212312 is dirty while 111222333 is clean.

To get the set of clean words of length `n` *in the alphabet* `A` *and set of 'mistakes'* `M`*, type, in* `Marko.txt`*,*

`CleanWords(A,M,n);` .

For example, to get the Fibonacci words of length 3 *type:*

`CleanWords({ 0,1 },{ [1,1] } , 3);`, *getting:*

{[0, 0, 0], [0, 0, 1], [0, 1, 0], [1, 0, 0], [1, 0, 1]} .

The problem of the straight enumeration of clean words is handled very efficiently via the Goulden-Jackson cluster algorithm [NZ], but it is not suitable for the present problem of *weighted* enumeration.

Definition: Two words of the same length in the alphabet A are neighbors if their Hamming distance is 1, in other words, $u = u_1 \dots u_n$ and $v = v_1 \dots v_n$ are neighbors if there exists a location r such $u_i = v_i$ if $i \neq r$ and $u_r \neq v_r$.

For example if $A = \{1, 2, 3\}$, the set of neighbors of 111 is

$$\{211, 311, 121, 131, 112, 113\} \quad .$$

Obviously every word of length n in the alphabet A has $n \cdot (|A| - 1)$ neighbors.

However, if w is a clean word, some of its neighbors may be dirty, so if there is one *typo*, it can become dirty, and that would be embarrassing (Oops, *embarrassing* is already dirty). While the word, *duckling* is clean, not all its neighbors are clean.

To see the number of clean neighbors of a word `w` *in the alphabet* `A` *and set of mistakes* `M`*, type*

```
NCN(w,A,M);
```

Output: Having fixed the (finite) alphabet A, and the finite set of forbidden substrings M (all of the same length), let $f_{n,k}$ be the number of clean words in the alphabet A of length n having k clean neighbors. Compute the bi-variate generating function

$$f(x,y) := \sum_{n,k \geq 0} f_{n,k} x^n \, y^k \quad .$$

It would follow from the algorithm (inspired by the methodology of [KMP], but vastly generalized) that this is always a **rational function** of x and y.

This is implemented in procedure

`WtEs(A,M,y,x,MaxK)`,

where `MaxK` *is a 'maximum complexity parameter'. See the beginning of this article for the case treated in [KMP]. For a more complicated example, where a word is clean if it avoids the substrings* 000 *and* 111*, type*

```
WtEs( { 0,1 }, { [1,1,1],[0,0,0] },y,x,5);
```

getting, immediately:

$$\frac{2x^5y^4 - 4x^5y^3 + 2x^5y^2 - 2x^4y^3 + 4x^4y^2 - 2x^4y - y^2x^3 + 2x^3y - 4x^2y^2 - x^3 + 2x^2y + x^2 - 2xy + x - 1}{y^2x^3 - x^3 + x^2 + x - 1} \, .$$

If you want to keep track of the individual letters, rather than just the length, use the more general procedure

`WtEg(A,M,x,y,t,MaxK)`.

Reverse-engineering the beautiful Klavžar-Mollard-Petkovšek proof and vastly generalizing it

In fact, the authors of [KMP] proved their results in two ways, and only the second way used *generating-functionology*. Even that part argued directly in terms of the (double) sequence $f_{n,k}$ itself, and only at the **end of the day**, took the (bi-variate) generating function.

A more efficient, and *streamlined*, approach is to forgo the actual bi-sequence and operate **directly** with weight-enumerators. Let $\mathcal{C}(A, M)$ be the ('infinite') set of words in the alphabet A, avoiding, as consecutive substrings, the members of M, and for each word w in $\mathcal{C}(A, M)$, define the **weight**, $Weight(w)$ by

$$Weight(w) = x^{length(w)}\, y^{NCN(w)} \quad .$$

For example, for the original case of $A = \{0, 1\}$ and $M = \{11\}$,

$$Weight(10101) \; = \; x^5\, y^3 \quad .$$

We are interested in the **weight-enumerator**

$$f(x, y) := Weight(\mathcal{C}(A, M)) = \sum_{w \in \mathcal{C}(A,M)} Weight(w) \quad .$$

Once you have it, and you are interested in a specific $f_{n,k}$, all you need is to take a Taylor expansion about $(0, 0)$ and extract the coefficient of $x^n y^k$.

