Ramanujan's tau function

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Abstract. Ramanujan's tau function is defined by

$$\sum_{n\geq 1} \tau(n)q^n = qE(q)^{24}$$

where $E(q) = \prod_{n\geq 1}(1-q^n)$.

It is known that if p is prime,

$$\tau(pn) = \tau(p)\tau(n) - p^{11}\tau\left(\frac{n}{p}\right),$$

where it is understood that $\tau\left(\frac{n}{p}\right) = 0$ if p does not divide n.

We give proofs of this relation which rely on nothing more than Jacobi's triple product identity for $p = 2$, 3, 5, 7 and 13.

1. Introduction

Ramanujan's tau function is defined by

$$\sum_{n\geq 1} \tau(n)q^n = qE(q)^{24}$$

where $E(q) = \prod_{n\geq 1}(1-q^n)$.

The tau function has many fascinating properties.

One of these is that if p is prime,

$$\tau(pn) = \tau(p)\tau(n) - p^{11}\tau\left(\frac{n}{p}\right), \tag{1.1}$$

where it is understood that $\tau\left(\frac{n}{p}\right) = 0$ if p does not divide n.

It follows easily from (1.1) that tau is multiplicative,

$$\tau(mn) = \tau(m)\tau(n) \tag{1.2}$$

provided m and n have no common divisor other than 1,

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and that, at least formally,

$$\sum_{n\geq 1}\frac{\tau(n)}{n^s}=\prod_{p\text{ prime}}\left(1-\frac{\tau(p)}{p^s}+\frac{1}{p^{2s-11}}\right)^{-1}. \tag{1.3}$$

I have found proofs of (1.1) for $p=2,\ 3,\ 5,\ 7$ and 13 which require nothing more than Jacobi's triple product identity,

$$\prod_{n\geq 1}(1+a^{-1}q^{2n-1})(1+aq^{2n-1})(1-q^{2n})=\sum_{-\infty}^{\infty}a^nq^{n^2}. \tag{1.4}$$

Completely different elementary proofs of (1.1) for $p=2$ and 3 have recently been given by Kenneth S. Williams [**3**].

A modern proof of (1.1) may be found in [**1**].

2. $p=2$

Let

$$\phi(q)=\sum_{-\infty}^{\infty}q^{n^2},\quad \psi(q)=\sum_{n\geq 0}q^{(n^2+n)/2}=\sum_{-\infty}^{\infty}q^{2n^2+n}.$$

It can be shown with, or even without, (1.4), [**2**], Chapter 1 that

$$\phi(q)\phi(-q)=\phi(-q^2)^2 \text{ and } \phi(q)\psi(q^2)=\psi(q)^2.$$

Also, by (1.4),

$$\phi(-q)=\frac{E(q)^2}{E(q^2)} \text{ and } \psi(q)=\frac{E(q^2)^2}{E(q)}.$$

It is easy to see that

$$\phi(q)=\phi(q^4)+2q\psi(q^8) \tag{2.1}$$

Put $-q$ for q in (2.1).

$$\phi(-q)=\phi(q^4)-2q\psi(q^8). \tag{2.2}$$

Multiply (2.1) by (2.2).

$$\phi(-q^2)^2=\phi(q^4)^2-4q^2\psi(q^8)^2. \tag{2.3}$$

Put q for q^2 in (2.3).

$$\phi(-q)^2=\phi(q^2)^2-4q\psi(q^4)^2. \tag{2.4}$$

Put $-q$ for q in (2.4).

$$\phi(q)^2=\phi(q^2)^2+4q\psi(q^4)^2. \tag{2.5}$$

Multiply (2.4) by (2.5).

$$\phi(-q^2)^4=\phi(q^2)^4-16q^2\psi(q^4)^4. \tag{2.6}$$

Put q for q^2 in (2.6) and rearrange.

$$\phi(q)^4-\phi(-q)^4=16q\psi(q^2)^4. \tag{2.7}$$

We have

$$\begin{aligned}
(2.8)\qquad \sum_{n\geq 1}\tau(n)q^n &= qE(q)^{24}\\
&= qE(q^2)^{12}\left(\frac{E(q)^2}{E(q^2)}\right)^{12}\\
&= qE(q^2)^{12}\phi(-q)^{12}\\
&= qE(q^2)^{12}\left(\phi(-q)^2\right)^6\\
&= qE(q^2)^{12}\left(\phi(q^2)^2-4q\psi(q^4)^2\right)^6\\
&= qE(q^2)^{12}\big(\phi(q^2)^{12}-24q\phi(q^2)^{10}\psi(q^4)^2+240q^2\phi(q^2)^8\psi(q^4)^4\\
&\quad -1280q^3\phi(q^2)^6\psi(q^4)^6+3840q^4\phi(q^2)^4\psi(q^4)^8-6144q^5\phi(q^2)^2\psi(q^4)^4\\
&\quad +4096q^6\psi(q^4)^6\big).
\end{aligned}$$

