THE ANDREWS-STANLEY PARTITION FUNCTION AND $p(n)$

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Abstract. In a recent paper, G. E. Andrews [And04] formulated a new partition function $t(n)$. This function counts the number of partitions π for which the number of odd parts of π is congruent to the number of odd parts in the conjugate partition π' (mod 4). This condition is due to R. Stanley [Sta]. In the first part of this paper we obtain an asymptotic formula for $t(n)$. From this we see that $t(n) \sim p(n)/2$, where $p(n)$ is the ordinary partition function. Moreover, we show that for sufficiently large n, the sign of $t(n) - p(n)/2$ depends only on $n \pmod 4$. In [And04], Andrews showed that the (mod 5) Ramanujan congruence for $p(n)$ also holds for $t(n)$. We extend his observation by showing that there are infinitely many arithmetic progressions $An + B$, such that for all $n \geq 0$,

$$t(An + B) \equiv p(An + B) \equiv 0 \pmod{l^j}$$

whenever $l \geq 5$ is prime and $j \geq 1$.

1. Introduction and statement of results

Let n be a nonnegative integer. Recall that $p(n)$ counts the number of partitions π of n, and $p(0)$ is defined to be 1. For a partition π, let $\mathcal{O}(\pi)$ be the number of odd parts in π. Let π' denote the conjugate partition, which is obtained by reading the columns (instead of the rows) of the Ferrer's diagram for π [And98]. The Andrews-Stanley partition function $t(n)$ counts the number of partitions π of n for which $\mathcal{O}(\pi) \equiv \mathcal{O}(\pi') \pmod 4$. The generating function for $p(n)$ is known to equal the following infinite product (throughout let $q := e^{2\pi i z}$)

$$F(q) := \sum_{q=0}^{\infty} p(n)q^n = \prod_{n=1}^{\infty} \frac{1}{(1-q^n)}. \tag{1}$$

Notice that $F(q)$ converges absolutely for all $z \in \mathcal{H}$, the upper half of the complex plane. In [And04], Andrews proves that the generating function for $t(n)$ can also be written as an infinite product

$$G(q) := \sum_{n=0}^{\infty} t(n)q^n = \frac{F(q)F(q^4)^5F(q^{32})^2}{F(q^2)^2F(q^{16})^5}. \tag{2}$$

It is natural to ask for the size of $t(n)$. Here are some values of $p(n)$, $t(n)$, and $t(n)/p(n)$, computed using Maple.

1991 *Mathematics Subject Classification.* 11P55, 11P81, 11P82, 11P83.

n	$t(n)$	$p(n)$	$t(n)/p(n)$
50	73852	204226	.3616189907
100	107883650	190569292	.5661124564
150	18895766111	40853235313	.4625280217
200	2078730441344	3972999029388	.5232144347
250	111883314327463	230793554364681	.4847765989
300	4722726799452756	9253082936723602	.5103949496
350	137637052365622088	279363328483702152	.4926811730
400	3399061241292811170	6727090051741041926	.5052795808
450	66731552815421191902	134508188001572923840	.4961151719
500	1156767741034860735121	2300165032574323995027	.5029064109

This data suggests that

$$\lim_{n\to\infty} t(n)/p(n) = \frac{1}{2}.$$

Theorem 1.1. *If n is a positive integer, then*

$$t(n) = \frac{1}{2}p(n) + S(n) + \epsilon(n),$$

where $|\epsilon(n)| = O\left(\exp\left(\frac{2\pi}{3\sqrt{6}}\sqrt{n}\right)\right)$, $|S(n)| = O\left(\exp\left(\frac{\sqrt{13}\pi}{2\sqrt{6}}\sqrt{n}\right)\right)$, *and* $S(n) \in \mathbb{R}$ *satisfies* $S(n) > 0$ *for* $n \equiv 0, 1 \pmod 4$, *and* $S(n) < 0$ *for* $n \equiv 2, 3 \pmod 4$.

Since it is known that $p(n) \sim \frac{1}{4n\sqrt{3}} \exp(\pi\sqrt{2n/3})$ [And98], we obtain the following corollary.

Corollary 1.2. *We have that*

$$\lim_{n\to\infty} \frac{t(n)}{p(n)} = \frac{1}{2}.$$

It turns out that $t(n) > \frac{1}{2}p(n)$ for $n = 1, 4, 5, 8, 9, 12, \ldots$ These are precisely the n for which $n \equiv 0, 1 \pmod 4$. The following can easily be seen from the sign of $S(n)$.

Corollary 1.3. *If n is sufficiently large, then the following are true:*

(i) If $n \equiv 0, 1 \pmod 4$, then $t(n) > \frac{1}{2}p(n)$.
(ii) If $n \equiv 2, 3 \pmod 4$, then $t(n) < \frac{1}{2}p(n)$.

Andrews proved in [And04] that for all $n \geq 0$,

$$t(5n + 4) \equiv 0 \pmod 5, \tag{3}$$

showing that the classical (mod 5) Ramanujan congruence for $p(n)$ also holds for $t(n)$. He did this by proving a certain partition identity using q-series. Yee, Sills, and Boulet have all proven this identity combinatorially, see [Yee], [Sil], and [Bou]. In addition, Berkovich and Garvan [BG] have proven (3) combinatorially by deriving statistics related to the Andrews-Garvan crank and the 5-core crank which divide $t(5n + 4)$ into 5 equinumerous classes. This extends the famous work of Garvan, Kim, and Stanton [GKS90].

It is natural to ask whether the (mod 5) congruence is just one of many congruences shared by $t(n)$ and $p(n)$. In recent years, it has been proven by Ono and Ahlgren that there are many other congruences for $p(n)$ (see [Ono00], [Ahl00], and [AO01]). Here we show that the (mod 5) congruence is not an isolated example.

Theorem 1.4. *Let $l \geq 5$ be prime, and j a positive integer. There are infinitely many arithmetic progressions $An + B$ such that for all $n \geq 0$ we have*

$$t(An+B) \equiv p(An+B) \equiv 0 \pmod{l^j}.$$

Remark. In fact, Theorem 1.4 follows from a more precise statement (see Theorem 5.4).

In Section 2 we will recall techniques of the "circle method" of Hardy and Ramanujan, used to prove the exact formula for $p(n)$, and use them to gain information about $t(n)$. In Section 3, we will give the proof of Theorem 1.1. In Section 4, we will recall some basic facts about integral and half-integral weight modular forms, the Shimura correspondence, and a theorem of Serre in order to prepare for the proof of Theorem 1.4. In the final section, we will give proof of Theorem 1.4.

2. Preliminaries for the proof of Theorem 1.1

Recall from (2) that

$$G(x) := \sum_{n=0}^{\infty} t(n)x^n = \frac{F(x)F(x^4)^5F(x^{32})^2}{F(x^2)^2F(x^{16})^5}.$$

Notice that $G(x)$ converges absolutely whenever $|x| < 1$ and has poles at all roots of unity.

Assume that n is a positive integer. Now $G(x)/x^{n+1}$ has a pole at 0 with residue $t(n)$, so by Cauchy's Residue Theorem we have that

$$t(n) = \frac{1}{2\pi i}\int_{\gamma} \frac{G(x)}{x^{n+1}}dx,$$

where γ is an oriented contour around 0 inside the unit circle. Let $x = e^{-\rho+2\pi i\phi}$ for $\rho > 0 \in \mathbb{R}$, and $0 \leq \phi \leq 1$. This change of variables gives that

$$t(n) = \int_0^1 e^{n\rho - 2\pi i n\phi} \cdot G(e^{-\rho+2\pi i\phi})d\phi. \tag{4}$$

Of course, this integral is not easy to evaluate. However, using the "circle method" of Hardy and Ramanujan we can get an adequate approximation which leads to Theorem 1.1. For brevity, we assume that the reader is familiar with the use of this method in the case of $p(n)$ (see [And98], [Apo90]).

2.1. **Farey fractions.** We begin by using Farey fractions to divide up the integral in (4). Recall that for a positive integer N, the set of Farey fractions of order N is the set of reduced fractions h/k, with $k \leq N$, in the interval $[0,1]$ listed in increasing order (see [Apo90] and [And98] for a full description of Farey fractions).

Definition 2.1. Let $\frac{h_0}{k_0}, \frac{h}{k}, \frac{h_1}{k_1}$ be successive Farey fractions of order N. When $0 < h < k$, we let

$$\Theta_{h,k}^0 := \frac{h}{k} - \frac{h_0+h}{k_0+k}$$

$$\Theta_{h,k}^1 := \frac{h+h_1}{k+k_1} - \frac{h}{k},$$

and let

$$\Theta_{0,1}^0 := \frac{1}{N+1} =: \Theta_{0,1}^1.$$

The following proposition follows easily from the definitions of $\Theta_{h,k}^i$ (see [Apo90]).

Proposition 2.2. *Let $\frac{h_0}{k_0}, \frac{h}{k}, \frac{h_1}{k_1}$ be successive Farey fractions of order N. For $i = 0, 1$, we have*

$$\frac{1}{2kN} \leq \Theta^i_{h,k} < \frac{1}{kN}.$$

Using standard methods of substitution and changing variables, for every positive integer N, (4) becomes

$$t(n) = \sum_{\substack{k=1 \\ (h,k)=1 \\ 0\leq h<k}}^{N} \frac{1}{2\pi i} \cdot e^{-\frac{2\pi i n h}{k}} \int_{\rho-2\pi i\Theta^1_{h,k}}^{\rho+2\pi i\Theta^0_{h,k}} e^{ny} \cdot G\left(\exp\left\{\frac{2\pi i h}{k} - y\right\}\right) dy. \tag{5}$$

We shall impose two conditions on ρ which will be useful in the evaluation of these integrals. For $n \geq 1$, we choose ρ and N so that

$$\rho = \frac{1}{n}, \tag{6}$$

and

$$6 \leq N^2\rho \leq 7. \tag{7}$$

2.2. **Transformation formula for** $F(x)$**.** Since $G(x)$ can be written in terms of $F(x)$, we can make use of the transformation properties of $F(x)$ to carry out the calculation required in (5). First we will recall the classical Dedekind sums.