Let $\mathcal{C}(A, M)^{(i)}$ be the subset of $\mathcal{C}(A, M)$ of words of length i, and pick a positive integer k. For any word $v \in \mathcal{C}(A, M)^{(k)}$, let $\mathcal{C}_v(A, M)$ be the set of words in $\mathcal{C}(A, M)$ of length $\geq k$ that start with v. Obviously

$$\mathcal{C}(A, M) = \bigcup_{i=0}^{k-1} \mathcal{C}(A, M)^{(i)} \;\cup \bigcup_{v \in \mathcal{C}(A,M)^{(k)}} \mathcal{C}(A, M)_v \quad .$$

We can decompose $\mathcal{C}(A, M)_v$ as follows

$$\mathcal{C}(A, M)_v \; = \bigcup_{a \in A} \mathcal{C}(A, M)_{va} \quad ,$$

where, of course $\mathcal{C}(A, M)_{va}$ is empty if appending the letter a turns the clean v into a dirty word. Now, writing $v = v_1 \dots v_k$, and for $a \in A$ the computer verifies whether the difference

$$NCN(v_1 \dots v_k a w) - NCN(v_2 \dots v_k a w)$$

is always the same, for any $v_1 \dots v_k a w \in \mathcal{C}(A_M)_{va}$. The way we implemented it is to test it for sufficiently long words, and then in *retrospect* have the computer check it 'logically', by looking the at the difference in the number of clean neighbors that happens by deleting the first letter v_1. Let's call this constant quantity $\alpha(v, a)$.

It follows that we have a system of $|\mathcal{C}(A, M)^{(k)}|$ equations with $|\mathcal{C}(A, M)^{(k)}|$ unknowns.

$$Weight(\mathcal{C}(A, M)_v) \; = \sum_{\substack{a \in A \\ va \in \mathcal{C}(A,M)}} x y^{\alpha(v,a)} Weight(\mathcal{C}(A, M)_{v_2 \dots v_{k-1} a}) \quad .$$

After the computer algebra system (Maple in our case) automatically found all the $\alpha(v, a)$, and set up the system of equations, we kindly asked it to `solve` it, getting certain rational functions of x and y. **Finally**, our object of desire, $f(x, y)$, is given by

$$Weight(\mathcal{C}(A, M)) \; = \sum_{i=0}^{k-1} Weight(\mathcal{C}(A, M)^{(i)}) \; + \sum_{v \in \mathcal{C}(A,M)^{(k)}} Weight(\mathcal{C}(A, M)_v) \quad .$$

This is implemented in procedure `WtEs(A,M,y,x,MaxK)`.

If you also want to keep track of the individual letters, having the variable t take care of the length, the equations are

$$Weight(\mathcal{C}(A,M)_v) \, = \, \sum_{\substack{a \in A \\ va \in \mathcal{C}(A,M)}} x_{v_1} t y^{\alpha(v,a)} Weight(\mathcal{C}(A,M)_{v_2 \dots v_{k-1} a}) \quad .$$

This is implemented in procedure `WtEg(A,M,x,y,t,MaxK)`.

Sample output

- If you want to see the bi-variate generating functions for words in the alphabet $\{0,1\}$, avoiding i consecutive occurrences of 1, for $2 \leq i \leq 6$, see

  `https://sites.math.rutgers.edu/~zeilberg/tokhniot/oMarko1.txt` .

  Note that the original case was $i = 2$.

- If you want to see the bi-variate generating functions for words in the alphabet $\{0,1\}$, avoiding i consecutive occurrences of 1, **and** i consecutive occurrences of 0, for $3 \leq i \leq 6$, see

  `https://sites.math.rutgers.edu/~zeilberg/tokhniot/oMarko2.txt` .

- If you want to see all such generating functions (still with BINARY words) for all possible SINGLE patterns of length 3,4,5 (up to symmetry), look at:

  `https://sites.math.rutgers.edu/~zeilberg/tokhniot/oMarko3.txt` .

The front of this article contains numerous other output files, but you dear reader, can generate much more!

