SEPARATION OF VARIABLES FOR QUANTIZED ENVELOPING ALGEBRAS

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1. Introduction. The base field k is assumed of characteristic zero, with $K = k(q)$. The notation is that of [11] but will be redefined where necessary.

1.1. Let $\mathfrak{g}$ be a semisimple Lie algebra and $U(\mathfrak{g})$ its enveloping algebra. A famous theorem of Kostant ([13]; [2], 8.2.4) asserts that $U(\mathfrak{g})$ is a free module over its centre $Z(\mathfrak{g})$. More precisely one can choose an *ad* $U(\mathfrak{g})$ invariant subspace $\mathbb{H}$ of $U(\mathfrak{g})$ such that the multiplication map $\mathbb{H} \otimes Z(\mathfrak{g}) \longrightarrow U(\mathfrak{g})$ is an isomorphism. Moreover $\mathbb{H}$ is a direct sum of simple finite dimensional $U(\mathfrak{g})$ modules, where the multiplicity of any isomorphism class E is just the dimension of the zero weight of E with respect to a Cartan subalgebra $\mathfrak{h}$ of $\mathfrak{g}$. Finally Kostant showed that one can make a particularly nice choice for $\mathbb{H}$, namely $(ad\ U(\mathfrak{g}))\ U(\mathfrak{n}^+)$ where $\mathfrak{g} = \mathfrak{n}^+ \oplus \mathfrak{h} \oplus \mathfrak{n}^-$ is a triangular decomposition for $\mathfrak{g}$.

1.2. Our present aim is to obtain the analogue of Kostant's theorem for the quantum group $U_q(\mathfrak{g})$ based on $\mathfrak{g}$ as defined by V.G. Drinfeld and M. Jimbo. We prefer to call this the quantized enveloping algebra for $\mathfrak{g}$ and we shall use the notation and conventions of [11], hereafter referred to as JL. Following Kostant we call the required tensor product decomposition, a separation of variables.

1.3. One cannot expect to deduce a separation of variables for $U_q(\mathfrak{g})$ by any limiting process $q \longrightarrow 1$, because this is unlikely to pin down the required free generators. Attempts to mimic Kostant's proof also failed. Thus setting $U = U_q(\mathfrak{g})$ Rosso obtained a triangular decomposition $U = U^- \otimes U^0 \otimes U^+$ see (JL, 4.8) but neither $(ad\ U)U^+$ or any obvious modification of it seems to give a correct choice of free generators. Again although Rosso provided ([20], Sect. II) a nondegenerate *ad* U invariant bilinear form (the Rosso form) on U, this does not specify a choice of free generators in any obvious way even in the case $\mathfrak{g} = \mathfrak{sl}(2)$. From all this it becomes clear that we need a completely new idea.

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1.4. Fix $Z_\chi \in Max\ Z(\mathfrak{g})$ and set $I_\chi = U(\mathfrak{g})Z_\chi$. Let $\mathcal{F}$ denote the canonical filtration ([2], 2.3.1) of $U(\mathfrak{g})$. Now let $S(\mathfrak{g})$ denote the symmetric algebra of $\mathfrak{g}$ which we identify with $gr_{\mathcal{F}}U(\mathfrak{g})$. Let $Y(\mathfrak{g})$ denote the *ad* $\mathfrak{g}$ invariant elements of $S(\mathfrak{g})$ and Y_+ the augmentation ideal of $S(\mathfrak{g})$. Set $J_+ = S(\mathfrak{g})Y_+$. The crucial and most difficult step in Kostant's proof of his separation theorem for $U(\mathfrak{g})$ was to establish the primeness of J_+. This approach is definitely excluded here. However, notice that $gr_{\mathcal{F}}I_\chi \supset J_+$. Then an easy argument based on dimension theory and the primeness of J_+ gives equality and hence the independence of $gr_{\mathcal{F}}I_\chi$ on χ. This independence leads quickly to the tensor product decomposition of $U(\mathfrak{g})$ over $Z(\mathfrak{g})$ by simply choosing $\mathbb{H}$ to be the symmetrization of a graded complement H to J_+. Finally H can be chosen to be *ad* $U(\mathfrak{g})$ stable; but instead of calculating its module structure from algebraic geometry we may calculate directly the module structure of $\mathbb{H}$ from the representation theory of $U(\mathfrak{g})$.

1.5. The thrust of this paper is that one can prove independence of χ in another fashion. An easy argument based on the density of the set of semisimple orbits and using the above form shows that $S(\mathfrak{g}) = (ad\ U(\mathfrak{g}))S(\mathfrak{h})$. In particular we can identify $U(\mathfrak{g})$ with $(ad\ U(\mathfrak{g}))U(\mathfrak{h})$. On the other hand it is easy to show that the multiplication map $S(\mathfrak{h}) \otimes Y(\mathfrak{g}) \longrightarrow S(\mathfrak{h})Y(\mathfrak{g})$ is bijective and so we obtain $gr(S(\mathfrak{h})Y_\chi) = S(\mathfrak{h})Y_+$ for all $Y_\chi \in Max\ Y$, which combined with the previous equality gives the required independence, if we can show that *gr* commutes with *ad* $U(\mathfrak{g})$. This is a very delicate point. In the enveloping algebra case it holds by Kostant's theorem. It may fail in the quantum case, and to get it to hold, we must augment the algebra to obtain a so called simply connected version $\check{U}$. Then we resolve it in this case by an entirely new reasoning based on distributativity laws (6.1, 6.4, 6.8) for sum and intersection of certain vector spaces which are only defined in the augmented algebra. This uses in particular 4.10 which does not have an immediate analogue in the enveloping algebra case and recent deep work of Lusztig [15-17]. This avoids the argument involving the primeness of J_+. However it does have the disadvantage that one cannot assert that I_χ is completely prime and hence (by easy Gelfand-Kirillov dimension arguments) equal to the annihilator of a Verma module. Possibly the above commutativity can be similarly established in the enveloping algebra case. However, in this respect the quantum case appears easier and more natural.

1.6. Our analysis in the quantum case is based on the existence of an *ad* U invariant filtration $\mathcal{F}$ of U. At first sight this does not seem too useful as the subspaces $\mathcal{F}^iU$ are infinite dimensional for all $i \in \mathbb{Z}$. However recall that a main result of *JL* was to exhibit the subalgebra $F(U)$ on U on which the action of *ad* U is locally finite. A beautiful fact (4.3, 4.4) is that $\mathcal{F}^i(F(U)) := \mathcal{F}^i(U) \cap F(U)$ is finite dimensional for all $i \in \mathbb{Z}$ and nonzero only if $i \geq 0$. From this we are able to give a presentation of $F(U)$ analogous to that for $U(\mathfrak{g})$ described in 1.5, although the proof is much harder and gives the quite unexpected decomposition

(4.10) of $F(U)$ referred to above. Then if $Z(\check{U})$ (or simply, Z) denotes the centre of $F(\check{U})$ one concludes (Sect. 7) as indicated in 1.5 that $F(\check{U}) = \mathbb{H} \otimes Z$ for an appropriate *ad* U invariant subspace $\mathbb{H}$ of $F(\check{U})$. Moreover $\mathbb{H}$ is a direct sum of simple finite dimensional U modules with the same multiplicities as in the classical (enveloping algebra) case. From (JL, 6.2) one can recover that $\check{U}$ itself is free over Z but this is less interesting. This approach does not immediately yield that the $I_\chi := F(U)Z_\chi : Z_\chi \in Max\ Z$ are completely prime. (Here we make a natural technical restriction and only consider those Z_χ that annihilate Verma modules with integral weights; see also Section 8.) However one can recover this (8.1 and 8.6) from the classical fact by using a $q \to 1$ argument devel̀oped in 6.10–6.17. This also states that I_χ is a Verma module annihilator and so we can present $F(U)/I_\chi$ as the space of endormorphisms of a Verma module which are locally finite for the diagonal action of U. This is the crucial step to establishing an analogue of Duflo's theorem for *Prim* $F(U)$. We also establish the rather surprising fact (6.17) that $Z(U)$ specializes to $Z(\mathfrak{g})$. In 6.18 we even find that the quantum point of view leads to a proof of the hard part of ([6], 4.12) characterizing $Z(\mathfrak{n}^-)$.

1.7. Though we could have avoided this, it turns out that the Rosso form also plays a significant role in the interpretation of $F(U)$ and its subsequent decomposition. An important question which remains (5.4) is to compute the multiplicity of a given simple module in each gradation level. This would give the analogues of Kostant's generalized exponents which were first determined by Hesselink [4] for the enveloping algebra.

The decomposition theorem 4.10 was reported at a seminar in the Weizmann Institute during September 1990. We would like to thank V. Hinich and S.P. Smith for useful discussions and Miriam Abraham for typing a lengthy paper under trying circumstances.

2. An *ad*-Invariant Filtration of $U_q(\mathfrak{g})$.

2.1. Fix a semisimple Lie algebra $\mathfrak{g}$ of rank ℓ and let $x_i, y_i, t_i, t_i^{-1} : i = 1, 2, \dots, \ell$ denote the canonical generators of $U := U_q(\mathfrak{g})$. We recall that the t_i commute and

$$t_j x_i t_j^{-1} = q^{(\alpha_i,\alpha_j)} x_i, \qquad t_j y_i t_j^{-1} = q^{-(\alpha_i,\alpha_j)} y_i$$

$$x_i y_{i'} - y_{i'} x_i = \begin{cases} 0 & : \quad i \neq i' \\ \frac{t_i^2 - t_i^{-2}}{q^{2d_i} - q^{-2d_i}} & : \quad \text{otherwise,} \end{cases}$$

where (,) denotes the Cartan inner product on $\mathfrak{h}^*$ and $2d_i = (\alpha_i, \alpha_i)$. The

$x_i : i = 1, 2, \ldots, \ell$ (resp. $y_i : i = 1, 2, \ldots, \ell$) further satisfy an analogue of the Serre relations. Also U can be given a Hopf algebra structure, so that $ad\ a : a \in U$ is defined (as an element of $End_{k(q)}U$). For further details we refer the reader to JL.

2.2. We define a filtration $\{\mathcal{F}^m U\}_{m\in\mathbb{Z}}$ on U which in some obvious sense is given by taking $x_i, y_i, t_i^{-1} : i = 1, 2, \ldots, \ell$ to have degree 1. To do this carefully let $\tilde{U}^+$ (resp. $\tilde{U}^-$) denote the free algebra (over K) generated by the x_i (resp. y_i) and U^0 the Laurent polynomial ring (over K) generated by the t_i, t_i^{-1}. Let U^+ (resp. U^-) denote the image of $\tilde{U}^+$ (resp. $\tilde{U}^-$) obtained by factoring out the corresponding Serre relations which we recall are homogeneous. Now $\tilde{U}^+$ (resp. $\tilde{U}^-$) is filtered (even graded) by degree and the filtration (gradation) passes to U^+ (resp. U^-). Again U^0 is filtered (even graded) by (the negative of) degree. Finally recall (JL, 4.8) that we have a triangular decomposition defined by the isomorphism $U^- \otimes U^0 \otimes U^+ \xrightarrow{\sim} U$ and given by the multiplication map. Hence each $u \in U$ can be uniquely written as a sum of terms of form $u^- u^0 u^+$ and we define u to belong to $\mathcal{F}^m U$ if $m \leq deg\ u^- - deg\ u^0 + deg\ u^+$ for each such term.