Definition 2.3. Let $h, k > 0$ be integers such that $(h, k) = 1$. Then the Dedekind sum $s(h, k)$ is defined by

$$s(h,k) := \sum_{r \mod k} \left(\left(\frac{r}{k}\right)\right)\left(\left(\frac{hr}{k}\right)\right),$$

where $((x))$ is defined to be $x - \lfloor x \rfloor - 1/2$ when $x \notin \mathbb{Z}$, and 0 for $x \in \mathbb{Z}$.

The following transformation formula for $F(x)$ can be obtained from Dedekind's functional equation for $\eta(z)$ (see Theorem 5.1 in [Apo90] and let $y = 2\pi z/k^2$).

Theorem 2.4. *If $Re(y) > 0$, $k > 0$, $(h, k) = 1$, and $hH \equiv -1 \pmod k$, then*

$$F\left(\exp\left\{\frac{2\pi i h}{k} - y\right\}\right) = e^{\pi i s(h,k)} \left(\frac{ky}{2\pi}\right)^{\frac{1}{2}} \exp\left(\frac{\pi^2}{6k^2y} - \frac{y}{24}\right) F\left(\exp\left\{\frac{2\pi i H}{k} - \frac{4\pi^2}{k^2 y}\right\}\right).$$

2.3. **Application of transformation formula to** $G(x)$**.** Now we would like to apply Theorem 2.4 to $G(x)$. However in doing so, we must raise x to certain powers of 2. Consequently, to satisfy the hypotheses we must know what power of 2 divides k. To deal with this, we will break up $t(n)$ into several sums depending on which power of 2 (up to 5) divides k. In this way we find that

$$t(n) = \sum_{m=0}^{5} S_m,$$

where for $m \in \{0, 1, 2, 3, 4\}$, S_m is defined as

$$S_m := \sum_{\substack{2^m k \leq N \\ (h,2^m k)=1 \\ 0\leq h<2^m k \\ k \text{ odd}}} \frac{1}{2\pi i} \cdot e^{-\frac{2\pi i n h}{2^m k}} \int_{\rho-2\pi i\Theta^1_{h,2^m k}}^{\rho+2\pi i\Theta^0_{h,2^m k}} e^{ny} \cdot G\left(\exp\left\{\frac{2\pi i h}{2^m k} - y\right\}\right) dy, \tag{8}$$

and

$$S_5 := \sum_{\substack{32k\le N\\ (h,32k)=1\\ 0\le h<32k}} \frac{1}{2\pi i}\cdot e^{-\frac{2\pi i n h}{32k}} \int_{\rho-2\pi i\Theta^1_{h,32k}}^{\rho+2\pi i\Theta^0_{h,32k}} e^{ny}\cdot G\left(\exp\left\{\frac{2\pi i h}{32k}-y\right\}\right) dy. \tag{9}$$

Note that we may replace k by $2^m k$ (and H by H_m) in Theorem 2.4 provided that $(h, 2^m k) = 1$, and $hH_m \equiv -1 \pmod{2^m k}$. Likewise, in Theorem 2.4 we may replace h by $2^l h$ (and H by H^l), provided that $(2^l h, k) = 1$, and $2^l hH^l \equiv -1 \pmod{k}$. We may also replace y by ny for any real $n > 0$. In this way, we may apply Theorem 2.4 to $G(x)$ inside each of the six summands. After a straightforward (but tedious and lengthy) computation we obtain the following formulas.

Lemma 2.5. *Let $m \in \{0, 1, 2, 3, 4, 5\}$, $Re(y) > 0$, $k > 0$, and $(h, 2^m k) = 1$. Then we have*

$$G\left(\exp\left\{\frac{2\pi i h}{2^m k}-y\right\}\right) = a_m\cdot e^{\pi i\sigma_m(h,k)}\cdot\sqrt{\frac{ky}{\pi}}\cdot\exp\left(b_m\left(\frac{\pi^2}{k^2y}\right)-\frac{y}{24}\right)\cdot\Psi_m(x_m,w_m),$$

where $a_m, b_m \in \mathbb{Q}$, and $\sigma_m(h, k)$ is a linear combination of Dedekind sums. Moreover, $\Psi_m(x_m, w_m)$ is a power series in the variables

$$x_m = \exp\left(\frac{2\pi i H_m}{2^m k}-\frac{4\pi^2}{2^{2m}k^2y}\right) \quad \text{and} \quad w_m = \exp\left(\frac{2\pi i H'_m}{k}-\frac{4\pi^2}{32k^2y}\right)$$

for H_m, H'_m satisfying $hH_m \equiv -1 \pmod{mk}$, and $\frac{32}{m}hH'_m \equiv -1 \pmod{k}$ respectively. In particular, the following two tables give the specific values:

m	a_m	b_m	$\Psi_m(x_m, w_m)$
0	$\frac{1}{\sqrt{8}}$	$\frac{1}{6}$	$F(x_0)F^5(w_0^8)F^2(w_0)F^{-2}(w_0^{16})F^{-5}(w_0^2)$
1	$\frac{1}{2}$	$\frac{1}{24}$	$F(x_1)F^5(w_1^8)F^2(w_1)F^{-2}(x_1^2)F^{-5}(w_1^2)$
2	$\frac{1}{\sqrt{8}}$	$\frac{13}{96}$	$F(x_2)F^5(x_2^4)F^2(w_2)F^{-2}(x_2^2)F^{-5}(w_2^2)$
3	$\sqrt{2}$	$\frac{1}{384}$	$F(x_3)F^5(x_3^4)F^2(w_3)F^{-2}(x_3^2)F^{-5}(w_3^2)$
4	$\sqrt{32}$	$\frac{-47}{1536}$	$F(x_4)F^5(x_4^4)F^2(w_4)F^{-2}(x_4^2)F^{-5}(x_4^{16})$
5	4	$\frac{1}{6144}$	$G(x_{32})$

m	$\sigma_m(h, k)$
0	$s(h, k) + 5s(4h, k) + 2s(32h, k) - 2s(2h, k) - 5s(16h, k)$
1	$s(h, 2k) + 5s(2h, k) + 2s(16h, k) - 2s(h, k) - 5s(8h, k)$
2	$s(h, 4k) + 5s(h, k) + 2s(8h, k) - 2s(h, 2k) - 5s(4h, k)$
3	$s(h, 8k) + 5s(h, 2k) + 2s(4h, k) - 2s(h, 4k) - 5s(2h, k)$
4	$s(h, 16k) + 5s(h, 4k) + 2s(2h, k) - 2s(h, 8k) - 5s(h, k)$
5	$s(h, 32k) + 5s(h, 8k) + 2s(h, k) - 2s(h, 16k) - 5s(h, 2k)$

The following proposition will be useful when we begin to bound the error.

Proposition 2.6. *For $m \in \{0, 1, \ldots, 5\}$ and y on the path of the integral in equations (8) and (9), we have that $|x_m| \le 1$ and $|w_m| \le 1$.*

Proof. We see from the definitions of x_m and w_m that

$$|x_m| = \exp\left(\frac{-4\pi^2}{2^{2m}k^2}\cdot \operatorname{Re}\left(\frac{1}{y}\right)\right)$$

and

$$|w_m| = \exp\left(\frac{-4\pi^2}{32k^2}\cdot \operatorname{Re}\left(\frac{1}{y}\right)\right).$$

On the path of the integral in equations (8) and (9) we have $y = \rho + 2\pi i\Theta$, where $-\Theta^1_{h,2^m k} \leq \Theta \leq \Theta^0_{h,2^m k}$. Using Proposition 2.2 and (7), we get that

$$\operatorname{Re}\left(\frac{1}{y}\right) > \frac{6k^2\cdot 2^{2m}}{49\cdot 2^{2m}+4\pi^2},$$

and thus the proposition follows. □

3. Proof of Theorem 1.1

In view of Lemma 2.5, we have that $t(n) = \sum_{m=0}^{5} S_m$, with

$$S_m = \sum_m a_m\sqrt{\frac{k}{\pi}}\cdot e^{\pi i\sigma_m(h,k)}\cdot e^{\frac{-2\pi inh}{2^m k}}\cdot \frac{1}{2\pi i}I_m, \tag{10}$$

$$I_m = \int_m e^{\left(n-\frac{1}{24}\right)y}\cdot y^{1/2}\cdot \exp\left(\frac{b_m\pi^2}{k^2 y}\right)\cdot \Psi_m(x_m,w_m), \tag{11}$$

and where $\sum_m$, $\int_m$ are over the same sets as in equations (8) and (9).

3.1. Finding and Bounding Error.

We now begin to analyze equations (10) and (9).