Conclusion

The value of the article [KMP], that inspired the present article, is not so much with the actual result, that in hindsight, thanks to our Maple package, is trivial, but in the human-generated ideas and **methodology** that enabled one of us to generalize it to a much more general framework.

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Marko Petkovšek 1955-2023

Marko Petkovšek

Marko Petkovšek was born on April 9th 1955 in Ljubljana. His father was professor of botanics in the biology department at the University of Ljubljana, his mother was a housewife. He had 12 years older brother. Marko attended elementary and high school in Ljubljana, both with latin classes. He finished his mathematical diploma and masters at the University of Ljubljana. In the year 1983 he met Prof. Dana Scott at the conference in Dubrovnik who invited him to do his PhD at the Carnegie Mellon University in Pittsburgh. We got married in 1978 (we were high school sweethearts) and had already two daughters, Ana and Kristina, when we went to Pittsburgh for the first time from 1983 to 1985. To finish his PhD we went to Pittsburgh for the second time in 1988 for another two years, this time with three children since in the meantime our son Peter was born. Marko's first position was at the "Josef Stefan Institute" but he soon got the job in the Department of mathematics at the University in Ljubljana which he kept until his retirement in 2022.

Beside mathematics which was always his first love he had great knowledge of mushrooms and plants which he had both learned from his father. Some of this knowledge he had passed to me so together with hiking in the mountains, hunting for flowers (our special love were wild orchids) was our main hobby.

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During the last months of his life, while still undergoing chemotherapy after a recent severe oncological operation, Marko remained very much engaged in his work. Actually, not only did he collaborate with one of his young students on an article which was later accepted for publication. He co-wrote as well another one on the subject of the representation of prime numbers by quadratic forms with a now deceased surveyor adept at computer experiments. Marko both came up with the formulation and proof of the corresponding hypotheses.

Marko never turned down anyone who sought his help, sparing no time or effort to assist them.

There is a story which is particularly telling of his singular generosity and discretion. Recently, during an online seminar, he was the only non-Russian attendee. To spare him the effort and fatigue of following the seminar in a tongue he did understand but didn't speak fluently, it was suggested that the seminar be held in English : Marko insisted that everyone speak Russian. Marko never allowed for any exceptions to be made on his account, as the man who had consistently avoided the foreground in the group pictures of the many conferences he had both organized and attended.

It was at one such conference, the ISSAC'91 conference in Bonn, that we met for the first time. Shortly after, we started working together and ever since we almost permanently had had some common research going on. Our first joint paper dealt with d'Alembert solutions of ordinary differential and difference equations : I had the privilege to witness what an earnest and profound thinker untiryingly seeking (and founding) the essence of the issue at hand, how indefatigably he rejected superficial and facile answers, how graciously he shared the product of his effort.

Marko was gifted with rare depth, honesty and generosity.

Marko was an extraordinarily thoughtful and kind person.

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Marko has always been a hero in my heart! Since I was a PhD student, I carefully studied all of his papers, from which I learnt a lot about symbolic algorithms for solving differential and difference equations. Marko was one of the three reviewers of my PhD dissertation, and I have kept his detailed and inspiring report ever since. Thanks to him for his help and encouragement in my early academic career! After that, I had more exchanges with Marko at many international conferences, and he became both mentor and friend of me. During the Moscow Computer Algebra Conference in 2019, I had a lot of discussions with Marko about the arithmetic theory of power series and stability problems in symbolic integration and summation. He suggested me to consider the stability of the Gosper algorithm. Last year, when I sent him my first paper on stability problems, he was very happy and encouraged me to keep moving forward as always. Marko has always been a gentleman of our community, helping and inspiring countless people. I believe everyone will remember him forever, and his spirit lives on!