Call $u \in U$ to be of degree $m \in \mathbb{Z}$ if $u \in \mathcal{F}^m U \setminus \mathcal{F}^{m-1} U$. From the above description and the relations in 2.1 we can calculate the degree of any monomial (not necessarily ordered) in the x_i, y_i, t_i, t_i^{-1}. Its degree turns out to be the sum of the exponents of the x_i, y_i, t_i^{-1}, which justifies our original intuitive definition of $\mathcal{F}^m U$. Again if $a, b \in U$ are monomials, then so is ab and $deg\ ab = deg\ a + deg\ b$. This last relation extends to all $a, b \in U$ if we take $deg\ 0 = -\infty$. We conclude that $gr_{\mathcal{F}} U$ is an integral domain (and hence so is U; but this was already known (JL, 4.10 (iv))).

Unless an ambiguity arises we shall also use x_i, y_i, t_i to denote their images in $gr_{\mathcal{F}} U$. It is clear that the relations in 2.1 still hold in $gr_{\mathcal{F}} U$ except in the last of these t_i^2 must be dropped from the right hand side. It is also clear that these and the Serre relations generate all the relations in $gr_{\mathcal{F}} U$. Indeed since the filtration respects the weight space decomposition of U^+, U^-, U^0 individually, the multiplication map continues to give an isomorphism of $U^- \otimes U^0 \otimes U^+$ onto $gr_{\mathcal{F}} U$. This presentation of $gr_{\mathcal{F}} U$ makes the above assertion quite obvious.

2.3. Taking account of the Hopf algebra structure of U one finds that

$$\begin{aligned} (ad\ x_i)a &= x_i a t_i - q^{-2d_i} t_i a x_i \\ (ad\ y_i)a &= y_i a t_i - q^{2d_i} t_i a y_i \\ (ad\ t_i)a &= t_i a t_i^{-1} \end{aligned}$$

for all $a \in U$. We conclude that $\mathcal{F}^m U$ is $ad\ U$ stable. Set $F(U) = \{a \in U \mid dim(ad\ U)a < \infty\}$. One checks (JL, 2.3) that $F(U)$ is a subalgebra of U which is obviously $ad\ U$ stable. Set $\mathcal{F}^m(F(U)) = \mathcal{F}^m(U) \cap F(U)$. We shall eventually prove that $dim\ \mathcal{F}^m(F(U)) < \infty$ for all $m \in \mathbb{Z}$. This is consistent with the fact $F(U)$

admits a locally finite action of *ad* U and indicates that the above filtration is the best analogue of the canonical filtration of $U(\mathfrak{g})$. Another indication is given by 6.6.

3. An Injectivity Property of the Harish-Chandra Map.

3.1. Let T be the free Abelian group generated by the $t_i : i = 1, 2, \dots, \ell$. Let $\pi := \{\alpha_1, \alpha_2, \dots, \alpha_\ell\}$ denote the set of simple roots. Let $P(\pi)$ denote the lattice of integral weights, $Q(\pi)$ the lattice of radical weights and $R(\pi) = 4P(\pi) \cap Q(\pi)$. Define a map $\tau : Q(\pi) \to T$ of Abelian groups by $\tau(\alpha_i) = t_i$. Set $T_\diamondsuit = \tau(R(\pi))$. (This is a slight departure from our notation in JL). Order $P(\pi)$ through $\lambda \geq \mu$ if $\lambda - \mu$ is a sum of simple roots with integer coefficients ≥ 0. It restricts to an order relation on $Q(\pi)$.

In general $P(\pi) \not\subset Q(\pi)$ but we can extend τ in an obvious fashion so that $\tau(P(\pi))$ can be viewed as an overgroup of T. Set $\check{U}^0 = K\tau(P(\pi))$ which is a commutative algebra containing U^0 as a subalgebra. Then we can define $\check{U} := U^- \otimes \check{U}^0 \otimes U^+$ as an over algebra of U by augmentating slightly the relations in 2.1, namely by setting

$$\tau(\lambda)x_i\tau(\lambda)^{-1} = q^{(\alpha_i,\lambda)}x_i \ , \quad \tau(\lambda)y_i\tau(\lambda)^{-1} = q^{-(\alpha_i,\lambda)}y_i \quad .$$

Similarly the Hopf algebra rules can be extended by $\varepsilon(\tau(\lambda)) = 1$ taking $\Delta(\tau(\lambda)) = \tau(\lambda) \otimes \tau(\lambda)$ and $\sigma(\tau(\lambda)) = \tau(\lambda)^{-1}$ for augmentation, comultiplication and antipode. In 4.2 it is convenient to introduce fractional powers, for example $t_i^{1/2} := \tau(\alpha_i/2)$ and for this we must extend K by corresponding fractional powers in q. Adjoin $q^{1/2}$ to K and for each $i \in \{1, 2, \dots, \ell\}$ set ${}^iU^0 = \check{U}^0[t_i^{1/2}, t_i^{1/2}]$ and ${}^iU = U^- \otimes^i U^0 \otimes U^+$ which are Hopf algebras over $K(q^{1/2})$.

We call $\check{U}$ the simply connected quantized enveloping algebra. All our results for U carry over to $\check{U}$ with $R(\pi)$ replaced by $4P(\pi)$ without change and without difficulty. However, our main result for $\check{U}$, namely the separation theorem (7.4) fails for U.

3.2. Identify U^0 with KT and let $\varphi : U \to KT$ denote the Harish-Chandra map defined by triangular decomposition (JL, 8.1). By (JL, 8.5) one has $\varphi(F(U)) \subset KT_\diamondsuit$. Given $t \in T$, $\lambda \in P(\pi)$ set $t(\lambda) = q^{(\tau^{-1}(t),\lambda)}$, where as before (,) denotes the Cartan inner product on $\mathfrak{h}^*$. Then define $a(\lambda) : a \in U^0$ by linearity. Call $N(\lambda)$ a highest weight module of highest weight $\lambda \in P(\pi)$ if it is generated by a vector e_λ satisfying $x_ie_\lambda = 0$, $\forall\ i$ and $ae_\lambda = a(\lambda)e_\lambda$, $\forall\ a \in U^0$. Observe that such a vector defines a one dimensional $B := U^0U^+$ module and that $M(\lambda) := U \otimes_B Ke_\lambda$ is the universal highest weight module of highest weight λ. It admits (JL, 5.4) a unique simple quotient which we denote by $L(\lambda)$. The following extends ([9], 5.4) to the quantum case.

LEMMA. *Let M be a finite dimensional ad U submodule of $F(U)$. Then for all $\lambda \in P(\pi)$*

$$\varphi(M)(\lambda) = 0 \Longleftrightarrow M \subset Ann\, L(\lambda) .$$

Consequently $\varphi(M) = 0$ implies $M = 0$.

Let $L(\lambda)^{<}$ denote the U^- submodule of $L(\lambda)$ spanned by the weight vectors of weight $< \lambda$. Observe that

$$(*) \qquad \varphi(M)(\lambda) = 0 \Longleftrightarrow Me_\lambda \subset L(\lambda)^{<} \quad .$$

Now let $m \in M$ be a weight vector. Since $(ad\ y_i)m \in M$ it follows that $my_i \in Kt_i^{-1}y_it_im + t_i^{-1}M$, $\forall_i$. Hence $MU^- \subset KTU^-M$. Then $ML(\lambda) = MU^-e_\lambda \subset KTU^-Me_\lambda$, so if $\varphi(M)(\lambda) = 0$ we conclude from $(*)$ that $ML(\lambda)$ is a proper subspace of $L(\lambda)$. Since M is *ad U* stable a similar calculation shows that $ML(\lambda)$ is a U submodule of $L(\lambda)$ hence zero, that is $M \subset Ann\ L(\lambda)$. The converse implication is immediate from $(*)$. The last part of the lemma results from the first part and (JL, 8.3.).

3.3. Let $\mathcal{R}(\ ,\ )$ denote the Rosso form ([20], Sect. II) on U. Recall that $\mathcal{R}$ is *ad U* invariant. Furthermore, for any subspace $M \subset U$, $\lambda \in R(\pi)$ one obtains using ([20], Thm. 6) that

$$(*) \qquad \mathcal{R}(M, \tau(\lambda)) = \mathcal{R}(\varphi(M), \tau(\lambda)) = \varphi(M)\left(-\frac{1}{4}\,\lambda\right) .$$

With respect to $\mathcal{R}$ let $M^\perp$ denote the orthogonal in $F(U)$ of a subspace M of U.

COROLLARY. *For all $\lambda \in -1/4\, R(\pi)$ one has*

$$Ann_{F(U)}L(\lambda) = [(ad\ U)\tau(-4\lambda)]^\perp \quad .$$

Let M be a finite dimensional *ad U* stable subspace of $F(U)$. Then by $(*)$

$$(**) \qquad M \subset [(ad\ U)\tau(-4\lambda)]^\perp \Longleftrightarrow \varphi(M)(\lambda) = 0$$

which by 3.2 is again equivalent to the assertion that $M \subset Ann_{F(U)}L(\lambda)$.

3.4. Let $P^+(\pi)$ (resp. $R^+(\pi)$) denote the dominant elements of $P(\pi)$ (resp. $R(\pi)$). Set

$$F_\diamond(U) = \sum_{-\mu \in R^+(\pi)} (ad\ U)\tau(\mu) \quad .$$

By (JL, 6.4) one has $F_{\diamondsuit}(U) \subset F(U)$.

LEMMA. $F_{\diamondsuit}(U)^{\perp} = 0$. *In particular the restriction of* $\mathcal{R}$ *to* $F(U)$ *is nondegenerate.*

Let M be a finite dimensional *ad* U stable subspace of $F_{\diamondsuit}(U)^{\perp}$. Then by 3.3(*) one has $\varphi(M)(-\frac{1}{4}\mu) = 0$ for all $-\mu \in R^+(\pi)$. Then by 3.2 we obtain

$$M \subset \bigcap_{-\mu \in R^+(\pi)} Ann\, L\left(-\frac{1}{4}\mu\right) = 0$$

where the last step follows by a slight variation on (JL, 8.3.). (Here the intersection is not over *all* the annihilators of finite dimensional modules).

Remark. We cannot yet deduce that $F_{\diamondsuit}(U) = F(U)$ because the spaces concerned are infinite dimensional. The analysis of section 4 was motivated by an attempt to prove this equality. It is proved in 4.10. Note that in 3.3 both sides can be of infinite codimension. Thus such a formula is only possible for quantum groups and not for enveloping algebras.

3.5. Take $\mu \in -R^+(\pi)$. Then $(ad\ U)\tau(\mu)$ is a finite dimensional subspace of $F(U)$. Again $L(-\frac{1}{4}\mu)$ is finite dimensional and from (JL, 6.2) one easily checks that any simple weight module in particular $L(-\frac{1}{4}\mu)$ is simple as an $F(U)$ module. Then by the Jacobson density theorem it follows that action of $F(U)$ on $L(-\frac{1}{4}\mu)$ defines an isomorphism $F(U)/Ann_{F(U)}L(-\frac{1}{4}\mu) \xrightarrow{\sim} End_K L(-\frac{1}{4}\mu)$ of *ad* U modules (even of $F(U)$ bimodules). Then by 3.3 we obtain $dim(ad\ U)\tau(\mu) = dim\ End_K L(-\frac{1}{4}\mu)$. Recall complete reducibility (JL, 5.12) and let M be the isotypical component of type E of $(ad\ U)\tau(\mu)$. By 3.3 we have $M^{\perp} \supset Ann_{F(U)}L(\lambda)$. By nondegeneracy 3.3 and the finite dimensionality of M we have $dim_K M = codim_K M^{\perp}$. Thus $M \cong (F(U)/M)^*$ by *ad* U invariance. We conclude that $[(ad\ U)\tau(\mu) : E] \leq [F(U)/Ann_{F(U)}L(-\frac{1}{4}\mu) : E^*]$ and then equality is obtained by addition of dimensions. Thus $(ad\ U)\tau(\mu) \cong (F(U)/Ann_{F(U)}L(-\frac{1}{4}\mu))^*$. By the self-duality of $End_K L(-\frac{1}{4}\mu)$ we deduce the

COROLLARY. *The Rosso form defines an isomorphism* $(ad\ U)\tau(\mu) \cong End_K L(-(1/4)\mu)$ *of ad U modules.*

Remarks. By complete reducibility (JL, 5.12) the (unique) copy of the trivial representation in the right hand side determines an element z_{μ} of the centre $Z(U)$ of U. This was essentially the key step in the construction of Z given in (JL, 8.6).

