A NEW PROOF OF THE REFINED ALTERNATING SIGN MATRIX THEOREM

Ilse Fischer

Fakultät für Mathematik, Universität Wien
Nordbergstrasse 15, A-1090 Wien, Austria
E-mail: Ilse.Fischer@univie.ac.at

Abstract. In the early 1980s, Mills, Robbins and Rumsey conjectured, and in 1996 Zeilberger proved a simple product formula for the number of $n \times n$ alternating sign matrices with a 1 at the top of the i-th column. We give an alternative proof of this formula using our operator formula for the number of monotone triangles with prescribed bottom row. In addition, we provide the enumeration of certain 0-1-(-1) matrices generalizing alternating sign matrices.

1. Introduction

An alternating sign matrix is a square matrix of 0s, 1s and -1s for which the sum of entries in each row and in each column is 1 and the non-zero entries of each row and of each column alternate in sign. For instance,

$$\begin{pmatrix} 0 & 0 & 1 & 0 & 0 \\ 1 & 0 & -1 & 1 & 0 \\ 0 & 1 & 0 & -1 & 1 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \end{pmatrix}$$

is an alternating sign matrix. In [7, 8] Mills, Robbins and Rumsey conjectured that there are

$$\prod_{j=1}^{n} \frac{(3j-2)!}{(n+j-1)!} \tag{1.1}$$

$n \times n$ alternating sign matrices. This was first proved by Zeilberger [9]. Another, shorter, proof was given by Kuperberg [6] using the equivalence of alternating sign matrices with the six-vertex model for "square ice", which was earlier introduced in statistical mechanices. Zeilberger [10] then used Kuperberg's observations to prove the following refinement generalizing (1.1).

Theorem 1. *The number of $n \times n$ alternating sign matrices where the unique* 1 *in the first row is at the top of the i-th column is*

$$\frac{(i)_{n-1}(1+n-i)_{n-1}}{(n-1)!} \prod_{j=1}^{n-1} \frac{(3j-2)!}{(n+j-1)!}. \tag{1.2}$$

In this formula, $(a)_n = \prod_{i=0}^{n} (a+i)$.

The task of this paper is to give an alternative proof of Theorem 1. As a byproduct, we obtain the enumeration of certain objects generalizing alternating sign matrices.

Theorem 2. *The number of $n \times k$ matrices of 0s, 1s and -1s for which the non-zero entries of each row and each column alternate in sign and the sum in each row and each column is 1, except for the columns $n, n+1, \ldots, k-1$, where we have sum 0 and the first non-zero element is a 1, is*

$$\prod_{j=1}^{n-1} \frac{(3j-2)!}{(n+j-1)!} \sum_{i=1}^{n} \frac{(i)_{n-1}(1+n-i)_{n-1}}{(n-1)!} \binom{i+k-n-1}{i-1}.$$

Our proofs are based on a formula [3, Theorem 1] for the number of monotone triangles with given bottom row $(k_1, k_2, \ldots, k_n)$, which we have recently derived. Monotone triangles with bottom row $(1, 2, \ldots, n)$ are in bijection with $n \times n$ alternating sign matrices. The monotone triangle corresponding to a given alternating sign matrix can be obtained as follows: Replace every entry in the matrix by the sum of elements in the same column above, the entry itself included. In our running example we have

$$\begin{pmatrix} 0 & 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 1 & 0 \\ 1 & 1 & 0 & 0 & 1 \\ 1 & 1 & 1 & 0 & 1 \\ 1 & 1 & 1 & 1 & 1 \end{pmatrix}.$$

Row by row we record the columns that contain a 1 and obtain the following triangular array.

$$\begin{array}{ccccccccc} & & & & 3 & & & & \\ & & & 1 & & 4 & & & \\ & & 1 & & 2 & & 5 & & \\ & 1 & & 2 & & 3 & & 5 & \\ 1 & & 2 & & 3 & & 4 & & 5 \end{array}$$

This is the monotone triangle corresponding to the alternating sign matrix above. Observe that it is weakly increasing in northeast direction and in southeast direction. Moreover, it is strictly increasing along rows. In general, a monotone triangle with n rows is a triangular array $(a_{i,j})_{1 \le j \le i \le n}$ of integers such that $a_{i,j} \le a_{i-1,j} \le a_{i,j+1}$ and $a_{i,j} < a_{i,j+1}$ for all i, j. It is not too hard to see that monotone triangles with n rows and $a_{n,j} = j$ are in bijection with $n \times n$ alternating sign matrices. Moreover, monotone triangles with $n-1$ rows and bottom row $(1, 2, \ldots, i-1, i+1, \ldots, n)$ are in bijection with $n \times n$ alternating sign matrices with a 1 at the bottom (or equivalently the top) of the i-th column. In [3] we gave the following operator formula for the number of monotone triangles with prescribed bottom row $(k_1, k_2, \ldots, k_n) \in \mathbb{Z}^n$.