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Marko Petkovsek will live on as an excellent counterexample to Stigler's law of eponymy, according to which no scientific discovery is ever named after its actual discoverer. Marko Petkovsek was really the discoverer of Petkovsek's algorithm, which he found as a Ph.D. student, and which immediately became classic. It is ironic and says much about Marko's personality that he himself was not really comfortable with his algorithm bearing his name. He preferred to call the algorithm 'Hyper'. It is well deserved and perfectly justified that Marko will be remembered for Petkovsek's algorithm and all his other contributions to computer algebra (several of which also acquired his name: the Gosper-Petkovsek form, the Abramov-Petkovsek reduction, ...). At the same time, we also want to remember him as a wonderful and generous colleague. For example, we had the pleasure to experience his overwhelming hospitality during visits to Ljubljana, where he cared not only for the scientific program but also enjoyed serving as a passionate tour guide in the beautiful surroundings. Marko has always been eager to support young colleagues by writing valuable letters of recommendation. He also served as member of many Ph.D. and promotion committees. We are saddened that he left us way too early and extend our condolences to his family.

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Marko Petkovšek was born in 1955 and died in 2023. After completing his PhD in 1991 at the School of Computer Science, Carnegie Mellon University, Pittsburgh, he worked as a professor and researcher at the University of Ljubljana until his retirement in 2021.

Marko Petkovšek has an outstanding worldwide reputation in the fields of discrete mathematics and theoretical computer science, which he has earned through his research and work in the field of symbolic computation; to mention only his well-known book A=B and the "Hyper" algorithm for solving linear recursive differential equations with polynomial coefficients in hypergeometric form, nowadays called Petkovšek's algorithm.

In addition to fundamental results and publications in symbolic computation, Marko's work in graph theory, where we have collaborated intermittently over several decades, also contributes to his visibility, so let me say a little more about his work in this area. In graph theory, he has worked on various classes of perfect graphs, graphs with non-empty intersections of longest paths, hereditary graph classes, Fibonacci and Lucas cubes, and several other problems. One of the first problems that Marko suggested as a problem worth of investigating was the problem of the intersection of longest paths in graphs. We wrote a joint paper that was then almost ignored for a quarter of a century, but in the last decade it has had a very wide resonance. Marko's mathematical breadth has been extremely welcome in the treatment of various problems, as it has often given us unexpected insights into the topics at hand. As an example let me mention his contributions to the enumeration of the vertex and edge orbits of Fibonacci cubes and Lucas cubes. Marko's work also established new directions of development. In his paper [Marko Petkovšek, Letter graphs and well-quasi-order by induced subgraphs, Discrete Mathematics 244 (2002) 375-388] he introduced the notion of letter graphs and proved that the class of k-letter graphs is well-quasi-ordered by the induced subgraph relation and that it has only finitely many minimal forbidden induced subgraphs. The paper was not noticed for the first decade after its publication, but in recent years it has received a tremendous response and constitutes a fundamental reference for the development of the field that Prof. Dr. Petkovšek had visionarily outlined more than a decade earlier.

Let me finish with a few personal thoughts about Marko. Our deep and unbroken friendship began almost 40 years ago, when I shared an office with him as a freshman assistant. To be assigned to his office was an extraordinary stroke of luck for me, because he introduced me to the world of research and transferred his enthusiasm for it to me. He was the best possible friend. Despite his broad mathematical knowledge and depth of thought, he was extremely modest and tried to hide his strengths as much as possible. His company was always extremely pleasant and relaxing, whether it was on mountain trails or on Saturday evenings playing bridge. In addition to mathematics, he had a broad general outlook. Let me just mention that he was able to name all the flowers along the mountain paths in three languages, his mother tongue Slovene, German, and Latin.

Unfortunately, Marko left us much too soon. We will always remember the beautiful moments we spent with him and keep him in our memories as a great man in every way.

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There are three sad moments I associate with Marko, each much sadder than the previous one. First, the announcement that he is retiring; second, hearing that he is ill; and last, his passing a few weeks ago.

But there are thousands of happy moments. What a marvelous colleague he was! Hard-working, well-read, always friendly and ready to help. An excellent mathematician, an avid hiker, botanist and bridge-player, and a calming presence in every situation. He always put other people first; senior people because they were senior and deserved respect; younger people because they were younger and deserved an opportunity.

Marko was a wonderful teacher, always extremely precise and able to engage his audience. He deeply influenced how we teach discrete mathematics, optimization and theoretical computer science courses at our department. His notes for these classes are amazing – essentially textbooks once TeXed up! He also deeply influenced my own career path, by showing me how beautiful and exciting combinatorics is.