We do not claim that the above isomorphism is the natural one, namely that it results from the action of $F(U)$ on $L(-\frac{1}{4}\mu)$. The latter is equivalent to

$$(*) \qquad (ad\ U)\tau(\mu) \cap ((ad\ U)\tau(\mu))^{\perp} = 0 \ .$$

This can probably be checked, but we shall not need it here. We shall eventually obtain (6.7, remark 2) a second proof of this corollary.

4. A Support Property.

4.1. Fix $i \in \{1, 2, \ldots, \ell\}$ and consider U^+ as a T module. Set $U_i^+ = K[x_i]$. For each $m \in \mathbb{N}$ the weight subspace of U^+ of weight $m\alpha_i$ is just Kx_i^m. We conclude that U_i^+ admits a unique T stable complement M_i^+ in U^+. Set $U_i^- = K[y_i]$. Similarly U_i^- admits a unique T stable complement M_i^- in U^-. Set $L_i = U_i^- U^0 U_i^+$ which is a Hopf subalgebra of U. Triangular decomposition (JL, 4.8) gives a direct sum decomposition

$$U = L_i \oplus (M_i^- U + UM_i^+)$$

into $ad\ L_i$ stable subspaces. (Although this is straightforward one needs a little more care than in the enveloping algebra case. For example only UM_i^+ is $ad\ y_i$ stable and not M_i^+ itself). Let φ^i denote the projection of U onto L_i defined by this decomposition. It is clear that we also have a triangular decomposition $U_i^- \otimes U^0 \otimes U_i^+ \xrightarrow{\sim} L_i$ and hence a direct sum decomposition

$$L_i = U^0 \oplus (U_i^- L_i + L_i U_i^+) \ .$$

Let φ_i denote the projection of L_i onto U^0 defined by this decomposition. It is clear that $\varphi = \varphi_i \varphi^i$.

4.2. Given $s \in KT = U^0$ we can write $s = \sum c_\lambda \tau(\lambda)$ and we set $Supp\ s = \{\lambda \mid c_\lambda \neq 0\}$. If S is a subset of KT we set $Supp\ S = \bigcup_{s \in S} Supp\ s$. Since $Q(\pi)$ is W stable, the isomorphism $\tau : Q(\pi) \xrightarrow{\sim} T$ of Abelian groups gives an action of the Weyl group W on T. For each $\lambda \in \mathfrak{h}^*$ we define the norm of λ by $\|\lambda\| := (\lambda, \lambda)^{1/2}$. For any U module E admitting a weight space decomposition, let $\Omega(E)$ denote its set of weights (cf. 3.2). Set $T_< = \tau(-R^+(\pi))$. For each $m \in \mathbb{Z}$, set $T_<^m = \mathcal{F}^m(F(U)) \cap T_<$. Note that $T_<^m = \phi$ for $m < 0$. Let w_0 denote the unique longest element of W. Recall that the simple finite dimensional U module with lowest weight $\lambda \in -P^+(\pi)$ has highest weight $w_0\lambda$.

Proposition. *Let M be a finite dimensional ad U stable subspace of $F(U)$.*

(i) If $\lambda \in Supp\ \varphi(M)$ has maximal norm then $W\lambda \subset Supp\ \varphi(M)$.

(ii) There exist $\lambda_1, \lambda_2, \ldots, \lambda_s \in -R^+(\pi) \cap Supp\ \varphi(M)$ *such that*

$$Supp\ \varphi(M) \subset \bigcup_{j=1}^{s} 4\Omega(E(\frac{1}{4}w_0\lambda_j)) .$$

(iii) If $M \subset \mathcal{F}^m(F(U))$, *then* $\varphi(M) \subset KWT^m_<$. *In particular,* $s \leq m\ell$.

Fix $i \in \{1, 2, \ldots, \ell\}$ and recall (3.1) the definition of ${}^iU$ and define similarly $\check{L}_i := U_i^- \otimes^i U^0 \otimes U_i^+$ as a Hopf algebra. Let ω_i be the fundamental weight corresponding to α_i. Then $\tau(w_j) : j \neq i$ is central in $\check{L}_i$ and s_i invariant. By construction the $t_i^{1/2}$ and the $\tau(\omega_j) : j \neq i$ generate ${}^iU^0$ and are algebraically independent. Again by complete reducibility (for $U_q(\mathfrak{sl}(2))$) we can write $\varphi^i(M)$ as a direct sum of simple *ad* $\check{L}_i$ submodules N_m and then $Supp\ \varphi(M) = \sum_m Supp\ \varphi_i(N_m)$. It is clear that each $\mu \in Supp\ \varphi_i(N_m)$ is a multiple of $\alpha_i/2$ plus a fixed sum of the $w_j : j \neq i$. Since $(\alpha_i, \omega_j) = 0$ for $j \neq i$ it follows that if $\lambda \in Supp\ \varphi(M)$ has maximal norm, then the corresponding term, namely $\frac{1}{2}(\lambda, \alpha_i^\vee)\alpha_i$, which occurs in a summand of some $Supp\ \varphi_i(N_m)$ also has maximal norm. Thus to prove (i) it suffices to show: a), for each simple $\check{U}_i \cong \check{U}_q(\mathfrak{sl}(2))$ module N that the maximal norm elements of $Supp\ \varphi(N)$ take the form $\{n\alpha_i, -n\alpha_i\}$ for some $n \in 2\mathbb{N}$. Moreover it is easy to see that (ii) will follow if we can furthermore show: b), $Supp\ \varphi(N) \subset \{n\alpha_i, (n-4)\alpha_i, \ldots, -n\alpha_i\}$. Indeed writing $\lambda = n\alpha_i + \mu$ with $\mu = \sum_{j\neq i} c_j\omega_j \in P(\pi)$, it is standard that $\frac{1}{4}(\mu + Supp\ \varphi(N))$ belongs to the set of weights of the simple U module with extreme weight $\frac{1}{4}\lambda$. Assertions a), b), are proved below and in this the i subscript is omitted. In particular, $U \cong U_q(\mathfrak{sl}(2))$.

Recall (JL, 3.3) that $Z(U) = K[z]$. One easily checks that $F(U) = F(\check{U})$ and in particular $Z(\check{U}) = Z(U)$. Consider N as a simple *ad* U submodule of $F(U)$. By (JL, 3.11) there exist $n \in \mathbb{N}$ and a polynomial p such that the zero weight space of N takes the form $Kp(z)(ad\ y)^n(xt^{-1})^n$. Yet up to a nonzero scalar

$$\begin{aligned}\varphi(ad\ y)^n(xt^{-1})^n) &= \varphi(x^ny^n) \\ &= \prod_{j=1}^{n}(t^2q^{-2(j-1)} - t^{-2}q^{2(j-1)})\end{aligned}$$

whilst $\varphi(p(z)) = p(\varphi(z)) = p(q^2t^2 + q^{-2}t^{-2})$. Now the restriction of φ to the zero weight space of U (and hence of N) is a homomorphism, so the required assertions follow from the above formulae. Hence (i), (ii) are proved.

To prove (iii) we remark that by triangular decomposition it is immediate that $\varphi(\mathcal{F}^m(U)) \subset \mathcal{F}^m(U) \cap T$. Thus for each $\lambda_j \in -R^+(\pi) \cap Supp\ \varphi(M)$ we have $\tau(\lambda_j) \in T^m_<$. Now take $\mu \in Supp\ \varphi(M)$. By (ii) we have $\mu \in 4\Omega(E(1/4\ w_0\lambda_j))$ for some $\lambda_j \in -R^+(\pi) \cap\ Supp\ \varphi(M)$. Choose $w \in W$ such that $\mu' := w\mu \in -P^+(\pi)$. Since $\lambda_j \in Q(\pi)$ it follows that $E(1/4w_0\lambda_j)$ has lowest weight $\frac{1}{4}\lambda_j$ and so $\mu' = \lambda_j + \sum k_r\alpha_r$, for some $k_r \in 4\mathbb{N}$. We conclude that $\mu' \in -R^+(\pi)$ and so $\tau(\mu) = w^{-1}\tau(\mu') = w^{-1}(\tau(\lambda_j)\Pi t_i^{k_i}) \subset w^{-1}(T_< \cap \mathcal{F}^{m-\Sigma k_r}(U)) \subset w^{-1}T_<^m$, as required.

Remarks. One can easily choose the polynomial p above so that *Supp* $\varphi(N)$ is not W stable. We shall eventually discuss (4.16) to what extent (for M simple) it suffices to take $s = 1$ in (ii) and furthermore to have equality.

COROLLARY. *$\mathcal{F}^m(F(U)) = 0$ if $m < 0$ and reduces to scalars if $m = 0$.*

Let M be a nonzero simple (finite dimensional) *ad U* submodule of $\mathcal{F}^m(F(U))$. If $m < 0$, then $\varphi(M) = 0$ by 4.2(iii) and so $M = 0$ by 3.2. If $m = 0$, then $\varphi(M)$ reduces to scalars by 4.2(iii). Then by 3.2, M does not annihilate any simple highest weight module and in particular not the trivial module. This implies that M reduces to scalars.

4.4. Given $\lambda \in P(\pi)$ we can write $\lambda = \sum n_i\omega_i : n_i \in \mathbb{Z}$, and we define $ht(\lambda) = \sum n_i$ to be the height of λ. Set

$$P_m^+(\pi) = \{\lambda \in P^+(\pi) \mid ht(\lambda) \leq m\} \quad .$$

PROPOSITION. *For each $m \in \mathbb{N}$ one has dim $\mathcal{F}^m(F(U)) < \infty$.*

Let M be a nonzero simple *ad U* submodule of $\mathcal{F}^m(F(U))$, and let $\lambda_1, \lambda_2, \ldots, \lambda_s$ be as in the conclusion of 4.2. Consider $\varphi(M)$ as a K subspace of rational functions in the $t_i : 1, 2, \ldots, \ell$. By 4.2(ii) every such function is a linear combination of form $\tau(\mu) : \mu \in 4\Omega(E(\frac{1}{4}w_0\lambda_j)), j = 1, 2, \ldots, s$. From the representation theory of such modules we can write $\tau(\mu)$ in the form $a^{-1}b$ where a, b are polynomials in the $t_i : i = 1, 2, \ldots, \ell$ and where a divides $\Pi_{j=1}^s \tau(\lambda_j)^{-1}$ and b divides $\Pi_{j=1}^s \tau(w_0\lambda_j)$. Since by 4.2(i) we have $\tau(\lambda_j) \in \mathcal{F}^m(U)$ and so $\tau(w_0\lambda_j)^{-1} \in \mathcal{F}^m(U)$ we may eliminate denominators and view $\varphi(M)$ as a K subspace of the space of polynomials in the $t_i : i = 1, 2, \ldots, \ell$ of degree $\leq 2ms$. Now suppose $p \in K[t_1, t_2, \ldots, t_\ell]$ of degree r satisfies $p(\lambda) = 0, \forall \ \lambda \in P_r^+(\pi)$. An easy induction argument on ℓ implies that $p = 0$. Since $\varphi(M) \neq 0$ we conclude that if $\varphi(M)(\lambda) = 0$ for all $\lambda \in P_n^+(\pi)$, then $n < 2ms$.

