COMPLETE SEQUENCES OF POLYNOMIAL VALUES

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Introduction. Let $f(x)$ be a polynomial with real coefficients. In 1947, R. Sprague [7] established the result that if $f(x) = x^n$, n an arbitrary positive integer, then every sufficiently large integer can be expressed in the form

$$\sum_{k=1}^{\infty} \epsilon_k f(k) \tag{1}$$

where ϵ_k is 0 or 1 and all but a finite number of the ϵ_k are 0. More recently K. F. Roth and G. Szekeres [5] have shown (using ingenious analytic techniques) that if $f(x)$ is assumed to map integers into integers, then the following conditions are necessary and sufficient in order for every sufficiently large integer to be written as (1):

(a) $f(x)$ has a positive leading coefficient.
(b) For any prime p there exists an integer m such that p does not divide $f(m)$.

It is the object of this paper to determine, in an elementary manner, *all* polynomials $f(x)$ with real coefficients for which every sufficiently large integer can be expressed as (1) (cf. Theorem 4).

Preliminary results. Let $S = (s_1, s_2, \cdots)$ be a sequence of real numbers.

Definition 1. $P(S)$ is defined to be the set of all sums of the form $\sum_{k=1}^{\infty} \epsilon_k s_k$ where ϵ_k is 0 or 1 and all but a finite number of ϵ_k are 0.

Definition 2. S is said to be *complete* if all sufficiently large integers belong to $P(S)$.

Definition 3. S is said to be *nearly complete* if for all integers k, $P(S)$ contains k consecutive positive integers.

Definition 4. S is said to be a Σ-*sequence* if there exist integers k and h such that

$$s_{h+m} < k + \sum_{n=0}^{m-1} s_{h+n}, \qquad m = 0, 1, 2, \cdots.$$

(where a sum of the form $\sum_{n=a}^{b}$ is 0 for $b < a$).

The following lemma is one of the main tools used in this paper:

Lemma 1. *Let* $S = (s_1, s_2, \cdots)$ *be a* Σ-*sequence and let* $T = (t_1, t_2, \cdots)$ *be nearly complete. Then the sequence* $U = (s_1, t_1, s_2, t_2, \cdots)$ *is complete.*

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Proof. Since S is a Σ-sequence then there exist k and h such that

$$s_{h+m} < k + \sum_{n=0}^{m-1} s_{h+n}, \qquad m = 0, 1, 2, \cdots . \tag{2}$$

Also, since T is nearly complete, there exists an integer c such that all the integers

$$c + j, \qquad 1 \leq j \leq k,$$

belong to $P(T)$. But (2) implies that

$$\begin{aligned} c + k &\geq c + s_h \\ c + k + s_h &\geq c + s_{h+1} \\ &\cdots \\ c + k + \sum_{n=0}^{m-1} s_{h+n} &\geq c + s_{h+m} \\ &\cdots . \end{aligned} \tag{3}$$

Thus, since all the integers

$$c + j + s_{h+m}, \qquad 1 \leq j \leq k, \qquad m \geq 0$$

belong to $P(U)$, as well as all the integers

$$c + j, \qquad 1 \leq j \leq k,$$

then by (3), all integers exceeding c belong to $P(U)$.

Hence U is complete and the lemma is proved.

LEMMA 2. *Let $S = (s_1, s_2, \cdots)$ be a sequence of real numbers such that for all sufficiently large n we have $s_{n+1} \leq 2s_n$. Then S is a Σ-sequence.*

Proof. By hypothesis there exists an h such that

$$n \geq h \Rightarrow s_{n+1} \leq 2s_n .$$

Therefore, for any $m \geq 0$ we have

$$\begin{aligned} s_{h+m} &\leq 2s_{h+m-1} = s_{h+m-1} + s_{h+m-1} \\ &\leq s_{h+m-1} + 2s_{h+m-2} \leq \cdots \\ &\leq \sum_{n=0}^{m-1} s_{h+n} + s_h \end{aligned}$$

and consequently S is a Σ-sequence.

