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Proof of a Conjecture by Dyson in the Statistical Theory of Energy Levels

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A conjectured identity relating the statistical properties of two types of ensembles occurring in a statistical theory of the distribution of energy levels in nuclei and other complex systems is proved.

In a recent series of papers,[1] Dyson has introduced new types of statistical ensembles in an attempt to provide a mathematically tractable description of statistical properties of energy levels in nuclei and other complex systems. They possess the probability distribution functions

$$P_{N\beta}(\theta_1, \cdots, \theta_N) = C_{N\beta} \prod_{\nu=1}^{N} \prod_{\mu=1}^{\nu-1} |\exp(i\theta_\mu) - \exp(i\theta_\nu)|^\beta, \qquad (1)$$

$$\beta > 0 \quad 0 \leq \theta_\mu \leq 2\pi$$

for N points $\{\exp(i\theta_\mu)\}$ distributed round the unit circle $|z| = 1$ in the complex z plane. Two important conjectures appear in the papers. The first gives the normalising constant $C_{N\beta}$ in Eq. (1) to be

$$C_{N\beta} = (2\pi)^{-N}[\Gamma(1 + \tfrac{1}{2}\beta)]^N/\Gamma(1 + \tfrac{1}{2}N\beta). \qquad (2)$$

This is proved to hold for all integers N and real positive numbers β by Wilson[2] and independently by the author. The second conjecture appears in part III of reference 1 and can be stated as the following theorem.

Theorem. Let $P_{NM}(\theta_1, \cdots, \theta_N)$ be the probability distribution function describing the statistical properties of a set of N points $\{\exp(i\theta_\mu)\}$ constructed as follows: Take two independent sets each consisting of N points distributed according to the probability distribution function of Eq. (1) with $\beta = 1$ (Dyson's "orthogonal" ensemble[1]), superimpose the two sets on the unit circle, and pick out a set of N alternate points. Then

$$P_{NM}(\theta_1, \cdots, \theta_N) = P_{N2}(\theta_1, \cdots, \theta_N), \qquad (3)$$

where P_{N2} is the distribution function of the "unitary" ensemble.[1]

Proof. From Eqs. (1) and (2) we obtain the probability of finding *any* point in the interval $[\alpha_1, \alpha_1 + d\alpha_1]$, *any other* point in the interval $[\alpha_2, \alpha_2 + d\alpha_2]$, etc., where $\alpha_1 < \alpha_2 < \cdots < \alpha_N < \alpha_1 + 2\pi$, to be

$$N!\, C_{N1} \prod_{\mu=1}^{N} \prod_{\nu=1}^{\mu-1} 2 \sin[(\alpha_\mu - \alpha_\nu)/2]\, d\alpha_1\, d\alpha_2 \cdots d\alpha_N. \qquad (4)$$

This is correctly normalised to unity over the domain $0 < \alpha_1 < \alpha_2 < \cdots < \alpha_N < 2\pi$. We can now express $N!\, P_{NM}(\theta_1, \cdots, \theta_N)\, d\theta_1 \cdots d\theta_N$ as a sum of partial probabilities over configurations, a particular configuration being specified by the assignment of the $2N$ angles $\alpha_1, \cdots, \alpha_N$ and $\beta_1, \cdots, \beta_N$ from the orthogonal ensembles A and B into the $2N$ intervals $I_\lambda = [\theta_\lambda, \theta_\lambda + d\theta_\lambda]$ and $G_\lambda = [\theta_\lambda + d\theta_\lambda, \theta_{\lambda+1}]$ $\lambda = 1, 2, \cdots, N$, one in each interval. A typical contribution can be written

$$(C_{N1}N!)^2 \int_{\theta_N}^{\theta_1+2\pi} d\phi_N \cdots \int_{\theta_2}^{\theta_3} d\phi_2 \int_{\theta_1}^{\theta_2} d\phi_1 \prod_{\nu=1}^{N} \prod_{\mu=1}^{\nu-1} \times |4 \sin[(\alpha_\mu - \alpha_\nu)/2] \sin[(\beta_\mu - \beta_\nu)/2]|\; d\theta_1 \cdots d\theta_N \qquad (5)$$

in which, for a given integer m, $0 \leq m \leq N$, we require: m of the angles α are identified with m of the angles ϕ, $N - m$ of the angles β are identified with the remainder of the ϕ's, The remainder of the angles α and β are identified with the angles θ, in a one-to-one manner. For a given m, we then have $\binom{N}{m}^2$ essentially different possible identifications and thus the same number of different contributions to P_{NM}. Each of these terms must be positive, each being an independent contribution to a probability, so we may write Eq. (5) in the form

$$(C_{N1}N!)^2 \int_{\theta_N}^{\theta_1+2\pi} d\phi_N \cdots \int_{\theta_1}^{\theta_2} d\phi_1 \det[e^{ik_\mu \alpha_\nu}] \times \det[e^{ik_\mu \beta_\nu}]\, d\theta_1 \cdots d\theta_N \qquad (6)$$