If we extract the even powers and replace q^2 by q, we obtain

$$\begin{aligned}
(2.9)\qquad \sum_{n\geq 1}\tau(2n)q^n &= -8qE(q)^{12}\phi(q)^2\psi(q^2)\left(3\phi(q)^8+160q\phi(q)^4\psi(q^2)^4+768q^2\psi(q^2)^8\right)\\
&= -8qE(q)^{12}\left(\phi(q)\psi(q^2)\right)^2\left(3\left(\phi(q)^4-16q\psi(q^2)^4\right)^2+256q\phi(q)^4\psi(q^2)^4\right)\\
&= -8qE(q)^{12}\psi(q)^4\left(3\left(\phi(-q)^4\right)^2+256q\psi(q)^8\right)\\
&= -8qE(q)^{12}\left(\frac{E(q^2)^2}{E(q)}\right)^4\left(3\left(\frac{E(q)^2}{E(q^2)}\right)^8+256\left(\frac{E(q^2)^2}{E(q)}\right)^8\right)\\
&= -24qE(q)^{24}-2^{11}q^2E(q^2)^{24}\\
&= -24\sum_{n\geq 1}\tau(n)q^n-2^{11}\sum_{n\geq 1}\tau(n)q^{2n}.
\end{aligned}$$

The term $n=1$ in (2.9) gives

$$\tau(2)=-24\tau(1)=-24,$$

so (2.9) becomes

$$\sum_{n\geq 1}\tau(2n)q^n=\tau(2)\sum_{n\geq 1}\tau(n)q^n-2^{11}\sum_{n\geq 1}\tau(n)q^{2n},$$

as claimed.

Aside: (2.7) can be written

$$\left(\prod_{n\geq 1}(1+q^{2n-1})^2(1-q^{2n})\right)^4-\left(\prod_{n\geq 1}(1-q^{2n-1})^2(1-q^{2n}\right)^4 = 16q\left(\prod_{n\geq 1}\frac{(1-q^{4n})^2}{(1-q^{2n})}\right)^4. \tag{2.10}$$

IF we divide (2.10) by $\prod_{n\geq 1}(1-q^{2n})^4$, we find

$$\prod_{n\geq 1}(1+q^{2n-1})^8-\prod_{n\geq 1}(1-q^{2n-1})^8=16q\prod_{n\geq 1}(1+q^{2n})^8. \tag{2.11}$$

Jacobi described (2.11) as *"aequatio identica satis abstrusa"*. ("A fairly obscure identity".)

(2.11) can perhaps most strikingly be written [**2**], Chapter 19

$$\mathbf{O}\Big(\prod_{\substack{n\geq 1\\ n\not\equiv 0 \pmod 4}}(1-q^n)^8\Big)=-8q.$$

$$\begin{aligned}\prod_{\substack{n\geq 1\\ n\not\equiv 0 \pmod 4}}(1-q^n)^8 &= 1-8q+20q^2-62q^4+216q^6-641q^8\\ &\quad +1636q^{10}-3778q^{12}+8248q^{14}+\ \cdots\ .\end{aligned}$$

3. $p=3$

We have the 3-dissection

(3.1)

$$\begin{aligned}E(q)^3&=\sum_{n\geq 0}(-1)^n(2n+1)q^{(n^2+n)/2}\\ &=1-3q+5q^3-7q^6+9q^{10}-11q^{15}+\ \cdots\\ &=A(q^3)-3qE(q^9)^3.\end{aligned}$$

We have

(3.2)

$$\begin{aligned}\sum_{n\geq 1}\tau(n)q^n &= qE(q)^{24}\\
&= q\left(E(q)^3\right)^8\\
&= q\left(A(q^3)-3qE(q^9)^3\right)^8\\
&= q\big(A(q^3)^8-24qA(q^3)^7E(q^9)^3+252q^2A(q^3)^6E(q^9)^6-1512q^3A(q^3)^5E(q^9)^9\\
&\quad +5670q^4A(q^3)^4E(q^9)^{12}-13608q^5A(q^3)^3E(q^9)^{15}+20412q^6A(q^3)^2E(q^9)^{18}\\
&\quad -17496q^7A(q^3)E(q^9)^{21}+6561q^8E(q^9)^{24}\big)\,.\end{aligned}$$