LEMMA 3. *Let $S = (s_1, s_2, \cdots)$ be a sequence of integers such that for any prime p, there exist infinitely many s_i in S such that p does not divide s_i. Then for any positive integer m, $P(S)$ contains a complete residue system modulo m.*

Proof. Let m be an arbitrary positive integer. If $m = 1$, then the lemma is immediate. Assume that $m > 1$. Then m can be written as

$$m = q_1^{a_1} q_2^{a_2} \cdots q_n^{a_n}$$

where the q_k are distinct primes and $a_k > 0$ for $1 \leq k \leq n$. For each q_k choose m^4 terms of S, say $t_k(j)$, such that

$$q_k \text{ divides } t_k(j) \quad \text{for} \quad 1 \leq j \leq m^4, \qquad 1 \leq k \leq n,$$

and such that all nm^4 of the integers $t_k(j)$ are distinctly indexed terms of S (by hypothesis, such a choice can be made). For each k, at least m^3 of the $t_k(j)$ are congruent modulo q_k to the same integer, say d_k, where $1 \leq d_k \leq q_k$. Denote the smallest m^3 of these $t_k(j)$ by $t_k'(j)$ for $1 \leq j \leq m^3$, $1 \leq k \leq n$. Now, for each k form the m^2 sums

$$t_k''(j) = \sum_{i=1}^{m(k)} t_k'((j-1)m + i), \qquad 1 \leq j \leq m^2, \qquad 1 \leq k \leq n,$$

where $m(k) = m/q_k^{a_k}$. Note that

$$t_k''(j) \equiv d_k q_1^{a_1} \cdots q_{k-1}^{a_{k-1}} q_{k+1}^{a_{k+1}} \cdots q_n^{a_n} \pmod{q_k}$$

for $1 \leq j \leq m^2$. Finally, let

$$u_j = \sum_{k=1}^{n} t_k''(j), \qquad 1 \leq j \leq m^2.$$

Thus we have $(u_j, m) = 1$. Now at least m of the u_j are congruent modulo m. Denote the smallest m of these by u_j', $1 \leq j \leq m$. Therefore, as r assumes the values $1, 2, \cdots, m$, then the integers $\sum_{j=1}^{r} u_j'$ run through a complete residue system modulo m. Since each of these integers belongs to $P(S)$ then the lemma is proved.

DEFINITION 5. *Let $S = (s_1, s_2, \cdots)$ be a sequence of real numbers. $A(S)$ is defined to be the set of all sums of the form $\sum_{k=1}^{\infty} \delta_k s_k$ where δ_k is -1, 0 or 1 and all but a finite number of the δ_k are 0.*

LEMMA 4. *Let $S = (s_1, s_2, \cdots)$ be a sequence of real numbers. Suppose there exists an integer m such that for all n, we have $m \in A((s_n, s_{n+1}, \cdots))$. Then for all k, $P(S)$ contains an arithmetic progression of k integers with common difference m.*

Proof. The proof will proceed by induction on k. The lemma is true for $k = 1$. Suppose the lemma is true for $k = r \geq 1$, i.e., there exists an integer c such that all the integers

$$c + jm, \qquad 1 \leq j \leq r,$$

belong to $P(S)$. Since each of these integers $c + jm$ is the sum of only finitely many terms of S then there is an h such that none of the terms s_i for $i \geq h$ is used in representing any of the integers

$$c + jm, \qquad 1 \leq j \leq r.$$

But by hypothesis $m \varepsilon A((s_{h+1}, s_{h+2}, \cdots))$. Thus, there exist distinct integers

$$i_1, i_2, \cdots, i_p, j_1, j_2, \cdots, j_q$$

all exceeding h such that

$$m = (s_{i_1} + \cdots + s_{i_p}) - (s_{j_1} + \cdots + s_{j_q}).$$

Let

$$w = s_{j_1} + \cdots + s_{j_q}.$$

Then all the integers

$$c + jm + w, \qquad 1 \leq j \leq r,$$

and

$$c + rm + (s_{i_1} + \cdots + s_{i_p})$$

belong to $P(S)$. But

$$c + rm + (s_{i_1} + \cdots + s_{i_p}) = c + rm + w + m$$
$$= c + (r + 1)m + w.$$

Thus, $P(S)$ contains an arithmetic progression of $r + 1$ integers with common difference m. This completes the induction step and the proof of the lemma.

We need a final lemma before proceeding to the main theorems.

LEMMA 5. *Let $S = (s_1, s_2, \cdots)$ and $T = (t_1, t_2, \cdots)$ be sequences of real numbers and suppose there exists a positive integer m such that:*

(1) *For all n, $P(S)$ contains an arithmetic progression of n integers with common difference m.*
(2) *$P(T)$ contains a complete residue system modulo m.*

Then the sequence $U = (s_1, t_1, s_2, t_2, \cdots)$ is nearly complete.