Lemma 3.1. *Let $\Psi(x,w) = \sum_{0\leq i+j} c_{ij}x^i w^j$ converge absolutely whenever $|x|,|w|<1$, and let $A(y)$ be a function on the complex plane for which $|A(y)|>0$ for all y. Suppose $x_m(y)$ and $w_m(y)$ have the property that $|x_m(y)|\leq|w_m(y)|<1$, and there exist nonnegative integers k and l such that for all y, $|A(y)|\cdot|w_m(y)|^k < C$ and $|A(y)|\cdot|x_m(y)|^l < C$ for some real constant C. Moreover, suppose that k and l are chosen minimally. Define*

$$\tilde{\Psi}(x,w) := \sum_{\substack{k\leq i+j\\ \text{or } l\leq i}} c_{ij}x^i w^j.$$

Then for all y,

$$|A(y)\cdot\tilde{\Psi}(x_m(y),w_m(y))| = \mathcal{O}(1).$$

Proof. One directly finds that

$$\begin{aligned}
|A(y)|\cdot\left|\tilde{\Psi}(x_m(y),w_m(y))\right| &\leq |A(y)|\cdot\sum_{\substack{k\leq i+j\\ \text{or } l\leq i}}|c_{ij}||x_m(y)|^i|w_m(y)|^j\\
&= |A(y)|\cdot\sum_{\substack{l\leq i\\ i+j<k}}|c_{ij}||x_m(y)|^i|w_m(y)|^j + |A(y)|\cdot\sum_{k\leq i+j}|c_{ij}||x_m(y)|^i|w_m(y)|^j\\
&\leq |A(y)|\cdot|x_m(y)|^l\sum_{\substack{l\leq i\\ i+j<k}}|c_{ij}||x_m(y)|^{i-l}|w_m(y)|^j + |A(y)|\cdot|w_m(y)|^k\sum_{k\leq i+j}|c_{ij}||w_m(y)|^{i+j-k}\\
&\leq C\sum_{\substack{l\leq i\\ i+j<k}}|c_{ij}||x_m(y)|^{i-l}|w_m(y)|^j + C\sum_{k\leq i+j}|c_{ij}||w_m(y)|^{i+j-k}
\end{aligned}$$

The first sum is finite, and the second converges by the hypotheses. Thus the lemma holds. ☐

Write

$$\Psi_m(x_m,w_m) = \begin{cases} 1+2w_m+\widetilde{\Psi}_m(x_m,w_m) & \text{if } m=0,2 \\ 1+\widetilde{\Psi}_m(x_m,w_m) & \text{if } m=1,3,5 \\ \widetilde{\Psi}_m(x_m,w_m) & \text{if } m=4. \end{cases} \tag{12}$$

From Lemma 2.5, we see that $\widetilde{\Psi}_m(x_m,w_m)$ satisfies the hypotheses of Lemma 3.1 with $A(y) = \exp(b_m(\pi^2/k^2y))$. So for each $0 \le m \le 5$,

$$\left|\exp\left(b_m\left(\frac{\pi^2}{k^2y}\right)\right)\widetilde{\Psi}_m(x_m,w_m)\right| \le c_m \tag{13}$$

for some $c_m \in \mathbb{R}$. We define the "error" integrals I_m^E as follows

$$I_m^E := \int_m e^{(n-\frac{1}{24})y}y^{1/2}\exp\left(\frac{b_m\pi^2}{k^2y}\right)\widetilde{\Psi}_m(x_m,w_m)dy.$$

We will show that the size of these integrals is negligible.

Proposition 3.2. *For y on the path of the integral I_m^E, we have*

$$|e^{(n-\frac{1}{24})y}\cdot y^{1/2}| \le e^{1/4}\left(\frac{49\cdot 2^{2m}+4\pi^2}{2^{2m}k^2N^2}\right)^{\frac{1}{4}}.$$

Proof. On the path of the integral I_m^E we have $y=\rho+2\pi i\Theta$, where $-\Theta^1_{h,2^mk} \le \Theta \le \Theta^0_{h,s^mk}$. Thus using (6), (7), and Proposition 2.2 we have that

$$\left|e^{(n-\frac{1}{24}y)}\cdot y^{1/2}\right| \le e^{n\rho}(\rho^2+4\pi^2\Theta^2)^{\frac{1}{4}} \le e^{1/4}\left(\frac{2^{2m}(N^2\rho)^2+4\pi^2}{2^{2m}k^2N^2}\right)^{\frac{1}{4}} \le e^{1/4}\left(\frac{49\cdot 2^{2m}+4\pi^2}{2^{2m}k^2N^2}\right)^{\frac{1}{4}}.$$

☐

By Proposition 2.2, it is easy to see that the length of the integral I_m^E is bounded by $\frac{4\pi}{2^mkN}$. Thus by equation (13), and Proposition 3.2 we obtain the following lemma.

Lemma 3.3. *For $m \in \{0,1,\dots,5\}$, we have*

$$|I_m^E| \le \frac{4\pi c_m\cdot e^{1/4}(49\cdot 2^{2m}+4\pi^2)^{1/4}}{2^{3m/2}k^{3/2}N^{3/2}} =: K_m k^{-3/2}N^{-3/2} = \mathcal{O}(k^{-3/2}N^{-3/2}).$$

So now we have that $I_m = I_m^* + I_m^E$, where for $m=4$

$$I_m = I_m^E,$$

for $m=1,3,5$

$$I_m^* = \int_m e^{(n-\frac{1}{24})y}\cdot y^{\frac{1}{2}}\cdot\exp\left(\frac{b_m\pi^2}{k^2y}\right)dy, \tag{14}$$

and for $m=0,2$

$$\begin{aligned} I_m^* &= \int_m e^{(n-\frac{1}{24})y}\cdot y^{\frac{1}{2}}\cdot\exp\left(\frac{b_m\pi^2}{k^2y}\right)(1+2w_m)dy \\ &= \int_m e^{(n-\frac{1}{24})y}\cdot y^{\frac{1}{2}}\cdot\exp\left(\frac{b_m\pi^2}{k^2y}\right)dy + 2e^{\frac{2\pi iH'_m}{k}}\int_m e^{(n-\frac{1}{24})y}\cdot y^{\frac{1}{2}}\cdot\exp\left(\frac{(b_m-1/8)\pi^2}{k^2y}\right)dy. \end{aligned} \tag{15}$$

3.2. **Evaluation of integral.** We consider in general integrals of the type

$$I(c_m) := \int_m g(y)dy := \int_m e^{\left(n-\frac{1}{24}\right)y} \cdot y^{\frac{1}{2}} \cdot \exp\left(\frac{c_m\pi^2}{k^2y}\right)dy,$$

where c_m is a positive real number. Let $0 < \epsilon < \rho$, and split $I(c_m)$ into pieces by using the following contour:

$$\begin{aligned} I(c_m) &= \left(\int_{\mathcal{L}} - \int_{-\infty}^{-\epsilon} - \int_{-\epsilon}^{-\epsilon-2\pi i\Theta^1} - \int_{-\epsilon-2\pi i\Theta^1}^{\rho-2\pi i\Theta^1} - \int_{\rho+2\pi i\Theta^0}^{-\epsilon+2\pi i\Theta^0} - \int_{-\epsilon+2\pi i\Theta^0}^{-\epsilon} - \int_{-\epsilon}^{-\infty}\right) g(y)dy \\ &= L - J_1 - J_2 - J_3 - J_4 - J_5 - J_6, \end{aligned}$$

where the integrals $L, J_1, \dots, J_6$ correspond in the obvious way to the integrals above.

Lemma 3.4. *Both J_2 and J_5 are $\mathcal{O}(k^{-3/2}N^{-3/2})$.*

Proof. By Proposition 2.2, the length of both integrals is less than $\frac{2\pi}{2^mkN}$, and in both integrals $y = -\epsilon + 2\pi i\Theta$, where

$$\frac{-1}{2^mkN} < \Theta < \frac{1}{2^mkN}.$$

Thus $\mathrm{Re}(y) = -\epsilon$, and $\mathrm{Re}(1/y) = -\epsilon/(\epsilon^2 + 4\pi^2\Theta^2) < 0$. So we have that $\left|e^{\left(n-\frac{1}{24}\right)y}\right| \leq 1$ and $\left|\exp\left(\frac{c_m\pi^2}{k^2y}\right)\right| \leq 1$. Also, it's easy to see that

$$\left|y^{\frac{1}{2}}\right| < \left(\epsilon^2 + \frac{4\pi^2}{2^{2m}k^2N^2}\right)^{\frac{1}{4}}.$$

Consequently, using (7) and the fact that $0 < \epsilon < \rho$, we get that

$$|J_2|, |J_5| < \frac{2\pi}{2^mkN}\left(\frac{49 \cdot 2^w m + 4\pi^2}{2^2mk^2N^2}\right)^{\frac{1}{4}}$$

and the lemma follows. □

Lemma 3.5. *Both J_3 and J_4 are $\mathcal{O}(k^{-1/2}N^{-5/2})$.*

Proof. The length of both of these integrals is less than 2ρ. Also, for both integrals $y = u + 2\pi i\Theta$, where $-\epsilon \leq u \leq \rho$, and Θ is fixed satisfying $\Theta^2 < \frac{1}{2^mkN}$. Thus

$$\left|e^{\left(n-\frac{1}{24}\right)y}\right| < e,$$

and

$$\left|y^{\frac{1}{2}}\right| \leq \left(\rho^2 + \frac{4\pi^2}{2^{2m}k^2N^2}\right)^{\frac{1}{4}} < \left(49 \cdot 2^{2m} + \frac{4\pi^2}{2^{2m}k^2N^2}\right)^{\frac{1}{4}}.$$

Also using (7), we see that

$$\mathrm{Re}\left(\frac{1}{y}\right) \leq \frac{2^{2m}k^2N^2\rho}{\pi^2} < \frac{7 \cdot 2^{2m}k^2}{\pi^2},$$

so we have

$$\left|\exp\left(\frac{c_m\pi^2}{k^2y}\right)\right| \leq \exp\left(7c_m \cdot 2^{2m}\right).$$

Thus we get that

$$|J_3|, |J_5| < \frac{14}{N^2} \cdot e\left(49 \cdot 2^{2m} + \frac{4\pi^2}{2^{2m}k^2N^2}\right)^{\frac{1}{4}} \exp\left(7c_m \cdot 2^{2m}\right),$$

and the lemma follows. □

We have shown the following.

Proposition 3.6. *We have that*

$$I(c_m) = L - (J_1 + J_6) - J,$$

where $|J| = \mathcal{O}(k^{-3/2}N^{-3/2})$.

