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Short proof of the ASM theorem avoiding the six-vertex model

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ABSTRACT

Alternating sign matrix (ASM) counting is fascinating because it pushes the limits of counting tools. Nowadays, the standard method to attack such problems is the six-vertex model approach which involves computing a certain generating function of ASMs with—at first sight—nonorthodox weights originating from statistical mechanics. Still nobody has been able to use this technique to reprove the generalization of the ASM theorem that Zeilberger has actually established in the first proof of the ASM theorem, where he showed that there is the same number of $n \times k$ Gog-trapezoids as there is of $n \times k$ Magog-trapezoids nor has anybody proved Krattenthaler's conjectural generalization of this result. In 2007 I have presented a proof of the ASM theorem in a 12 page paper which does not involve the six-vertex model, but relies on another 19 page paper as well as Andrew's determinant evaluation that he used to enumerated descending plane partitions. Over the years I have discovered many simplifications of my original proof and it is the main purpose of this paper to present now a 9 page self-contained proof of the ASM theorem. In addition, I speculate on how to possibly transform this computational proof into a more combinatorial

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proof and I also provide a new constant term expression for the number of monotone triangles with prescribed bottom row.

1. Introduction

An alternating sign matrix (ASM) is a square matrix in which each entry is either 1, −1 or 0, and, along each row and each column, non-zero elements alternate and add up to 1, see Fig. 1 (left) for an example. Mills, Robbins and Rumsey [33,22,24] defined ASMs in the course of generalizing the determinant and conjectured that there are $\prod_{j=0}^{n-1} \frac{(3j+1)!}{(n+j)!}$ ASMs of order n. After considerable efforts, Zeilberger [35] provided the first proof of the ASM theorem, and soon after that, Kuperberg [19] used six-vertex model techniques to provide a shorter proof. Accounts on the history of ASM counting are given in [6,7].

There are several different directions of related research that were followed after that, many of them concerning exact enumerations of subclasses of ASMs. For instance, already Mills, Robbins and Rumsey [24] conjectured that the number of $n \times n$ ASMs where the unique 1 in the top row is situated in column i is also given by a simple product formula, namely by

$$\frac{\binom{n+i-2}{n-1}\binom{2n-i-1}{n-1}}{\binom{3n-2}{n-1}} \prod_{j=0}^{n-1} \frac{(3j+1)!}{(n+j)!}. \tag{1.1}$$

The first proof of this result was again provided by Zeilberger [36], then also employing six-vertex model techniques. The ASM theorem then follows by summing over all i and using the Chu–Vandermonde summation. Several other results on doubly and triply refined enumerations (where the 1's on two or three boundary rows and/or columns are fixed) were obtained until finally three years ago Ayyer and Romik [3] and Behrend [4] derived formulas for the quadruply refined enumeration of ASMs fixing the 1's in all four boundary rows and columns. Behrend's result involves in addition two so-called bulk statistics, namely the number of −1's in the ASM and the inversion number of ASMs.

On the other hand, Stanley suggested in the 1980s the systematic study of *symmetry classes of ASMs*, see [31,32], which led Robbins [32] to conjecture that several symmetry classes of ASMs are also counted by simple product formulas. All the conjectures are proven now by using the six-vertex model approach. More precisely, Kuperberg [20] dealt with *vertically symmetric ASMs*, *half-turn symmetry ASMs* of even order and *quarter-turn symmetric ASMs* of even order, Razumov and Stroganov proved the odd order cases of half-turn symmetry ASMs [28] and quarter-turn symmetric ASMs [27], Okada [25] enumerated *vertically and horizontally symmetric ASMs*, while Behrend,

$$
\begin{pmatrix} 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 1 & 0 & -1 & 1 & 0 \\ 1 & -1 & 0 & 1 & -1 & 1 \\ 0 & 1 & 0 & -1 & 1 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \end{pmatrix} \to \begin{pmatrix} 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 & 1 & 0 \\ 1 & 0 & 0 & 1 & 0 & 1 \\ 1 & 1 & 0 & 0 & 1 & 1 \\ 1 & 1 & 0 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 & 1 & 1 \end{pmatrix} \to \begin{array}{ccccccccccc} & & & & & 4 & & & & & \\ & & & & 2 & & 5 & & & & \\ & & & 1 & & 4 & & 6 & & & \\ & & 1 & & 2 & & 5 & & 6 & & \\ & 1 & & 2 & & 4 & & 5 & & 6 & \\ 1 & & 2 & & 3 & & 4 & & 5 & & 6 \end{array}
$$

Fig. 1. ASM $\to$ partial column sums $\to$ monotone triangle.

Fischer and Konvalinka [5] recently dealt with *diagonally and antidiagonally symmetric ASMs* of odd order.

To mention briefly a third direction of related research, the Razumov–Stroganov (ex-)Conjecture [26], proved by Cantini and Sportiello [8], provides a fascinating connection between the $O(1)$ loop model and fully packed loop configurations (and thus ASMs because they are in bijective correspondence with fully packed loop configurations). However, there are several ASM mysteries that have still not been resolved, two of which are certainly the unknown bijections between order n ASMs and $2n \times 2n \times 2n$ totally symmetric self-complementary plane partitions (TSSCPP) [23,2], respectively descending plane partitions with parts no greater than n [24,1]. By defining objects that generalize ASMs (GOG-trapezoids) and TSSCPPs (MAGOG-trapezoids) respectively, and proving that also these generalizations are equinumerous, Zeilberger [35] provided progress concerning the first bijection, and Krattenthaler [18] generalized these objects further and provided a pair of statistics on the two types of objects that seem to have the same distribution. However, to prove (bijectively or not) that this is indeed the case is an open problem up to this day and it is unclear whether the six-vertex model approach is the right tool.

In 2007 I have given an alternative proof of the ASM theorem [11] which does not involve the six-vertex model. However, this proof relies heavily on another paper [10], where an *operator formula* for monotone triangles with prescribed bottom row was derived, and also on Andrew's evaluation of the determinant that counts descending plane partitions [1]. In the past few years, I have discovered many shortcuts (some of which appeared in [12,13]) and it is the main purpose of this paper to present the most concise version of this proof (see also [30]). This is accomplished on about 9 pages in Section 2. The proof relies at one place on the famous *Lindström–Gessel–Viennot Theorem* [21,16,17] and is otherwise self-contained. Hopefully this makes this alternative approach to ASMs complementing six-vertex model techniques easier to digest. In the final section, we present some thoughts on how to possibly "combinatorialize" this computational proof as well as a new constant term expression that counts monotone triangles with prescribed bottom row.