Now take $\lambda \in P_r^+(\pi)$. View $End_K L(\lambda) \cong L(\lambda) \otimes_K L(-w_0\lambda)$ as a quotient of $L(\lambda) \otimes_K M(-w_0\lambda) \cong U \otimes_B (L(\lambda) \mid_B \otimes_K Ke_{-w_0\lambda})$ which has a Verma flag with factors $M(\mu - w_0\lambda) : \mu \in \Omega(L(\lambda))$. It follows that the simple factors of $End_K L(\lambda)$ belong to the set of dominant weights lying in $\Omega(L(\lambda)) - w_0\lambda$. Obviously $ht(-w_0\lambda) = ht(\lambda)$ and we claim for some u depending only on $\mathfrak{g}$ that $ht(\mu) \leq (u-1)ht(\lambda)$, for all $\mu \in \Omega(L(\lambda))$. For this we remark that $ht(\alpha_i) = \{1, 0, -1\}$ and for $\mathfrak{g}$ simple there is at most one α_i such that $ht(\alpha_i) = -1$. Let w_h denote the unique shortest element of W such that $ht(w_h\alpha_j) \geq 0, \forall \ j = 1, 2, \ldots, \ell$ and let

$u \neq 1 = max_i ht(w_h w_i)$. An easy representation theory argument shows that $w_h\lambda$ has maximal height in $\Omega(L(\lambda))$. Then $ht(\mu) \leq ht(w_h\lambda) \leq (u-1)ht(\lambda)$ as required. Consequently, if $M \cong L(\mu)$ with $\mu \in P^+(\pi) \setminus P^+_{ur}(\pi)$ it follows that $ML(\lambda) = 0$ equivalently by 3.2 that $\varphi(M)(\lambda) = 0$ and this must hold for all $\lambda \in P^+_r(\pi)$. Then by our previous observation $r \leq 2ms$. We conclude that $M \cong L(\mu)$ for some $\mu \in P^+_{2um^2\ell}$. In particular only finitely many nonisomorphic simple modules can occur in $\mathcal{F}^m(F(U))$.

Finally suppose that E is an isomorphism class of a simple module occurring in $\mathcal{F}^m(F(U))$. Let E_0 denote the zero weight space of E. Since $\varphi(\mathcal{F}^m(F(U)) \subset KW\ T^m_<$ which is finite dimensional and since $\varphi(M) = 0$ implies $M = 0$ by 3.2, we conclude that the multiplicity of E in $\mathcal{F}^m(F(U))$ is at most

$$dim_K KW\ T^m_<\ dim_K E_0$$

which is finite. Combined with our previous observation, this proves the proposition.

4.5. For each $m \in \mathbb{N}$, set

$$T_m = \{t \in T^m_< \mid t \notin T^{m-1}_<\} \quad .$$

LEMMA. *Take $\tau(\lambda) \in T_m$. Then $dim(ad\ U)\tau(\lambda) = (dim\, L(-\frac{1}{4}\lambda))^2$ holds in both $F(U)$ and $gr_{\mathcal{F}}F(U)$.*

By the hypothesis $\lambda \in -R^+(\pi)$. By 3.5 we already have $dim(ad\ U)\tau(\lambda) = (dim\ L(-\frac{1}{4}\lambda))^2$ in $F(U)$. So it remains to show that this dimensionality estimate holds in $gr_{\mathcal{F}}F(U)$. Actually we show that the above equalities hold without needing 3.5.

Recall that U^+ (resp. U^-) is actually *graded* by $\mathcal{F}$ and indeed let U^+_m (resp. U^-_m) denote the linear span of the monomials in U^+ (resp. U^-) of degree m. (These gradations of U^+, U^- are also given by weight space decomposition).

Observe that

$$(ad\ x_j)(ad\ U^-_m)\tau(\lambda) = (ad\ U^-_m)(ad\ x_j)\tau(\lambda) \quad \mathrm{mod}\ (ad\ U^-_{m-1})\tau(\lambda)$$

and so by induction

$$(ad\ U^+)(ad\ U^-_m)\tau(\lambda) = (ad\ U^-_m)(ad\ U^+)\tau(\lambda) \quad \mathrm{mod} \sum_{n=0}^{m-1} (ad\ U^-_n)(ad\ U^+)\tau(\lambda)\ .$$

Set $I_m^- = Ann_{ad\ U_m^-}\tau(\lambda)$. Then by the above reasoning we conclude that

$$(*)\qquad I_m^-(ad\ U^+)\tau(\lambda) \subset \sum_{n=0}^{m-1}(ad\ U_n^-)(ad\ U^+)\tau(\lambda)\ .$$

Hence

$$\begin{aligned} dim\ (ad\ U)\tau(\lambda) &= dim(ad\ U^-)(ad\ U^+)\tau(\lambda)\ ,\ \text{by triangular decomposition,}\\ &\le \sum_{n=0}^{\infty} dim\ ad(U_m^-/I_m^-)dim(ad\ U^+)\tau(\lambda)\ ,\quad \text{by}\ \ (*)\ ,\\ &= dim(ad\ U^-)\tau(\lambda)\ dim(ad\ U^+)\tau(\lambda)\ , \end{aligned}$$

since $Ann_{(ad\ U^-)}\tau(\lambda) = \oplus_{m=0}^{\infty} I_m^-$ by weight space decomposition. We show below that the opposite inequality holds in $gr_{\mathcal{F}}U$. This will prove equality throughout and so establish the lemma. In view of the Chevalley antiautomorphism (JL, 4.8) we shall also obtain

$$(**)\qquad dim\ L\left(-\frac{1}{4}\ \lambda\right) = dim(ad\ U^-)\tau(\lambda) = dim(ad\ U^+)\tau(\lambda)$$

for all $\lambda \in -R^+(\pi)$.

4.6. Set $G = gr_{\mathcal{F}}(U)$, $G^0 = gr_{\mathcal{F}}U^0$. Let G^+ (resp. G^-) denote the subalgebra of G generated by the x_it_i (resp. y_it_i) : $i = 1, 2, \ldots, \ell$. By triangular decomposition (2.2) the multiplicity map gives an isomorphism

$$(*)\qquad G^- \otimes G^0 \otimes G^+ \xrightarrow{\ \sim\ } G\ .$$

Now set $G(\lambda) = G^- \otimes K\tau(\lambda) \otimes G^+$, $\forall\ \lambda \in Q(\pi)$. Then the above gives a direct sum decomposition

$$(**)\qquad G = \bigoplus_{\lambda\in Q(\pi)} G(\lambda)\ .$$

Observe the rather nice fact that each $G(\lambda)$ is $ad\ U$ stable (for the induced action of $ad\ U$ on $gr_{\mathcal{F}}U$). Again $G(\lambda)G(\mu) \subset G(\lambda+\mu)$ and so $(**)$ is a gradation of G with respect to the Abelian group $Q(\pi)$. We can recover the gradation of G defined by its identification with $gr_{\mathcal{F}}(U)$ by setting $Q_m(\pi) = \{\lambda \in Q(\pi) \mid deg\ \tau(\lambda) = m\}$. Then

$$\mathcal{F}^m(U)/\mathcal{F}^{m-1}(U) = \bigoplus_{\lambda\in Q_m(\pi)} G(\lambda)\ .$$

Now let G_m^+ (resp. G_m^-) denote the subspace of elements of G^+ (resp G^-) of degree exactly m in the x_it_i (resp. y_it_i) : $i = 1, 2, \dots, \ell$. (This is well-defined and corresponds to an obvious weight space decomposition). Now in $gr_{\mathcal{F}}U$ we have

$$(***) \qquad \begin{aligned}(ad\ x_j)y_kt_k &= x_jy_kt_kt_j - q^{-2d_j}t_jy_kt_kx_j \\ &= (x_jy_k - y_kx_j)t_jt_k = -\delta_{jk}/(q^{2d_j} - q^{-2d_j})\end{aligned}$$

where δ is the Kronecker delta. This shows that G^- admits a U module structure. This is a purely quantum group phenomenon—see also 6.5, 6.6.

Fix $\lambda \in Q(\pi)$, $m \in \mathbb{N}$ and consider $(ad\ U_m^-)\tau(\lambda)$. Note that we can define a unique *ad T* stable subspace $G(\lambda)_m^-$ of G_m^- such that the multiplicity map defines an isomorphism of $G(\lambda)_m^- \otimes K\tau(\lambda)$ onto $(ad\ U_m^-)\tau(\lambda)$. (This follows from inspection of the formulae for the $ad\ y_i$). Set

$$G(\lambda)^- = \bigoplus_{m\in\mathbb{N}} G(\lambda)_m^- \ .$$

Obviously $dim\ G(\lambda)^- = dim(ad\ U^-)\tau(\lambda)$ and we remark that the latter is independent of whether the computation is carried out in U or in $gr_{\mathcal{F}}U$ since it involves no commutators between the x_i and y_j. Define $G(\lambda)^+$ similarly. From the above computations and the Hopf algebra rules in U (cf. JL, 9.2 for a similar computation) we obtain (in $gr_{\mathcal{F}}U$) that

$$\begin{aligned}(ad\ U^+)(ad\ U_m^-)\tau(\lambda) &= (ad\ U^+)(G(\lambda)_m^- \otimes K\tau(\lambda)) \\ &= G(\lambda)_m^- \otimes G(\lambda)^+ \otimes K\tau(\lambda)\ mod \sum_{n=0}^{m-1} G_n^- \otimes G(\lambda)^+ \otimes K\tau(\lambda).\end{aligned}$$

Hence in $gr_{\mathcal{F}}U$ one has

$$\begin{aligned}dim(ad\ U)\tau(\lambda) &\geq \left(\sum_{m\in\mathbb{N}} dim\ G(\lambda)_m^-\right) dim\ G(\lambda)^+ \\ &= dim\ G(\lambda)^-\ dim\ G(\lambda)^+ \\ &= dim(ad\ U^-)\tau(\lambda)\ dim(ad\ U^+)\tau(\lambda)\ ,\end{aligned}$$

as required. This completes the proof of lemma 4.5.

4.7. The dimensionality formulae in $(**)$ of 4.5 have a rather nice interpretation. First recall that we showed in 4.6 that G^- is an *ad U* module. Thus, for $\lambda \in Q(\pi)$, we can give $G^- \otimes K\tau(\lambda)$ a U module structure if we can specify how the generators act on $\tau(\lambda)$. We set

$$(ad\ x_i).\tau(\lambda) = 0\ ,\quad (ad\ t_i).\tau(\lambda) = q^{-\frac{1}{4}\ (\lambda,\alpha_i)}\tau(\lambda)$$

$$(ad\ y_i).\tau(\lambda) = q^{\frac{1}{4}(\lambda,\alpha_i)}(1 - q^{-(\lambda,\alpha_i)})\ (y_i t_i \otimes \tau(\lambda))\ .$$

Notice that these relations do not quite coincide with what one would get by applying $ad\ x_i$ etc. to $\tau(\lambda)$. The first two relations express the fact that $\tau(\lambda)$ is a highest weight vector of weight $-\frac{1}{4}\ \lambda$. This choice of highest weight is not arbitrary; but is just what is required to ensure that $G^- \otimes K\tau(\lambda)$ does inherit a U module structure and in particular that $[ad\ x_i,\ ad\ y_j]$ coincides as it should with $ad[x_i, y_j]$ on $\tau(\lambda)$. All this is easily checked.

Recall (JL, 5.12) that we have a duality functor δ on the subcategory $\mathcal{O}_{P(\pi)}$ of weight modules whose simple factors are amongst the $L(\mu) : \mu \in P(\pi)$.

LEMMA.

(i) $(G^-)^{U^+}$ *reduces to scalars.*

(ii) *As a U module* $M := G^- \otimes K\tau(\lambda)$ *is isomorphic to* $\delta M(-\frac{1}{4}\lambda)$.