Theorem 3 ([3], Theorem 1). *The number of monotone triangles with n rows and prescribed bottom row $(k_1, k_2, \ldots, k_n)$ is given by*

$$\left(\prod_{1 \le p < q \le n} \left(\mathrm{id} + E_{k_p} \Delta_{k_q} \right) \right) \prod_{1 \le i < j \le n} \frac{k_j - k_i}{j - i},$$

where E_x denotes the shift operator, defined by $E_x \, p(x) = p(x+1)$, and $\Delta_x := E_x - \mathrm{id}$ denotes the difference operator. (In this formula, the product of operators is understood as the composition.)

Thus, for instance, the number of monotone triangles with bottom row (k_1, k_2, k_3) is

$$\begin{aligned}(\mathrm{id} + E_{k_1}\Delta_{k_2})(\mathrm{id} + E_{k_1}\Delta_{k_3})(\mathrm{id} + E_{k_2}\Delta_{k_3}) \frac{1}{2}(k_2 - k_1)(k_3 - k_1)(k_3 - k_2) \\ = \frac{1}{2}(-3k_1 + k_1^2 + 2k_1k_2 - k_1^2k_2 - 2k_2^2 + k_1k_2^2 + 3k_3 - 4k_1k_3 \\ + k_1^2k_3 + 2k_2k_3 - k_2^2k_3 + k_3^2 - k_1k_3^2 + k_2k_3^2).\end{aligned}$$

We outline our proof of Theorem 1. To prove the theorem using the formula in Theorem 3 clearly means that we have to evaluate the formula at $(k_1, k_2, \ldots, k_n) = (1, 2, \ldots, i-1, i+1, \ldots, n+1)$. Let $A_{n,i}$ denote the number of $n \times n$ alternating sign matrices with a 1 at the top of the i-th column and let $\alpha(n; k_1, \ldots, k_n)$ denote the number of monotone triangles with bottom row $(k_1, \ldots, k_n)$. Using the formula in Theorem 3, we extend the interpretation of $\alpha(n; k_1, \ldots, k_n)$ to arbitrary $(k_1, \ldots, k_n) \in \mathbb{Z}^n$. In our proof, we first give a formula for $\alpha(n; 1, 2, \ldots, n-1, k)$ in terms of $A_{n,i}$. Next we show that $\alpha(n; 1, 2, \ldots, n-1, k)$ is an even polynomial in k if n is odd and an odd polynomial if n is even. These two facts will then imply that $(A_{n,i})_{1 \le i \le n}$ is an eigenvector with respect to the eigenvalue 1 of a certain matrix with binomial coefficients as entries. Finally we see that this determines $A_{n,i}$ up to a constant, which can easily be computed by induction with respect to n.

We believe that this second approach to prove the refined alternating sign matrix theorem not only provides us with a better understanding of known theorems, but will also enable us to obtain new results in the field of plane partition and alternating sign matrix enumeration in the future. A general strategy might be to derive analogous multivariate operators formulas for other (triangular) arrays of integers, which correspond to certain classes of plane partitions and alternating sign matrices and which simplify to nice product formulas if we specialize the parameters in the right way. Hopefully these operator formulas can then also be used to derive these product formulas.

2. A formula for $\alpha(n; 1, 2, \ldots, n-1, k)$

We start by stating a fundamental recursion for $\alpha(n; k_1, \ldots, k_n)$. If we delete the last row of a monotone triangle with bottom row $(k_1, k_2, \ldots, k_n)$, we obtain a monotone triangle with $n-1$ rows and bottom row, say, $(l_1, l_2, \ldots, l_{n-1})$. By the definition of a monotone triangle, we have $k_1 \le l_1 \le k_2 \le l_2 \le \ldots \le k_{n-1} \le l_{n-1} \le k_n$ and $l_i \ne l_{i+1}$.

Thus

$$\alpha(n;k_1,\ldots,k_n)=\sum_{\substack{(l_1,\ldots,l_{n-1})\in\mathbb{Z}^{n-1},\\ k_1\le l_1\le k_2\le\ldots\le k_{n-1}\le l_{n-1}\le k_n, l_i\ne l_{i+1}}}\alpha(n-1;l_1,\ldots,l_{n-1}). \tag{2.1}$$

In the following lemma, we explain the action of operators, which are symmetric polynomials in $E_{k_1}, E_{k_2},\ldots,E_{k_n}$, on $\alpha(n;k_1,\ldots,k_n)$. It will be used twice in our proof of Theorem 1.