1.1. Monotone triangles

The proof makes use of the well-known equivalence between order n ASMs and *monotone triangles* with bottom row $1, 2, \ldots, n$. A *Gelfand–Tsetlin pattern* is a triangular array $(a_{i,j})_{1 \le j \le i \le n}$ of integers, where the elements are usually arranged as follows

$$\begin{array}{ccccccccccccc} & & & & & & a_{1,1} & & & & & & \\ & & & & & a_{2,1} & & a_{2,2} & & & & & \\ & & & & \dots & & \dots & & \dots & & & & \\ & & & a_{n-2,1} & & \dots & & \dots & & a_{n-2,n-2} & & & \\ & & a_{n-1,1} & & a_{n-1,2} & & \dots & & \dots & & a_{n-1,n-1} & & \\ & a_{n,1} & & a_{n,2} & & a_{n,3} & & \dots & & \dots & & a_{n,n} & \end{array}, \tag{1.2}$$

and increase in northeast and in southeast direction, that is $a_{i+1,j} \le a_{i,j} \le a_{i+1,j+1}$ for all i,j with $1 \le j \le i < n$. A monotone triangle is a Gelfand–Tsetlin pattern with strictly increasing rows. To transform an ASM into the corresponding monotone triangle, compute the partial column sums, that is add to each entry all the entries that are in the same column above the entry, and record then row by row the positions of the 1's, see Fig. 1 for an example.

We say that the integer partitions $\lambda = (\lambda_1, \dots, \lambda_n)$ and $\mu = (\mu_1, \dots, \mu_{n-1})$ are *interlacing* (in symbols: $\mu \prec \lambda$), if

$$\lambda_1 \ge \mu_1 \ge \lambda_2 \ge \mu_2 \ge \dots \ge \mu_{n-1} \ge \lambda_n,$$

that is the skew shape λ/μ is a *horizontal strip*. Consecutive rows of monotone triangles are obviously interlacing partitions if read them from right to left. In fact, monotone triangle of order n with bottom row λ can be seen as a sequence of n *strict* partitions $\lambda^{(1)}, \dots, \lambda^{(n)}$ with $\lambda^{(i)} \prec \lambda^{(i+1)}$, $i = 1, 2, \dots, n-1$, and $\lambda^{(n)} = \lambda$.

2. The proof

2.1. Monotone triangles with prescribed bottom row

We fix notation that is needed in this paper: We use the *shift operator* E_x, defined as $\mathrm{E}_x\, p(x) = p(x+1)$, and set

$$\mathrm{Strict}_{x,y} = \mathrm{E}_x + \mathrm{E}_y^{-1} - \mathrm{E}_x \mathrm{E}_y^{-1}.$$

The *forward difference* is defined as $\overline{\Delta}_x = E_x - \mathrm{Id}$, while the *backward difference* is defined as $\underline{\Delta}_x = \mathrm{Id} - E_x^{-1}$. For a vector $\mathbf{x} = (x_1, \dots, x_n)$, we let $\overline{\Delta}_{\mathbf{x}} = \prod_{i=1}^{n} \overline{\Delta}_{x_i}$ and $\underline{\Delta}_{\mathbf{x}} = \prod_{i=1}^{n} \underline{\Delta}_{x_i}$. Note that shift operators with respect to different variables commute, which has the important consequence that all operators used in this paper commute. We define two polynomials as follows:

$$\mathrm{GT}_n(\mathbf{x}) = \prod_{1 \le i < j \le n} \frac{x_i - x_j + j - i}{j - i} \quad \text{and} \quad M_n(\mathbf{x}) = \prod_{1 \le p < q \le n} \mathrm{Strict}_{x_q, x_p} \mathrm{GT}_n(\mathbf{x}).$$

Multivariate Laurent polynomials in shift operators with respect to several variables also act on functions in these variables in the obvious way.

Theorem 2.1 *([10,12]). Suppose $\lambda = (\lambda_1, \ldots, \lambda_n)$ is a strict partition, then the number of MTs with bottom row λ is the evaluation of the polynomial $M_n(\mathbf{x})$ at $(x_1, \ldots, x_n) = (\lambda_1, \ldots, \lambda_n)$.*

For $n = 3$, we have

$$\begin{aligned}\prod_{1 \le p < q \le 3} (\mathrm{E}_{x_p}^{-1} + \mathrm{E}_{x_q} - \mathrm{E}_{x_p}^{-1} \mathrm{E}_{x_q}) = & - \mathrm{E}_{x_1}^{-2} + \mathrm{E}_{x_1}^{-1} + \mathrm{E}_{x_1}^{-2} \mathrm{E}_{x_2}^{-1} + \mathrm{E}_{x_3} + 3 \mathrm{E}_{x_1}^{-2} \mathrm{E}_{x_3} - 3 \mathrm{E}_{x_1}^{-1} \mathrm{E}_{x_3} \\ & - 2 \mathrm{E}_{x_1}^{-2} \mathrm{E}_{x_2}^{-1} \mathrm{E}_{x_3} + \mathrm{E}_{x_1}^{-1} \mathrm{E}_{x_2}^{-1} \mathrm{E}_{x_3} - \mathrm{E}_{x_1}^{-2} \mathrm{E}_{x_2} \mathrm{E}_{x_3} \\ & + \mathrm{E}_{x_1}^{-1} \mathrm{E}_{x_2} \mathrm{E}_{x_3} - 2 \mathrm{E}_{x_1}^{-2} \mathrm{E}_{x_3}^{2} + 3 \mathrm{E}_{x_3}^{2} \mathrm{E}_{x_1}^{-1} + \mathrm{E}_{x_1}^{-2} \mathrm{E}_{x_2}^{-1} \mathrm{E}_{x_3}^{2} \\ & - \mathrm{E}_{x_1}^{-1} \mathrm{E}_{x_2}^{-1} \mathrm{E}_{x_3}^{2} + \mathrm{E}_{x_2} \mathrm{E}_{x_3}^{2} + \mathrm{E}_{x_1}^{-2} \mathrm{E}_{x_2} \mathrm{E}_{x_3}^{2} \\ & - 2 \mathrm{E}_{x_1}^{-1} \mathrm{E}_{x_2} \mathrm{E}_{x_3}^{2}\end{aligned}$$

and applying this operator to the polynomial $\frac{1}{2}(x_1 - x_2 + 1)(x_1 - x_3 + 2)(x_2 - x_3 + 1)$ results in

$$\begin{aligned}&\frac{1}{2}(3x_1 + x_1^2 + 2x_1x_2 + x_1^2x_2 - 2x_2^2 - x_1x_2^2 - 3x_3 - 4x_1x_3 - x_1^2x_3 + 2x_2x_3 \\ &\quad + x_2^2x_3 + x_3^2 + x_1x_3^2 - x_2x_3^2).\end{aligned}$$

Evaluating at $(x_1, x_2, x_3) = (3, 2, 1)$ gives 7, and this is, by the correspondence between monotone triangles and ASMs, the number of 3×3 ASMs.

For strict partitions λ, we let

$$\mathrm{MT}_\lambda = \# \text{ of monotone triangles with bottom row } \lambda.$$

To compute MT_λ, we can use the recursion

$$\mathrm{MT}_\lambda = \sum_{\substack{\mu \prec \lambda \\ \mu \text{ strict}}} \mathrm{MT}_\mu \tag{2.1}$$

and the initial condition $\mathrm{MT}_\lambda = 1$ if $\ell(\lambda) = 1$. This is the approach we use to prove Theorem 2.1. Two lemmas are necessary. In the first lemma, we apply the recursion in (2.1) to a particular class of polynomials. As the polynomial in Theorem 2.1 belongs to this class, this will be enough to prove the formula by induction on n.