Proof. By hypothesis, $P(T)$ contains a complete residue system modulo m, say

$$k_1 < k_2 < \cdots < k_m.$$

Let r be an arbitrary positive integer and suppose that n is chosen greater than $r + k_m$. By hypothesis, there is an integer c such that all the integers

$$c + jm, \qquad 1 \leq j \leq n,$$

belong to $P(S)$. Now, note that if we let

$$n_j = \left[\frac{k_m - k_j}{m}\right] + 1, \qquad 1 \leq j \leq m,$$

(where [ ] is the greatest integer function), then

$$1 \leq n_j \leq k_m$$

and

$$c + k_m < c + n_j m + k_j \leq c + m + k_m.$$

Since no two of the $c + n_j m + k_j$ are congruent modulo m, then the set of integers $\{c + n_j m + k_j : 1 \leq j \leq m\}$ is exactly the set $\{c + k_m + j : 1 \leq j \leq m\}$. Since $p \leq r - 1$ implies that

$$n_j + p < n_j + r \leq k_m + r < n,$$

then in the expression $c + n_j m + k_j$, we can replace n_j by $n_j + p$ for $1 \leq p \leq r - 1$ and conclude that all the integers

$$c + k_m + pm + j, \quad \text{for} \quad 1 \leq j \leq m, \qquad 1 \leq p \leq r - 1,$$

belong to $P(U)$. Therefore, all the integers

$$c + k_m + j, \qquad 1 \leq j \leq rm,$$

belong to $P(U)$. Since r was arbitrary, then U is nearly complete and the lemma is proved.

The main theorems. Let $f(x)$ be a polynomial with real coefficients and let $S(f)$ denote the sequence $(f(1), f(2), f(3), \cdots)$. In this section we shall characterize those f for which $S(f)$ is complete. We first consider those $f(x)$ which map integers into integers.

THEOREM 1. *Let*

$$f(x) = \alpha_n x^n + \cdots + \alpha_1 x + \alpha_0, \qquad \alpha_n \neq 0$$

be a polynomial which maps integers into integers. (Thus all the α_k are rational numbers.) Then $S(f)$ is complete if and only if:

(1) $\alpha_n > 0$.
(2) *For any prime p, there exists an integer m such that p does not divide $f(m)$.*

Proof. The necessity of Conditions (1) and (2) is immediate. We proceed with sufficiency. Let $g(x)$ be any polynomial which maps integers into integers. Define Δ_k (mapping polynomials into polynomials) by:

$$\Delta_1(g(x)) = g(4x + 2) - g(4x),$$
$$\Delta_k(g(x)) = \Delta_1(\Delta_{k-1}(g(x))), \qquad 2 \leq k \leq n.$$

Note that

$$\begin{aligned} \Delta_2(f(x)) &= \Delta_1(f(4x + 2) - f(4x)) \\ &= \Delta_1(f(4x + 2) - \Delta_1(f(4x)) \\ &= f(16x + 10) - f(16x + 8) - f(16x + 2) + f(16x), \cdots \text{etc.} \end{aligned}$$

Thus, for all positive integers m,

$$\Delta_k(f(m)) \in A(S(f)), \qquad 1 \leq k \leq n.$$

It follows from the definition of Δ_k that for $1 \leq k \leq n$, $\Delta_k(f(x))$ is a polynomial of degree $n - k$ which maps integers into integers and which has a positive leading coefficient. For,

$$\begin{aligned}\Delta_1(f(x)) &= f(4x + 2) - f(4x)\\ &= (\alpha_n(4x + 2)^n + \alpha_{n-1}(4x + 2)^{n-1} + \cdots) - (\alpha_n(4x)^n + \alpha_{n-1}(4x)^{n-1} + \cdots)\\ &= (4^n\alpha_n x^n + n\cdot 2^{2n-1}\alpha_n x^{n-1} + \cdots + 4^{n-1}\alpha_{n-1}x^{n-1} + \cdots)\\ &\qquad\qquad - (4^n\alpha_n x^n + 4^{n-1}\alpha_{n-1}x^{n-1} + \cdots)\\ &= n\cdot 2^{2n-1}\alpha_n x^{n-1} + \text{terms of lower degree}\end{aligned}$$