Lemma 1. *Let $P(X_1,\ldots,X_n)$ be a symmetric polynomial in $(X_1,\ldots,X_n)$ over $\mathbb{C}$. Then*

$$P(E_{k_1},\ldots,E_{k_n})\alpha(n;k_1,\ldots,k_n)=P(1,1,\ldots,1)\cdot\alpha(n;k_1,\ldots,k_n).$$

Proof. By Theorem 3 and the fact that shift operators with respect to different variables commute, it suffices to show that

$$P(E_{k_1},\ldots,E_{k_n})\prod_{1\le i<j\le n}\frac{k_j-k_i}{j-i}=P(1,1,\ldots,1)\cdot\prod_{1\le i<j\le n}\frac{k_j-k_i}{j-i}.$$

Let $(m_1,\ldots,m_n)\in\mathbb{Z}^n$ be with $m_i\ge 0$ for all i and $m_i\ne 0$ for at least one i. It suffices to show that

$$\sum_{\pi\in\mathcal{S}_n}\Delta_{k_1}^{m_{\pi(1)}}\Delta_{k_2}^{m_{\pi(2)}}\ldots\Delta_{k_n}^{m_{\pi(n)}}\prod_{1\le i<j\le n}\frac{k_j-k_i}{j-i}=0.$$

By the Vandermonde determinant evaluation, we have

$$\prod_{1\le i<j\le n}\frac{k_j-k_i}{j-i}=\det_{1\le i,j\le n}\left(\binom{k_i}{j-1}\right).$$

Therefore, it suffices to show that

$$\sum_{\pi,\sigma\in\mathcal{S}_n}\operatorname{sgn}\sigma\binom{k_1}{\sigma(1)-m_{\pi(1)}-1}\binom{k_2}{\sigma(2)-m_{\pi(2)}-1}\cdots\binom{k_n}{\sigma(n)-m_{\pi(n)}-1}=0.$$

If, for fixed $\pi,\sigma\in\mathcal{S}_n$, there exists an i with $\sigma(i)-m_{\pi(i)}-1<0$ then the corresponding summand vanishes. We define a sign reversing involution on the set of non-zero summands. Fix $\pi,\sigma\in\mathcal{S}_n$ such that the summand corresponding to π and σ does not vanish. Consequently, $\{\sigma(1)-m_{\pi(1)}-1,\sigma(2)-m_{\pi(2)}-1,\ldots,\sigma(n)-m_{\pi(n)}-1\}\subseteq\{0,1,\ldots,n-1\}$ and since $(m_1,\ldots,m_n)\ne(0,\ldots,0)$, there are i,j, $1\le i<j\le n$, with $\sigma(i)-m_{\pi(i)}-1=\sigma(j)-m_{\pi(j)}-1$. Among all pairs (i,j) with this property, let (i',j') be the pair, which is minimal with respect to the lexicographic order. Then the summand corresponding to $\pi\circ(i',j')$ and $\sigma\circ(i',j')$ is the negativ of the summand corresponding to π and σ. $\square$

Let

$$e_p(X_1,\ldots,X_n)=\sum_{1\le i_1<i_2<\ldots i_p\le n}X_{i_1}X_{i_2}\ldots X_{i_p}$$

denote the p-th elementary symmetric function. Lemma 1 will be used to deduce a formula for

$$e_{p-j}(E_{k_1},E_{k_2},\ldots,E_{k_{n-1}})\alpha(n;k_1,\ldots,k_n)\big|_{(k_1,\ldots,k_n)=(1,2,\ldots,n-1,n+j)} \tag{2.2}$$

in terms of $A_{n,i}$ if $0 \le j \le p \le n-1$ (Lemma 3). If we specialize $p=j$ in this identity we obtain the desired formula for $\alpha(n;1,2,\dots,n-1,k)$, which we have mentioned in the outline of our proof. The formula for (2.2) will be shown by induction with respect to j. In the following lemma, we deal with the initial case of the induction.

Lemma 2. *Let $0 \le p \le n-1$. Then we have*

$$e_p(E_{k_1},E_{k_2},\dots,E_{k_{n-1}})\,\alpha(n;k_1,\dots,k_n)\big|_{(k_1,\dots,k_n)=(1,2,\dots,n)} = \sum_{i=1}^{n} \binom{n-i}{p} A_{n,i}.$$