Lemma 2.2. *Let $n \ge 2$. Suppose $P(\mathbf{x}), Q(\mathbf{x})$ are polynomials in $\mathbf{x} = (x_1, \ldots, x_{n-1})$ with $P(\mathbf{x}) = \overline{\Delta}_{\mathbf{x}} Q(\mathbf{x})$ and, for each $i = 1, 2, \ldots, n-2$, $\mathrm{Strict}_{x_i, x_{i+1}} Q(\mathbf{x})$ vanishes if we specialize $x_{i+1} = x_i + 1$. If λ is a partition with n parts, then*

$$\sum_{\substack{\mu \prec \lambda \\ \mu \ strict}} P(\mu) = \sum_{r=1}^{n} (-1)^{r+n} Q(\lambda_1 + 1, \dots, \lambda_{r-1} + 1, \lambda_{r+1}, \dots, \lambda_n). \tag{2.2}$$

Proof. The crucial observation is the following identity which is an immediate consequence of the definition of $\prec$ and $\operatorname{Strict}_{\lambda_i^{(1)},\lambda_i^{(2)}}$.

$$\sum_{\substack{(\mu_{i-1},\mu_i)\prec(\lambda_{i-1},\lambda_i,\lambda_{i+1}) \\ \mu_{i-1}>\mu_i}} f(\mu_{i-1},\mu_i)$$
$$= \left[\operatorname{Strict}_{\lambda_i^{(1)},\lambda_i^{(2)}} \sum_{\mu_{i-1}=\lambda_i^{(1)}}^{\lambda_{i-1}} \sum_{\mu_i=\lambda_{i+1}}^{\lambda_i^{(2)}} f(\mu_{i-1},\mu_i) \right] \Bigg|_{\lambda_i^{(1)}=\lambda_i^{(2)}=\lambda_i}$$

Consequently, the left-hand side in (2.2) is equal to

$$\Bigg[\operatorname{Strict}_{\lambda_2^{(1)},\lambda_2^{(2)}} \operatorname{Strict}_{\lambda_3^{(1)},\lambda_3^{(2)}} \cdots$$
$$\operatorname{Strict}_{\lambda_{n-1}^{(1)},\lambda_{n-1}^{(2)}} \sum_{\mu_1=\lambda_2^{(1)}}^{\lambda_1} \sum_{\mu_2=\lambda_3^{(1)}}^{\lambda_2^{(2)}} \cdots \sum_{\mu_{n-2}=\lambda_{n-1}^{(1)}}^{\lambda_{n-2}^{(2)}} \sum_{\mu_{n-1}=\lambda_n}^{\lambda_{n-1}^{(2)}} P(\mu) \Bigg] \Bigg|_{\lambda_i^{(1)}=\lambda_i^{(2)}=\lambda_i} . \tag{2.3}$$

We use $P(\mu) = \overline{\Delta}_\mu Q(\mu)$ and see that, by telescoping, the multiple sum

$$\sum_{\mu_1=\lambda_2^{(1)}}^{\lambda_1} \sum_{\mu_2=\lambda_3^{(1)}}^{\lambda_2^{(2)}} \cdots \sum_{\mu_{n-2}=\lambda_{n-1}^{(1)}}^{\lambda_{n-1}^{(2)}} \sum_{\mu_{n-1}=\lambda_n}^{\lambda_{n-1}^{(2)}} \overline{\Delta}_\mu Q(\mu)$$

can be written as a sum of 2^{n-1} terms, where each term corresponds to the choice of either the upper or the lower bound in each of the $n-1$ sums. However, those terms where we choose the lower bound in the i-th sum and the upper bound in the $(i+1)$-st sum, for some $i = 1, 2, \dots, n-2$, vanish after the application of $\operatorname{Strict}_{\lambda_{i+1}^{(1)},\lambda_{i+1}^{(2)}}$ and setting $\lambda_{i+1}^{(1)} = \lambda_{i+1}^{(2)}$, by the assumption on $Q(\mathbf{x})$.

Thus, there is an $r = 1, \dots, n$ such that we choose the upper bound in the first $r-1$ sums and the lower bound in the remaining sums; each of the $n-r$ choices of the lower bound contributes a -1. Using the fact that $\operatorname{Strict}_{x,y}$ acts like the identity on functions that depend only on either x or y, we obtain the right-hand side of (2.2). $\square$

Lemma 2.3. *Let $d_1, d_2, \dots, d_{n-1} \geq 0$ be integers and set $d_n = -1$. If λ is a partition with n parts, then*

$$\sum_{\substack{\mu \prec \lambda \\ \mu \text{ strict}}} \prod_{1 \le p < q \le n-1} \operatorname{Strict}_{\mu_q,\mu_p} \det_{1 \le i,j \le n-1} \binom{\mu_i - i + n - 1}{d_j}$$
$$= \prod_{1 \le p < q \le n} \operatorname{Strict}_{\lambda_q,\lambda_p} \det_{1 \le i,j \le n} \binom{\lambda_i - i + n}{d_j + 1}.$$

Proof. We apply Lemma 2.2 to

$$P(\mathbf{x}) = \prod_{1 \le p < q \le n-1} \operatorname{Strict}_{x_q,x_p} \det_{1 \le i,j \le n-1} \binom{x_i - i + n - 1}{d_j},$$
$$Q(\mathbf{x}) = \prod_{1 \le p < q \le n-1} \operatorname{Strict}_{x_q,x_p} \det_{1 \le i,j \le n-1} \binom{x_i - i + n - 1}{d_j + 1}.$$

The polynomials fulfill the requirements: First, $\overline{\Delta}_x \binom{x}{d+1} = \binom{x}{d}$ implies $\overline{\Delta}_{\mathbf{x}} Q(\mathbf{x}) = P(\mathbf{x})$. Second, we use that

$$\operatorname{Strict}_{x_i,x_{i+1}} \prod_{1 \le p < q \le n-1} \operatorname{Strict}_{x_q,x_p}$$

is symmetric in x_i, x_{i+1} and

$$\operatorname{E}_{x_{i+1}} \det_{1 \le i,j \le n-1} \binom{x_i - i + n - 1}{d_j + 1}$$

is antisymmetric in x_i, x_{i+1} to deduce that also $\operatorname{E}_{x_{i+1}} \operatorname{Strict}_{x_i,x_{i+1}} Q(\mathbf{x})$ is antisymmetric in x_i, x_{i+1}, and this shows that also the second requirement is fulfilled as antisymmetric polynomials in x_i, x_{i+1} need to have the factor $x_{i+1} - x_i$. Lemma 2.2 now implies that the left-hand side of (2.3) is equal to

$$\sum_{r=1}^{n} (-1)^{r+n} Q(\lambda_1 + 1, \ldots, \lambda_{r-1} + 1, \lambda_{r+1}, \ldots, \lambda_n)$$
$$= \sum_{r=1}^{n} (-1)^{r+n} \prod_{\substack{1 \le p < q \le n \\ p,q \ne r}} \operatorname{Strict}_{\lambda_q,\lambda_p}$$
$$\left[\det_{1 \le i,j \le n-1} \binom{\mu_i - i + n - 1}{d_j + 1} \right] \Bigg|_{(\mu_1,\ldots,\mu_{n-1}) = (\lambda_1+1,\ldots,\lambda_{r-1}+1,\lambda_{r+1},\ldots,\lambda_n)}.$$