If we extract those terms in which the power of q is a multiple of 3, and replace q^3 by q, we obtain

$$\sum_{n\geq 1}\tau(3n)q^n = 252qA(q)^6E(q^3)^6-13608q^2A(q)^3E(q^3)^{15}+6561q^3E(q^3)^{24}. \tag{3.3}$$

If we put q, ωq, $\omega^2 q$ for q in (3.1) and multiply the three results, we find

$$E(q)^3E(\omega q)^3E(\omega^2q)^3 = A(q^3)^3-27q^3E(q^9)^9, \tag{3.4}$$

or,

$$\left(\frac{E(q^3)^4}{E(q^9)}\right)^3 = A(q^3)^3-27q^3E(q^9)^9. \tag{3.5}$$

If in (3.5) we replace q^3 by q and rearrange, we obtain

$$A(q)^3 = \frac{E(q)^{12}}{E(q^3)^3}+27qE(q^3)^9. \tag{3.6}$$

If we substitute (3.6) into (3.3), we obtain

$$\begin{aligned}\sum_{n\geq 1}\tau(3n)q^n &= 252qE(q^3)^6\left(\frac{E(q)^{12}}{E(q^3)^3}+27qE(q^3)^9\right)^2\\
&\quad -13608q^2E(q^3)^{15}\left(\frac{E(q)^{12}}{E(q^3)^3}+27qE(q^3)^9\right)\\
&\quad +6561q^3E(q^3)^{24}\\
&= 252qE(q)^{24}-3^{11}q^3E(q^3)^{24}\\
&= 252\sum_{n\geq 1}\tau(n)q^n-3^{11}\sum_{n\geq 1}\tau(n)q^{3n}.\end{aligned} \tag{3.7}$$

The term $n=1$ in (3.7) gives

$$\tau(3)=252\tau(1)=252,$$

so (3.7) becomes

$$\sum_{n\geq 0}\tau(3n)q^n = \tau(3)\sum_{n\geq 1}\tau(n)q^n-3^{11}\sum_{n\geq 1}\tau(n)q^{3n},$$

as claimed.

Aside: It can be shown [**2**], Chapter 21 that

$$\begin{aligned} A(q) &= E(q)\left(1+6\sum_{n\geq 0}\left(\frac{q^{3n+1}}{1-q^{3n+1}}-\frac{q^{3n+2}}{1-q^{3n+2}}\right)\right) \\ &= E(q)\sum_{m,n=-\infty}^{\infty} q^{m^2+mn+n^2}. \end{aligned}$$

4. $p = 5$

We have

$$\begin{aligned} E(q) &= 1-q-q^2+q^5+q^7-q^{12}-q^{15}+q^{22}+q^{26}--++\ \cdots \\ &= E_0+E_1+E_2 \end{aligned} \tag{4.1}$$

where E_i is the sum of those terms in $E(q)$ in which the power of q is congruent to i modulo 5. ($i = 0,\ 2,\ 3.$)

It is easy to prove that

$$E_1 = -qE(q^{25}) \tag{4.2}$$

and, using Jacobi's expansion of the cube of Euler's product, namely

$$\prod_{n\geq 1}(1-q^n)^3 = \sum_{n\geq 0}(-1)^n(2n+1)q^{(n^2+n)/2} \tag{4.3}$$

that

$$E_0E_2 = -E_1^2. \tag{4.4}$$

If we write

$$\alpha = -\frac{E_0}{E_1} \quad \text{and} \quad \beta = -\frac{E_2}{E_1} \tag{4.5}$$

then $\alpha\beta = -1$ and

$$E(q) = qE(q^{25})\,(\alpha - 1 + \beta)\,. \tag{4.6}$$

We have

(4.7)
$$\begin{aligned}\sum_{n\geq 1}\tau(n)q^n &= qE(q)^{24}\\ &= q^{25}E(q^{25})^{24}\left(\alpha-1+\beta\right)^{24}\\ &= q^{25}E(q^{25})^{24}\big(\alpha^{24}-24\alpha^{23}+252\alpha^{22}-1472\alpha^{21}+4830\alpha^{20}-6072\alpha^{19}\\ &\quad -16192\alpha^{18}+78936\alpha^{17}-82731\alpha^{16}-212520\alpha^{15}+649704\alpha^{14}\\ &\quad -73416\alpha^{13}-1977862\alpha^{12}+2034672\alpha^{11}+3487260\alpha^{10}\\ &\quad -7072408\alpha^{9}-3432198\alpha^{8}+15343944\alpha^{7}+134596\alpha^{6}\\ &\quad -25077360\alpha^{5}+6067446\alpha^{4}+33474936\alpha^{3}-12286968\alpha^{2}\\ &\quad -38228232\alpha+14903725-38228232\beta-12286968\beta^{2}+33474936\beta^{3}\\ &\quad +6067446\beta^{4}-25077360\beta^{5}+134596\beta^{6}+15343944\beta^{7}-3432198\beta^{8}\\ &\quad -7072408\beta^{9}+3487260\beta^{10}+2034672\beta^{11}-1977862\beta^{12}-73416\beta^{13}\\ &\quad +649704\beta^{14}-212520\beta^{15}-82731\beta^{16}+78936\beta^{17}-16192\beta^{18}\\ &\quad -6072\beta^{19}+4830\beta^{20}-1472\beta^{21}+252\beta^{22}-24\beta^{23}+\beta^{24}\big).\end{aligned}$$