Proof. First observe that for $1 \le i_1 < i_2 < \dots i_p \le n-1$ we have

$$\begin{aligned} &E_{k_{i_1}} E_{k_{i_2}} \dots E_{k_{i_p}} \alpha(n;k_1,\dots,k_n)\Big|_{(k_1,\dots,k_n)=(1,2,\dots,n)} \\ &= \alpha(n;1,2,\dots,i_1-1,i_1+1,i_1+1,\dots,i_2-1,i_2+1,i_2+1,\dots,i_p-1,i_p+1,i_p+1,\dots,n) \\ &= \sum_{\substack{1\le j_1\le 2\le j_2\le\dots\le i_1-1\le j_{i_1-1}\le i_1+1 \\ j_1<j_2<\dots<j_{i_1-1}}} \alpha(n-1;j_1,j_2,\dots,j_{i_1-1},i_1+1,i_2+1,\dots,n) \\ &= \alpha(n;1,2,\dots,i_1-1,i_1+1,i_1+1,i_1+2,i_1+3,\dots,n), \end{aligned}$$

where the second equality follow from (2.1). In particular, we see that

$$E_{k_{i_1}} E_{k_{i_2}} \dots E_{k_{i_p}} \alpha(n;k_1,\dots,k_n)\Big|_{(k_1,\dots,k_n)=(1,2,\dots,n)}$$

does not depend on $i_2,\dots,i_p$. Consequently, the left-hand-side in the statement of the lemma is equal to

$$\sum_{j=1}^{n-1} \binom{n-1-j}{p-1} \alpha(n;1,2,\dots,j-1,j+1,j+1,j+2,\dots,n). \tag{2.3}$$

Next observe that by (2.1)

$$\begin{aligned} &\alpha(n;1,2,\dots,j-1,j+1,j+1,j+2,\dots,n) \\ &= \sum_{i=1}^{j} \alpha(n-1;1,2,\dots,i-1,i+1,\dots,n) = \sum_{i=1}^{j} A_{n,i}. \end{aligned}$$

Thus, (2.3) is equal to

$$\sum_{j=1}^{n-1} \binom{n-1-j}{p-1} \sum_{i=1}^{j} A_{n,i} = \sum_{i=1}^{n} \sum_{j=i}^{n} \binom{n-1-j}{p-1} A_{n,i} - \binom{-1}{p-1} \sum_{i=1}^{n} A_{n,i}.$$

We complete the proof by using the following summation formula

$$\sum_{j=a}^{b} \binom{x+j}{n} = \sum_{j=a}^{b} \left(\binom{x+j+1}{n+1} - \binom{x+j}{n+1} \right) = \binom{x+b+1}{n+1} - \binom{x+a}{n+1}.$$

$\square$

Lemma 3. *Let $0 \le j \le p \le n-1$. Then*

$$e_{p-j}(E_{k_1}, E_{k_2}, \ldots, E_{k_{n-1}}) \alpha(n; k_1, \ldots, k_n)\big|_{(k_1,\ldots,k_n)=(1,2,\ldots,n-1,n+j)} \\ = (-1)^j \sum_{i=1}^{n} A_{n,i} \left(\binom{n-i}{p} + \sum_{l=0}^{j-1} \binom{n}{p-l} \binom{i+l-1}{l} (-1)^{l-1} \right)$$

For $p=j$ this simplifies to

$$\alpha(n; 1, \ldots, n-1, n+j) = \sum_{i=1}^{n} A_{n,i} \binom{i+j-1}{i-1}.$$

Proof. First we show how the first formula implies the second. For this purpose, we have to consider

$$(-1)^j \left(\binom{n-i}{j} + \sum_{l=0}^{j-1} \binom{n}{j-l} \binom{i+l-1}{l} (-1)^{l-1} \right).$$

Using $\binom{i+l-1}{l} = \binom{-i}{l}(-1)^l$, we see that this is equal to

$$(-1)^j \left(\binom{n-i}{j} + (-1)^j \binom{i+j-1}{j} - \sum_{l=0}^{j} \binom{n}{j-l} \binom{-i}{l} \right).$$

We apply the Chu-Vandermonde summation [4, p. 169, (5.22)], in order to see that this is simplifies to $\binom{i+j-1}{j} = \binom{i+j-1}{i-1}$.

Now we consider the first formula. By Lemma 1, we have

$$e_{p-j}(E_{k_1}, \ldots, E_{k_n}) \alpha(n; k_1, \ldots, k_n)\big|_{(k_1,\ldots,k_n)=(1,2,\ldots,n-1,n+j)} \\ = \binom{n}{p-j} \alpha(n; 1, 2, \ldots, n-1, n+j).$$

On the other hand, we have

$$e_{p-j}(E_{k_1}, \ldots, E_{k_n}) \alpha(n; k_1, \ldots, k_n)\big|_{(k_1,\ldots,k_n)=(1,2,\ldots,n-1,n+j)} \\ = e_{p-j-1}(E_{k_1}, \ldots, E_{k_{n-1}}) \alpha(n; k_1, \ldots, k_n)\big|_{(k_1,\ldots,k_n)=(1,2,\ldots,n-1,n+j+1)} \\ + e_{p-j}(E_{k_1}, \ldots, E_{k_{n-1}}) \alpha(n; k_1, \ldots, k_n)\big|_{(k_1,\ldots,k_n)=(1,2,\ldots,n-1,n+j)}.$$