Let us investigate $J_1 + J_6$.

$$J_1 + J_6 = \left(\int_{-\infty}^{-\epsilon} + \int_{-\epsilon}^{-\infty}\right) e^{\left(n-\frac{1}{24}\right)y} \cdot y^{\frac{1}{2}} \cdot \exp\left(\frac{c_m\pi^2}{k^2 y}\right) dy.$$

We choose opposite branches of the square root so that

$$J_1 + J_6 = \left(\int_{-\infty}^{-\epsilon} + \int_{-\epsilon}^{-\infty}\right) e^{\left(n-\frac{1}{24}\right)y} \cdot \sqrt{|y|} \cdot e^{\frac{-\pi i}{2}} \cdot \exp\left(\frac{c_m\pi^2}{k^2 y}\right) dy$$
$$+ \left(\int_{-\infty}^{-\epsilon} + \int_{-\epsilon}^{-\infty}\right) e^{\left(n-\frac{1}{24}\right)y} \cdot \sqrt{|y|} \cdot e^{\frac{\pi i}{2}} \cdot \exp\left(\frac{c_m\pi^2}{k^2 y}\right) dy.$$

Substituting $u = -y$, and combining the two integrals, we get that

$$J_1 + J_6 = -2i\int_{\epsilon}^{\infty} e^{-\left(n-\frac{1}{24}\right)u}\sqrt{u}\exp\left(\frac{-c_m\pi^2}{k^2 u}\right) du =: -2iH.$$

We can use the fact that (see (5.2.31) in [And98])

$$\int_0^{\infty} t^{\frac{1}{2}} \exp(-c^2 t - a^2 t^{-1}) dt = \frac{-\sqrt{\pi}}{2c} \cdot \frac{d}{du}\left(\frac{\exp(-2au)}{u}\right)_{u=c}$$

to evaluate H by letting $c = \sqrt{n - \frac{1}{24}}$ and $a = (\sqrt{c_m}\pi)/k$. Thus

$$H = \frac{-\sqrt{\pi}}{2\sqrt{n-\frac{1}{24}}} \cdot \frac{d}{du}\left(\frac{\exp\left(\frac{-2\sqrt{c_m}\pi u}{k}\right)}{u}\right)_{u=\sqrt{n-\frac{1}{24}}}.$$

Letting $x = u^2 + 1/24$ yields that

$$H = -\sqrt{\pi} \cdot \frac{d}{dx}\left(\frac{\exp\left(\frac{-2\sqrt{c_m}\pi\sqrt{x-1/24}}{k}\right)}{\sqrt{x-1/24}}\right)_{x=n}.$$

In particular, we have now shown the following.

Proposition 3.7. *We have that*

$$\frac{1}{2\pi i} I(c_m) = \frac{1}{2\pi i} L - \frac{1}{\sqrt{\pi}} \cdot \frac{d}{dx}\left(\frac{\exp\left(\frac{-2\sqrt{c_m}\,\pi\sqrt{x-1/24}}{k}\right)}{\sqrt{x-1/24}}\right)_{x=n} - J,$$

where $|J| = \mathcal{O}(k^{-3/2}N^{-3/2})$.

Now all that remains is to evaluate $L/2\pi i$. It is easy to see that

$$\frac{1}{2\pi i} L = \frac{1}{2\pi i}\int_{\mathcal{L}} e^{\left(n-\frac{1}{24}\right)y} \cdot y^{\frac{1}{2}} \cdot \exp\left(\frac{c_m\pi^2}{k^2 y}\right) dy = \sum_{s=0}^{\infty} \frac{1}{s!}\left(\frac{c_m\pi^2}{k^2}\right)^s \cdot \frac{1}{2\pi i}\int_{\mathcal{L}} e^{\left(n-\frac{1}{24}\right)y} y^{\left(\frac{1}{2}-s\right)} dy.$$

Thus, if $z = (n - 1/24)y$, then the integral becomes

$$\frac{1}{2\pi i}L = \sum_{s=0}^{\infty}\left(n - \frac{1}{24}\right)^{\left(s-\frac{3}{2}\right)}\frac{1}{s!}\left(\frac{c_m\pi^2}{k^2}\right)^s \cdot \frac{1}{2\pi i}\int_{\mathcal{L}} e^z z^{\left(\frac{1}{2}-s\right)}dz.$$

By Hankel's loop integral formula (see Ch.3, section 24 of [Rad73]),

$$\frac{1}{2\pi i}\int_{\mathcal{L}} e^z z^{\left(\frac{1}{2}-s\right)}dz = \frac{1}{\Gamma(s-\frac{1}{2})}.$$

Thus we have that

$$\frac{L}{2\pi i} = \left(n - \frac{1}{24}\right)^{-\frac{3}{2}}\sum_{s=0}^{\infty}\frac{\left(\frac{c_m\pi^2}{k^2}\left(n-\frac{1}{24}\right)\right)^s}{s!\cdot\Gamma(s-\frac{1}{2})}. \tag{16}$$

The following fact is easy to show, and can be found in [And98] (page 80).

$$\sum_{s=0}^{\infty}\frac{(Y^2/4)^s}{s!\cdot\Gamma(s-\frac{1}{2})} = \frac{Y^2}{2\sqrt{\pi}}\cdot\frac{d}{dY}\left(\frac{\cosh(Y)}{Y}\right). \tag{17}$$

For future reference we let

$$M(a) := \frac{d}{dx}\left(\frac{\sinh\left(a\pi\sqrt{x-\frac{1}{24}}\right)}{\sqrt{x-\frac{1}{24}}}\right)_{x=n}. \tag{18}$$

By (16), (17), and Proposition 3.7, we obtain the following.

Theorem 3.8. *For M defined in (18), we have*

$$\frac{1}{2\pi i}I(c_m) = \frac{1}{\sqrt{\pi}}\cdot M\left(\frac{2\sqrt{c_m}}{k}\right) + J(m)$$

where $|J(m)| = \mathcal{O}(k^{-3/2}N^{-3/2})$.

3.3. **Asymptotic formula for $t(n)$.** Here we combine the previous results to find an asymptotic formula for $t(n)$. Define

$$A_m(n) := \sum_{\substack{h=0\\(h,2^mk)=1}}^{2^mk-1} e^{\pi i\sigma_m(h,k)}e^{\frac{-2\pi inh}{2^mk}}. \tag{19}$$

Then by equation (10) we now have that $t(n) = \sum_{m=0}^{5} S_m$, where for $m \in \{0,\ldots,4\}$,

$$S_m = \sum_{\substack{2^mk\leq N\\ k \text{ odd}}} a_m\sqrt{\frac{k}{\pi}}A_m(n)\cdot\frac{1}{2\pi i}\left(I_m^* + I_m^E\right) \tag{20}$$

and

$$S_5 = \sum_{2^5k\leq N} a_5\sqrt{\frac{k}{\pi}}A_5(n)\cdot\frac{1}{2\pi i}\left(I_5^* + I_5^E\right). \tag{21}$$

We write

$$E_m := \sum_{m'} a_m\sqrt{\frac{k}{\pi}}A_m(n)\cdot\frac{1}{2\pi i}\ I_m^E$$

for the error arising from I_m^E, where the summation $\sum_{m'}$ is the same as in equations (20) or (21).

Proposition 3.9. *If a_m and K_m are the constants defined in Propositions 2.5 and 3.3, then*

$$|E_m| \leq \frac{a_m K_m}{2\pi^{3/2}} \cdot N^{-1/2}.$$

In particular, the size of E_m approaches 0 as $n \to \infty$.

Proof. Note that since A_m is a sum of terms with size 1, we have

$$|A_m(n)| \leq 2^m k. \tag{22}$$

Using this fact, Lemma 3.3, and that the a_m are all positive, the proposition follows. $\square$

By Proposition 3.9, equations (20) and (21) become

$$S_m = \sum_{m'} a_m \sqrt{\frac{k}{\pi}} A_m(n) \cdot \frac{1}{2\pi i}\ I_m^* + \mathcal{O}(N^{-1/2}). \tag{23}$$

From equations (14) and (15), we have

$$I_m^* = \begin{cases} I(b_m) + 2e^{\frac{2\pi i H_m'}{k}} I(b_m - 1/8) & \text{if } m = 0, 2 \\ I(b_m) & \text{if } m = 1, 3, 5 \\ I_4^E & \text{if } m = 4. \end{cases}$$

Thus by Theorem 3.8,

$$\frac{1}{2\pi i} I_m^* = \begin{cases} \frac{1}{\sqrt{\pi}} \cdot M\left(\frac{2\sqrt{b_m}}{k}\right) + 2e^{\frac{2\pi i H_m'}{k}} \frac{1}{\sqrt{\pi}} \cdot M\left(\frac{2\sqrt{b_m - 1/8}}{k}\right) + J(m) & \text{if } m = 0, 2, \\ \frac{1}{\sqrt{\pi}} \cdot M\left(\frac{2\sqrt{b_m}}{k}\right) + J(m) & \text{if } m = 1, 3, 5 \\ J(4) & \text{if } m = 4, \end{cases} \tag{24}$$

where $|J(m)| = \mathcal{O}(k^{-3/2} N^{-3/2})$.

The following lemma gauges the size of $M(b)$ for positive real numbers b.