Since $\operatorname{Strict}_{\lambda_r,\lambda_p}$ and $\operatorname{Strict}_{\lambda_q,\lambda_r}$ have no effect on a function that is independent of λ_r, we can extend $\prod_{\substack{1 \le p < q \le n \\ p,q \ne r}} \operatorname{Strict}_{\lambda_q,\lambda_p}$ to $\prod_{1 \le p < q \le n} \operatorname{Strict}_{\lambda_q,\lambda_p}$ and now, since the latter does not depend on r, we can pull it out of the sum. Now the assertion follows as

$$\det_{1\le i,j\le n}\binom{\lambda_i-i+n}{d_j+1}$$
$$=\sum_{r=1}^{n}(-1)^{r+n}\left[\det_{1\le i,j\le n-1}\binom{\mu_i-i+n-1}{d_j+1}\right]\Bigg|_{(\mu_1,\ldots,\mu_{n-1})=(\lambda_1+1,\ldots,\lambda_{r-1}+1,\lambda_{r+1},\ldots,\lambda_n)},$$

by expanding the determinant on the left-hand side along the last column and using $d_n=-1$. □

Theorem 2.1 now follows by induction on n from Lemma 2.3 and (2.1) as

$$\prod_{1\le i<j\le n}\frac{x_i-x_j+j-i}{j-i}=\det_{1\le i,j\le n}\binom{x_i-i+n}{n-j}.$$

Indeed, this identity is a consequence of the Vandermonde determinant evaluation $\det_{1\le i,j\le n}\left(x_i^{j-1}\right)=\prod_{1\le i<j\le n}(x_j-x_i)$ as, by elementary column operations, $\det_{1\le i,j\le n}(p_j(x_i))=\det_{1\le i,j\le n}\left(x_i^{j-1}\right)$ for any sequence of polynomials $(p_j(x))_{1\le j\le n}$, where $p_j(x)$ is of degree $j-1$ and the leading coefficient of $p_j(x)$ is 1.

2.2. Rotating

So far, MT_λ is only defined for strict partitions λ, however, Theorem 2.1 allows us to extend the definition of MT_λ to all integer vectors λ by setting $\mathrm{MT}_\lambda=M_n(\lambda_1,\lambda_2,\ldots,\lambda_n)$. For an integer vector $\lambda=(\lambda_1,\ldots,\lambda_n)$, we define

$$\operatorname{rot}(\lambda)=(\lambda_n-n,\lambda_1,\ldots,\lambda_{n-1}).$$

Theorem 2.4 *([11]). Suppose λ is an integer vector of length n, then*

$$\mathrm{MT}_\lambda=(-1)^{n-1}\,\mathrm{MT}_{\operatorname{rot}(\lambda)}\,. \tag{2.4}$$

Recall that the r-th *elementary symmetric polynomial* is defined as

$$\mathrm{e}_r(x_1,\ldots,x_n)=\sum_{\substack{T\subseteq\{1,2,\ldots,n\}\\|T|=r}}\prod_{t\in T}x_t.$$

The proof of the theorem is based on the following lemma.

Lemma 2.5. *Let $n\ge 1$ and $1\le r\le n$. Then*

$$\mathrm{e}_r(\overline{\Delta}_{x_1},\ldots,\overline{\Delta}_{x_n})\prod_{1\le i<j\le n}\frac{x_i-x_j+j-i}{j-i}$$
$$=\mathrm{e}_r(\underline{\Delta}_{x_1},\ldots,\underline{\Delta}_{x_n})\prod_{1\le i<j\le n}\frac{x_i-x_j+j-i}{j-i}=0 \tag{2.5}$$

Proof. As

$$\mathrm{E}_{x_1} \mathrm{E}_{x_2}^2 \ldots \mathrm{E}_{x_n}^n \mathrm{e}_r(\overline{\Delta}_{x_1}, \ldots, \overline{\Delta}_{x_n}) \prod_{1 \le i < j \le n} \frac{x_i - x_j + j - i}{j - i}$$
$$= \mathrm{e}_r(\overline{\Delta}_{x_1}, \ldots, \overline{\Delta}_{x_n}) \prod_{1 \le i < j \le n} \frac{x_i - x_j}{j - i},$$

it suffices to show that the right-hand side vanishes in order to see that the first term in (2.5) vanishes. Now, $\prod_{1 \le i < j \le n} \frac{x_i - x_j}{j-i}$ is—up to a constant—the non-zero antisymmetric polynomial with the smallest total degree. Since $\mathrm{e}_r(\overline{\Delta}_{x_1}, \ldots, \overline{\Delta}_{x_n})$ is symmetric in $x_1, \ldots, x_n$, $\mathrm{e}_r(\overline{\Delta}_{x_1}, \ldots, \overline{\Delta}_{x_n}) \prod_{1 \le i < j \le n} \frac{x_i - x_j}{j-i}$ is antisymmetric as well, however, the total degree has been decreased as $\mathrm{e}_r(\overline{\Delta}_{x_1}, \ldots, \overline{\Delta}_{x_n})$ is homogeneous of degree greater than zero, and thus it must be zero.

The proof that the second term of (2.5) vanishes too is analogous. □

Proof of Theorem 2.4. It suffices to show that

$$M_n(x_1, \ldots, x_n) + (-1)^n M_n(x_n - n, x_1, \ldots, x_{n-1}). \tag{2.6}$$

Observe that the operator in $M_n(x_1, \ldots, x_n)$ can also be expressed as

$$E_{x_p}^{-1} + E_{x_q} - E_{x_p}^{-1} E_{x_q} = \mathrm{Id} + \underline{\Delta}_{x_p} \overline{\Delta}_{x_q}.$$

It follows that (2.6) is equal to

$$\prod_{1 \le p < q \le n} (\mathrm{Id} + \underline{\Delta}_{x_p} \overline{\Delta}_{x_q}) \prod_{1 \le i < j \le n} \frac{x_i - x_j + j - i}{j - i}$$
$$+ (-1)^n \mathrm{E}_{x_n}^{-n} \prod_{1 \le p < q \le n-1} (\mathrm{Id} + \underline{\Delta}_{x_p} \overline{\Delta}_{x_q}) \prod_{q=1}^{n-1} (\mathrm{Id} + \underline{\Delta}_{x_n} \overline{\Delta}_{x_q})$$
$$\prod_{1 \le i < j \le n-1} \frac{x_i - x_j + j - i}{j - i} \prod_{j=1}^{n-1} \frac{x_n - x_j + j}{j}$$
$$= \prod_{1 \le p < q \le n-1} (\mathrm{Id} + \underline{\Delta}_{x_p} \overline{\Delta}_{x_q}) \left(\prod_{p=1}^{n-1} (\mathrm{Id} + \underline{\Delta}_{x_p} \overline{\Delta}_{x_n}) - \prod_{q=1}^{n-1} (\mathrm{Id} + \underline{\Delta}_{x_n} \overline{\Delta}_{x_q}) \right)$$
$$\prod_{1 \le i < j \le n} \frac{x_i - x_j + j - i}{j - i}.$$