If we extract those terms in which the power of q is a multiple of 5, we obtain

(4.8)
$$\begin{aligned}\sum_{n\geq 1}\tau(5n)q^{5n} &= q^{25}E(q^{25})^{24}\big(4830\alpha^{20}-212520\alpha^{15}+3487260\alpha^{10}-25077360\alpha^{5}\\ &\quad +14903725-25077360\beta^{5}+3487260\beta^{10}-212520\beta^{15}+4830\beta^{20}\big).\end{aligned}$$

Miraculously, this can be written

$$\sum_{n\geq 1}\tau(5n)q^{5n} = q^{25}E(q^{25})^{24}\left(4830(\alpha^5-11+\beta^5)^4-5^{11}\right). \tag{4.9}$$

If in (4.6) we replace q by q, ηq, $\eta^2 q$, $\eta^3 q$ and $\eta^4 q$ where η is a fifth root of unity other than 1, and multiply the five results, we obtain

$$E(q)E(\eta q)E(\eta^2 q)E(\eta^3 q)E(\eta^4 q) = q^5E(q^{25})^5(\alpha^5-11+\beta^5). \tag{4.10}$$

or,

$$\alpha^5-11+\beta^5 = \frac{E(q^5)^6}{q^5E(q^{25})^6}. \tag{4.11}$$

If we substitute (4.11) into (4.9) we find

(4.12)
$$\begin{aligned}\sum_{n\geq 1}\tau(5n)q^{5n} &= q^{25}E(q^{25})^{24}\left(4830\left(\frac{E(q^5)^6}{q^5E(q^{25})^6}\right)^4-5^{11}\right)\\ &= 4830q^5E(q^5)^{24}-5^{11}q^{25}E(q^{25})^{24}.\end{aligned}$$

If in (4.12) we replace q^5 by q, we obtain

(4.13)
$$\sum_{n\ge1}\tau(5n)q^n = 4830qE(q)^{24} - 5^{11}q^5E(q^5)^{24}$$
$$= 4830\sum_{n\ge1}\tau(n)q^n - 5^{11}\sum_{n\ge1}\tau(n)q^{5n}.$$

The term $n=1$ in (4.13) gives

$$\tau(5) = 4830\tau(1) = 4830,$$

so (4.13) becomes

$$\sum_{n\ge1}\tau(5n)q^n = \tau(5)\sum_{n\ge1}\tau(n)q^n - 5^{11}\sum_{n\ge1}\tau(n)q^{5n},$$

as claimed.

Aside: It can be shown [**2**], Chapter 8 that

$$\alpha = r(q^5)^{-1}, \quad \beta = -r(q^5),$$

where

$$r(q) = \cfrac{q^{\frac15}}{1+\cfrac{q}{1+\cfrac{q^2}{1+\cfrac{q^3}{1+\ddots}}}}$$
$$= q^{\frac15}\prod_{n\ge0}\frac{(1-q^{5n+1})(1-q^{5n+4})}{(1-q^{5n+2})(1-q^{5n+3})}.$$

5. $p=7$

We can write

(5.1) $$E(q) = E_0 + E_1 + E_2 + E_5$$

where E_i is the sum of those terms in $E(q)$ in which the power of q is congruent to i modulo 7. ($i = 0,\ 1,\ 2,\ 5.$)

It is easy to show that

(5.2) $$E_2 = -q^2E(q^{49}).$$

If we write

(5.3) $$\alpha = -\frac{E_0}{E_2}, \quad \beta = -\frac{E_1}{E_2}, \quad \gamma = -\frac{E_5}{E_2},$$

then

(5.4) $$E(q) = q^2E(q^{49})(\alpha+\beta-1+\gamma).$$

Jacobi's identity (4.1) yields

(5.5) $$\alpha\beta\gamma = -1,$$

$$-\alpha^2+\alpha\beta^2+\gamma=0, \tag{5.6}$$

$$\alpha-\beta^2+\beta\gamma^2=0 \tag{5.7}$$

and

$$\alpha^2\gamma+\beta-\gamma^2=0. \tag{5.8}$$

We have

$$\begin{aligned}\sum_{n\geq 1}\tau(n)q^n &= qE(q)^{24} \\ &= q^{49}E(q^{49})^{24}(\alpha+\beta-1+\gamma)^{24}.\end{aligned} \tag{5.9}$$