There is now palpable sadness at the Department of Mathematics of the University of Ljubljana. We miss him, and we are sad that he didn't get to experience many more years with his family.

Ziming Li

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Marko Petkovsěk was not only an outstanding researcher in symbolic combinatorics and computer algebra, but also an influential educator.

His seminal book "A=B", coauthored with Herbert Wilf and Doron Zeilberger, has been studied and discussed in great detail at several reading seminars held in our laboratory. Marko generously provided a PDF copy of the book on his home page, making it easily accessible to many of us.

During his visit to the Symbolic Computation Group (SCG) at the University of Waterloo in 2003, Marko was invited by Eugene Zima to teach Gosper's algorithm as part of the "Computer Algebra" course for graduate students. As a postdoc at SCG, I had the privilege of auditing Marko's lecture, and was highly impressed by his fluent and elegant teaching style. In particular, his lucid explanation of the ingenuity behind Gosper's algorithm left a lasting impression on me. To this day, I still remember this lecture.

Peter Paule

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A couple of associations with Marko Petkovsek:
a brilliant PhD thesis (his algorithm "hyper") with Dana Scott,

a pioneer in computer algebra and symbolic summation (e.g., the Gosper-Petkovsek normal form),
a coauthor (with H.S. Wilf and D. Zeilberger) of the iconic book "A=B",
a generous host,
an ORCCA Research Chair,
a highly welcome visitor to RISC,
enjoyable evenings (e.g., with the Wilfs at the CAMTP Maribor, 2010),
his presence made the duration of a transatlantic flight shrink,
a wonderful colleague as one can only wish for. -
Marko: we miss you badly!

Carsten Schneider

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The first time I got in touch with Marko was at the Séminare Lotharingien de Combinatoire at Kloster Schöntal (Germany) in March 1999.There I got the chance to present my prototype implementation of Karr's indefinite summation algorithm that I explored within my first year of PhD studies at RISC. As a novice, it was exciting for me to meet someone of the famous $A = B$-gang. In particular, when I listened to Marko's three keynote lectures "Exact solutions of linear recurrence equations" I got really thrilled. I already tried some innocent attempts to solve linear recurrences in general difference fields. But his crystal clear presentation and his vision of future developments showed me his beautiful point of view how recurrences can and should be solved in an elegant way. Needless to say, this was a true corner stone of my future academic life and paved my way for future scientific investigations.

Luckily, I had many other opportunities to spend time with Marko. One of the most enjoyable meetings was in January 2008 in San Diego where we both presented a talk in a Special Session of the AMS. We were both happy that we found us in this gigantic gathering and used the chance to spend together a full day in order to explore the sightseeing places of San Diego. Among them we also visited the aircraft carrier at the USS Midway museum. Here Marko told me fascinating and entertaining stories from his youth and his experience of military service at the marine in Slovenia. As in all our other meetings, I got again new scientific inspirations during our chats. I still remember this day as it was yesterday, and I dare to say that we made friends with ourselves.

I am very grateful to Marko for all the highly inspiring discussions and that he was willing to share so much wisdom and knowledge with me. In this regard, I am proud that I had the chance to work with Marko together with one of the most challenging problems that I have struggled so far: to develop a complete algorithm that solves linear recurrences with coefficients in terms of Karr's $\Pi\Sigma$-fields, a general class of difference fields in which one can one can represent indefinite nested sums and products. Already in my PhD thesis I obtained interesting algorithmic results that led to a useful but still heuristic methods to find solutions within such difference fields. When Manuel Bronstein visited RISC in April 2004, I showed him my recent implementation to find hypergeometric solutions in $\Pi\Sigma$-fields. At this occasion I learned that Manuel together with Marko and Sergei Abramov also worked on this challenge, and it was obvious that merging our ideas and results would open up new synergies. But with Manuel Bronstein's unexpected death this promising cooperation drifted off. But luckily, Marko did not give up: In summer 2009 he presented the existing results of him, Manuel and Sergei at a satellite workshop of the FPSAC at RISC. This was the trigger moment for a very fruitful long-term cooperation. For instance, I still remember with pleasure my stay at his home university in Ljubljana in spring 2011 where we discussed one week many highly technical details that are strongly related to Michael Singer's outstanding linear differential equation solver in the setting of Liouvillian field extensions. This was one of my most productive visits and I enjoyed tremendously Marko's great hospitality. It was still a long way to bring all these derived ideas to a formal correct but readable form. At last, our common project with Marko, Manuel, Sergei and myself could be finished in 2020.