By Lemma 2.5 and as

$$\prod_{p=1}^{n-1} (\mathrm{Id} + \underline{\Delta}_{x_p} \overline{\Delta}_{x_n}) - \prod_{q=1}^{n-1} (\mathrm{Id} + \underline{\Delta}_{x_n} \overline{\Delta}_{x_q})$$
$$= \sum_{r=0}^{n-1} \left(\overline{\Delta}_{x_n}^{r} \, \mathrm{e}_r(\underline{\Delta}_{x_1}, \ldots, \underline{\Delta}_{x_{n-1}}) - \underline{\Delta}_{x_n}^{r} \, \mathrm{e}_r(\overline{\Delta}_{x_1}, \ldots, \overline{\Delta}_{x_{n-1}}) \right),$$

it suffices to show that

$$\overline{\Delta}_{x_n}^{r} \, \mathrm{e}_r(\underline{\Delta}_{x_1}, \ldots, \underline{\Delta}_{x_{n-1}}) - \underline{\Delta}_{x_n}^{r} \, \mathrm{e}_r(\overline{\Delta}_{x_1}, \ldots, \overline{\Delta}_{x_{n-1}})$$
$$= \sum_{s=1}^{r} (-1)^{r+s} \overline{\Delta}_{n}^{r} \underline{\Delta}_{n}^{r-s} \, \mathrm{e}_s(\underline{\Delta}_1, \ldots, \underline{\Delta}_n) + \sum_{s=1}^{r} (-1)^{r+s+1} \underline{\Delta}_{n}^{r} \overline{\Delta}_{n}^{r-s} \, \mathrm{e}_s(\overline{\Delta}_1, \ldots, \overline{\Delta}_n), \quad (2.7)$$

since the right-hand side is in the ideal of $\mathbb{C}[\underline{\Delta}_1, \ldots, \underline{\Delta}_n, \overline{\Delta}_1, \ldots, \overline{\Delta}_n]$ generated by $\mathrm{e}_i(\underline{\Delta}_1, \ldots, \underline{\Delta}_n)$ and $\mathrm{e}_i(\overline{\Delta}_1, \ldots, \overline{\Delta}_n)$, $i = 1, 2, \ldots, n$. Equation (2.7) is obvious for $r = 0$ and it follows for $r > 0$ by induction with respect to r, after observing that

$$\begin{aligned} &\overline{\Delta}_{n}^{r} \, \mathrm{e}_r(\underline{\Delta}_1, \ldots, \underline{\Delta}_{n-1}) - \underline{\Delta}_{n}^{r} \, \mathrm{e}_r(\overline{\Delta}_1, \ldots, \overline{\Delta}_{n-1}) \\ &\quad = \overline{\Delta}_{n}^{r} (\mathrm{e}_r(\underline{\Delta}_1, \ldots, \underline{\Delta}_n) - \underline{\Delta}_n \, \mathrm{e}_{r-1}(\underline{\Delta}_1, \ldots, \underline{\Delta}_{n-1})) \\ &\qquad - \underline{\Delta}_{n}^{r} (\mathrm{e}_r(\overline{\Delta}_1, \ldots, \overline{\Delta}_n) - \overline{\Delta}_n \, \mathrm{e}_{r-1}(\overline{\Delta}_1, \ldots, \overline{\Delta}_{n-1})) \\ &\quad = \left(\overline{\Delta}_{n}^{r} \, \mathrm{e}_r(\underline{\Delta}_1, \ldots, \underline{\Delta}_n) - \underline{\Delta}_{n}^{r} \, \mathrm{e}_r(\overline{\Delta}_1, \ldots, \overline{\Delta}_n) \right) \\ &\qquad - \overline{\Delta}_n \underline{\Delta}_n \left(\overline{\Delta}_{n}^{r-1} \, \mathrm{e}_{r-1}(\underline{\Delta}_1, \ldots, \underline{\Delta}_{n-1}) - \underline{\Delta}_{n}^{r-1} \, \mathrm{e}_{r-1}(\overline{\Delta}_1, \ldots, \overline{\Delta}_{n-1}) \right) \end{aligned}$$

and applying then the induction hypothesis to

$$\overline{\Delta}_{n}^{r-1} \, \mathrm{e}_{r-1}(\underline{\Delta}_1, \ldots, \underline{\Delta}_{n-1}) - \underline{\Delta}_{n}^{r-1} \, \mathrm{e}_{r-1}(\overline{\Delta}_1, \ldots, \overline{\Delta}_{n-1}). \qquad \square$$

2.3. Refined ASMs numbers and partial MTs

The following proposition contains certain expressions for the refined ASM numbers, i.e. the number $A_{n,i}$ of $n \times n$ ASMs with a 1 in the first row and i-th column.

Proposition 2.6 *([13]). Let n, i be positive integers and $M_n(\mathbf{x})$ denote the polynomial in (2.1).*

(1) *The number of MTs with bottom row $1, 2, \ldots, n$ and i occurrences of 1 is equal to the evaluation of the polynomial $(-\overline{\Delta}_{x_n})^{i-1} M_n(x_1, \ldots, x_n)$ at $(x_1, \ldots, x_n) = (n, n-1, \ldots, 3, 2, 2)$.*
(2) *The number of MTs with bottom row $1, 2, \ldots, n$ and i occurrences of n is equal to the evaluation of the polynomial $\underline{\Delta}_{x_1}^{i-1} M_n(x_1, \ldots, x_n)$ at $(x_1, \ldots, x_n) = (n-1, n-1, n-2, \ldots, 2, 1)$.*

Order n ASMs that have the 1 in the top row in column i correspond to either of the two objects in the proposition. To see this, rotate the ASM counterclockwise by 90° (resp. rotate clockwise by 90° and reflect then along the horizontal symmetry axis) and use the well-known bijection between ASMs and MTs.

The proposition is a consequence of the following simple observations. A *left-partial monotone triangle* of order n and depth i is an array of the form as a monotone triangle $(a_{i,j})_{1 \le j \le i \le n}$ of order n with the bottom $i-1$ elements of the first NE-diagonal deleted, that is $a_{n,1}, a_{n-1,1}, \dots, a_{n-i+2,1}$, see also (1.2). As usual we require weak increase along NE-diagonals and SE-diagonals, and strict increase along rows, except for $a_{n-i+1,1} < a_{n-i+1,2}$ does not have to be fulfilled. The number of such arrays with $(a_{n,n}, \dots, a_{n,2}) = (\lambda_1, \dots, \lambda_{n-1})$ and $\lambda_n = a_{n-i+1,1}$ is equal to the evaluation of $(-\overline{\Delta}_{x_n})^{i-1} M_n(x_1, \dots, x_n)$ at $(x_1, \dots, x_n) = (\lambda_1, \dots, \lambda_n)$. This follows by induction with respect to i and applying

$$\sum_{\substack{(\mu_1,\dots,\mu_{n-2}) \prec (\lambda_1,\dots,\lambda_{n-1}) \\ (\mu_1,\dots,\mu_{n-2}) \text{ strict}}} P(\mu_1, \dots, \mu_{n-2}, \lambda_n)$$
$$= -\overline{\Delta}_{\lambda_n} \left(\sum_{\substack{(\mu_1,\dots,\mu_{n-1}) \prec (\lambda_1,\dots,\lambda_n) \\ (\mu_1,\dots,\mu_{n-1}) \text{ strict}}} P(\mu_1, \dots, \mu_{n-1}) \right).$$