(i) Notice that the action of U^+ on M is independent of λ. In particular M^{U^+} identifies with $(G^-)^{U^+} \otimes K\tau(\lambda)$. Assume (i) fails. Then for any choice λ, M admits a proper submodule (generated by the T stable complement of $K\tau(\lambda)$ in $(G^- \otimes K\tau(\lambda))^{U^+}$) such that $\tau(\lambda)$ has a nonzero image in the quotient. Now take $\lambda \in 4P^+(\pi)$. Choose a submodule M' of M maximal with property that $(M/M')^{U^+}$ is a one-dimensional subspace of T weight λ, which we identify with $K\tau(\lambda)$. Such a submodule exists because all objects in $\mathcal{O}_{P(\pi)}$ have finite length and their simples have the above property. Moreover M' is proper by our previous observation. Set $N = \delta(M/M')$ and let U^-_+ denote the augmentation ideal of U^-. The above property translates by duality to give

$$N = K\tau(\lambda) + U^-_+ N\quad .$$

A standard argument based on induction and weight space decomposition shows that $N = U^-\tau(\lambda)$, that is N is a cyclic U^- module with cyclic vector $\tau(\lambda)$. Yet N is a U module, hence a highest weight module of highest weight $-\frac{1}{4}\lambda$ and so an image of $M(-\frac{1}{4}\lambda)$. Our choice of λ implies that $M(-\frac{1}{4}\lambda)$ is simple (cf. 8.2(i)). Hence N is isomorphic to $M(-\frac{1}{4}\lambda)$. Using ch to denote formal character (JL, 5.2), we obtain

$$ch\ M/M' = ch\ N = ch\ M\left(-\frac{1}{4}\lambda\right) = (ch\ U^-)e^{-\frac{1}{4}\lambda}\ .$$

Now let U_ν^- (resp. G_ν^-) denote the *ad T* weight subspace of U^- (resp. G^-) of weight ν. It is immediate that $G_\nu^- = U_\nu^- \tau(\nu)$ and so $ch\ G^- = ch\ U^-$. Hence $ch\ M = (ch\ U^-)e^{-\frac{1}{4}\lambda} = ch(M/M') = ch\ M - ch\ M'$ and so $M' = 0$. This contradiction gives (i).

By (i) we have $M^{U^+} = K\tau(\lambda)$ for *any* λ. Then taking $N = \delta M$, we conclude as above that N is an image of $M(-\frac{1}{4}\lambda)$ with the same formal character. Hence $M(-\frac{1}{4}\lambda) \xrightarrow{\sim} N$. Since $\delta^2 = Id$, (ii) results.

Remark. The calculation hides some deep mathematics. Thus (i) would fail if we replaced G^- by the free algebra $\tilde{G}^-$ generated by the $y_i t_i$. Indeed by say the calculation in (JL, 4.6) the Serre relations provide additional invariants in $\tilde{G}^-$. Effectively what we are saying is that these relations generate all the invariants. To prove this we need a simple Verma module. The existence of the latter depends on the characterization of $Z(U)$ given in (JL, 8.6)—see 8.2(i). Our argument would therefore fail in the Kac-Moody case. Moreover in the Kac-Moody case, there is an analogous question (cf. [12], 9.11), this time for the Lie algebra whose solution is only easy in the symmetrizable case. In the nonsymmetrizable one also does not even know if quantization is possible.

4.8. Set $L(\lambda)^- = \{m \in G^- \otimes K\tau(\lambda) \mid dim U^- m < \infty\}$.

COROLLARY.

(i) $G^- \otimes K\tau(\lambda)$ *admits a unique finite dimensional submodule. This is nonzero if and only if* $\lambda \in -R^+(\pi)$ *and then is isomorphic to* $L(-\frac{1}{4}\lambda)$.

(ii) $L(\lambda)^- \neq 0$ *if and only if* $\lambda \in -R^+(\pi)$ *and then coincides with* $(ad\ U^-)\tau(\lambda)$.

(i) follows from 4.7 and the classification theory of simple finite dimensional modules. (In particular every such module in $\mathcal{O}_{P(\pi)}$ is the unique simple quotient of some $M(\mu) : \mu \in P^+(\pi)$. See JL, Sect. 5, for example.)

(ii) Obviously $L(\lambda)^-$ is U^- stable. Take $m \in L(\lambda)^-$. Then $(ad\ U_i^-)m$ is finite dimensional. By weight space decomposition, this forces y_i to act nilpotently on each term in the decomposition of m as a linear combination of weight vectors. Since this holds for each i it follows from (JL, 4.5) that $ad\ U^-$ acts locally finitely on each such vector. Hence $L(\lambda)^-$ is U^0 stable. From this a straightforward computation shows that $L(\lambda)^-$ is also U^+ stable. Hence $L(\lambda)^-$ is a U submodule of $G^- \otimes K\tau(\lambda)$. By hypothesis the action of U^- is locally finite. By weight space considerations this also holds (trivially) for the U^0 and U^+ actions. By triangular decomposition, it follows that $L(\lambda)^-$ is a sum of finite dimensional modules. Consequently the first part of (ii) follows from (i).

Assume $\lambda \in -R^+(\pi)$. Then $dim(ad\ U^-)\tau(\lambda) = dim\ L(-\frac{1}{4}\lambda)$ by 4.5(**).

Now observe that

$$(ad\ y_i)\tau(\lambda) = (1 - q^{-(\lambda,\alpha_i)})y_i t_i \tau(\lambda)$$

which viewed as an element of $G^- \otimes K\tau(\lambda)$ coincides with $q^{-\frac{1}{4}(\lambda,\alpha_i)}(ad\ y_i)\cdot\tau(\lambda)$. (We shall see in 6.7 that this apparently anomolous factor of q does have a role to play). We conclude that $(ad\ U^-)\tau(\lambda)$ can be viewed as a finite dimensional U^- stable subspace of $G^- \otimes K\tau(\lambda)$ and hence as a subspace of $L(\lambda)^- \cong L(-\frac{1}{4}\lambda)$. Coincidence of dimension proves equality.

4.9. Fix $\lambda \in Q(\pi)$ and set

$$F(\lambda) = \{m \in G(\lambda) \mid dim(ad\ U)m < \infty\} \ .$$

PROPOSITION. *$F(\lambda) \neq 0$ if and only if $\lambda \in -R^+(\pi)$. In this case*

$$F(\lambda) = (ad\ U)\tau(\lambda) \ .$$

We can write $L(\lambda)^-$ in the form $K(\lambda)^- \otimes K\tau(\lambda)$ for some uniquely determined $ad\ T$ stable subspace $K(\lambda)^-$ of G^-. Since $L(\lambda)^-$ is a U module with $\tau(\lambda)$ its highest weight vector, it follows that $K(\lambda)^-$ is an $ad\ U^+$ submodule of G^-. Define $L(\lambda)^+$, $K(\lambda)^+$ by replacing G^- by G^+. If $\lambda \notin -R^+(\pi)$ then $K(\lambda)^{\pm} = 0$ by 4.8(ii). If $\lambda \in -R^+(\pi)$ then by 4.8(ii) we obtain

$$(ad\ U^-)\tau(\lambda) = K(\lambda)^- \otimes K\tau(\lambda) \ .$$

Then by the above

$$\begin{aligned}(ad\ U)\tau(\lambda) = (ad\ U^+)(ad\ U^-)\tau(\lambda) &= (ad\ U^+)(K(\lambda)^- \otimes K\tau(\lambda)) \ ,\\ &= K(\lambda)^- \otimes (ad\ U^+)K\tau(\lambda) \ ,\\ &= K(\lambda)^- \otimes K(\lambda)^+ \otimes K\tau(\lambda) \ ,\end{aligned}$$

by 4.8(ii) again (applied with + and − reversed).

Now let $m \in F(\lambda)$ be an $ad\ T$ weight vector. We can write m in the form

$$m = \sum a_j \otimes b_j \otimes \tau(\lambda)$$

with the $a_j \in G^-$ (resp. $b_j \in G^+$) weight vectors and linearly independent. Now take i so that a_i has highest degree as a polynomial in the $y_r t_r : r = 1, 2, \ldots, \ell$. Let us say $a_i \in G_m^-$. The calculation in 4.6 shows that

$$(ad\ U^+)(a_i \otimes b_i \otimes \tau(\lambda)) = a_i \otimes (ad\ U^+)(b_i \otimes \tau(\lambda)) \ ,$$

$$mod \sum_{n=0}^{m-1} G_n^- \otimes G^+ \otimes K\tau(\lambda) \ .$$

We conclude that the hypothesis $dim(ad\ U^+)m < \infty$ and our choice of i and the a_j imply that

$$dim(ad\ U^+)(b_i \otimes \tau(\lambda)) < \infty$$

and so $b_i \in K(\lambda)^+$. Similarly $a_i \in K(\lambda)^-$ if b_i has the highest degree. Since $deg\ a_i - deg\ b_i$ is constant in the sum we conclude that there is a choice of i such that $a_i \otimes b_i \in K(\lambda)^- \otimes K(\lambda)^+$. If $\lambda \notin -R^+(\pi)$ we conclude that $a_i = b_i = 0$ and so m cannot be nonzero. If $\lambda \in -R^+(\pi)$, then from the obvious inclusion

$$(ad\ U)\tau(\lambda) \subset F(\lambda)$$

and from the above observations we obtain the required equality.

4.10. Since the $G(\lambda) : \lambda \in Q(\pi)$ form a direct sum decomposition of G we conclude from 4.9 that

$$gr_{\mathcal{F}}F(U) = \bigoplus_{\lambda \in -R^+(\pi)} (ad\ U)\tau(\lambda)$$

Then by 4.5 it follows that $(ad\ U)\tau(\lambda) : \lambda \in -R^+(\pi)$ also form a direct sum in $F(U)$ and this equals $F(U)$. That is we have proved the

THEOREM.

$$F(U) = \bigoplus_{\lambda \in -R^+(\pi)} (ad\ U)\tau(\lambda) \quad .$$

Remark. In particular $F_{\diamondsuit}(U) = F(U)$.

4.11. For each $m \in \mathbb{N}$, set $F_m(U) = (ad\ U)T_m$. Then by 4.10 we have the

COROLLARY.

$$F(U) = \bigoplus_{m \in \mathbb{N}} F_m(U) = (ad\ U)T_< \quad .$$

Remark. This is not a ring grading; but each $F_m(U)$ is $ad\ U$ stable and isomorphic to $\mathcal{F}^m(F(U))/\mathcal{F}^{m-1}(F(U))$ as a U module.

4.12. Consider $(ad\ U)\tau(\lambda) : \lambda \in -R^+(\pi)$ as a subspace of the ring $gr_{\mathcal{F}}F(U)$. Recall the definition of $K(\lambda)^{\pm}$ given in 4.9 and that $F(\lambda) = (ad\ U)\tau(\lambda) = K(\lambda)^-\tau(\lambda)K(\lambda)^+$.

PROPOSITION. *For all* $\lambda, \mu \in -R^+(\pi)$ *one has*

(i) $K(\lambda)^- K(\mu)^- = K(\lambda+\mu)^-$.

(ii) $K(\lambda)^- K(\mu)^+ = K(\mu)^+ K(\lambda)^-$.

(iii) $F(\lambda)F(\mu) = F(\lambda+\mu)$.