This implies the recursion

$$e_{p-j-1}(E_{k_1}, \ldots, E_{k_{n-1}}) \alpha(n; k_1, \ldots, k_n)\big|_{(k_1,\ldots,k_n)=(1,2,\ldots,n-1,n+j+1)} \\ = \binom{n}{p-j} \alpha(n; 1, \ldots, n-1, n+j) \\ - e_{p-j}(E_{k_1}, \ldots, E_{k_{n-1}}) \alpha(n; k_1, \ldots, k_n)\big|_{(k_1,\ldots,k_n)=(1,2,\ldots,n-1,n+j)}.$$

Now we can prove the first formula in the lemma by induction with respect to j. The case $j=0$ was dealt with in Lemma 2. $\square$

By Theorem 3, $\alpha(n;1,2,\ldots,n-1,n+j)$ is a polynomial in j of degree no greater than $n-1$. By Lemma 3, it coincides with $\sum_{i=1}^{n} A_{n,i}\binom{i+j-1}{i-1}$ for $j=0,1,\ldots,n-1$ and, since $\sum_{i=1}^{n} A_{n,i}\binom{i+j-1}{i-1}$ is a polynomial in j of degree no greater than $n-1$ as well, the two polynomials must be equal. This constitutes the following.

Lemma 4. *Let $n \geq 1$ and $k \in \mathbb{Z}$. Then we have*

$$\alpha(n;1,2,\ldots,n-1,k) = \sum_{i=1}^{n} A_{n,i} \binom{i+k-n-1}{i-1}.$$

3. The symmetry of $k \to \alpha(n;1,2,\ldots,n-1,k)$

In the following lemma we prove a transformation formula for $\alpha(n;k_1,\ldots,k_n)$, which implies as a special case that $k \to \alpha(n;1,2,\ldots,n-1,k)$ is even if n is odd and odd otherwise.

Lemma 5. *Let $n \geq 1$. Then we have*

$$\alpha(n;k_1,\ldots,k_n) = (-1)^{n-1}\alpha(n;k_2,\ldots,k_n,k_1-n).$$

Proof. By Theorem 3

$$\begin{aligned}
&(-1)^{n-1}\alpha(n;k_2,\ldots,k_n,k_1-n) \\
&\quad = (-1)^{n-1} \prod_{2\leq p<q\leq n} (\mathrm{id}+E_{k_p}\Delta_{k_q}) \prod_{p=2}^{n} (\mathrm{id}+E_{k_p}\Delta_{k_1}) \prod_{2\leq i<j\leq n} \frac{k_j-k_i}{j-i} \prod_{i=2}^{n} \frac{k_1-k_i-n}{i-1} \\
&\qquad = \prod_{2\leq p<q\leq n} (\mathrm{id}+E_{k_p}\Delta_{k_q}) \prod_{p=2}^{n} (\mathrm{id}+E_{k_p}\Delta_{k_1}) \prod_{2\leq i<j\leq n} \frac{k_j-k_i}{j-i} \prod_{i=2}^{n} \frac{k_i+n-k_1}{i-1} \\
&\qquad\qquad = \prod_{2\leq p<q\leq n} (\mathrm{id}+E_{k_p}\Delta_{k_q}) \prod_{p=2}^{n} (\mathrm{id}+E_{k_p}\Delta_{k_1}) E_{k_1}^{-n} \prod_{1\leq i<j\leq n} \frac{k_j-k_i}{j-i}.
\end{aligned}$$

Thus, we have to show that

$$\prod_{2\leq p<q\leq n} (\mathrm{id}+E_{k_p}\Delta_{k_q}) \left(\prod_{q=2}^{n} \left(\mathrm{id}+E_{k_1}\Delta_{k_q}\right) - E_{k_1}^{-n} \prod_{q=2}^{n} (\mathrm{id}+E_{k_q}\Delta_{k_1}) \right) \prod_{1\leq i<j\leq n} \frac{k_j-k_i}{j-i} = 0.$$

Clearly, it suffices to prove that

$$\left(E_{k_1}^{n} \prod_{q=2}^{n} \left(\mathrm{id}+E_{k_1}\Delta_{k_q}\right) - \prod_{q=2}^{n} (\mathrm{id}+E_{k_q}\Delta_{k_1}) \right) \prod_{1\leq i<j\leq n} \frac{k_j-k_i}{j-i} = 0.$$

We replace Δ_{k_i} by X_i and, accordingly, E_{k_i} by (X_i+1) in the operator in this expression and obtain

$$(X_1+1)^n \prod_{q=2}^{n} (1+(X_1+1)X_q) - \prod_{q=2}^{n} (1+(X_q+1)X_1). \tag{3.1}$$