So to speak, $-\overline{\Delta}_{\lambda_n}$ "eats" the leftmost NE-diagonal. Analogously, we define *right-partial monotone triangles* of order n and depth i as arrays $(a_{i,j})_{1 \le j \le i \le n}$ with $a_{n,n}$, $a_{n-1,n-1}, \dots, a_{n-i+2,n-i+2}$ deleted, and the usual monotonicity requirements except for $a_{n-i+1,n-i} < a_{n-i+1,n-i+1}$. The number of such arrays with $\lambda_1 = a_{n-i+1,n-i+1}$ and $(a_{n,n-1}, \dots, a_{n,1}) = (\lambda_2, \dots, \lambda_n)$ is equal to the evaluation of $(\underline{\Delta}_{x_1})^{i-1} M_n(x_1, \dots, x_n)$ at $(x_1, \dots, x_n) = (\lambda_1, \dots, \lambda_n)$ as

$$\sum_{\substack{(\mu_2,\dots,\mu_{n-1}) \prec (\lambda_2,\dots,\lambda_n) \\ (\mu_2,\dots,\mu_{n-1}) \text{ strict}}} P(\lambda_1, \mu_2, \dots, \mu_{n-1})$$
$$= \underline{\Delta}_{\lambda_1} \left(\sum_{\substack{(\mu_1,\dots,\mu_{n-1}) \prec (\lambda_1,\dots,\lambda_n) \\ (\mu_1,\dots,\mu_{n-1}) \text{ strict}}} P(\mu_1, \dots, \mu_{n-1}) \right).$$

2.4. Linear equation system for the refined ASM numbers

Proposition 2.7. *Let $n \ge 1$. Then*

$$A_{n,i} = \sum_{j=1}^{n} \binom{2n-i-1}{n-i-j+1} (-1)^{j+1} A_{n,j}, \quad i = 1, 2, \dots, n. \tag{2.8}$$

Proof. By Proposition 2.6 (1) and Theorem 2.4,

$$\begin{aligned} A_{n,i} &= (-\overline{\Delta}_{x_n})^{i-1} M_n(n, n-1, \ldots, 2, x_n)\big|_{x_n=2} \\ &= (-1)^{n+i} \overline{\Delta}_{x_n}^{i-1} M_n(x_n - n, n, n-1, \ldots, 2)\Big|_{x_n=2} \end{aligned}$$

As $\overline{\Delta}_{x_n} = \mathrm{E}_{x_n} \underline{\Delta}_{x_n}$, this is equal to

$$\begin{aligned} &(-1)^{n+i} \, \mathrm{E}_{x_n}^{-2n+i+1} \, \underline{\Delta}_{x_n}^{i-1} M_n(x_n + 1, n, n-1, \ldots, 2)\Big|_{x_n=n-1} \\ &\qquad = (-1)^{n+i} (\mathrm{Id} - \underline{\Delta}_{x_n})^{2n-i-1} \underline{\Delta}_{x_n}^{i-1} M_n(x_n, n-1, n-2, \ldots, 1)\Big|_{x_n=n-1}, \end{aligned}$$

where we use $\mathrm{E}_{x_n}^{-1} = (\mathrm{Id} - \underline{\Delta}_{x_n})$ and $M_n(\mathbf{x} + \mathbf{1}) = M_n(\mathbf{x})$. Now we expand $(\mathrm{Id} - \underline{\Delta}_{x_n})^{2n-i-1}$ using the Binomial Theorem, and then employ Proposition 2.6 (2), to obtain

$$\sum_{j \geq 0} \binom{2n-i-1}{j} (-1)^{n+i+j} A_{n,i+j} = \sum_{j=1}^{n} \binom{2n-i-1}{j-i} (-1)^{n+j} A_{n,j},$$

where we use that $A_{n,j} = 0$ if $j > n$ and the binomial coefficient vanishes if $j < i$. This is now equal to the right-hand side in the statement as $A_{n,j} = A_{n,n+1-j}$. □

2.5. The linear equation system determines $A_{n,i}$

To show that the linear equation system in (2.8) determines the numbers $A_{n,i}$ for fixed n up to a multiplicative constant independent of i, it suffices to show that the rank of the $n \times n$ matrix $(\binom{2n-i-1}{n-i-j+1}(-1)^j + \delta_{i,j})_{1 \leq i,j \leq n}$ is $n-1$. For $n = 1$, this is easy to see, and otherwise this is accomplished by showing that the matrix B_n obtained by deleting the first row and the first column is non-singular. Let $S_n = \left(\binom{n}{j-i}(-1)^{i+j}\right)_{1 \leq i,j \leq n-1}$ and use the Chu–Vandermonde summation to see that $S_n^{-1} = \left(\binom{n+j-i-1}{j-i}\right)_{1 \leq i,j \leq n-1}$. Chu–Vandermonde summation also shows that $S_n B_n S_n^{-1} = \left(\binom{i+j}{j-1}(1 - \delta_{i,n-1}) + \delta_{i,j}\right)_{1 \leq i,j \leq n-1}$. Now

$$\det B_n = \det_{1 \leq i,j \leq n-1} \left(\binom{i+j}{j-1}(1 - \delta_{i,n-1}) + \delta_{i,j} \right) = \det_{1 \leq i,j \leq n-2} \left(\binom{i+j}{j-1} + \delta_{i,j} \right),$$

where the second equality follows from expanding with respect to the last row. The latter determinant is equal to

$$\sum_{k=0}^{n-2} \sum_{1 \leq t_1 < t_2 < \ldots < t_k \leq n-2} \det_{1 \leq i,j \leq k} \left(\binom{t_i + t_j}{t_j - 1} \right),$$

and thus positive as $\det_{1\le i,j\le k}\left(\binom{t_i+t_j}{t_j-1}\right)$ is by the Lindström–Gessel–Viennot theorem [21,16,17] just the number of families of k non-intersection lattice paths with unit north steps and east steps, and starting points $(0,-t_1-1),(0,-t_2-1),\ldots,(0,-t_k-1)$ and ending points $(t_1-1,0),(t_2-1,0),\ldots,(t_k-1,0)$.

Showing that the proposed numbers in (1.1) fulfill (2.8) (using again the Chu–Vandermonde summation) implies that we have the right numbers up to a constant independent of i but still possibly dependence on n. That this constant is in fact 1 follows then by induction with respect to n basically by showing (using Chu–Vandermonde summation) that the proposed numbers fulfill also the identity $\sum\limits_{i=1}^{n-1} A_{n-1,i} = A_{n,1}$.

3. Miscellaneous

3.1. Combinatorializing

A drawback of all existing proofs of the ASM theorem is that they are computational proofs. Combinatorial proofs are more desirable. For instance, it would be very interesting to have a bijective explanation for (2.8). We present some thoughts on "combinatorializing" the proof presented in this paper, see also [14].