Lemma 3.10. *Let $b \in \mathbb{R}$ be positive. Then*

$$|M(b)| \leq (2\pi\sqrt{b} + 1/2) e^{\frac{2\pi\sqrt{b}}{k}\sqrt{n}}.$$

Proof. Calculating the derivative from the definition of $M(b)$ yields

$$M(b) = \left(\frac{\pi\sqrt{b}}{k(n - 1/24)}\right) \cosh\left(\frac{2\pi\sqrt{b}}{k}\sqrt{x - 1/24}\right) - \left(\frac{1}{2(n - 1/24)^{3/2}}\right) \sinh\left(\frac{2\pi\sqrt{b}}{k}\sqrt{x - 1/24}\right).$$

Using the fact that

$$a\cosh(x) - b\sinh(x) = \frac{1}{2}(a - b)e^x + \frac{1}{2}(a + b)e^{-x}$$

for all real numbers a and b, we get that

$$M(b) = \frac{1}{2}\left(\frac{2\pi\sqrt{b}\sqrt{n - 1/24} - k}{2k(n - 1/24)^{3/2}}\right) e^{\frac{2\pi\sqrt{b}}{k}\sqrt{n - 1/24}} + \frac{1}{2}\left(\frac{2\pi\sqrt{b}\sqrt{n - 1/24} + k}{2k(n - 1/24)^{3/2}}\right) e^{\frac{-2\pi\sqrt{b}}{k}\sqrt{n - 1/24}}.$$

Using the simple facts that $(n - 1/24) \geq 23/24$ and $2(23/24)^{3/2} > 1$, it follows that

$$|M(b)| \leq \pi\sqrt{b} \cdot e^{\frac{2\pi\sqrt{b}}{k}\sqrt{n}} + \frac{1}{2}(2\pi\sqrt{b} + 1) \leq (2\pi\sqrt{b} + 1/2) e^{\frac{2\pi\sqrt{b}}{k}\sqrt{n}}.$$

$\square$

In order to evaluate the sums in equation (23), we make the following definitions. Let $S(a,b)$ denote the following general sum for real numbers a and b

$$S(a,b) := \sum_{m'} \frac{ak^{\frac{1}{2}}}{\pi} A_m(n) M(b). \tag{25}$$

Let $J(m)^E$ denote the error term in (23)

$$J(m)^E := \sum_{m'} a_m \sqrt{\frac{k}{\pi}} A_m(n) J(m) + \mathcal{O}(N^{-1/2}). \tag{26}$$

By (22), and the fact that $|J(m)| = \mathcal{O}(k^{-3/2}N^{-3/2})$, we get the following proposition.

Proposition 3.11. *Let $J(m)^E$ be defined as above. Then*

$$\left|J(m)^E\right| = \mathcal{O}(N^{-1/2}).$$

Using (23), (24), and Proposition 3.11, we have now established the following theorem, giving the asymptotic formula for $t(n)$.

Theorem 3.12. *For S defined as before, we have that*

$$t(n) = S\left(a_0, \frac{2\sqrt{b_0}}{k}\right) + S\left(2e^{\frac{2\pi i H_0'}{k}} a_0, \frac{2\sqrt{b_0 - 1/8}}{k}\right) + S\left(a_1, \frac{2\sqrt{b_1}}{k}\right) + S\left(a_2, \frac{2\sqrt{b_2}}{k}\right)$$
$$+ S\left(2e^{\frac{2\pi i H_2'}{k}} a_2, \frac{2\sqrt{b_2 - 1/8}}{k}\right) + S\left(a_3, \frac{2\sqrt{b_3}}{k}\right) + S\left(a_5, \frac{2\sqrt{b_5}}{k}\right) + \mathcal{O}(N^{-1/2}).$$

3.4. **Proof of Theorem 1.1.** We proceed by analyzing the size of the terms comprising the sums in S_m and comparing them with $p(n)$.

Proposition 3.13. *Let $S_k(a,b)$ denote the term of the sum $S(a,b)$ corresponding to k. Then*

$$|S_k(a,b)| \le \left(\frac{2^m a k^{3/2}(2\pi\sqrt{b} + 1/2)}{\pi}\right) \exp\left(\frac{2\pi\sqrt{b}}{k}\sqrt{n}\right) = \mathcal{O}\left(\exp\left\{\frac{2\pi\sqrt{b}}{k}\sqrt{n}\right\}\right).$$

Proof. Fix a positive integer k. The proposition follows easily from equations (25), (22), and by Lemma 3.10. □

Proposition 3.13 shows that we can approximate $S(a,b)$ by it's first few terms.

Using the exact formula for $p(n)$ (see [And98]), we find that

$$p(n) = \frac{1}{2\pi}\left[M\left(\frac{2}{\sqrt{6}}\right) + \sqrt{2}e^{-\pi i n} M\left(\frac{1}{\sqrt{6}}\right)\right] + \mathcal{O}\left(\exp\frac{2\pi}{3\sqrt{6}}\sqrt{n}\right). \tag{27}$$

To compare $t(n)$ with $p(n)$, we check the first few terms of each sum $S(a,b)$ to determine when the order of magnitude is greater than $\mathcal{O}\left(\exp\frac{2\pi}{3\sqrt{6}}\sqrt{n}\right)$. Using the definitions of b_m and Proposition 3.13, we get that

$$t(n) = S_1\left(a_0, \frac{2\sqrt{b_0}}{k}\right) + S_1\left(2e^{\frac{2\pi i H_0'}{k}} a_0, \frac{2\sqrt{b_0 - 1/8}}{k}\right)$$
$$+ S_1\left(a_1, \frac{2\sqrt{b_1}}{k}\right) + S_2\left(a_2, \frac{2\sqrt{b_2}}{k}\right) + \mathcal{O}\left(\exp\frac{2\pi}{3\sqrt{6}}\sqrt{n}\right).$$

By the definition of $S(a,b)$, this gives

$$t(n) = \frac{A_0(n)}{2\sqrt{2}\pi} M\left(\frac{2}{\sqrt{6}}\right) + \frac{A_0(n)}{\sqrt{2}\pi} M\left(\frac{1}{\sqrt{6}}\right) + \frac{A_1(n)}{2\pi} M\left(\frac{1}{\sqrt{6}}\right) + \frac{A_2(n)}{2\sqrt{2}\pi} M\left(\frac{\sqrt{13}}{2\sqrt{6}}\right) + \mathcal{O}\left(\exp \frac{2\pi}{3\sqrt{6}}\sqrt{n}\right). \tag{28}$$

A calculation of Dedekind sums yields

$$A_0(n) = 1$$
$$A_1(n) = e^{-\pi i n}$$
$$A_2(n) = e^{\pi i(1/8 - n/2)} + e^{\pi i(-1/8 - 3n/2)}.$$

Plugging these values in to equation (28) gives that

$$t(n) = \frac{1}{2\sqrt{2}\pi}\left[M\left(\frac{2}{\sqrt{6}}\right) + 2M\left(\frac{1}{\sqrt{6}}\right)\right] + \frac{1}{2\pi}\left[e^{-\pi i n} M\left(\frac{1}{\sqrt{6}}\right)\right] + \frac{1}{2\sqrt{2}\pi}\left[\left(e^{\pi i(1/8 - n/2)} + e^{-\pi i(1/8 + 3n/2)}\right) M\left(\frac{\sqrt{13}}{2\sqrt{6}}\right)\right] + \mathcal{O}\left(\exp \frac{2\pi}{3\sqrt{6}}\sqrt{n}\right). \tag{29}$$

Combining this with equation (27), we see that

$$t(n) = \frac{1}{2}p(n) + S(n) + \mathcal{O}\left(\exp \frac{2\pi}{3\sqrt{6}}\sqrt{n}\right),$$

where

$$S(n) = \frac{1}{\sqrt{2}\pi} M\left(\frac{1}{\sqrt{6}}\right) + \frac{1}{2\sqrt{2}\pi}\left(e^{\pi i(1/8 - n/2)} + e^{-\pi i(1/8 + 3n/2)}\right) M\left(\frac{\sqrt{13}}{2\sqrt{6}}\right).$$

In order to finish the proof of Theorem 1.1, we need to analyze $S(n)$. First we will show that $S(n)$ is real. By the definition of $M(a)$ it is clear that when $a \in \mathbb{R}$, we have $M(a) \in \mathbb{R}$. Thus to show that $S(n)$ is real, it suffices to note the following.

Proposition 3.14. *For all positive integers n, $e^{\pi i(1/8 - n/2)} + e^{-\pi i(1/8 + 3n/2)} \in \mathbb{R}$. In particular,*

$$e^{\pi i(1/8 - n/2)} + e^{-\pi i(1/8 + 3n/2)} = \begin{cases} 2\cos(\pi/8) & \text{if } n \equiv 0 \pmod 4 \\ 2\cos(3\pi/8) & \text{if } n \equiv 1 \pmod 4 \\ -2\cos(\pi/8) & \text{if } n \equiv 2 \pmod 4 \\ -2\cos(3\pi/8) & \text{if } n \equiv 3 \pmod 4. \end{cases}$$

In light of Proposition 3.14, we have shown that

$$S(n) = \begin{cases} \frac{1}{\pi\sqrt{2}}\left[M\left(\frac{1}{\sqrt{6}}\right) + \cos(\pi/8) M\left(\frac{\sqrt{13}}{2\sqrt{6}}\right)\right] & \text{if } n \equiv 0 \pmod 4 \\ \frac{1}{\pi\sqrt{2}}\left[M\left(\frac{1}{\sqrt{6}}\right) + \cos(3\pi/8) M\left(\frac{\sqrt{13}}{2\sqrt{6}}\right)\right] & \text{if } n \equiv 1 \pmod 4 \\ \frac{1}{\pi\sqrt{2}}\left[M\left(\frac{1}{\sqrt{6}}\right) - \cos(\pi/8) M\left(\frac{\sqrt{13}}{2\sqrt{6}}\right)\right] & \text{if } n \equiv 2 \pmod 4 \\ \frac{1}{\pi\sqrt{2}}\left[M\left(\frac{1}{\sqrt{6}}\right) - \cos(3\pi/8) M\left(\frac{\sqrt{13}}{2\sqrt{6}}\right)\right] & \text{if } n \equiv 3 \pmod 4. \end{cases} \tag{30}$$

We now examine $S(n)$ in terms of $n \pmod 4$ to determine when it is greater than or less than 0. A simple calculation shows that

$$(31)\quad M(a) = \left(\frac{a\pi\sqrt{n-1/24}-1}{4(n-1/24)^{3/2}}\right)\exp(a\pi\sqrt{n-1/24}) + \left(\frac{a\pi\sqrt{n-1/24}+1}{4(n-1/24)^{3/2}}\right)\exp(-a\pi\sqrt{n-1/24}).$$

We see that $M(a)$ is clearly positive whenever $a > 1/(\pi\sqrt{n-1/24})$, thus it is positive for all $a > \sqrt{24}/(\pi\sqrt{23})$. So for $a = 1/\sqrt{6}$ and $a = \sqrt{13}/2\sqrt{6}$, we have $M(a) > 0$. With (30), this shows that $S(n) > 0$ for $n \equiv 0, 1 \pmod 4$. Noting that $\cos(\pi/8) > \cos(3\pi/8)$, we show that the opposite is true for $n \equiv 2, 3 \pmod 4$ by proving the following proposition.