(which certainly maps integers into integers and has a positive leading coefficient) and

$$\Delta_k(f(x)) = \Delta_1(\Delta_{k-1}(f(x))), \qquad 2 \leq k \leq n.$$

Therefore $\Delta_n(f(x))$ is a polynomial of degree 0 which maps integers into integers and has a positive leading coefficient, i.e., $\Delta_n(f(x))$ is just a positive integer which we shall denote by m. Note that m is independent of x. Now, by hypothesis, for any prime p, there exists an h such that p does not divide $f(m)$. But

$$f(h) \equiv f(h + k\,dp) \pmod p$$

where d is the product of all the denominators of the α_i and k is an arbitrary integer. For,

$$\begin{aligned}\alpha_j(h + k\,dp)^j &= \alpha_j h^j + d\alpha_j p(jkh^{j-1} + \cdots)\\ &\equiv \alpha_j h^j \pmod p\end{aligned}$$

since $d\alpha_j$ is an integer. Thus there are infinitely many integers t such that p does not divide $f(t)$. Hence, by Lemma 3, $P(S(f))$ contains a complete residue system modulo m. Of course, we need only a finite number of terms of $S(f)$ to obtain the complete residue system, so that there exists some integer r such that if we denote the sequence $(f(1), f(2), \cdots, f(r))$ by S, then $P(S)$ contains a complete residue system modulo m. Let T denote the sequence

$$(f(2r), f(2r + 2), f(2r + 4), \cdots).$$

Since $m = \Delta_n(f(x))$ uses only terms of $S(f)$ of the form $f(2t)$ and is independent of x, then by Lemma 4, for all k, $P(T)$ contains an arithmetic progression of k integers with common difference m. Thus, by Lemma 5, the sequence

$$U = (f(1), f(2), \cdots, f(r), f(2r), f(2r + 2), f(2r + 4), \cdots)$$

is nearly complete. But the sequence

$$W = (f(2r + 1), f(2r + 3), f(2r + 5), \cdots)$$

has

$$\lim_{k\to\infty} \frac{f(2r + 2k + 1)}{f(2r + 2k - 1)} = 1$$

so that for all sufficiently large k we have

$$f(2r + 2k + 1) \leq 2f(2r + 2k - 1).$$

Hence, by Lemma 2, W is a Σ-sequence. Therefore, by applying Lemma 1, we see that the sequence formed by combining U and W, namely

$$S(f) = (f(1), f(2), f(3), \cdots),$$

is complete. This proves the theorem.

We now consider polynomials $f(x)$ which have rational coefficients but are not restricted to map integers into integers. It is well known (cf. [1]) that any polynomial $f(x)$ of degree n which has rational coefficients can be uniquely expressed in the form

$$f(x) = \alpha_0 + \alpha_1 \begin{bmatrix} x \\ 1 \end{bmatrix} + \cdots + \alpha_n \begin{bmatrix} x \\ n \end{bmatrix}$$

where the α_k are rational, $\alpha_n \neq 0$ and $\binom{x}{k}$ denotes the expression

$$\frac{x(x-1) \cdots (x-k+1)}{n!}, \qquad 0 \leq k \leq n.$$

THEOREM 2. *Let*

$$f(x) = \frac{p_0}{q_0} + \frac{p_1}{q_1}\binom{x}{1} + \cdots + \frac{p_n}{q_n}\binom{x}{n}$$

where the p_k *and* q_k *are integers such that*

$$(p_k, q_k) = 1, \qquad p_n \neq 0 \quad \text{and} \quad q_k \neq 0, \qquad 0 \leq k \leq n.$$

Then $S(f)$ *is complete if and only if*:

(1) $\frac{p_n}{q_n} > 0$.

(2) $\text{g.c.d.}\,(p_0, p_1, \cdots, p_n) = 1$.

Proof. Suppose $S(f)$ is complete. Condition (1) is immediate. To show Condition (2), suppose that

$$\text{g.c.d.}\,(p_0, p_1, \cdots, p_n) = a > 1.$$

Let $q = \text{l.c.m.}\,(q_0, q_1, \cdots, q_n)$. Then

$$h(x) = \frac{q}{a} \cdot f(x)$$

has integer coefficients. Now we must have $(q, a) = 1$. For if $(q, a) = c > 1$, then there exists a prime p such that $p \mid c$. Thus $p \mid q$ and $p \mid a$. Hence, there exists an i such that $p \mid q_i$. Since $p \mid a$ then $p \mid p_i$. Therefore $p \mid (p_i, q_i)$, which is impossible, since $(p_i, q_i) = 1$. Thus, we must have $(q, a) = 1$. Consequently every term in $S(f)$ is of the form ak/q for some integer k. Hence, every integer in $P(S(f))$ is a multiple of $a > 1$, which is a contradiction to the hypothesis that $S(f)$ is complete. This establishes the necessity of (1) and (2).