3.1.1. Combinatorial proof of Theorem 2.1?

Theorem 2.1 is an important ingredient in our proof and so it would be interesting to provide a combinatorial proof. To this end, we point out that the evaluation of the polynomial $M_n(\mathbf{x})$ at $(x_1,\ldots,x_n)=(\lambda_1,\ldots,\lambda_n)$ has a quite obvious combinatorial interpretation (besides the one proved in Theorem 2.1) in terms of a signed enumeration. A combinatorial proof of Theorem 2.1 could then consist in bijectively explaining the equivalence of the two combinatorial interpretations.

First, let us note that $\mathrm{GT}_n(\mathbf{x})$ is the number of *semistandard tableaux* of shape $\mathbf{x}=(x_1,\ldots,x_n)$, or, equivalently, the number of *Gelfand–Tsetlin patterns* with bottom row $(x_n,x_{n-1},\ldots,x_1)$ if $x_1\ge x_2\ge\ldots\ge x_n\ge 0$. If we expand the operator $\prod\limits_{1\le p<q\le n}\left(\mathrm{E}_{x_p}^{-1}+\mathrm{E}_{x_q}-\mathrm{E}_{x_p}^{-1}\mathrm{E}_{x_q}\right)$ into monomials in $\mathrm{E}_{x_1}^{\pm1},\mathrm{E}_{x_2}^{\pm1},\ldots,\mathrm{E}_{x_n}^{\pm1}$ and apply the shift operators to $\mathrm{GT}_n(\mathbf{x})$, we obtain a signed sum of expressions each of which count Gelfand–Tsetlin patterns with a prescribed bottom row that is a "deformation" of $(x_n,x_{n-1},\ldots,x_1)$. (In fact, these deformations count what we call "generalized" Gelfand–Tsetlin patterns because after the deformation the bottom row does not have to be increasing, see below.)

The signed sum and the deformations have an easy description as follows: Let us define a *direction pattern* d of order n to be a function d assigning each element in $\{(i,j)|1\le j\le i\le n-1\}$ an element in $\{\leftarrow,\rightarrow,\leftrightarrow\}$, and define the sign of d to be $\mathrm{sgn}(d)=(-1)^{\#\text{ of }\leftrightarrow}$. So the domain of an order n direction pattern is just the "index set" of the elements of an order n Gelfand–Tsetlin pattern, where the bottom row is excluded, see also (1.2). The background is that we identify (i,j) in the domain of the direction pattern with the factor corresponding to the pair $(i-j+1, n-j+1) =: (p,q)$ in $\prod_{1 \le p < q \le n} \left(\mathrm{E}_{x_p}^{-1} + \mathrm{E}_{x_q} - \mathrm{E}_{x_p}^{-1} \mathrm{E}_{x_q} \right)$, where the assignment "$d(i,j) = \rightarrow$" corresponds to choosing $\mathrm{E}_{x_p}^{-1}$ from the factor $\left(\mathrm{E}_{x_p}^{-1} + \mathrm{E}_{x_q} - \mathrm{E}_{x_p}^{-1} \mathrm{E}_{x_q} \right)$ when expanding the product, "$d(i,j) = \leftarrow$" corresponds to choosing E_{x_q} and "$d(i,j) = \leftrightarrow$" corresponds to choosing $\mathrm{E}_{x_p}^{-1} \mathrm{E}_{x_q}$.

Now given a direction pattern p of order n, the corresponding deformation $(x_1, \ldots, x_n)_d =: (y_1, \ldots, y_n)$ is computed as follows: We arrange the direction pattern in the form of a Gelfand–Tsetlin pattern (1.2) and add $x_n, x_{n-1}, \ldots, x_1$ as bottom row. Then

$$\begin{aligned} y_i = x_i &- (\#\text{ of } \rightarrow, \leftrightarrow \text{ in the NW diagonal of } x_i) \\ &+ (\#\text{ of } \leftarrow, \leftrightarrow \text{ in the NE diagonal of } x_i). \end{aligned}$$

In conclusion, we can write $M_n(\mathbf{x})$ as

$$\sum_{d \text{ direction pattern of order n}} \operatorname{sgn}(d) \operatorname{GT}(\mathbf{x}_d).$$

One important final observation in this respect is that $\operatorname{GT}(\mathbf{x}_d)$ is not necessarily the number of Gelfand–Tsetlin patterns with bottom row $\mathbf{x}_d$, since, because of the deformation, the sequence $\mathbf{x}_d$ does not have to be non-increasing (even if $\mathbf{x}$ was decreasing). However, there exists a combinatorial interpretation of $\operatorname{GT}(\mathbf{x})$ for all $\mathbf{x} \in \mathbb{Z}^n$ which extends Gelfand–Tsetlin patterns, see [9, Section 5.1]: a *generalized Gelfand–Tsetlin pattern* is a triangular array of integers $(a_{i,j})_{1 \le j \le i \le n}$ such that the following is fulfilled for each entry $a_{i,j}$ with $i < n$:

- If $a_{i+1,j} \le a_{i+1,j+1}$, then $a_{i+1,j} \le a_{i,j} \le a_{i+1,j+1}$.
- If $a_{i+1,j} > a_{i+1,j+1}$, then $a_{i+1,j} > a_{i,j} > a_{i+1,j+1}$.

(This implies that there is no generalized Gelfand–Tsetlin pattern with $a_{i+1,j} = a_{i+1,j+1} + 1$.) Whenever we are in the second case, we say that $a_{i,j}$ is an *inversion*. The sign of a generalized Gelfand–Tsetlin pattern is $(-1)^{\#\text{ of inversions}}$ and the signed enumeration of a generalized Gelfand–Tsetlin pattern with bottom row $\mathbf{x}$ is equal to $\operatorname{GT}(\mathbf{x})$.

3.1.2. Combinatorial proof of Lemma 2.4?

On the other hand, Theorem 2.1 is only needed to prove Lemma 2.4, so one could immediately go for a combinatorial proof of Lemma 2.4. Since $\operatorname{rot}(\lambda)$ is not a strict partition if λ is, one prerequisite is a combinatorial interpretation of MT_λ for all finite integer sequences λ. (Recall that MT_λ is for arbitrary finite integer sequences defined as the evaluation of the polynomial $M_n(\mathbf{x})$ at λ.) I have provided several such interpretations in [14] (see also [29]) in terms of a signed enumeration and repeat my favorite interpretation here. (The sign seems unavoidable, because (2.4) involves a sign.)

We start by providing an alternative combinatorial interpretation for MT_λ if λ is a strict partition. Here a *direction pattern* d of order n is a function d assigning each element in $\{(i,j) | 1 \le j \le i \le n\}$ an element in $\{\leftarrow, \rightarrow, \leftrightarrow\}$ and its sign is $\operatorname{sgn}(d) = (-1)^{\# \text{ of } \leftrightarrow}$. So the domain of an order n direction pattern is now the entire index set of the elements of Gelfand–Tsetlin patterns of order n. We say that a Gelfand–Tsetlin pattern $(a_{i,j})_{1 \le j \le i \le n}$ *respects the direction pattern* if the following is fulfilled: For all entries $a_{i,j}$ with $i < n$, we need to have the following.