Finally, I would like to state that all my current investigations to simplify, e.g., complicated 3-loop Feynman integrals to indefinite nested sums by means of symbolic summation would be unthinkable without all his pioneering work and inspiration. I am grateful for all his scientific advice that had a major impact on my academic career and I am very happy that I have met Marko as a real friend.

Min Wu

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It was in 2002 that I first met Marko, in the beautiful INRIA Sophia Antipolis. At that time, Manuel Bronstein was also healthy and sound.

For me, Manuel, Sergei and Marko were like a perfect golden triangle. They had been collaborated in differential and difference algebras for many years, though with different interest and personalities. During my PhD study, I have received many encouragement and valuable comments from Marko on my thesis work.

Although having got to know Marko early in 2002, we haven't had many personal communications until 2016. It was during the first Moscow Computer Algebra conference, we had many opportunities to talk a lot, and our conversations involved our common old friends, the current state of Russia, the Canon La Chapelle melody, etc.

What I feel most impressive about Marko is his gentlemanship. He always talked and behaved in a gentle, modest and considerate way. When talking to him, you are sure that Marko will be fully present and listen to your carefully and you never feel nervous or stressed, and therefore could express with ease. Now I could understand what a valuable virtue it is.

So it's really sad we lost again a good friend, and a gentleman in the computer algebra community.

There will be no pain and suffering in heaven and I believe that Marko is there in heaven. May Marko rest at peace.

Helena Zakrajšek

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After a successful dissertation defense at the Carnegie Mellon University in 1991, Marko returned to Slovenia. At that time, I was a postgraduate Mathematics student at the University of Ljubljana and so I had an opportunity to attend his lectures on symbolic computation which were quite a novelty then. He got me interested in that field of study and he later mentored my master thesis and dissertation defense.

Marko was a great professor. His lectures were precise and clear, he was always well-prepared and ready to answer any questions we might have had. In case of not having the answer, he would have suggested or found a possible reference. Whenever I found myself at a dead end, he was there encouraging me proposing different approaches to cope with the situation.

Marko was not only an excellent scientist – he was above all a great man. Generous, kind, helpful and always smiling. And that is exactly how he will be remembered.

Doron Zeilberger

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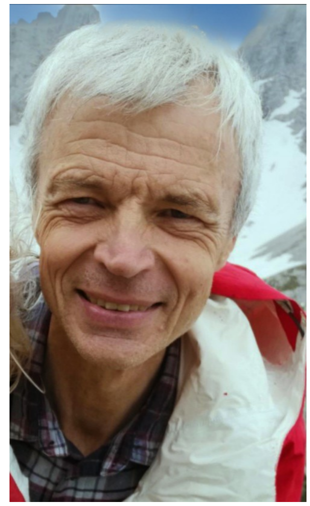
Marko Petkovšek (1955-2023), My A=B Mate



I have only met Marko a few times in my life, but he was a constant presence.

The first time was in 1991, in Philadelphia, shortly after he defended his Carnegie-Mellon PhD under the logical giant Dana Scott (https://www.mathgenealogy.org/id.php?id=8024). Herb Wilf invited Marko to speak at UPenn's combinatorics seminar, about the amazing *Petkovšek algorithm*. Both Herb and I were very excited about Marko's algorithm, since that was the *missing link* needed by us to solve the at-the-time *wide open* **decision problem** for the existence of a closed-form solution for **definite hypergeometric summation**.

