RESEARCH ARTICLE | JUNE 01 1970

Short Proof of a Conjecture by Dyson

I. J. Good

J. Math. Phys. 11, 1884 (1970)
https://doi.org/10.1063/1.1665339

[17] O. Penrose and L. Onsager, Phys. Rev. **104**, 576 (1956).
[18] C. N. Yang, Rev. Mod. Phys. **34**, 694 (1962).
[19] As has been pointed out to the author, it might be desirable to weaken the above assumption (ii) to a form that does not imply the existence of the automorphisms $\{\tau_t\}$. Thus, it may be of interest to note that our subsequent theory of Gibbs states will still hold if we replace assumption (ii) by the following weaker postulate: $\exists$ a 1-parameter group $\{V_\phi(t)\}$ of unitary transformations of $\mathfrak{H}_\phi$, such that

$$V_\phi(t)\Omega_\phi = \Omega_\phi, \quad \forall t \in T,$$

and

$$\phi^{(n)}(B\tau_t(n)A) \to (R_\phi(B^*)\Omega_\phi, V_\phi(t)R_\phi(A)\Omega_\phi) \quad \text{as} \quad n \to \infty, \quad \forall A, B \in \mathfrak{U}_L, \quad t \in T.$$

In this case, the subsequent theory can be carried through provided that, for $K \in \mathfrak{U} \cup \mathfrak{Q}_L$, one always replaces $R_\phi(\tau_t K)$ by

$$(R_\phi(K))_t \equiv V_\phi(t)R_\phi(K)V_\phi(t)^{-1}.$$

[20] The term "Gibbs state" is usually reserved for a state ϕ, constructed according to the above procedure, in cases where the $\rho^{(n)}$ are grand canonical density matrices corresponding to the regions Λ_n.
[21] Cf. D. Kastler, D. W. Robinson, and A. Swieca, Commun. Math. Phys. **2**, 108 (1966) and A. Swieca, Commun. Math. Phys. **4**, 1 (1967). In these papers, Goldstone's theorem was derived within the framework of local field theory, on the basis of an *a priori* assumption that the relevant local conservation laws could be represented on a domain of the GNS space that includes the cyclical vector.

JOURNAL OF MATHEMATICAL PHYSICS VOLUME 11, NUMBER 6 JUNE 1970

Short Proof of a Conjecture by Dyson

I. J. Good
Department of Statistics, Virginia Polytechnic Institute, Blacksburg, Virginia

(Received 26 December 1969)

Dyson made a mathematical conjecture in his work on the distribution of energy levels in complex systems. A proof is given, which is much shorter than two that have been published before.

Let $G(\mathbf{a})$ denote the constant term in the expansion of

$$F(\mathbf{x}; \mathbf{a}) = \prod_{i \neq j} \left(1 - \frac{x_j}{x_i}\right)^{a_j}, \quad i, j = 1, 2, \cdots, n,$$

where $a_1, a_2, \cdots, a_n$ are nonnegative integers and where $F(\mathbf{x}; \mathbf{a})$ is expanded in positive and negative powers of $x_1, x_2, \cdots, x_n$. Dyson[1] conjectured that $G(\mathbf{a}) = M(\mathbf{a})$, where $M(\mathbf{a})$ is the multinomial coefficient $(a_1 + \cdots + a_n)!/(a_1! \cdots a_n!)$. This was proved by Gunson[2] and by Wilson.[3] A much shorter proof is given here.

By applying Lagrange's interpolation formula (see, for example, Kopal[4]) to the function of x that is identically equal to 1 and then putting $x = 0$, we see that

$$\sum_j \prod_i \left(1 - \frac{x_j}{x_i}\right)^{-1} = 1, i = j.$$

By multiplying $F(\mathbf{x}; \mathbf{a})$ by this function we see that, if $a_j \neq 0, j = 1, \cdots, n$, then

$$F(\mathbf{x}; \mathbf{a}) = \sum_j F(\mathbf{x}; a_1, a_2, \cdots, a_{j-1}, a_j - 1, a_{j+1}, \cdots, a_n),$$

so that

$$G(\mathbf{a}) = \sum_j G(a_1, \cdots, a_{j-1}, a_j - 1, a_{j+1}, \cdots, a_n). \tag{1}$$

If $a_j = 0$, then x_j occurs only to negative powers in $F(\mathbf{x}; \mathbf{a})$ so that $G(\mathbf{a})$ is then equal to the constant term in

$$F(x_1, \cdots, x_{j-1}, x_{j+1}, \cdots, x_n; a_1, \cdots, a_{j-1}, a_{j+1}, \cdots, a_n),$$

that is,

$$G(\mathbf{a}) = G(a_1, \cdots, a_{j-1}, a_{j+1}, \cdots, a_n), \quad \text{if } a_j = 0. \tag{2}$$

Also, of course,

$$G(\mathbf{0}) = 1. \tag{3}$$

Equations (1)–(3) clearly uniquely define $G(\mathbf{a})$ recursively. Moreover, they are satisfied by putting $G(\mathbf{a}) = M(\mathbf{a})$. Therefore $G(\mathbf{a}) = M(\mathbf{a})$, as conjectured by Dyson.

[1] F. J. Dyson, J. Math. Phys. **3**, 140, 157, 166 (1962).
[2] J. Gunson, J. Math. Phys. **3**, 752 (1962).
[3] K. G. Wilson, J. Math. Phys. **3**, 1040 (1962).
[4] Z. Kopal, *Numerical Analysis* (Chapman and Hall, London, 1955), p. 21.