Since $K(\mu)^-$ is *ad* T stable one has $K(\mu)^-\tau(\lambda) = \tau(\lambda)K(\mu)^-$. Then

$$G^-\tau(\lambda+\mu) \supset K(\lambda)^-K(\mu)^-\tau(\lambda+\mu) = K(\lambda)^-\tau(\lambda)K(\mu)^-\tau(\mu) = L(\lambda)^-L(\mu)^- \quad .$$

Thus the inclusion $K(\lambda)^-K(\mu)^- \subset K(\lambda+\mu)^-$ obtains from (JL, 2.3) applied to the defining property (4.8) of $L(\lambda+\mu)^-$. For equality observe again from (JL, 2.3) that $L(\lambda)^-\ L(\mu)^-$ is a U submodule of $L(\lambda+\mu)^-$ which is simple by 4.8(i).

Recall that $(ad\ x_i)y_jt_j$ is a scalar (in $gr_{\mathcal{F}}F(U)$). Consequently $(ad\ x_i)K(\lambda)^- \subset G^-$. Then from the defining property of $L(\lambda)^-$ we conclude that $K(\lambda)^-$ is *ad* U^+ stable. Let $z \in K(\lambda)^-$ be a weight vector of weight ξ. Then

$$\begin{aligned}(ad\ x_j)z &= x_jzt_j - q^{-2d_j}t_jzx_j \\ &= x_jt_jq^{-(\xi,\alpha_j)}z - q^{(\xi,\alpha_j)}zx_jt_j \in K(\lambda)^- \quad .\end{aligned}$$

Hence $x_jt_jz \in K(\lambda)^-x_jt_j + K(\lambda)^-$. Since $K(\lambda)^-$ is spanned by its weight vectors and j is arbitrary, we conclude that $G^+K(\lambda)^- \subset K(\lambda)^-G^+$. The reverse inclusion follows similarly. Again a similar property holds with + and − interchanged. From the isomorphism $G^- \otimes G^+ \xrightarrow{\sim} G^-G^+$ we obtain $K(\lambda)^-G^+ \cap G^-K(\mu)^+ = K(\lambda)^-K(\mu)^+$. All this combined gives (ii).

Finally

$$\begin{aligned}F(\lambda)F(\mu) &= K(\lambda)^-\tau(\lambda)K(\lambda)^+K(\mu)^-\tau(\mu)K(\mu)^+ \\ &= K(\lambda)^-K(\mu)^-\tau(\lambda)\tau(\mu)K(\lambda)^+K(\mu)^+ \quad , \quad \text{by (ii)} \\ &= K(\lambda+\mu)^-\tau(\lambda+\mu)K(\lambda+\mu)^+ \quad , \quad \text{by (i)} \\ &= F(\lambda+\mu) \ , \quad \text{as required} \ .\end{aligned}$$

4.13. Take $\lambda \in -R^+(\pi)$. By 3.5 there exists a unique up to scalars element $z_\lambda \in F(\lambda) \cap Z(U)$. Again we can view $F(\lambda)$ as lying in $gr_{\mathcal{F}}F(U)$. By 4.5 it is isomorphic to its image and we let y_λ denote the corresponding element of $F(\lambda) \subset gr_{\mathcal{F}}F(U)$ transforming by the trivial representation. Obviously $y_\lambda = gr\ z_\lambda$. Recall also that $gr_{\mathcal{F}}F(U)$ is an integral domain (2.2).

COROLLARY. *In* $gr_{\mathcal{F}}F(U)$ *one has*

$$y_\mu F(\lambda) \subset F(\lambda+\mu)$$

for all $\lambda, \mu \in -R^+(\pi)$. *In particular* $y_\mu y_\lambda = y_{\mu+\lambda}$, *up to a nonzero scalar.*

4.14. Define a new order relation on $P^+(\pi)$ through $\mu \succ \lambda$ if $\mu - \lambda \in P^+(\pi)$. Given $\mu, \lambda \in P^+(\pi)$ observe that there is a unique maximal element, which we denote by $\mu \cap \lambda$, less than both μ, λ. Given $-\lambda, \ -\mu \in P^+(\pi)$ we define $\lambda \cap \mu = -(- \lambda \cap -\mu)$.

When $\lambda, \ \mu \in -R^+(\pi)$ we do not necessarily have $\mu \cap \lambda \in -R^+(\pi)$. This simple combinatorial failure renders the results in sections 6.8, 6.9 and 7.2–7.4 valid only for the simply connected case.

LEMMA. *For all* $\mu, \lambda \in -4P^+(\pi)$, *one has* $K(\mu)^- \cap K(\lambda)^- = K(\lambda \cap \mu)^-$.

Fix $i \in \{1, 2, \dots, \ell\}$. Observe that $-(\alpha_i, \mu \cap \lambda) = min\{-(\alpha_i, \mu), -(\alpha_i, \lambda)\}$. Suppose that $-(\alpha_i, \mu) \leq -(\alpha_i, \lambda)$ and take $\xi \in K(\mu)^- \cap K(\lambda)^-$. One easily checks that the condition $(ad\ y_i)^n \xi \tau(\mu) = 0$ is equivalent to the condition $(ad\ y_i)^n \xi \tau(\mu \cap \lambda) = 0$. Then by the defining property (4.8) of $L(\lambda)^-$ and weight space decomposition, it follows that $ad\ y_i$ acts nilpotently on $\xi \tau(\mu \cap \lambda)$. This holds for all i and so by (JL, 5.9) we conclude that $(ad\ U^-)\xi \tau(\mu \cap \lambda)$ is finite dimensional. Comparison of actions shows that $\xi \tau(\mu \cap \lambda) \in L(\lambda \cap \mu)^-$ and so $\xi \in K(\mu \cap \lambda)^-$, that is $K(\mu)^- \cap K(\lambda)^- \subset K(\mu \cap \lambda)^-$. The opposite inclusion is immediate from 4.12(i) and the fact that $1 \in K(\mu)$, $\forall\ \mu \in -4P^+(\pi)$.

4.15. The following will settle an issue raised by 4.2(ii).

LEMMA. *For each* $\lambda \in -R^+(\pi)$ *one has*

$$\varphi(F(\lambda)) = K\{\tau(4\mu) : \mu \in \Omega(E(1/4\ w_0\lambda))\} .$$

We have $(ad\ U)\tau(\lambda) = (ad\ U^+)(ad\ U^-)\tau(\lambda)$. Moreover we can write $(ad\ U^-)\tau(\lambda) = K(\lambda)^- \tau(\lambda)$ since the commutation relations in $U^- \otimes U^0$ are unchanged by $gr_{\mathcal{F}}$. Let $K(\lambda)^-_\mu$ denote the space of vectors of weight $-\mu$ in $K(\lambda)^-$. By 4.8 and the definition of $K(\lambda)^-$ we have $K(\lambda)^-_\mu \neq 0 \iff 1/4\ \lambda - \mu \in \Omega(E(1/4\ \lambda)) \iff \lambda + 4\mu \in 4\Omega(E(1/4\ w_0\lambda))$. Set $I_\mu = \sum K\{\tau(\nu) | \nu \in 4\Omega(E(1/4\ w_0\lambda))$ and $\nu < \lambda + 4\mu\}$. To prove the lemma it suffices to show for all $\mu \in \Omega(E(1/4\ w_0\lambda)) - 1/4\ \lambda$ that $F_{\lambda,\mu} := \varphi((ad\ U^+)(K(\lambda)^-_\mu \tau(\lambda)) = K\tau(\lambda + 4\mu) mod\ I_\mu$. This will be proved by induction with respect to order relation $\geq$. It is clear for $\mu = 0$, which corresponds to $\varphi((ad\ U^+)\tau(\lambda)) = \tau(\lambda)$.

Analogous to 4.6(***) we have

$$(*) \qquad (ad\ x_i)y_j t_j = \frac{\delta_{ij}}{q^{2d_i} - q^{-2d_i}} \{t_i^4 - 1\} .$$

It follows that if we calculate $(ad\ x_i)K(\lambda)^-_\mu \tau(\lambda)\ mod\ UU^+_+$ then the contribution from the second term on the right hand side of (*) exactly corresponds to the action of U on G^- defined in 4.7. We conclude that the resulting term lies in $K(\lambda)^-_{\mu+\alpha_i}\tau(\lambda)$. By our induction hypothesis this gives a contribution to $F_{\lambda,\mu}$ lying in I_μ. Consequently we can ignore this term in (*). With this modified rule, it is clear that $\varphi((ad\ U^+)p\tau(\lambda)) \subset K\tau(\lambda + 4\mu)$, for all $p \in K(\lambda)^-_\mu$.

It remains to show that this term is nonzero if $p \neq 0$. We shall deduce this from 4.7(i). Indeed it is easy to see that it is enough to show for $a \in G^-$ that $(ad\ x_i)a = 0, \forall\ i$ implies $a \in K$. Yet here we must set $(ad\ x_i)y_jt_j = \delta_{ij}t_i^4/(q^{2d_i} - q^{-2d_i})$ which differs from the rule used in establishing 4.7(i).

We can assume that a is a weight vector of weight $-\mu < 0$. Then $(ad\ x_i)a = 0$ translates to $x_ia\ t_i = q^{-(\mu,\alpha_i)}ax_it_i$ and hence to $x_i(a\tau(-2\mu))t_i^{-1} - q^{2d_i}t_i^{-1}(a\tau(-2\mu))x_i = 0$. Observe that $a\tau(-2\mu)$ is a polynomial in the $y_jt_j^{-1}$ and that

$$(**) \qquad x_i(y_jt_j^{-1})t_i^{-1} - q^{2d_i}t_i^{-1}(y_jt_j^{-1})x_i = (x_iy_j - y_jx_i)t_i^{-1}t_j^{-1} = \frac{\delta_{ij}}{q^{2d_i} - q^{-2d_i}}$$

by our (new) rule. Finally note that the map $x_i \mapsto y_i$, $y_i \mapsto x_i$, $t_i \mapsto t_i^{-1}$ extends to an automorphism θ of U taking in particular $y_jt_j^{-1}$ to x_jt_j and $x_ibt_i^{-1} - q^{2d_i}t_i^{-1}bx_i$ to $y_i\theta(b)t_i - q^{2d_i}t_i\theta(b)y_i = (ad\ y_i)\theta(b)$. Consequently we have only to show for $a \in G^+$ that $(ad\ y_i)a = 0, \forall\ i$ implies $a \in K$. By (**) we have $(ad\ y_i)x_jt_j = \delta_{ij}/(q^{2d_i} - q^{-2d_i})$ which is just the (old) rule used in establishing 4.7(i) (with +, $-$ interchanged). Thus our assertion follows from 4.7(i).

4.16. Corollary. *Take $\lambda \in -R^+(\pi)$ and let M be an ad U submodule of $(ad\ U)\tau(\lambda)$. Then*

$$\{W\lambda\} \subset Supp\ \varphi(M) \subset 4\Omega(E(1/4\ w_0\lambda))$$

that is it suffices to take $s = 1$ in 4.2(ii).

The second inclusion follows from 4.15. Then by 4.2(i) for the first inclusion it suffices to show that $\lambda \in Supp\ \varphi(M)$. By 4.5, $gr_{\mathcal{F}}M$ is a nonzero submodule of $G(\lambda)$ and hence of $(ad\ U)\tau(\lambda)$ considered as an $(ad\ U)$ submodule of $G(U)$. Writing $(ad\ U)\tau(\lambda) = K(\lambda)^- \otimes K\tau(\lambda) \otimes K(\lambda)^+ \cong L(-1/4\lambda) \otimes L(1/4\lambda)^*$, we see that it suffices to show that every U submodule N lying in a $T \times T$ complement to $e_{-1/4\lambda} \otimes e_{1/4\lambda}$ is zero. Indeed if $a \in N$ is a nonzero highest weight vector it must take the form $a = e_{-1/4\lambda} \otimes f\ mod\ L(-1/4\lambda) \otimes L(1/4\lambda)^*$ (notation 3.2) for some nonzero weight vector $f \in L(1/4\lambda)^*$. (An easy calculation shows that only such vectors can satisfy $x_ia = 0, \forall\ i$). Then up to nonzero scalar $N \ni y_ia = e_{-1/4\lambda} \otimes y_if\ mod\ L(-1/4\lambda) \otimes L(1/4\lambda)^*$ and so some $b \in N$ takes the above form with $f \neq 0$ and satisfying $y_if = 0, \forall\ i$. Then $f = e_{1/4\ \lambda}$, up to a nonzero scalar.