We now show that (1) and (2) are sufficient. Suppose Conditions (1) and (2) hold. Then $q = \text{l.c.m.}\ (q_0, q_1, \cdots, q_n)$ is the smallest positive integer such that qp_j/q_j is an integer for $0 \leq j \leq n$. Now we must have

$$d = \text{g.c.d.}\left(\frac{qp_0}{q_0}, \cdots, \frac{qp_n}{q_n}\right) = 1.$$

For, suppose $d > 1$ and let d' be a prime factor of d. Then

$$d' \left| \frac{qp_j}{q_j}, \qquad 0 \leq j \leq n. \right.$$

Thus, for each j, either

$$d' \left| \frac{q}{q_j} \quad \text{or} \quad d' \mid p_j . \right.$$

But g.c.d. $(p_0, p_1, \cdots, p_n) = 1$ by hypothesis. Thus for some i we must have $d' \mid q/q_i$. Therefore $d' \mid q$ and consequently $q' = q/d'$ is a positive integer less than q which has the property that $q'p_j/q_j$ is an integer for $0 \leq j \leq n$. This is impossible since q is the smallest positive integer which has this property. Hence, if we let r_j denote qp_j/q_j for $0 \leq j \leq n$, then we have g.c.d. $(r_0, r_1, \cdots, r_n) = 1$. Now let

$$h(x) = qf(x) = r_0 + r_1\binom{x}{1} + \cdots + r_n\binom{x}{n}.$$

Suppose there exists a prime t such that t divides $h(m)$ for all m. Then

$$\begin{aligned}
&t \text{ divides } h(0) = r_0 ,\\
&t \text{ divides } h(1) = r_0 + r_1 ,\\
&t \text{ divides } h(2) = r_0 + 2r_1 + r_2 ,\\
&\qquad\cdots\\
&t \text{ divides } h(n) = r_0 + \binom{n}{1}r_1 + \binom{n}{2}r_2 + \cdots + \binom{n}{1}r_{n-1} + r_n .
\end{aligned}$$

Thus, t divides g.c.d. $(r_0, r_1, \cdots, r_n) = 1$, which is impossible. Therefore, for any prime t, there is an m such that t does not divide $h(m)$. Hence, by Theorem 1, $S(h) = (h(1), h(2), \cdots)$ is complete and consequently $P(S(h))$ contains all sufficiently large multiples of q. Since

$$f(x) = \frac{1}{q}\cdot h(x),$$

then the sequence $S(f) = (f(1), f(2), \cdots)$ is complete. This proves the theorem.

Finally, if not all the coefficients of $f(x)$ are rational, then we have

THEOREM 3. *Let*

$$f(x) = \alpha_n x^n + \cdots + \alpha_1 x + \alpha_0 , \qquad \alpha_n \neq 0,$$

and suppose that at least one α_k is irrational. Then $S(f)$ is not complete.

Proof. Let A denote the vector space over the rational numbers generated by the set $\{1, \alpha_0 , \alpha_1 , \cdots , \alpha_n\}$. Since not all the α_k are rational, we have

$$2 \leq \dim A \leq n + 2.$$

The set $\{1\}$ is linearly independent over the rational numbers so that we can extend $\{1\}$ to a basis $\{\beta_1 , \beta_2 , \cdots , \beta_t\}$ of A where $\beta_1 = 1$ and $2 \leq t \leq n + 2$ (cf. [3]). Thus, we have

$$\alpha_k = \sum_{i=1}^{t} r(k, i)\beta_1 , \qquad 0 \leq k \leq n,$$

where the $r(k, i)$ are rational. Therefore,

$$f(x) = \sum_{k=0}^{n} \alpha_k x^k = \sum_{k=0}^{n} \sum_{i=1}^{t} r(k, i)\beta_i x^k \tag{4}$$