We can expand the right side of (5.9) and extract those terms in which the power of q is a multiple of 7. Thus, if H is the Huffing operator modulo 7, given by

$$H\left(\sum_n a(n)q^n\right)=\sum_n a(7n)q^{7n}, \tag{5.10}$$

and if we apply H to (5.9), we find

$$\sum_{n\geq 1}\tau(7n)q^{7n}=q^{49}E(q^{49})^{24}H\left((\alpha+\beta-1+\gamma)^{24}\right). \tag{5.11}$$

Let

$$\zeta=\alpha+\beta-1+\gamma=\frac{E(q)}{q^2E(q^{49})}. \tag{5.12}$$

Then (5.11) becomes

$$\sum_{n\geq 1}\tau(7n)q^{7n}=q^{49}E(q^{49})^{24}H(\zeta^{24}). \tag{5.13}$$

Now,

$$H(\zeta^0)=H(1)=1, \tag{5.14}$$

$$H(\zeta)=H(\alpha+\beta-1+\gamma)=-1, \tag{5.15}$$

$$H(\zeta^2)=H(\alpha^2+2\alpha\beta+(\beta^2-2\alpha)-2\beta+1+2\alpha\gamma+2\beta\gamma+2\gamma+\gamma^2)=1, \tag{5.16}$$

and it can be shown [**2**], Chapter 7 that

$$H(\zeta^3)=-7, \tag{5.17}$$

$$H(\zeta^4)=-4T-7, \tag{5.18}$$

$$H(\zeta^5)=10T+49 \tag{5.19}$$

and

$$H(\zeta^6)=49 \tag{5.20}$$

where

$$T=\frac{E(q^7)^4}{q^7E(q^{49})^4}. \tag{5.21}$$

It can then be shown that ζ satisfies the so-called modular equation

(5.22)

$$\zeta^7+7\zeta^6+21\zeta^5+49\zeta^4+(7T+147)\zeta^3+(35T+343)\zeta^2+(49T+343)\zeta-T^2=0.$$

It follows that for $i \geq 0$,

(5.23)

$$H(\zeta^{i+7})+7H(\zeta^{i+6})+21H(\zeta^{i+5})+49\zeta^{i+4})+(7T+147)H(\zeta^{i+3}) \\ +(35T+343)H(\zeta^{i+2})+(49T+343)H(\zeta^{i+1})-T^2H(\zeta^i)=0.$$

Now write

(5.24) $$u_i = H(\zeta^i).$$

Then

(5.25) $u_0=1,\ u_1=-1,\ u_2=1,\ u_3=7,\ u_4=-4T-7,\ u_5=10T+49,\ u_6=49$

and for $i \geq 0$,

(5.26)

$$u_{i+7}+7u_{i+6}+21u_{i+5}+49u_{i+4}+(7T+147)u_{i+3} \\ +(35T+343)u_{i+2}+(49T+343)u_{i+1}-T^2u_i=0.$$

If we write

(5.27) $$U=\sum_{i\geq 0} u_i z^i$$

it follows from (5.25) and (5.26) that

(5.28)

$$(1+7z+21z^2+49z^3+(7T+147)z^4+(35T+343)z^5+(49T+343)z^6-T^2z^7)U \\ =1+6z+15z^2+28z^3+(3T+63)z^4+(10T+98)z^5+(7T+49)z^6$$

and so

(5.29)

$$U=\frac{1+6z+15z^2+28z^3+(3T+63)z^4+(10T+98)z^5+(7T+49)z^6}{1+7z+21z^2+49z^3+(7T+147)z^4+(35T+343T)z^5+(49T+343T)z^6-T^2z^7}.$$

If we expand the right side of (5.29) as a series, we find

(5.30) $$u_{24}=-16744T^6-7^{11}.$$

That is,

(5.31) $$H(\zeta^{24})=-16744\left(\frac{E(q^7)^4}{q^7E(q^{49})^4}\right)^6-7^{11}.$$