Proposition 3.15. *For all positive integers n, we have*

$$\cos(3\pi/8)M\left(\frac{\sqrt{13}}{2\sqrt{6}}\right) > M\left(\frac{1}{\sqrt{6}}\right).$$

Proof. Consider the function $f(n) = \cos(3\pi/8)M\left(\frac{\sqrt{13}}{2\sqrt{6}}\right) - M\left(\frac{1}{\sqrt{6}}\right)$. By (31), it is clear that $f(n)$ grows as $n \to \infty$. Thus it has a minimum on the interval $[1, \infty)$. A calculation shows that the derivative of f is never zero, so the minimum value is attained at $n = 1$. As $f(1) > 0$, the lemma holds. □

By Proposition 3.15 we have shown that $S(n) < 0$ for $n \equiv 2, 3 \pmod 4$. Also, from equation (31) it is easy to see that $|S(n)| = \mathcal{O}\left(\exp(\frac{\sqrt{13}\pi}{2\sqrt{6}}\sqrt{n})\right)$. This finishes the proof of Theorem 1.1 . □

4. Preliminaries for proof of Theorem 1.4

First we fix some notation. Suppose $w \in \frac{1}{2}\mathbb{Z}$, N is a positive integer (which is divisible by 4 if $w \notin \mathbb{Z}$), and χ is a Dirichlet character $\pmod N$. Let $M_w(\Gamma_0(N), \chi)$ (resp. $S_w(\Gamma_0(N), \chi)$) be the usual space of holomorphic modular forms (resp. cusp forms) on the congruence subgroup $\Gamma_0(N)$, with Nebentypus character χ. There are several operators which act on spaces of modular forms. The first we will discuss are Hecke operators. The results in this section, and more information on modular forms can be found in [Ono04].

4.1. **Hecke operators.** We will define Hecke operators for both integer and half-integer weight modular forms. We begin with the integer weight case.

Definition 4.1. With the notation above, we let $f(z) := \sum_{n=0}^{\infty} a(n)q^n \in M_k(\Gamma_0(N), \chi)$ where k is an integer, and $p \nmid N$ is prime. The action of the Hecke operator T_p is defined by

$$f(z)|T_p := \sum_{n=0}^{\infty}\left(a(pn) + \chi(p)p^{k-1}a(n/p)\right)q^n,$$

where $a(n/p) = 0$ when $p \nmid n$.

We see that the half-integer weight case is somewhat different.

Definition 4.2. Let $f(z) := \sum_{n=0}^{\infty} a(n)q^n \in M_{\lambda+\frac{1}{2}}(\Gamma_0(4N), \chi)$ where λ is an integer and $p \nmid 4N$ is prime. The action of the half-integral Hecke operator $T(p^2)$ is defined by

$$f(z)|T(p^2) := \sum_{n=0}^{\infty}\left(a(p^2n) + \chi(p)\left(\frac{(-1)^\lambda}{p}\right)\left(\frac{n}{p}\right)p^{\lambda-1}a(n) + \chi(p^2)\left(\frac{(-1)^\lambda}{p^2}\right)p^{2\lambda-1}a(n/p^2)\right)q^n,$$

where $a(n/p^2) = 0$ when $p^2 \nmid n$.

These operators are useful due to the following.

Proposition 4.3. *If $f(z) \in M_k(\Gamma_0(N), \chi)$, where $k \in \mathbb{Z}$, then*

$$f(z)|T_p \in M_k(\Gamma_0(N), \chi).$$

Similarly, if $f(z) \in M_{\lambda+\frac{1}{2}}(\Gamma_0(4N), \chi)$, where $\lambda \in \mathbb{Z}$, then

$$f(z)|T(p^2) \in M_{\lambda+\frac{1}{2}}(\Gamma_0(4N), \chi).$$

Furthermore, both T_p and $T(p^2)$ take cusp forms to cusp forms.

4.2. **Other operators.** We define two other operators U and V which act on formal power series. If d is a positive integer, then we define the U-operator $U(d)$ by

$$\left(\sum_{n=0}^{\infty} c(n)q^n\right) |U(d) := \sum_{n=0}^{\infty} c(dn)q^n, \tag{32}$$

and the V-operator $V(d)$ by

$$\left(\sum_{n=0}^{\infty} c(n)q^n\right) |V(d) := \sum_{n=0}^{\infty} c(n)q^{dn}. \tag{33}$$

These two operators also act on spaces of modular forms. First we consider the integer weight case.

Proposition 4.4. *Suppose $f(z) \in M_k(\Gamma_0(N), \chi)$ where $k \in \mathbb{Z}$.*

(1) If d is a positive integer and $d|N$, then

$$f(z)|U(d) \in M_k(\Gamma_0(N), \chi)$$

(2) If d is any positive integer, then

$$f(z)|V(d) \in M_k(\Gamma_0(Nd), \chi).$$

Moreover, both $U(d)$ and $V(d)$ take cusp forms to cusp forms.

The half-integer weight case is slightly more complicated.

Proposition 4.5. *Suppose $f(z) \in M_{\lambda+\frac{1}{2}}(\Gamma_0(4N), \chi)$ where $\lambda \in \mathbb{Z}$.*

(1) If d is a positive integer and $d|N$, then

$$f(z)|U(d) \in M_{\lambda+\frac{1}{2}}\left((\Gamma_0(4N), \left(\frac{4d}{\bullet}\right)\chi\right).$$

(2) If d is any positive integer, then

$$f(z)|V(d) \in M_{\lambda+\frac{1}{2}}\left((\Gamma_0(4Nd), \left(\frac{4d}{\bullet}\right)\chi\right).$$

Furthermore, again for the half-integral case both $U(d)$ and $V(d)$ take cusp forms to cusp forms.

4.3. **Shimura correspondence.** Here we recall the famous "Shimura correspondences" [Shi73] which give a means of mapping half-integer weight cusp forms to even integer weight modular forms.

Definition 4.6. Let $f(z) = \sum_{n=1}^{\infty} c(n)q^n \in S_{\lambda+\frac{1}{2}}(\Gamma_0(4N), \chi)$, with $\lambda \geq 1$, and let t be a square-free integer. Define the Dirichlet character ψ_t by $\psi_t(n) = \chi(n) \left(\frac{-1}{n}\right)^{\lambda} \left(\frac{t}{n}\right)$. If we define complex numbers $A_t(n)$ by

$$\sum_{n=1}^{\infty} \frac{A_t(n)}{n^s} := L(s - \lambda + 1, \psi_t) \cdot \sum_{n=1}^{\infty} \frac{c(tn^2)}{n^s},$$

then

$$S_t(f(z)) := \sum_{n=1}^{\infty} A_t(n)q^n$$

is a modular form in $M_{2\lambda}(\Gamma_0(2N), \chi^2)$. Furthermore, if $\lambda \geq 2$ then $S_t(f(z))$ is a cusp form. When $\lambda = 1$ there are conditions (here omitted) which guarantee $S_t(f(z))$ is a cusp form.

From the definition above, it is not hard to show that the Shimura correspondences commute with the Hecke operators in the following way.

Proposition 4.7. *Let $f(z) \in S_{\lambda+\frac{1}{2}}(\Gamma_0(4N), \chi)$ with $\lambda \geq 1$. If t is a square-free integer and $p \nmid 4Nt$ is prime, then*

$$S_t(f(z)|T(p^2)) = S_t(f(z))|T_p.$$

4.4. **A Theorem of Serre.** The following theorem due to Serre (see [Ono00] and [Ser76]) is a crucial component to the proof of Theorem 1.4. The theorem arises from the existence of certain Galois representations with special properties. More details can be found in [Ono04], and [Ser76].

Theorem 4.8. *Let $l \geq 5$ be prime, and $k \in \mathbb{Z}$. A positive proportion of the primes $p \equiv -1 \pmod{N}$ have the property that*

$$f(z)|T_p \equiv 0 \pmod{l}$$

for every $f(z)$ that is the reduction modulo l of a cusp form in $S_k(\Gamma_0(N), \chi) \cap \mathbb{Z}[[q]]$.

5. Proof of Theorem 1.4

We will see that using the works of Serre and Shimura, the proof of Theorem 1.4 boils down to the existence of a half-integral weight cusp form satisfying certain key properties. This is given in the following theorem. We make the following notation for ease of exposition

$$G_l(z) := \sum_{\substack{n \geq 0 \\ ln \equiv -1 \pmod{24}}}^{\infty} t\left(\frac{ln+1}{24}\right) q^n.$$

Theorem 5.1. *Let j be a positive integer and $l \geq 5$ prime. There is a cusp form*

$$g_{l,j}(z) \in S_{\frac{l^{j+1}-l^{j}-1}{2}}\left(\Gamma_0(2304l), \left(\frac{12}{\bullet}\right)\right)$$

having integer coefficients, such that

$$G_l(z) \equiv g_{l,j}(z) \pmod{l^j}.$$

5.1. **Deduction of Theorem 1.4 from Theorem 5.1.** Before we prove Theorem 5.1, we will show how Theorem 1.4 is proved from it. We use techniques of Ono and Ahlgren previously used on $p(n)$ (see [Ono00] and [Ahl00]) and extend them to work on $t(n)$ and $p(n)$ simultaneously.