Remark. It is clear from 4.15 that one cannot take $s = 1$ in 4.2(ii) for an arbitrary simple *ad* U submodule M of U. Finally if $M = Kz_\lambda$, then by (JL, 8.6, eq.6) one has $Supp\ \varphi(M) = 4\Omega(E(1/4\ w_0\lambda))$. Yet as noted in loc. cit. one may choose $z'_\lambda \in Z(U)$ such that $Supp\ \varphi(Kz'_\lambda) = \{W\lambda\}$.

5. Multiplicities.

5.1. We establish a number of results below which are not all strictly necessary for our main theorems. However, we want to leave open the possibility of an alternative proof of 6.12 not based on taking specialization at $q = 1$ and using Kostant's theorem.

Fix an isomorphism class E of finite dimensional simple U modules and let $F(U)_E$ denote the E isotypical component of $F(U)$ considered as a U module for adjoint action.

LEMMA. *$F(U)_E$ is a finitely generated $Z(U)$ module.*

Fix a basis $\{e_s\}$ for the zero weight space E_0 of E. By 4.2 we have $\varphi(e_s) \in KWT_<$. Set $r = dim\ E_0 < \infty$. Consider $\{\varphi(e_s)\}_{s=1}^r$ as an r-tuple in $(KWT_<)^r$. By (JL, 8.6) it follows that $KWT_<$ is a finitely generated module over $\varphi(Z(U))$ and hence so is $(KWT_<)^r$. Now let ε_s denote the subspace of $F(U)_E$ spanned by the copies of the vector e_s in the various submodules of $F(U)$ isomorphic to E. It is clear that we may regard ε_s as a $Z(U)$ module. Since the restriction of φ to the zero weight space is a homomorphism (JL, 8.1) we conclude that $\{\varphi(\varepsilon_s)\}_{s=1}^r$ is a finitely generated module over $\varphi(Z(U))$. By the injectivity assertion of 3.2 we conclude that $F(U)_E$ is finitely generated over $Z(U)$.

5.2. Given any weight module E we let $\Omega(E)$ denote its set of weights. Take $\lambda, \mu \in P^+(\pi)$. We call λ sufficiently large relative to μ, noted $\lambda >> \mu$, if $\lambda + \Omega(L(\mu)) \subset P^+(\pi)$. The following is well-known but we give the proof for completion. Let $L(\mu)_0$ denote the zero weight space of $L(\mu)$.

LEMMA. *For all $\lambda, \mu \in P^+(\pi)$ one has $dim\, Hom_U(End_K L(\lambda), L(\mu)) \leq dim\, L(\mu)_0$ with equality if $\lambda >> \mu$.*

A standard isomorphism gives

$$(*) \qquad dim_K Hom_U(End_K L(\lambda), L(\mu)) = dim_K Hom_U(L(\mu) \otimes_K L(\lambda), L(\lambda))$$

for all $\lambda, \mu \in P^+(\pi)$. Now assume $\lambda >> \mu$. By say the isomorphism discussed in 4.4 we have

$$L(\lambda) \otimes L(\mu) = \bigoplus_{\nu \in \Omega(L(\mu))} L(\lambda + \nu) \quad .$$

It is immediate from this that the right hand side of $(*)$ is just $dim\ L(\mu)_0$ as required.

For the general case, observe that by 4.13 we have for each $\lambda, \xi \in R^+(\pi)$ an injection

$$(**) \qquad End_K L(\lambda) \hookrightarrow End_K L(\lambda + \xi)$$

of U modules (for the adjoint action). Warning $-(**)$ is not a ring embedding. This result can be extended to $P^+(\pi)$ by augmenting U as in 4.2. Alternatively one can just prove the corresponding result for enveloping algebras. This is essentially well-known; but for completion we give a proof below.

Set $\mathfrak{b} = \mathfrak{h} \oplus \mathfrak{n}^+$ and let $k_\xi : \xi \in \mathfrak{h}^*$ denote the one dimensional $\mathfrak{b}$ module of weight λ. Recall that w_0 is the unique longest element of W. Let $e_{w_0\lambda} \in L(\lambda)$: $\lambda \in P^+(\pi)$ denote a choice of lowest weight vector for $L(\lambda)$. Assume $\xi \in P^+(\pi)$. From the well-known formula for $Ann_{U(\mathfrak{b})} e_{w_0\lambda}$ (see [7], 2.13 for example) we obtain a surjection

$$k_{w_0\xi} \otimes_k L(\lambda) \twoheadleftarrow L(\lambda + \xi)$$

of $\mathfrak{b}$ modules. Let $\lambda \mapsto w \cdot \lambda$ denote the translated action of W—see 6.15. Dualizing and tensoring by $k_{w_0 \cdot (\lambda+\xi)}$ gives an injection

$$L(\lambda + \xi)^* \otimes k_{w_0 \cdot (\lambda+\xi)} \hookleftarrow L(\lambda)^* \otimes k_{w_0 \cdot \lambda}$$

of $\mathfrak{b}$ modules. Set $n = dim\ \mathfrak{n}^+$. Now apply the derived functor $\mathcal{D}^n_{w_0}$ defined in ([7], 2.9, 2.15, 5.5). It is left exact, commutes with tensoring over a $\mathfrak{g}$ module and satisfies $\mathcal{D}^n_{w_0} k_{w_0 \cdot \lambda} = L(\lambda)$ for $\lambda \in P^+(\pi)$, by ([7], 5.6). Hence we obtain

$$End_K L(\lambda + \xi) \cong L(\lambda + \xi)^* \otimes L(\lambda + \xi) \hookleftarrow L(\lambda)^* \otimes L(\lambda)$$

as required. Instead of the functor $\mathcal{D}_{w_0}$ one can take global sections of an appropriate sheaf over the flag variety and apply the Borel-Weil theorem to get the asserted claim. In any case it is clear that the quantum group proof is more elementary. We shall see in 6.2(i) that this elementary proof has an enveloping algebra analogue.

5.3. We shall need the following properties of Poincaré series $R_V(q)$ defined for any graded K vector space V by

$$R_V(q) = \sum_{m=0}^{\infty} (dim\ V_m) q^m \quad \text{where} \quad V = \bigoplus_{m=0}^{\infty} V_i \ .$$

Let A be a commutative graded K-algebra (K infinite) with generators $y_1, y_2, \ldots, y_n$ homogeneous of degrees $1 \leq d_1, \ldots, d_n < \infty$. Taking appropriate powers of the y_i we can find a subalgebra B with finitely many generators homogeneous of the same degree such that A is a finitely generated B module. By the normalization lemma we can assume (up to a linear transformation) that the first r of these generate a polynomial subring S over which B and hence A is finitely generated. Without loss of generality we can assume these generators to be $y_1, y_2, \dots, y_r$. Let S_+ denote the augmentation ideal of S.

Let M be a graded A module generated by finitely many homogeneous elements of degree ≥ 0. Let L (resp. C) denote the kernel (resp. cokernel) of the endomorphism $m \mapsto y_1 m$ of M. Then

$$(*) \qquad R_M(q)(1-q^{d_1}) = R_C(q) - q^{d_1} R_L(q) \ .$$

Noting that L and C are finitely generated over $K[y_2, \dots, y_n]$ an easy induction argument using (*) shows that

$$Q_M(q) := R_M(q)/R_S(q)$$

is a polynomial. Let $Fr(S)$ denote the fraction field of S. Let $rk_S M$ denote the dimension of $Fr(S) \otimes_S M$ over $Fr(S)$ and choose homogeneous generators $m_1, m_2, \dots, m_r \in M$ with $r = rk_Q(M)$, to form a basis for this vector space. (For some calculations it is useful to note that the m_i can be chosen from a graded complement of S_+M in M). Then $N := \sum S m_i$ is a free S module of rank r, whilst M/N is torsion. We conclude that

$$Q_M(1) = Q_N(1) + Q_{M/N}(1) = rk_S M \ .$$

In particular,

$$P_M(q) := R_M(q)/R_A(q) = Q_M(q)/Q_A(q)$$

has no pole at $q = 1$. In general $P_M(1)$ need not be an integer.

Now assume that A in an integral domain and let $Fr(A)$, $rk_A(M)$ be defined as for S above. Then a similar argument (using say 8.4(*) to show that $P_M(1)$ vanishes on torsion modules) gives

$$P_M(1) = rk_A M \ .$$

Let I_+ denote the augmentation ideal of A. Any graded complement V to I_+M in M generates M over A, from which we conclude that

$$dim_K M/I_+M \geq rk_A M = P_M(1) \ .$$

More generally let I_χ be an ideal of codimension 1 in A. Choose a complementary subspace V_χ to $I_\chi M$ in M and set $N = AV_\chi$. One has $M = N + I_\chi M$ and that V_χ complements $I_\chi N$ in N. It follows that the composite map $N \hookrightarrow M \longrightarrow M/I_\chi M$ factors to an isomorphism $N/I_\chi N \xrightarrow{\sim} M/I_\chi M$. Moreover $I_\chi(M/N) = M/N$.

Since M/N is a finitely generated A module, it follows by Nakayama's lemma that M/N is torsion with respect to the Ore set $A \setminus I_\chi$ and so by the above $P_{M/N}(1) = 0$. Consequently

$$P_M(1) = P_N(1) \leq dim\ V_\chi = dim(M/I_\chi M) \quad .$$

5.4. Fix $\mu \in P^+(\pi)$ and define the Poincaré series

$$R_\mu(q) := \sum_{m=0}^{\infty} [F_m(U) : L(\mu)] q^m \quad .$$

Obviously $R_0(q)$ corresponds to the Poincaré series for $Z(U)$ considered as a graded vector space. Recall (2.2) that $gr_{\mathcal{F}}(U)$ is an integral domain. Fix an isomorphism class E as in 5.1. By 4.13 and the uniform bound implied by 5.2 we conclude for each $i \in \{1, 2, \dots, \ell\}$ that $[y_{\omega_i} F(\mu) : E] = [F(\mu + \omega_i) : E]$ if $(\mu, \alpha_i^\vee)$ is sufficiently large. This implies that $gr_{\mathcal{F}} F(\check{U})$ is a finitely generated (graded) $Y(\check{U}) := gr_{\mathcal{F}} Z(\check{U})$ module. Since $Y(\check{U})$ is finite over the noetherian ring $Y(U)$ we conclude it holds with $\check{U}$ replaced by U. Set $P_\mu(q) = R_\mu(q)/R_0(q)$. By 5.3, $P_\mu(q)$ has no pole at $q = 1$ and since $Y(U)$ is an integral domain $P_\mu(1) \in \mathbb{N}$. Should $Y(U)$ be a polynomial ring then P_μ is a polynomial. Should $F(U)$ be free over $Y(U)$, then the coefficients of $P_\mu(q)$ are integers ≥ 0. The corresponding exponents are the natural analogues of Kostant's generalized exponents which were computed for enveloping algebras by Hesselink [4]. We remark that

LEMMA. *For all $\mu \in P^+(\pi)$ one has*

$$P_\mu(1) = dim\ L(\mu)_0 \quad .$$

Let $R_\mu^n(q)$ be the sum of the first n terms in the expansion of $R_\mu(q)$. By 5.2 we obtain $R_\mu(q) - R_\mu^n(q) = (R_0(q) - R_0^n(q))\ dim\ L(\mu)_0$, for all n sufficiently large. Dividing by $R_0(q)$ and evaluating at $q = 1$ proves the required assertion.