Recall that the famous Gosper algorithm [G] decides whether an *indefinite* sum

$$F(n) := \sum_{k=1}^{n} f(k) \quad ,$$

where $f(k)$ is a *hypergeometric term* (i.e. $f(k+1)/f(k)$ is **rational function** of k), is again hypergeometric. But what about **definite summation**, i.e.

$$F(n) := \sum_{k=0}^{n} f(n,k) \quad ,$$

where $f(n,k)$ is (proper) hypergeometric in **both** discrete variables n and k? So-called **Wilf-Zeilberger Algorithmic Proof Theory**, and the **Zeilberger algorithm** (see [PWZ]) can always find the next-best thing to a closed-form solution, a **linear recurrence equation with polynomial coefficients**, of *some* **order** L. In other words come up with $L+1$ polynomials: $p_0(n), p_1(n), \ldots, p_L(n)$; such that

$$\sum_{i=0}^{L} p_i(n)\, F(n+i) \, = \, 0 \quad .$$

If the order, L, happens to be 1, then we know right away that $F(n)$ is closed form (in the sense of being hypergeometric), but what if $L > 1$?

Zeilberger's algorithm guarantees to output *some* recurrence of *some order*, but does **not** always give you the **minimal order**. In order to know for **sure** that there is no first-order recurrence, we need the amazing **Petkovšek algorithm Hyper** [P1] [P2] [P3] [PWZ] (Ch. 8). That was exactly the missing ingredient that we needed to settle this important **decision problem**.

As Marko describes so charmingly in his reminiscences about Herb Wilf [CGH], this connection lead to our collaboration A=B [PWZ], whose *table of contents* was drafted in the Moosewood vegetarian restaurant, in Itacha, New York.

Marko then went on to become a leader in **symbolic summation** and **difference equations**, with very deep work in collaboration with Sergei A. Abramov, and others. He also wrote a charming paper with Herb Wilf about a 'high-tech' proof of an important identity in enumerative combinatorics [PW].

When Bill Chen organized a conference in Tianjin to celebrate my 60^{th} birthday, in the summer of 2010, he asked me whom to invite, and of course I suggested Marko, whom Bill dully invited. Unfortunately, Marko was unable to come, but instead wrote me this nice email.

```
From Marko.Petkovsek@fmf.uni-lj.si Sun Aug 8 17:30:52 2010

    Dear Doron,

    I was looking forward very much to the Zeilberger-fest at Nankai.  Alas, as it
    turned out, I will not be able to attend it.  Last week I had a small operation
    on my leg (nothing serious - removal of a large carbuncle probably caused by an
    insect bite), but the wound has not healed yet and my doctor advised against
    traveling.

    So, I just wish you a very happy |A_5|-th birthday!

    Best regards,

    --Marko
```

Last time I met Marko in person was during Herb Wilf's 80^{th} Birthday conference, held May 26-29, 2011, in Wilfrid Laurier University, Waterloo, Canada, and organized by Eugene Zima and Ilias Kotsireas `https://cargo.wlu.ca/W80/`. Marko gave a great talk on *enumeration of structures with no forbidden substructures* that clearly showed that he had a very broad perspective of combinatorial enumeration. In his own words
(from `https://sites.math.rutgers.edu/~zeilberg/akherim/W80abstracts.pdf`):

Many interesting classes of combinatorial structures are defined by restricting some general class of structures to those structures that avoid certain "forbidden" substructures. Examples include words avoiding forbidden subwords or subsequences, permutations avoiding forbidden patterns, matrices avoiding forbidden submatrices, graphs avoiding forbidden subgraphs, induced subgraphs, minors, or topological minors. We will try to look at the abundance of enumeration problems (solved and unsolved) presented by such classes.

Marko will be sorely missed, but his mathematics and algorithms guarantee his **immortality**.

References

[CGH] Fan Chung, Curtis Greene, and Joan Hutchinson [coordinating editors], *Herbert S. Wilf (1931-2012)*, Notices of the American Mathematical Society **62(4)** (April, 2015).
`https://www.ams.org/notices/201504/rnoti-p346.pdf` .

[G] Robert W. Gosper, Jr., *Decision procedure for indefinite hypergeometric summation*, Proc. Natl. Acad. Sci. USA, **75** (1978), 40-42.
`https://sites.math.rutgers.edu/~zeilberg/akherim/gosper1978.pdf` .

[P1] Marko Petkovšek, *Finding closed-form solutions of difference equations by symbolic methods*, PhD. thesis, Carnegie-Mellon University, CMU0-CS-91-103, 1991.
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[P3] Marko Petkovšek, *A generalization of Gosper's algorithm*, Discrete Mathematics **134** (1994), 125-131.