Consider the Shimura lift of $g_{l,j}(z)$ to an integer weight cusp form with integer coefficients. We have that

$$S_t(g_{l,j}) \in S_{5l^{j+1}-5l^j-3}(\Gamma_0(2304l)) \cap \mathbb{Z}[[q]]. \tag{34}$$

By Theorem 4.8, there are infinitely many primes $p \equiv -1 \pmod{2304l}$ such that every reduction $\pmod{l^j}$ of every form in the space $S_{5l^{j+1}-5l^j-3}(\Gamma_0(2304l)) \cap \mathbb{Z}[[q]]$ gets annihilated $\pmod{l^j}$ by the (integer weight) Hecke operator T_p. In particular, by equation (34) and Proposition 4.7, it follows that there are infinitely many primes $p \equiv -1 \pmod{2304l}$ such that for any square-free integer t

$$S_t(g_{l,j}(z)|T(p^2)) = S_t(g_{l,j}(z))|T_p \equiv 0 \pmod{l^j}.$$

Thus in particular,

$$g_{l,j}(z)|T(p^2) \equiv 0 \pmod{l^j}. \tag{35}$$

If we write $g_{l,j}(z) = \sum_{n=1}^{\infty} c(n)q^n$, and let $\lambda_{l,j} = (5l^{j+1} - 5l^j - 2)/2$, then Definition 4.2 and (35) say that

$$\sum_{n=1}^{\infty} \left(c(p^2n) + \left(\frac{(-1)^{\lambda_{l,j}}}{p}\right)\left(\frac{n}{p}\right) p^{\lambda_{l,j}-1}c(n) + \left(\frac{(-1)^{\lambda_{l,j}}}{p^2}\right) p^{2\lambda_{l,j}-1} c\left(\frac{n}{p^2}\right)\right) q^n \equiv 0 \pmod{l^j}.$$

Thus for each of the infinitely many primes p for which equation (35) holds, we have

$$c(p^2n) + \left(\frac{(-1)^{\lambda_{l,j}}}{p}\right)\left(\frac{n}{p}\right) p^{\lambda_{l,j}-1}c(n) + \left(\frac{(-1)^{\lambda_{l,j}}}{p^2}\right) p^{2\lambda_{l,j}-1} c\left(\frac{n}{p^2}\right) \equiv 0 \pmod{l^j}$$

for all positive integers n. Replacing n by np, the middle term vanishes to give us that

$$c(p^3n) + \left(\frac{(-1)^{\lambda_{l,j}}}{p^2}\right) p^{2\lambda_{l,j}-1} c\left(\frac{n}{p}\right) \equiv 0 \pmod{l^j}. \tag{36}$$

We restrict our attention further by only considering n which are not divisible by p. For these n, $c(n/p)$ is defined to be 0, so equation (36) becomes

$$c(p^3n) \equiv 0 \pmod{l^j}.$$

Combining this with Theorem 5.1 we obtain the following proposition.

Proposition 5.2. *A positive proportion of the primes* $p \equiv -1 \pmod{2304l}$ *have the property that*

$$t\left(\frac{p^3ln+1}{24}\right) \equiv 0 \pmod{l^j}$$

for all n*, where* $ln \equiv -1 \pmod{24}$ *and* $(n,p) = 1$.

The work of Ono and Ahlgren intersects nicely with this analysis of $t(n)$. Define $F_l(z)$ by

$$F_l(z) := \sum_{\substack{n \geq 0 \\ ln \equiv -1 \pmod{24}}}^{\infty} p\left(\frac{ln+1}{24}\right) q^n.$$

In [Ahl00] (see Theorem 1), Ahlgren proves the following theorem.

Theorem 5.3. *Let $l \geq 5$ be prime, and j a positive integer. There exists a cusp form*

$$f_{l,j}(z) \in S_{\frac{l^{j+1}-l^j-1}{2}}\left(\Gamma_0(576l), \left(\frac{12}{\bullet}\right)\right)$$

such that

$$f_{l,j}(z) \equiv F_l(z) \pmod{l^{j+1}}.$$

Note that we can realize $f_{l,j}(z)$ as a cusp form on the group $\Gamma_0(2304l)$. Serre's Theorem gives a statement about every reduction $\pmod{l^j}$ of a cusp form in $S_{5l^{j+1}-5l^j-3}(\Gamma_0(2304l)) \cap \mathbb{Z}[[q]]$. Since the character $\left(\frac{12}{\bullet}\right)$ becomes trivial when we lift to an integral weight form, we can apply Theorem 4.8 to $S_t(f_{l,j}(z))$ and $S_t(g_{l,j}(z))$ simultaneously. Thus we conclude the following

Theorem 5.4. *A positive proportion of the primes $p \equiv -1 \pmod{2304l}$ have the property that*

$$t\left(\frac{p^3 ln+1}{24}\right) \equiv p\left(\frac{p^3 ln+1}{24}\right) \equiv 0 \pmod{l^j}$$

for all positive integers n, where $(p,n)=1$.

Theorem 1.4 follows easily from Theorem 5.4.

5.2. **Preliminaries for the proof of Theorem 5.1.** Now it only remains to prove Theorem 5.1. Recall Dedekind's eta-function

(37) $$\eta(z) := q^{\frac{1}{24}} \Pi_{n=1}^{\infty}(1-q^n).$$

From equations (1) and (2) we deduce that

$$g(z) = \sum_{n=0}^{\infty} t(n) q^{n-\frac{1}{24}} = \frac{\eta(2z)^2 \eta(16z)^5}{\eta(z)\eta(4z)^5\eta(32z)^2}.$$

For ease of notation, we make the following abbreviations. Let δ_l be the positive integer

$$\delta_l := \frac{l^2-1}{24},$$

and define $1 \leq \beta_l \leq l-1$ such that

$$24\beta_l \equiv 1 \pmod{l}.$$

Thus we have that

$$\eta(lz)^l \cdot g(z) = \Pi_{n=1}^{\infty}(1-q^{ln})^l \sum_{n=0}^{\infty} t(n) q^{n+\delta_l}.$$

Using the definitions of the U and V-operators, we obtain the following

(38) $$\left[\frac{\eta(lz)^l \cdot g(z) | U(l)}{\eta(z)^l}\right] \Bigg| V(24) = \sum_{n=0}^{\infty} t(ln+\beta_l) q^{24n+\frac{24\beta_l-1}{l}}.$$

Letting $k = 24n + (24\beta_l - 1)/l$, we see that

$$ln + \beta_l = \frac{lk+1}{24}.$$

Thus we can show that in fact

(39) $$\left[\frac{\eta(lz)^l \cdot g(z) | U(l)}{\eta(z)^l}\right] \Bigg| V(24) = G_l(z).$$

Notice that $\eta(z)^l/\eta(lz) \equiv 1 \pmod{l}$. By induction we can argue that for any $j \geq 0$,

(40) $$\frac{\eta(z)^{l^{j+1}}}{\eta(lz)^{l^j}} \equiv 1 \pmod{l^{j+1}}.$$

In light of this fact consider for any integer $j \geq 0$,

$$h_j(z) := \eta(lz)^l \cdot g(z) \cdot \left(\frac{\eta(z)^{l^j+1}}{\eta(lz)^{l^j}}\right)^5 = \frac{\eta(2z)^2\eta(16z)^5\eta(z)^{5l^{j+1}-1}}{\eta(4z)^5\eta(32z)^2\eta(lz)^{5l^j-l}}.$$

It is clear from the definitions of the U and V-operators that they preserve congruences. Thus from the definition of $h_j(z)$, and equations (39) and (40) we have the following proposition.

Proposition 5.5. *If $l \geq 5$ is prime, and $j \geq 0$, then*

$$G_l(z) \equiv \left[\frac{h_j(z)|U(l)}{\eta(z)^l}\right] \Bigg| V(24) =: H_{l,j}(z)|V(24) \pmod{l^{j+1}}.$$

We can see from the definition of $\eta(z)$ that $H_{l,j}(z)|V(24)$ has integer coefficients. Thus to finish the proof of Theorem 5.1, we only need to prove that $H_{l,j}(z)|V(24)$ is a cusp form.

Theorem 5.6. *Let $l \geq 5$ be prime, and $j \geq 0$ an integer. Using the notation from above, $H_{l,j}(z)|V(24)$ is a half-integral weight cusp form on the group $\Gamma_0(2304l)$. In particular,*

$$H_{l,j}(z)|V(24) \in S_{\frac{5l^{j+1}-5l^j-1}{2}}\left(\Gamma_0(2304l), \left(\frac{12}{\bullet}\right)\right).$$

Proving Theorem 5.6 is the final step to prove Theorem 5.1. First we note that using the well-known formulas for determining if an eta-quotient is a modular form (see [Ono04], Theorems 1.64 and 1.65), we can compute that for every integer $j \geq 0$, $h_j(z) \in S_{\frac{5l^{j+1}-5l^j+l-1}{2}}(\Gamma_0(32l), 1)$. Notice that $h_j(z)$ is an integer weight cusp form. By Propositions 4.4 and 4.5 we see that

$$(h_j(z)|U(l))|V(24) \in S_{\frac{5l^{j+1}-5l^j+l-1}{2}}(\Gamma_0(768l), 1).$$

Also, it is well known that $\eta(24z)^l \in S_{\frac{l}{2}}(\Gamma_0(576), (\frac{12}{\bullet}))$. So we can view both on the group $\Gamma_0(2304l)$, and thus the quotient $H_{l,j}(z)$ as modular over the group $\Gamma_0(2304l)$ with character $(\frac{12}{\bullet})$. We need to show that $H_{l,j}(z)$ vanishes at every cusp of $\Gamma_0(2304l)$. However, $V(24)$ cannot introduce poles. So to prove Theorem 5.6 it suffices to show that

(41) $$H_{l,j}(z) = \left[\frac{h_j(z)|U(l)}{\eta(z)^l}\right]$$

vanishes at every cusp of $\Gamma_0(32l)$. Recall [Ono04] that a cusp of a congruence subgroup Γ is an equivalence class in $\mathbb{Q} \cup \{\infty\}$ under the action of Γ.