By the proof of Lemma 1, the assertion follows if we show that this polynomial is in the ideal, which is generated by the symmetric polynomials in $X_1, X_2, \ldots, X_n$ without constant term. Observe that (3.1) is equal to

$$\sum_{j=0}^{n-1} (X_1+1)^{n+j} e_j(X_2, \ldots, X_n) - \sum_{j=0}^{n-1} X_1^j \, e_j(X_2+1, X_3+1, \ldots, X_n+1)$$
$$= \sum_{j=0}^{n-1} (X_1+1)^{n+j} e_j(X_2, \ldots, X_n) - \sum_{j=0}^{n-1} X_1^j \sum_{i=0}^{j} \binom{n-i-1}{j-i} e_i(X_2, \ldots, X_n)$$
$$= \sum_{j=0}^{n-1} \left((X_1+1)^{n+j} - X_1^j (X_1+1)^{n-j-1} \right) e_j(X_2, \ldots, X_n). \quad (3.2)$$

We recursively define a sequence $(q_j(X))_{j \geq 0}$ of Laurent polynomials. Let $q_0(X) = 0$ and

$$q_{j+1}(X) = (X+1)^{2j+1} - X^j - q_j(X) - q_j(X) \, X^{-1}. \quad (3.3)$$

We want to show that this is in fact a sequence of polynomials having a zero at $X = 0$. For this purpose, we consider

$$Q(X, Y) := \sum_{j \geq 0} q_j(X) Y^j.$$

Using (3.3) and the initial condition, we obtain the following

$$Q(X, Y) = \frac{X \, Y}{(1 - X \, Y)(1 - (X+1)^2 Y)},$$

which immediately implies the assertion. We set

$$p_j(X) = q_j(X)(X+1)^{n-j} X^{-1}$$

and observe that, for all j with $j \leq n$, $p_j(X)$ is a polynomial in X. The recursion (3.3) clearly implies

$$p_{j+1}(X) X = (X+1)^{n+j} - X^j (X+1)^{n-j-1} - p_j(X).$$

Thus, (3.2) is equal to

$$\sum_{j=0}^{n-1} \left(p_j(X_1) + p_{j+1}(X_1) X_1 \right) e_j(X_2, \ldots, X_n) = \sum_{j=0}^{n} p_j(X_1) e_j(X_1, \ldots, X_n).$$

Since $p_0(X) = 0$, this expression is in the ideal generated by the symmetric polynomials in $(X_1, \ldots, X_n)$ without constant term and the assertion of the lemma is proved. □

If $(a_{i,j})_{1 \leq j \leq i \leq n}$ is a monotone triangle with bottom row $(k_1, \ldots, k_n)$ then $(-a_{i,n+1-j})_{1 \leq j \leq i \leq n}$ is a monotone triangle with bottom row $(-k_n, \ldots, -k_1)$. This implies the following identity.

$$\alpha(n; k_1, k_2, \ldots, k_n) = \alpha(n; -k_n, -k_{n-1}, \ldots, -k_1) \quad (3.4)$$

Similarly, it is easy to see that

$$\alpha(n; k_1, k_2, \ldots, k_n) = \alpha(n; k_1 + c, k_2 + c, \ldots, k_n + c) \quad (3.5)$$

for every integer constant c. Therefore

$$\begin{aligned}&\alpha(n;1,2,\ldots,n-1,k)=\alpha(n;-k,-n+1,-n+2,\ldots,-1)\\&=(-1)^{n-1}\alpha(n;-n+1,-n+2,\ldots,-1,-k-n)=(-1)^{n-1}\alpha(n;1,2,\ldots,n-1,-k),\end{aligned}$$

where the first equality follows from (3.4), the second from Lemma 5 and the third from (3.5) with $c=n$. This, together with Lemma 4, implies the following identity

$$\sum_{i=1}^{n} A_{n,i}\binom{i+k-n-1}{i-1}=(-1)^{n-1}\sum_{i=1}^{n} A_{n,i}\binom{i-k-n-1}{i-1}$$

for all integers k. In this identity, we replace $\binom{i-k-n-1}{i-1}$ by $(-1)^{i-1}\binom{k+n-1}{i-1}$ and $\binom{i+k-n-1}{i-1}$ by

$$\sum_{j=1}^{n}\binom{i-2n}{i-j}\binom{k+n-1}{j-1},$$

which is possible by the Chu-Vandermonde summation [4, p. 169, (5.22)] if $i\le n$. We interchange the role of i and j on the left-hand-side and obtain

$$\sum_{i=1}^{n}\sum_{j=1}^{n} A_{n,j}\binom{j-2n}{j-i}\binom{k+n-1}{i-1}=\sum_{i=1}^{n} A_{n,i}(-1)^{n+i}\binom{k+n-1}{i-1}$$