[PW] Marko Petkovšek and Herbert S. Wilf, *A high-tech proof of the Mills-Robbins-Rumsey determinant identity*, Electronic Journal of Combinatorics **3** (1996) no. 2, #R19.
`https://www.combinatorics.org/ojs/index.php/eljc/article/view/v3i2r19/pdf` .

[PWZ] Marko Petkovšek, Herbert S. Wilf, and Doron Zeilberger, *"A=B"*, A.K. Peters, 1996.
`https://www2.math.upenn.edu/~wilf/AeqB.pdf` .

Recent and Upcoming Events

Recent Trends in Computer Algebra 2023
France

PROGRAM **March 6–10, Luminy**

- Preparatory School during the French Computer Algebra Days

Spring 2023, Lyon
three one-week workshops.

- March 27–31: Effective Aspects in Diophantine Approximation
  Organized by B. Adamczewski, A. Bostan, B. Salvy, W. Zudilin.
- May 22–26: Certified and Symbolic-Numeric Computation
  Organized by N. Brisebarre, A. Mahboubi, D. Pous, B. Salvy.
- June 26–30: Mathematical Software and High Performance Algebraic Computing
  Organized by W. Decker, J.-G. Dumas, C. Pernet, E. Thomé, G. Villard.

Fall 2023, Paris
courses, topical days, seminars, general audience talks and three one-week workshops.

- September 25–29: Fundamental Algorithms and Algorithmic Complexity
  Organized by J. van der Hoeven, M. Giesbrecht, P. Koiran, G. Villard.
- October 16–20: Geometry of Polynomial System Solving, Optimization and Topology
  Organized by C. D'Andrea, P. Lairez, M. Safey El Din, É. Schost, L. Zhi.
- December 4–8: Computer Algebra for Functional Equations in Combinatorics and Physics
  Organized by A. Bostan, J. Bouttier, T. Cluzeau, L. Di Vizio, C. Krattenthaler, P. Lairez, J.-M. Maillard.

ORGANIZERS Alin Bostan, Mark Giesbrecht, Christoph Koutschan, Marni Mishna, Mohab Safey El Din, Bruno Salvy, Gilles Villard
WEBSITE `https://rtca2023.github.io/`

June 12–21, 2023
FoCM 2023: Foundations of Computational Mathematics
Paris, France

WEBSITE `https://focm2023.org/`

June 12–23, 2023
CIMPA School: Algebraic and Tropical Methods for Solving Differential Equations
Oaxaca, Mexico

WEBSITE `https://www.matem.unam.mx/~lara/cimpa23.html`

June 26–28, 2023
Computer Algebra: 5th International Conference (online)
Moscow, Russia

WEBSITE `http://www.ccas.ru/ca/conference`

July 9–15, 2023
ICLP 2023: 39th International Conference on Logic Programming
London, UK

WEBSITE https://iclp2023.imperial.ac.uk/

July 10–14, 2023
SIAM AG23: SIAM Conference on Applied Algebraic Geometry
Hybrid Event, Eindhoven University of Technology, Eindhoven, The Netherlands

WEBSITE https://www.siam.org/conferences/cm/conference/ag23

July 17–21, 2023
ACA 2023: 28th International Conference on Applications of Computer Algebra
Warsaw, Poland

WEBSITE https://iit.sggw.edu.pl/instytut-informatyki-technicznej/aca2023/

July 24–27, 2023
ISSAC 2023: 48th International Symposium on Symbolic and Algebraic Computation
Tromsø, Norway

WEBSITE https://www.issac-conference.org/2023/

July 28, 2023
SC-square 2023: 8th International Workshop on Satisfiability Checking and Symbolic Computation
Tromsø, Norway

WEBSITE http://www.sc-square.org/CSA/workshop8.html

August 28–September 1, 2023
CASC 2023: 25th International Workshop on Computer Algebra in Scientific Computing
Havana, Cuba

WEBSITE http://www.casc-conference.org/

October 26–27, 2023
Maple Conference 2023
Free Virtual Event

WEBSITE https://fr.maplesoft.com/mapleconference/2023