[1] F. J. Dyson, J. Math. Phys. 3, 140 (1962); 3, 157 (1962); 3, 166 (1962).
[2] K. Wilson, J. Math. Phys. (to be published).

on using the identities

$$\prod_{\mu=1}^{N}\prod_{\nu=1}^{\mu-1}(2\sin[(\alpha_\mu-\alpha_\nu)/2]) = i^{N(N+1)/2}\det(e^{ik_\mu\alpha_\nu})$$

$$k_\mu = \mu-(N-1)/2;\quad \mu,\nu = 1,2,\cdots N. \tag{7}$$

We choose the ordering in the assignment of the angles α and β in such a manner as to make the integrand positive at some point in the region of integration. It is then positive over the whole region of integration and so we are justified in dropping the modulus signs. Let us define normal ordering as that in which the angles α (and likewise the angles β) are identified successively as one of the angles θ or ϕ in increasing order of value. The integral in Eq. (6) may then be either positive or negative and we assert that its sign is given by the expression

$$(-1)^{P_\theta}(-1)^{P_\phi} \tag{8}$$

in which $P_{\theta,\phi}$ is the number of interchanges of the angles θ, ϕ between the two determinants in the integrand required to transform a particular positive term, taken as standard, into the term under consideration. To see this, consider the interchange of the integrations over the interval G of the unit circle in ensemble A with that over G' in ensemble B (Fig. 1).

This interchange produces a new term in which the normal ordering has been destroyed, but it can be restored by interchanging columns within each determinant in the manner suggested by the solid arrows in Fig. 1. The essential point about the latter is that this permutation involves in all an *odd* number of interchanges of columns in the two determinants, as the variable in every interval appears either in the A or B determinants and there are always an odd number of intervals (and hence θ's and ϕ's) between two intervals of type G. The same argument holds, with the necessary changes made, for interchanges of the θ's.

The next step is to sum the integrands of all the $\binom{N}{m}^2$ terms when written in normal form with appropriate sign factors included. To do this, we express the determinants in terms of alternating functions and use the identity

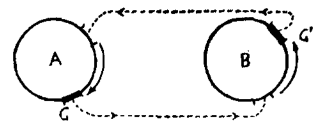

FIG. 1. Interchange of two integrations over the intervals G and G' in a contribution to $P_{NM}(\theta_1, \cdots, \theta_N)$.

$$\sum(-1)^{P_x}(-1)^{P_y}\,\Delta_N(x_1,\cdots,x_r,y_{r+1},\cdots,y_N)\times\Delta_N(y_1,\cdots,y_r,x_{r+1},\cdots,x_N) = \binom{N}{r}\Delta_N(x_1,\cdots,x_N)\,\Delta_N(y_1,\cdots,y_N), \tag{9}$$

in which

$$\Delta_N(z_1,\cdots,z_N) = \prod_{\nu=1}^{N}\prod_{\mu=1}^{\nu-1}(z_\mu-z_\nu) = \det[z_\mu^{N-\nu}]$$

and where the sum $\sum$ is taken over all partitions of the x's and y's between the arguments of the alternating functions Δ. This identity is an immediate consequence of Cauchy's result that any polynomial in N variables which is completely antisymmetric in the variables and contains powers of degree no higher than $(N-1)$ in each variable is a multiple of Δ_N. The sum of all the $\binom{N}{m}^2$ terms then becomes

$$N!\,C_{N1})^2\int_{\theta_N}^{\theta_1+2\pi}d\phi_N\cdots\int_{\theta_1}^{\theta_2}d\phi_1\binom{N}{m}\det[e^{ik_\mu\theta_\nu}]\,\det[e^{ik_\mu\phi_\nu}]\,d\theta_1\,d\theta_2\cdots d\theta_N. \tag{10}$$

The integrals over the ϕ's can now be evaluated to give

$$\begin{aligned} N!\,P_{Nm}(\theta_1,\cdots\theta_N) &= (N!\,C_{N1})^2\sum_{m=0}^{N}\binom{N}{m}2 \times\left(\frac{2^N\Gamma(1+\frac{1}{2}N)}{N!}\right)^2(\det[e^{ik_\mu\theta_\nu}])^2\cdot\tfrac{1}{2} \\ &= (2\pi)^{-N}(\det[e^{ik_\mu\theta_\nu}])^2 \\ &= N!\,P_{N2}(\theta_1,\theta_2,\cdots,\theta_N) \end{aligned} \tag{11}$$

as required. The factor $\frac{1}{2}$ in the second expression is introduced to express the probability that a particular alternate series of N points out of the two possible choices is taken.

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