for all integers k. Since $(\binom{k+n-1}{i-1})_{i\ge 1}$ is a basis of $\mathbb{C}[k]$ as a vectorspace over $\mathbb{C}$, this implies

$$\sum_{j=1}^{n} A_{n,j}\binom{j-2n}{j-i}=A_{n,i}(-1)^{n+i}$$

for $i=1,2,\ldots,n$. We replace $\binom{j-2n}{j-i}$ by $(-1)^{j-i}\binom{2n-i-1}{j-i}$. Moreover we replace j by $n+1-j$ and use the fact that $A_{n,j}=A_{n,n+1-j}$ in order to obtain

$$\sum_{j=1}^{n} A_{n,j}(-1)^{j+1}\binom{2n-i-1}{n-i-j+1}=A_{n,i}. \tag{3.6}$$

Phrased differently, $(A_{n,1},A_{n,2},\ldots,A_{n,n})$ is an eigenvector of $((-1)^{j+1}\binom{2n-i-1}{n-i-j+1})_{1\le i,j\le n}$ with respect to the eigenvalue 1. In the following section we see that this determines $(A_{n,1},A_{n,2},\ldots,A_{n,n})$ up to a multiplicative constant, which we are able to compute.

4. The eigenspace of $\left((-1)^{j+1}\binom{2n-i-1}{n-i-j+1}\right)_{1\le i,j\le n}$ with respect to the eigenwert 1 is one dimensional.

Since $(A_{n,1},\ldots,A_{n,n})$ is an eigenvector of $\left((-1)^{j+1}\binom{2n-i-1}{n-i-j+1}\right)_{1\le i,j\le n}$, it suffices to show that the dimension of the eigenspace with respect to 1 is no greater than 1. Thus we have to show that the rank of

$$\left((-1)^{j}\binom{2n-i-1}{n-i-j+1}+\delta_{i,j}\right)_{1\le i,j\le n}$$

is at least $n-1$. It suffices to show that

$$\det_{2\le i,j\le n}\left((-1)^j\binom{2n-i-1}{n-i-j+1}+\delta_{i,j}\right)\ne 0.$$

We shift i and j by one in this determinant and obtain

$$\det_{1\le i,j\le n-1}\left((-1)^{j+1}\binom{2n-i-2}{n-i-j-1}+\delta_{i,j}\right). \tag{4.1}$$

Let B_n denote the matrix underlying the determinant. We define $R_n=\left(\binom{n+j-i-1}{j-i}\right)_{1\le i,j\le n-1}$. Observe that $R_n^{-1}=\left((-1)^{i+j}\binom{n}{j-i}\right)_{1\le i,j\le n-1}$. Moreover, we have $R_n^{-1}B_nR_n=B_n^*+I_{n-1}$, where I_{n-1} denotes the $(n-1)\times(n-1)$ identity matrix and B_n^* is the $(n-1)\times(n-1)$ matrix with $\binom{i+j}{j-1}$ as entry in the i-th row and j-th column except for the last row, where we have all zeros. (This transformation is due to Mills, Robbins and Rumsey [7].) Thus the determinant in (4.1) is equal to

$$\det(B_n^*+I_{n-1})=\det_{1\le i,j\le n-2}\left(\binom{i+j}{j-1}+I_{n-2}\right),$$

where we have expanded $B_n^*+I_{n-1}$ with respect to the last row. Andrews [1] has shown that this determinant gives the number of desecending plane partitions with no part greater than $n-1$ and, therefore, the determinant does not vanish. Recently, Krattenthaler [5] showed that descending plane partitions can be geometrically realized as cyclically symmetric rhombus tilings of a certain hexagon of which a centrally located equilateral triangle of side length 2 has been removed.

In order to complete our proof, we have to show that $(B_{n,i})_{1\le i\le n}$ with

$$B_{n,i}=\frac{(i)_{n-1}(1+n-i)_{n-1}}{(n-1)!}\prod_{k=1}^{n-1}\frac{(3k-2)!}{(n+k-1)!}$$

is an eigenvector of $\left((-1)^{j+1}\binom{2n-i-1}{n-i-j+1}\right)_{1\le i,j\le n}$ with respect to the eigenwert 1, i.e. we have to show that

$$\sum_{j=1}^{n}(-1)^{j+1}\binom{2n-i-1}{n-i-j+1}\frac{(j)_{n-1}(1+n-j)_{n-1}}{(n-1)!}\prod_{k=1}^{n-1}\frac{(3k-2)!}{(n+k-1)!}=\frac{(i)_{n-1}(1+n-i)_{n-1}}{(n-1)!}\prod_{k=1}^{n-1}\frac{(3k-2)!}{(n+k-1)!}.$$