If we substitute (5.31) into (5.13), we find

(5.32)
$$\begin{aligned}\sum_{n\geq 1}\tau(7n)q^{7n} &= q^{49}E(q^{49})^{24}\left(-16744\left(\frac{E(q^7)^4}{q^7E(q^{49})^4}\right)^6 - 7^{11}\right)\\ &= -16744q^7E(q^7)^{24} - 7^{11}q^{49}E(q^{49})^{24}.\end{aligned}$$

If in (5.32) we replace q^7 by q, we obtain

(5.33)
$$\begin{aligned}\sum_{n\geq 1}\tau(7n)q^{n} &= -16744qE(q)^{24} - 7^{11}q^7E(q^7)^{24}\\ &= -16744\sum_{n\geq 1}\tau(n)q^n - 7^{11}\sum_{n\geq 1}\tau(n)q^{7n}\end{aligned}$$

The term $n = 1$ in (5.33) gives

$$\tau(7) = -16744\tau(1) = -16744,$$

so (5.33) becomes

$$\sum_{n\geq 1}\tau(7n)q^n = \tau(7)\sum_{n\geq 1}\tau(n)q^n - 7^{11}\sum_{n\geq 1}\tau(n)q^{7n},$$

as claimed.

Aside: It can be shown [**2**], Chapter 10 that

$$\begin{aligned}\alpha &= q^{-2}\prod_{n\geq 0}\frac{(1-q^{49n+14})(1-q^{49n+35})}{(1-q^{49n+7})(1-q^{49n+42})},\\ \beta &= -q^{-1}\prod_{n\geq 0}\frac{(1-q^{49n+21})(1-q^{49n+28})}{(1-q^{49n+14})(1-q^{49n+35})},\\ \gamma &= q^{3}\prod_{n\geq 0}\frac{(1-q^{49n+7})(1-q^{49n+42})}{(1-q^{49n+21})(1-q^{49n+28})}.\end{aligned}$$

6. $p = 13$

Define

(6.1)
$$\zeta = \frac{E(q)}{q^7E(q^{169})}, \qquad T = \frac{E(q^{13})^2}{q^{13}E(q^{169})^2}.$$

Then

$$H(\zeta) = 1, \tag{6.2}$$

$$H(\zeta^2) = -2T - 1, \tag{6.3}$$

$$H(\zeta^3) = 13, \tag{6.4}$$

$$H(\zeta^4) = 2T^2 - 13, \tag{6.5}$$

$$H(\zeta^5) = -20T^2 - 10 \times 13T - 13^2, \tag{6.6}$$

$$H(\zeta^6) = 10T^3 - 13^2, \tag{6.7}$$

$$H(\zeta^7) = 98T^3 + 28 \times 13T^2 - 13^3, \tag{6.8}$$

$$H(\zeta^8) = -70T^4 - 13^3, \tag{6.9}$$

$$\begin{aligned} H(\zeta^9) = {} & -162T^4 + 108 \times 13T^3 + 72 \times 13^2T^2 \\ & +18 \times 13^3T + 13^4, \end{aligned} \tag{6.10}$$

$$H(\zeta^{10}) = 238T^5 - 13^4, \tag{6.11}$$

$$\begin{aligned} H(\zeta^{11}) = {} & -902T^5 - 1672 \times 13T^4 - 792 \times 13^2T^3 \\ & -198 \times 13^3T^2 - 22 \times 13^4T - 13^5, \end{aligned} \tag{6.12}$$

$$H(\zeta^{12}) = -418T^6 - 13^5. \tag{6.13}$$

For $0 \le i \le 12$ let

$$\zeta_i = \zeta(\eta^i q) \tag{6.14}$$

where η is a 13th root of unity other than 1.

Then

$$\sum_i \zeta_i = 13, \tag{6.15}$$

$$\sum_i \zeta_i^2 = -2 \times 13T - 13, \tag{6.16}$$

$$\sum_i \zeta_i^3 = 13^2, \tag{6.17}$$

$$\sum_i \zeta_i^4 = 2 \times 13T^2 - 13^2, \tag{6.18}$$

$$\sum_i \zeta_i^5 = -20 \times 13T^2 - 10 \times 13^2T - 13^3, \tag{6.19}$$

$$\sum_i \zeta_i^6 = 10 \times 13T^3 - 13^3, \tag{6.20}$$

$$\sum_i \zeta_i^7 = 98 \times 13T^3 + 28 \times 13^2T^2 - 13^4, \tag{6.21}$$

$$\sum_i \zeta_i^8 = -70 \times 13T^4 - 13^4, \tag{6.22}$$

$$\begin{aligned} \sum_i \zeta_i^9 = {} & -162 \times 13T^4 + 108 \times 13^2T^3 \\ & +72 \times 13^3T^2 + 18 \times 13^4T + 13^5, \end{aligned} \tag{6.23}$$