$$= \sum_{i=1}^{t} \beta_i \sum_{k=0}^{n} r(k, i)x^k$$

$$= \sum_{i=1}^{t} \beta_i g_i(x)$$

where

$$g_i(x) = \sum_{k=0}^{n} r(k, i)x^k, \qquad 1 \leq i \leq t.$$

Now, suppose r is a rational number which belongs to $P(S(f))$. Then there exists a set $\{x_1 , \cdots , x_m\}$ of distinct positive integers such that

$$r = \sum_{j=1}^{m} f(x_j).$$

Thus, we have by (4),

$$r = \sum_{j=1}^{m} \sum_{i=1}^{t} \beta_i g_i(x_j)$$

$$= \sum_{i=1}^{t} \beta_i \sum_{j=1}^{m} g_i(x_j).$$

Since the β_i are linearly independent over the rationals, we have

$$r = \sum_{j=1}^{m} g_1(x_j),$$

$$0 = \sum_{j=1}^{m} g_i(x_j), \qquad 2 \leq i \leq t.$$

By hypothesis, there must be at least one h, $2 \leq h \leq t$, such that $g_h(x)$ is not identically zero. Hence, for each rational $r \varepsilon P(S(f))$, there exists a set $\{x_1, \cdots, x_m\}$ of distinct positive integers such that

$$0 = \sum_{i=1}^{m} g_h(x_i). \tag{5}$$

But this implies that there can be only finitely many rational numbers in $P(S(f))$. For suppose that there are infinitely many finite sets of distinct positive integers $\{x_1 \cdots x_m\}$ such that $\sum_{i=1}^{m} f(x_i)$ is rational. Suppose further that the leading coefficient of $g_h(x)$ is positive. (A similar argument can be applied if it is negative.) Then there are only finitely many positive integers y, say $y_1, \cdots, y_u$, for which $g_h(y) < 0$. Also, there exists an N so that $x > N$ implies that

$$g_h(x) > -\sum_{i=1}^{u} g_h(y_i). \tag{6}$$

Since we have assumed that there are infinitely many sets $\{x_1, \cdots, x_m\}$ for which $\sum_{i=1}^{m} f(x_i)$ is rational, then one of these sets, say $\{x_1', \cdots, x_{m'}\}$ must contain an integer $x_d' > N$. Thus by (5) and (6),

$$\begin{aligned} 0 &= \sum_{i=1}^{m'} g_h(x_i') \\ &= g_h(x_d') + \sum_{\substack{i=1 \\ i \neq d}}^{m'} g_h(x_i') \\ &\geq g_h(x_d') + \sum_{i=1}^{u} g_h(y_i) > 0, \end{aligned}$$

which is impossible. Thus, there can only be finitely many rational numbers in $P(S(f))$ and consequently $S(f)$ cannot be complete. This proves the theorem.

We can combine Theorems 2 and 3 to obtain the main result of the paper:

THEOREM 4. *Let $f(x)$ be a polynomial with real coefficients expressed in the form*

$$f(x) = \alpha_0 + \alpha_1\binom{x}{1} + \cdots + \alpha_n\binom{x}{n}, \qquad \alpha_n \neq 0.$$

Then the sequence

$$S(f) = (f(1), f(2), \cdots)$$

is complete if and only if:

(1) $\alpha_k = p_k/q_k$ *for some integers* p_k *and* q_k *with* $(p_k, q_k) = 1$ *and* $q_k \neq 0$ *for* $0 \leq k \leq n$.
(2) $\alpha_n > 0$.
(3) g.c.d. $(p_0, p_1, \cdots, p_n) = 1$.

Concluding remarks. It follows at once that the sequence of polynomial values $(f(1), f(2), f(3), \cdots)$ is complete if and only if for any n the sequence

$(f(n), f(n+1), f(n+2), \cdots)$ is complete. It might be noted that even for the simplest polynomials f, the exact determination of the largest integer $\lambda(f)$ which does not belong to $P(S(f))$ is not easy. While an upper bound for $\lambda(f)$ can be obtained from the proofs of the preceding theorems, it is too crude to be of much use. It is known that:

$$\lambda\left(\frac{x^2+x}{2}\right) = 33 \quad [4],$$

$$\lambda(x^2) = 128 \quad [6],$$

$$\lambda(x^3) = 12758 \quad [2],$$

$$\lambda(x^4) > 2400000 \quad [2],$$

$$\lambda(ax - a + 1) = \frac{a^2(a-1)}{2} \quad [2],$$

where a is an arbitrary positive integer.

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