This is equivalent to showing that

$$\sum_{j=1}^{n}(-1)^{j+1}\binom{2n-i-1}{n-j-i+1}\binom{n+j-2}{n-1}\binom{2n-j-1}{n-1}=\binom{n+i-2}{n-1}\binom{2n-i-1}{n-1}. \tag{4.2}$$

Observe that the left hand side of this identity is equal to

$$\binom{2n-i-1}{n-1}\binom{n+i-2}{n-1}\binom{n-1}{i-1}^{-1}\sum_{j=1}^{n}(-1)^{j+1}\binom{2n-j-1}{n-j-i+1}\binom{n-1}{j-1}=$$
$$\binom{2n-i-1}{n-1}\binom{n+i-2}{n-1}\binom{n-1}{i-1}^{-1}\sum_{j=1}^{n}(-1)^{n+i}\binom{-n-i+1}{n-j-i+1}\binom{n-1}{j-1}=$$
$$\binom{2n-i-1}{n-1}\binom{n+i-2}{n-1}\binom{n-1}{i-1}^{-1}(-1)^{n+i}\binom{-i}{n-i}=\binom{2n-i-1}{n-1}\binom{n+i-2}{n-1},$$

where the first and third equality follows from $\binom{a}{b}=(-1)^b\binom{b-a-1}{b}$ and the second equality follows from the Chu-Vandermonde identity; see [4, p. 169, (5.26)]. Consequently, $A_{n,i}=C_n\cdot B_{n,i}$ for $C_n\in\mathbb{Q}$. It is easy to check that $C_1=1$. Observe that $\sum_{i=1}^{n-1}A_{n-1,i}=A_{n,1}$, since $(n-1)\times(n-1)$ alternating sign matrices are bijectively related to $n\times n$ alternating sign matrices with a 1 at the top of the first column. Moreover, observe that we also have $\sum_{i=1}^{n-1}B_{n-1,i}=B_{n,1}$, since

$$\sum_{i=1}^{n-1}B_{n-1,i}=\prod_{k=1}^{n-2}\frac{(3k-2)!}{(n+k-2)!}\sum_{i=1}^{n-1}\frac{(i)_{n-2}(n-i)_{n-2}}{(n-2)!}=$$
$$\prod_{k=1}^{n-2}\frac{(3k-2)!}{(n+k-2)!}(n-2)!\sum_{i=1}^{n-1}\binom{i+n-3}{n-2}\binom{2n-i-3}{n-2}=$$
$$\prod_{k=1}^{n-2}\frac{(3k-2)!}{(n+k-2)!}(n-2)!\binom{3n-5}{2n-3}=B_{n,1},$$

where the second equality follows from an identity, which is equivalent to the Chu-Vandermonde identity; see [4, p. 169, (5.26)]. Therefore, by induction with respect to n, we have $C_n=1$ for all n. This completes our proof of the refined alternating sign matrix theorem. Theorem 2 follows if we combine Theorem 1 and Lemma 4, since a careful analysis of the bijection between alternating sign matrices and monotone triangles shows that $\alpha(n;1,2,\ldots,n-1,k)$ is the number of objects described in the statement of the theorem.

References

[1] G. E. Andrews, Plane Partitions (III): The weak Macdonald conjecture. *Invent. math.* **53** (1979), 193 – 225.

[2] D. M. Bressoud, *Proofs and Confirmations, The Story of the Alternating Sign Matrix Conjecture*, Cambridge University Press, Cambridge, 1999.

[3] I. Fischer, The number of monotone triangles with prescribed bottom row, preprint, `math.CO/0501102`, to appear in *Adv. Appl. Math.*

[4] R. L. Graham, D. E. Knuth, O. Patashnik, *Concrete Mathematics*, 2nd ed., Addison-Wesley, Reading, MA, 1994.

[5] C. Krattenthaler, Descending plane partitions and rhombus tilings of a hexagon with triangular hole, preprint, `math.CO/0310188`, to appear in *Disc. Math.*

[6] G. Kuperberg, Another proof of the alternating sign matrix conjecture, *Internat. Math. Res. Notices* **1996** (1996), 139 – 150.

[7] W. H. Mills, D. P. Robbins and H. Rumsey, Proof of the Macdonald Conjecture, *Invent. math.* **66** (1982), 73 – 87.

[8] W. H. Mills, D. P. Robbins and H. Rumsey, Alternating sign matrices and descending plane partitions, *J. Combin. Theory Ser. A* **34** (1983), no. 3, 340 – 359.

[9] D. Zeilberger, Proof of the alternating sign matrix conjecture, The Foata Festschrift, *Electron. J. Combin.* **3** (1996), no. 2, R13, 84 pp. (electronic)

[10] D. Zeilberger, Proof of the refined alternating sign matrix conjecture, *New York J. Math.* **2** (1996), 59 – 68, electronic.