$$\sum_i \zeta_i^{10} = 238 \times 13T^5 - 13^5, \tag{6.24}$$

$$\sum_i \zeta_i^{11} = -902 \times 13T^5 - 1672 \times 13^2T^4 \qquad (6.25)$$
$$-792 \times 13^3T^3 - 198 \times 13^4T^2 - 22 \times 13^5T - 13^6,$$

$$\sum_i \zeta_i^{12} = -418 \times 13T^6 - 13^6. \qquad (6.26)$$

From these we obtain the symmetric functions,

$$\Sigma_1 = \sum_i \zeta_i = 13, \qquad (6.27)$$

$$\Sigma_2 = \sum_{i<j} \zeta_i\zeta_j = 13T + 7 \times 13, \qquad (6.28)$$

$$\Sigma_3 = \sum_{i<j<k} \zeta_i\zeta_j\zeta_k = 13^2T + 3 \times 13^2, \qquad (6.29)$$

$$\Sigma_4 = 6 \times 13T^2 + 7 \times 13^2T + 15 \times 13^2, \qquad (6.30)$$

$$\Sigma_5 = 74 \times 13T^2 + 37 \times 13^2T + 5 \times 13^3, \qquad (6.31)$$

$$\Sigma_6 = 20 \times 13T^3 + 38 \times 13^2T^2 + 13^4T + 19 \times 13^3, \qquad (6.32)$$

$$\Sigma_7 = 222 \times 13T^3 + 184 \times 13^2T^2 \qquad (6.33)$$
$$+51 \times 13^3T + 5 \times 13^4,$$

$$\Sigma_8 = 38 \times 13T^4 + 102 \times 13^2T^3 + 56 \times 13^3T^2 \qquad (6.34)$$
$$+13^5T + 15 \times 13^4,$$

$$\Sigma_9 = 346 \times 13T^4 + 422 \times 13^2T^3 + 184 \times 13^3T^2 \qquad (6.35)$$
$$+37 \times 13^4T + 3 \times 13^5,$$

$$\Sigma_{10} = 36 \times 13T^5 + 126 \times 13^2T^4 + 102 \times 13^3T^3 \qquad (6.36)$$
$$+38 \times 13^4T^2 + 7 \times 13^5T + 7 \times 13^5,$$

$$\Sigma_{11} = 204 \times 13T^5 + 346 \times 13^2T^4 + 222 \times 13^3T^3 \qquad (6.37)$$
$$+74 \times 13^4T^2 + 13^6T + 13^6$$

$$\Sigma_{12} = 11 \times 13T^6 + 36 \times 13^2T^5 + 38 \times 13^3T^4 \qquad (6.38)$$
$$+20 \times 13^4T^3 + 6 \times 13^5T^2 + 13^6T + 13^6$$

and

$$\Sigma_{13} = \prod_i \zeta_i = \frac{E(q^{13})^{14}}{q^{91}E(q^{169})^{14}} = T^7. \qquad (6.39)$$

It follows that the modular equation is

(6.40)

$$\zeta^{13} - 13\zeta^{12} + (13T + 7 \times 13)\zeta^{11} - (13^2T + 3 \times 13^2)\zeta^{10} + (6 \times 13T^2 + 7 \times 13^2T + 15 \times 13^2)\zeta^9$$
$$-(74 \times 13T^2 + 37 \times 13^2T + 5 \times 13^3)\zeta^8 + (20 \times 13T^3 + 38 \times 13^2T^2 + 13^4T + 19 \times 13^3)\zeta^7$$
$$-(222 \times 13T^3 + 184 \times 13^2T^2 + 51 \times 13^3T + 5 \times 13^4)\zeta^6$$
$$+(38 \times 13T^4 + 102 \times 13^2T^3 + 56 \times 13^3T^2 + 13^5T + 15 \times 13^4)\zeta^5$$
$$-(346 \times 13T^4 + 422 \times 13^2T^3 + 184 \times 13^3T^2 + 37 \times 13^4T + 3 \times 13^5)\zeta^4$$
$$+(36 \times 13T^5 + 126 \times 13^2T^4 + 102 \times 13^3T^3 + 38 \times 13^4T^2 + 7 \times 13^5T + 7 \times 13^5)\zeta^3$$

$$-(204\times 13T^5+346\times 13^2T^4+222\times 13^3T^3+74\times 13^4T^2+13^6T+13^6)\zeta^2$$
$$+(11\times 13T^6+36\times 13^2T^5+38\times 13^3T^4+20\times 13^4T^3+6\times 13^5T^2+13^6T+13^6)\zeta-T^7=0.$$