- If $d(i+1,j) \in \{\rightarrow, \leftrightarrow\}$, then $a_{i+1,j} < a_{i,j}$.
- If $d(i+1,j+1) \in \{\leftarrow, \leftrightarrow\}$, then $a_{i,j} < a_{i+1,j+1}$.

Then MT_λ is equal to

$$\sum_{d \text{ direction pattern of order n}} \operatorname{sgn}(d) \, (\# \text{ of Gelfand–Tsetlin patterns with bottom row } \lambda \text{ respecting } d).$$

This follows basically from (2.1) and the fact that (2.3) is equal to the left-hand side in (2.2). See [14, Subsection 6.1.2] for a more detailed explanation.[2]

This can be extended to all finite integer sequences λ. Given a direction pattern d of order n, then a triangular array $(a_{i,j})_{1 \le j \le i \le n}$ respects d if, for each entry $a_{i,j}$ with $i < n$, the following is fulfilled.

(1) If $d(i+1,j) = \leftarrow$ and $d(i+1,j+1) \in \{\leftarrow, \leftrightarrow\}$, then $a_{i+1,j} \le a_{i,j} < a_{i+1,j+1}$ or $a_{i+1,j} > a_{i,j} \ge a_{i+1,j+1}$.
(2) If $d(i+1,j) = \leftarrow$ and $d(i+1,j+1) = \rightarrow$, then $a_{i+1,j} \le a_{i,j} \le a_{i+1,j+1}$ or $a_{i+1,j} > a_{i,j} > a_{i+1,j+1}$.
(3) If $d(i+1,j) \in \{\leftrightarrow, \rightarrow\}$ and $d(i+1,j+1) \in \{\leftarrow, \leftrightarrow\}$, then $a_{i+1,j} < a_{i,j} < a_{i+1,j+1}$ or $a_{i+1,j} \ge a_{i,j} \ge a_{i+1,j+1}$.
(4) If $d(i+1,j) \in \{\leftrightarrow, \rightarrow\}$ and $d(i+1,j+1) = \rightarrow$, then $a_{i+1,j} < a_{i,j} \le a_{i+1,j+1}$ or $a_{i+1,j} \ge a_{i,j} > a_{i+1,j+1}$.

In each case, we say that $a_{i,j}$ is an *inversion* if the second possibility applies. (If the bottom row of such a triangular array is an increasing sequence, then there can be no inversion and $(a_{i,j})_{1 \le j \le i \le n}$ is just a Gelfand–Tsetlin pattern that respects the direction pattern d in the above defined sense.) Now MT_λ has the following combinatorial interpretation.

[2] Note that this combinatorial interpretation of MT_λ could also be useful in providing the combinatorial proof asked for in the previous subsection. In fact, there it would also have been possible to work with direction patterns of order n the domain of which include also the entire index set of the elements of Gelfand–Tsetlin patterns of order n.

$$\sum_{\substack{d \text{ direction pattern of order n} \\ (a_{i,j})_{1 \le j \le i \le n} \text{ array with bottom row } \lambda \text{ respecting } d}} \operatorname{sgn}(d)\,(-1)^{\#\text{ of inversions of } (a_{i,j})_{1\le j\le i\le n}}.$$

3.2. Constant term formulation of Theorem 2.1

Corollary 3.1. *Suppose $\lambda = (\lambda_1, \ldots, \lambda_n)$ is a strict partition, then the number of MTs with bottom row λ is the constant term of the following Laurent polynomial.*

$$\prod_{i=1}^{n} (1+x_i)^{\lambda_i} x_i^{-n+1} \prod_{1\le i<j\le n} (x_j - x_i)(1 + x_i + x_i x_j)$$

Proof. Applying the operator $\prod_{i=1}^{n} E_{x_i}^{\lambda_i}$ to $M_n(\mathbf{x})$ and computing the constant term of the resulting polynomial gives the number of MTs with bottom row λ. As

$$\prod_{1\le p<q\le n} E_{x_p}^{-1} \prod_{1\le i<j\le n} \frac{x_i - x_j + j - i}{j-i} = \prod_{1\le i<j\le n} \frac{x_i - x_j}{j-i} = \det_{1\le i,j\le n} \binom{x_i}{n-j},$$

and by expressing the shift operators by difference operators, i.e. using $E_x = \operatorname{Id} + \overline{\Delta}_x$, this number is also the constant term of the following polynomial,

$$\sum_{\sigma\in\mathcal{S}_n} \operatorname{sgn}\sigma \prod_{i=1}^{n} (1+\overline{\Delta}_{x_i})^{\lambda_i} \prod_{1\le p<q\le n} (\operatorname{Id} + \overline{\Delta}_{x_p} + \overline{\Delta}_{x_p}\overline{\Delta}_{x_q}) \prod_{i=1}^{n} \binom{x_i}{n-\sigma(i)},$$

where we have used the Leibniz formula for the determinant. Now, since

$$\overline{\Delta}_x^s \binom{x}{t}\bigg|_{x=0} = \binom{x}{t-s}\bigg|_{x=0} = \delta_{s,t},$$

where $\delta_{s,t}$ is the Kronecker delta, this number is also

$$\sum_{\sigma\in\mathcal{S}_n} \operatorname{sgn}\sigma\, \langle x_1^{n-\sigma(1)} \cdots x_n^{n-\sigma(n)} \rangle \prod_{i=1}^{n} (1+x_i)^{\lambda_i} \prod_{1\le i<j\le n} (1 + x_i + x_i x_j),$$

where $\langle x_1^{m_1} \cdots x_n^{m_n} \rangle P(x_1, \ldots, x_n)$ denotes the coefficient of $x_1^{m_1} \cdots x_n^{m_n}$ in the polynomial $P(x_1, \ldots, x_n)$. But this is also the constant term of

$$\sum_{\sigma\in\mathcal{S}_n} \operatorname{sgn}\sigma\, x_1^{\sigma(1)-n} \cdots x_n^{\sigma(n)-n} \prod_{i=1}^{n} (1+x_i)^{\lambda_i} \prod_{1\le i<j\le n} (1 + x_i + x_i x_j),$$

and since

$$\sum_{\sigma\in\mathcal{S}_n} \operatorname{sgn}\sigma\, x_1^{\sigma(1)-1} \cdots x_n^{\sigma(n)-1} = \prod_{1\le i<j\le n} (x_j - x_i)$$

this is the expression in the statement. $\square$

In particular, this shows that the constant term of

$$\prod_{i=1}^{n}(1+x_i)^{n-i}x_i^{-n+1}\prod_{1\le i<j\le n}(x_j-x_i)(1+x_i+x_ix_j)$$

is the number of $n \times n$ ASMs. Similar identities have appeared before in the work of Di Francesco, Fonseca and Zinn–Justin, for instance, the constant term of

$$\prod_{i=1}^{n}(1+x_i)^{2}x_i^{2i-2n-1}\prod_{1\le i<j\le n}(x_i-x_j)(1+x_i+x_ix_j)$$

is also the number of $n \times n$ ASMs, see [15, (4.9)]. Compare also with [34, Theorem 1.13].

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