5.3. **Proof of Theorem 5.6.** A complete set of representatives for the cusps of $\Gamma_0(N)$, where N is a positive integer, is given by [Mar96]

(42) $$\left\{\frac{a_c}{c} \in \mathbb{Q} : c \mid N\,,\ 1 \leq a_c \leq N\,,\ \gcd(a_c, N) = 1\,,\ a_c \text{ distinct modulo } \gcd\left(c, \frac{N}{c}\right)\right\}.$$

We recall the definition of the slash operator [Kob93]. If $f(z)$ is a function on the upper half-plane, $\lambda \in \frac{1}{2}\mathbb{Z}$, and $\begin{pmatrix} a & b \\ c & d \end{pmatrix} \in GL_2^+(\mathbb{R})$, then

(43) $$f(z) \mid_\lambda \begin{pmatrix} a & b \\ c & d \end{pmatrix} := (ad-bc)^{\frac{\lambda}{2}} \cdot (cz+d)^{-\lambda} \cdot f\left(\frac{az+b}{cz+d}\right).$$

Moreover, let $\gamma_{\frac{a}{c}}$ be the matrix in $SL_2(\mathbb{Z})$ that takes ∞ to $\frac{a}{c}$. We know from [Mar96] that the expansion of a modular form $f(z)$ of weight $\lambda \in \mathbb{R}$ at the cusp $\frac{a}{c}$ is of the form

$$f(z)|_\lambda \, \gamma_{\frac{a}{c}} = k \cdot q^\alpha + \cdots$$

for some nonzero constant k and $\alpha \in \mathbb{Q}$. Thus α is the order of vanishing of $f(z)$ at the cusp $\frac{a}{c}$. We are interested in the expansion of $H_{l,j}(z)$ at the cusps of $\Gamma_0(32l)$. From equation(43) it is easy to show that

$$H_{l,j} \, |_{\frac{5l^{j+1}-5l^j-1}{2}} \, \gamma = \frac{(h_j(z)|U(l)) \, |_{\frac{5l^{j+1}-5l^j+l-1}{2}} \, \gamma}{\eta(z)^l|_{\frac{l}{2}}\gamma}.$$

Recall the fact [Mar96] that for all $\gamma \in SL_2(\mathbb{Z})$,

$$\eta(z)^l|_{\frac{l}{2}}\gamma = k \cdot q^{\frac{l}{24}} + \cdots$$

It is now clear that to prove Theorem 5.6 it suffices to show that if $\gamma_{\frac{a}{c}} \in SL_2(\mathbb{Z})$ with $\gamma_{\frac{a}{c}}\infty = \frac{a}{c}$, where $\frac{a}{c}$ is a cusp of $\Gamma_0(32l)$, then

$$(h_j(z)|U(l))|_{\frac{5l^{j+1}-5l^j+l-1}{2}}\gamma_{\frac{a}{c}} = k \cdot q^\alpha + \cdots \,, \tag{44}$$

where $\alpha > \frac{l}{24}$. We prove this now.

Proof of Theorem 5.6. By equation (42), we see that there are the following 16 cusps of $\Gamma_0(32l)$:

$$\frac{1}{c} \text{ for each } c|32l, \text{ and } \frac{3}{4}, \frac{3}{8}, \frac{3}{4l}, \frac{3}{8l}.$$

We consider what happens when the slash operator acts on $h_j(z)$, making note of the fact that $[h_j(z)|U(l)]|_\lambda\gamma$ and $[h_j(z)|\lambda\gamma]|U(l)$ will have the same orders of vanishing at the cusps of $\Gamma_0(32l)$.

For convenience, let $k := \frac{5l^{j+1}-5l^j+l-1}{2} \in \mathbb{R}$ denote the weight of $h_j(z)$. From [Mar96] (page 4827), we can immediately deduce the following fact (since $h_j(z)$ has trivial character). Let $\gamma = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in SL_2(\mathbb{Z})$. Then,

$$h_j(z)|_k\gamma = \sum_{n=0}^{\infty} b_\gamma(n) q^{\frac{n}{h_c}}, \tag{45}$$

where

$$h_c = \frac{32l}{\gcd(c^2, 32l)}. \tag{46}$$

Calculating the h_c for each cusp we get that

$$h_c = \begin{cases} 32l & \text{if } c = 1 \\ 8l & \text{if } c = 2 \\ 2l & \text{if } c = 4 \\ l & \text{if } c = 8, 16, 32 \\ 32 & \text{if } c = l \\ 8 & \text{if } c = 2l \\ 2 & \text{if } c = 4l \\ 1 & \text{if } c = 8l, 16l, 32l. \end{cases} \tag{47}$$

Since $h_j(z)$ is an eta-quotient, we can use the formula from Theorem 1.65 in [Ono04] to calculate the order of vanishing of $h_j(z)$ at each cusp. We find that

$$ord_{\frac{a}{c}}(h_j(z)) = \begin{cases} 160\delta_l l^j & \text{if } c = 1 \\ 40\delta_l l^j & \text{if } c = 2 \\ 10\delta_l l^j - l & \text{if } c = 4 \\ 5\delta_l l^j & \text{if } c = 8, 32 \\ 5\delta_l l^j + 2l & \text{if } c = 16 \\ 32\delta_l & \text{if } c = l \\ 8\delta_l & \text{if } c = 2l \\ 2\delta_l - 1 & \text{if } c = 4l \\ \delta_l & \text{if } c = 8l, 32l \\ \delta_l + 2 & \text{if } c = 16l. \end{cases} \tag{48}$$

Let $H(l,\gamma,z) := h_j(z)|_k\gamma$. Putting together equations (47) and (48), we see that

$$H(l,\gamma,z) = \begin{cases} \star q^{160\delta_l l^j} + \cdots = \sum_{n\geq 1} a_c(l,\gamma,n) q^{\frac{n}{32l}} & \text{if } c = 1 \\ \star q^{40\delta_l l^j} + \cdots = \sum_{n\geq 1} a_c(l,\gamma,n) q^{\frac{n}{8l}} & \text{if } c = 2 \\ \star q^{10\delta_l l^j - l} + \cdots = \sum_{n\geq 1} a_c(l,\gamma,n) q^{\frac{n}{2l}} & \text{if } c = 4 \\ \star q^{5\delta_l l^j} + \cdots = \sum_{n\geq 1} a_c(l,\gamma,n) q^{\frac{n}{l}} & \text{if } c = 8, 32 \\ \star q^{5\delta_l l^j + 2l} + \cdots = \sum_{n\geq 1} a_c(l,\gamma,n) q^{\frac{n}{l}} & \text{if } c = 16 \\ \star q^{32\delta_l} + \cdots = \sum_{n\geq 1} a_c(l,\gamma,n) q^{\frac{n}{32}} & \text{if } c = l \\ \star q^{8\delta_l} + \cdots = \sum_{n\geq 1} a_c(l,\gamma,n) q^{\frac{n}{8}} & \text{if } c = 2l \\ \star q^{2\delta_l - 1} + \cdots = \sum_{n\geq 1} a_c(l,\gamma,n) q^{\frac{n}{2}} & \text{if } c = 4l \\ \star q^{\delta_l} + \cdots = \sum_{n\geq 1} a_c(l,\gamma,n) q^{n} & \text{if } c = 8l, 32l \\ \star q^{\delta_l + 2} + \cdots = \sum_{n\geq 1} a_c(l,\gamma,n) q^{n} & \text{if } c = 16l, \end{cases} \tag{49}$$

where $\star$ denotes an arbitrary constant. The following proposition can be obtained easily from the definition of the U-operator.

Proposition 5.7. *If $P(z) = \sum_{n\geq 1} c(n)q^n$ is a formal power series, and l is prime, then*

$$P(z)|U(l) = \frac{1}{l}\sum_{j=0}^{l-1} P\left(\frac{z+j}{l}\right).$$

Applying Proposition 5.7 to $H(l,\gamma,z) = \sum_{n\geq 1} a_c(l,\gamma,n) q^{\frac{n}{h_c}}$, and letting $\zeta_m := e^{\frac{2\pi i}{m}}$, we easily see that

$$H(l,\gamma,z)|U(l) = \frac{1}{l}\sum_{n\geq 1} a_c(l,\gamma,n) q^{\frac{n}{lh_c}} \left[\sum_{j=0}^{l-1} \zeta_{lh_c}^{nj}\right]. \tag{50}$$

We must work through each case to show that if $H(l,\gamma,z)|U(l) = \star q^{\alpha} + \cdots$, then $\alpha > \frac{l}{24}$. Since the calculations for the cases are very similar, we will only show the details of the first case here, when $c = 1$. When $c = 1$, then $\gamma = \begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix}$. Equations 50 and (47) give that

$$H(l,\gamma,z)|U(l) = \frac{1}{l}\sum_{n\geq 1} a_c(l,\gamma,n) q^{\frac{n}{32l^2}} \left[\sum_{j=0}^{l-1} \zeta_{32l^2}^{nj}\right].$$

From equation (49), we see that $a_c(l,\gamma,n)=0$ whenever $\frac{N}{32l}<160\delta_l l^j$. Also, by equations (45) and (46) we know that if $H(l,\gamma,z)|U(l)=\star q^{\alpha}+\cdots$, then $\alpha=\frac{n}{32l}$ for some integer n. Thus

$$\alpha \geq \frac{m}{32l^2},$$

where m is the smallest multiple of l such that

$$\frac{m}{32l} \geq 160\delta_l l^j = \frac{160l^j(l^2-1)}{24}.$$

In other words, m is the least integer multiple of l such that

$$m \geq \frac{5120l^{j+1}(l^2-1)}{24}.$$

Noting that $\frac{5120l^{j+1}(l^2-1)}{24}\in\mathbb{Z}$, let x_l be the least nonnegative integer such that

$$m = \frac{5120l^{j+1}(l^2-1)}{24}+x_l \equiv 0 \pmod{l}.$$

Thus

$$\alpha \geq \frac{m}{32l^2} = \frac{5120l^{j+1}(l^2-1)+24x_l}{768l^2} = \frac{l}{24}+\left[\frac{5120j^{j+3}-5120l^{j+1}-32l^3+24x_l}{768l^2}\right],$$

and so we are done if

$$\left[\frac{5120j^{j+3}-5120l^{j+1}-32l^3+24x_l}{768l^2}\right] > 0.$$

This is equivalent to showing that

$$160l^j(l^2-1)-l^2>0,$$

which follows from a simple induction on l. □

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