If, as before, we let $u_i=H(\zeta^i)$ and $U=\sum_{i\geq 0}u_iz^i$, then

(6.41)
$$U=\frac{N}{D}$$

where

(6.42)
$$\begin{aligned}D=&\,1-13z+(13T+7\times 13)z^2-(13^2T+3\times 13^2)z^3+(6\times 13T^2+7\times 13^2T+15\times 13^2)z^4\\&-(74\times 13T^2+37\times 13^2T+5\times 13^3)z^5+(20\times 13T^3+38\times 13^2T^2+13^4T+19\times 13^3)z^6\\&-(222\times 13T^3+184\times 13^2T^2+51\times 13^3T+5\times 13^4)z^7\\&+(38\times 13T^4+102\times 13^2T^3+56\times 13^3T^2+13^5T+15\times 13^4)z^8\\&-(346\times 13T^4+422\times 13^2T^3+184\times 13^3T^2+37\times 13^4T+3\times 13^5)z^9\\&+(36\times 13T^5+126\times 13^2T^4+102\times 13^3T^3+38\times 13^4T^2+7\times 13^5T+7\times 13^5)z^{10}\\&-(204\times 13T^5+346\times 13^2T^4+222\times 13^3T^3+74\times 13^4T^2+13^6T+13^6)z^{11}\\&+(11\times 13T^6+36\times 13^2T^5+38\times 13^3T^4+20\times 13^4T^3+6\times 13^5T^2+13^6T+13^6)z^{12}-T^7z^{13}\end{aligned}$$

and

(6.43)
$$\begin{aligned}N=&\,1-12z+(11T+77)z^2-(10\times 13T+30\times 13)z^3+(54T^2+63\times 13T+135\times 13)z^4\\&-(592T^2+296\times 13T+40\times 13^2)z^5+(140T^3+266\times 13T^2+7\times 13^3T+133\times 13^2)z^6\\&-(1332T^3+1104\times 13T^2+306\times 13^2T+30\times 13^3)z^7\\&+(190T^4+510\times 13T^3+280\times 13^2T^2+5\times 13^4T+75\times 13^3)z^8\\&-(1384T^4+1688\times 13T^3+736\times 13^2T^2+148\times 13^3T+12\times 13^4)z^9\\&+(108T^5+378\times 13T^4+306\times 13^2T^3+114\times 13^3T^2+21\times 13^4T+21\times 13^4)z^{10}\\&-(408T^5+692\times 13T^4+144\times 13^2T^3+148\times 13^3T^2+2\times 13^5T+2\times 13^5)z^{11}\\&+(11T^6+36\times 13T^5+38\times 13^2T^4+20\times 13^3T^3+6\times 13^4T^2+13^5T+13^5)z^{12}.\end{aligned}$$

If we expand U to the power 24, we find that

(6.44)
$$H(\zeta^{24})=u_{24}=-577738T^{12}-13^{11}.$$

We then have

(6.45)
$$\begin{aligned}\sum_{n\geq 1}\tau(n)q^n&=qE(q)^{24}\\&=q^{169}E(q^{169})^{24}\left(\frac{E(q)}{q^7E(q^{169})}\right)^{24}\\&=q^{169}E(q^{169})^{24}\zeta^{24},\end{aligned}$$

(6.46)
$$\begin{aligned}\sum_{n\geq 1}\tau(13n)q^{13n}&=q^{169}E(q^{169})^{24}H(\zeta^{24})\\&=q^{169}E(q^{169})^{24}\left(-577738T^{12}-13^{11}\right)\end{aligned}$$

$$= q^{169}E(q^{169})^{24}\left(-577738\left(\frac{E(q^{13})^2}{q^{13}E(q^{169})^2}\right)^{12} - 13^{11}\right)$$
$$= -577738q^{13}E(q^{13})^{24} - 13^{11}q^{169}E(q^{169})^{24}$$

and

$$\sum_{n\geq 1}\tau(13n)q^n = -577738qE(q)^{24} - 13^{11}q^{13}E(q^{13})^{24} \tag{6.47}$$
$$= -577738\sum_{n\geq 1}\tau(n)q^n - 13^{11}\sum_{n\geq 1}\tau(n)q^{13n}.$$

The term $n = 1$ gives

$$\tau(13) = -577738\tau(1) = -577738,$$

so (6.47) becomes

$$\sum_{n\geq 1}\tau(13n)q^n = \tau(13)\sum_{n\geq 1}\tau(n)q^n - 13^{11}\sum_{n\geq 1}\tau(n)q^{13n}, \tag{6.48}$$

as claimed.

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