5.5. For a separation theorem we actually need a refinement of 5.4. Set $G(U) = gr_{\mathcal{F}} F(U)$. Let $Y(U)$ denote the (central) subalgebra of $G(U)$ of elements of $G(U)$ which transform by the trivial representation of U (under adjoint action). By 4.10, $Y(U)$ is spanned by the $y_\xi : \xi \in -R^+(\pi)$. Let Y_+ denote the ideal of $Y(U)$ generated by the $y_\xi : \xi \neq 0$. It is a graded ideal of $Y(U)$ complemented by the scalars. Set $J_+ = G(U)Y_+$ which is a two-sided ideal of $G(U)$. By 5.3 and 5.4 we have $[G(U)/J_+ : L(\mu)] \geq dim\ L(\mu)_0$ for all $\mu \in P^+(\pi)$. What we need is equality. This question motivated Section 4 where equality is proved in the simply connected case. The following example shows that equality cannot hold in general. Moreover, a similar phenomenon can be expected whenever R^+ is not stable under the cap operation of 4.14.

Example. Take $U = U_q(\mathfrak{sl}(3))$. The "adjoint representation" with highest weight $\alpha_1 + \alpha_2$ occurs in $(ad\ U)\tau(-4\omega_i) \subset G(\check{U}) : i = 1,2$ with highest weight vector e_i given by

$$e_1 = (x_2t_2x_1t_1 - q^{-2}x_1t_1x_2t_2)\tau(-4\omega_1), \qquad \text{for } i = 1 ,$$
$$e_2 = (x_1t_1x_2t_2 - q^{-2}x_2t_2x_1t_1)\tau(-4\omega_2), \qquad \text{for } i = 2 .$$

Note that $e_1,\ e_2$ are distinct and do not lie in U. Let $y_i : i = 1,2$ be a basis vector for the trivial one dimensional submodule of $(ad\ U)\tau(-4\omega_i)$. Since $4(\omega_1 + \omega_2)$, $12\omega_1$, $12\omega_2 \in R^+(\pi)$ it follows that e_1y_2, $e_1y_1^2$, e_2y_1, $e_2y_2^2 \in G(\check{U}) \cap gr_{\mathcal{F}}U = G(U)$. Taking account that the above weights are minimal nonzero elements of $R^+(\pi)$ it easily follows that $[G(U)/J_+ : E(\alpha_1 + \alpha_2)] \geq 4$. Moreover the relations $(e_1y_2)(y_1^3) = (e_1y_1^2)(y_1y_2)$ and similarly with e_2 show explicitly that $G(U)$ is not free over $Y(U)$. On the other hand $e_1,\ e_2$ are free generators of the highest weight space of $G(\check{U})_E$ considered as a module over $Y(\check{U})$.

6. Specialization.

6.1. It is well known (cf. [8], 3.5 for example) that $\mathfrak{g}$ acts by derivations on $S(\mathfrak{n}^-)$ in such a manner that $S(\mathfrak{n}^-)$ is isomorphic to $\delta M(0)$ as a $U(\mathfrak{g})$ module. Moreover one may twist this action in the following fashion. Under action by derivations, the identity 1 of $S(\mathfrak{n}^-)$ has zero weight. For any $\lambda \in \mathfrak{h}^*$ we can arrange for $\mathfrak{g}$ to act by inhomogeneous derivations (i.e. allow also multiplication by functions) so that the identity has weight λ. In this case $S(\mathfrak{n}^-)$ is isomorphic to $\delta M(\lambda)$. We call λ the twisting parameter. Given $\lambda \in P^+(\pi)$, then the unique simple submodule $L(\lambda)$ of $\delta M(\lambda)$ is finite dimensional. For consistency with the notation of Sect. 4 we denote by $S(-4\lambda)^-$ the subspace of $\delta M(\lambda)$ which identifies with $L(\lambda)$. All this is quite elementary; but may also be viewed geometrically as follows. Given $\lambda \in P(\pi)$ one may define the sheaf $G \times_B k_\lambda$ on the flag variety G/B. Then $\delta M(\lambda)$ identifies with the space of local sections on the open Bruhat cell and for $\lambda \in P^+(\pi)$, its socle $L(\lambda)$ identifies with the space of global sections. This point of view is quite unnecessary for us; but may benefit some readers.

Lemma. *For all $\lambda, \mu \in -4P^+(\pi)$ one has*

(i) $S(\lambda)^-S(\mu)^- = S(\lambda + \mu)^-$, *in particular* $S(\lambda)^- \subset S(\mu)^-$ *if* $\lambda \leq \mu$.

(ii) $S(\lambda)^- \cap S(\mu)^- = S(\lambda \cap \mu)^-$.

(iii) $\left(\sum_{j=1}^t S(\lambda_j)^-\right) \cap S(\mu)^- = \sum_{j=1}^t (S(\lambda_j)^- \cap S(\mu)^-)$. $\forall\, t \in \mathbb{N}^+$, $\forall\, \lambda_i,\ \mu \in -4P^+(\pi)$.

(i) and (ii) follow as in 4.12(i) and 4.14.

(iii) As pointed out to us by P. Polo this is an easy consequence of recent work of G. Lusztig who in a series of papers [15-17] has studied a canonical basis for U^+. Here we shall only require the canonical basis $\mathcal{B} := \{b_i\}_{i\in I}$ of

$U(\mathfrak{n}^+)$ which we obtain on specialization. This consists of weight vectors ([17], Prop. 11.2) and has following the basic property ([17], Cor. 11.10). Let $f_{-\lambda}$ denote a lowest weight vector for $L(\lambda)^* : \lambda \in P^+(\pi)$. Set $I_\lambda = \{i \in I \mid b_i f_{-\lambda} \neq 0\}$. Then $\{b_i f_{-\lambda} \mid i \in I_\lambda\}$ is a basis for $L(\lambda)^*$. We remark that the result is stated more exactly in this form in ([15], Introduction) where it is proved in the simply-laced case.

It remains to observe that $\mathcal{B}$ gives rise to a basis $\{a_i\}_{i \in I}$, indexed by I, of $S(\mathfrak{n}^-)$ such that for each $\lambda \in P^+(\pi)$ the subset $\{a_i\}_{i \in I_\lambda}$ is a basis for $L(\lambda)$, the latter being identified as above with a subspace of $S(\mathfrak{n}^-)$.

Let $\varepsilon : S(\mathfrak{n}^-) \to k$ denote the augmentation of $S(\mathfrak{n}^-)$ and $b \mapsto \sigma(b)$ the principal antiautomorphism of $U(\mathfrak{n}^+)$. Recall that $\mathfrak{n}^+$ acts on $S(\mathfrak{n}^-)$ by derivations and so we have a bilinear form $F(a,b) := \varepsilon(\sigma(a)b)$ on $U(\mathfrak{n}^+) \times S(\mathfrak{n}^-)$. Recall the well-known fact that $S(\mathfrak{n}^-)^{\mathfrak{n}^+}$ reduces to scalars—the proof can be made to follow 4.7(i). It easily follows $\{b \in S(\mathfrak{n}^-) \mid F(a,b) = 0, \forall\ a \in U(\mathfrak{n}^+)\} = 0$. Since $F(a,b) = 0$ whenever $a \in U(\mathfrak{n}^+)_\mu$, $b \in S(\mathfrak{n}^-)_{-\nu}$ with $\mu, \nu \in Q^+(\pi)$ distinct, we conclude that $\{b \in S(\mathfrak{n}^-)_{-\mu} \mid (a,b) = 0, \forall\ a \in U(\mathfrak{n}^+)_\mu\} = 0$. Since $dim\ U(\mathfrak{n}^+)_\mu = dim\ S(\mathfrak{n}^-)_{-\mu} < \infty$ it follows that F restricts to a nondegenerate pairing F_μ of $U(\mathfrak{n}^+)_\mu$ with $S(\mathfrak{n}^-)_{-\mu}$. Let $\{a_j\}_{j \in I}$ be the basis of $S(\mathfrak{n}^-)$ obtained from B through this pairing. That is $F(b_i, a_j) = \delta_{ij}$, $\forall\ i,j \in I$, where δ denotes the Kronecker delta.

To show that the above basis has the required property is straightforward. Let $\{\eta_j : j \in I_\lambda\}$ be a basis for $L(\lambda)$ dual to the basis $\{b_i f_{-\lambda} : i \in I_\lambda\}$ of $L(\lambda)^*$. Then $(\eta_j, b_i f_{-\lambda}) = \delta_{ij}$, $\forall\ i,j \in I_\lambda$. Moreover this equality extends to all $i \in I$ by the fundamental fact that $b_i f_{-\lambda} = 0$ if $i \in I \setminus I_\lambda$. Now identify the η_j with elements of $S(\mathfrak{n}^-)$ through the embedding $L(\lambda) \hookrightarrow \delta M(\lambda) \xrightarrow{\sim} S(\mathfrak{n}^-)$ where we recall that the last isomorphism is a map of $U(\mathfrak{n}^+)$ modules. Then we can write $\delta_{ij} = (\eta_j, b_i f_{-\lambda}) = (\sigma(b_i)\eta_j, f_{-\lambda}) = \varepsilon(\sigma(b_i)\eta_j) = F(b_i, \eta_j)$, $\forall\ i \in I,\ j \in I_\lambda$. Consequently $\eta_j = a_j$, $\forall\ j \in I_\lambda$, as required.

Remarks. We see from the above analysis that $L(\lambda) : \lambda \in P^+(\pi)$ can be embedded in $S(\mathfrak{n}^-)$ as a $U(\mathfrak{n}^+)$ submodule in only one way. Actually this is an elementary fact and indeed

$$L(\lambda) = \{b \in S(\mathfrak{n}^-) \mid F(a,b) = 0, \quad \forall\ a \in Ann_{U(\mathfrak{n}^+)} f_{-\lambda}\}$$

for any such embedding.

It is also possible that (iii) can be recovered from ([19], 4.1); but this is less immediate.

Finally via 6.6 we shall see that a result analogous to (iii) holds also for the $K(\mu)^- : \mu \in -4P^+(\pi)$ defined in Sect. 4. This may also be proved directly by the above method from Lusztig's basis for U^+ and using 4.7(i) to obtain a nondegenerate pairing $U^+ \times G^- \to K$.

6.2. The results of 6.1 of course hold with—replaced by + and in particular we let $S(\lambda)^+ : \lambda \in -4P^+(\pi)$ denote the corresponding finite dimensional subspace of $S(\mathfrak{n}^+)$. One should of course realize that $S(\lambda)^+$ is the submodule of the dual of a Verma module defined with respect to the opposed Borel $\mathfrak{b}^-$ and has lowest weight $-\lambda$. Then $S(\lambda)^+$ identifies with $(S(\lambda)^-)^*$ as a $U(\mathfrak{g})$ module. Consequently under the diagonal action of $U(\mathfrak{g})$ we have a unique up to scalars invariant element $y^0_\lambda \in S(\lambda)^- \otimes S(\lambda)^+ : \lambda \in -4P^+(\pi)$.

LEMMA. *For all* $\lambda, \mu \in -4P^+(\pi)$ *one has*

(i) $y^0_\lambda(S(\mu)^- \otimes S(\mu)^+) \subset S(\lambda+\mu)^- \otimes S(\lambda+\mu)^+$,

(ii) $y^0_\lambda y^0_\mu = y^0_{\lambda+\mu}$,

up to a nonzero scalar.

Apply 6.1(i).

6.3. Let $\omega \in P^+(\pi)$ be the fundamental weight corresponding to $\alpha \in \pi$.

LEMMA. $y^0_{-4\omega}$ *is an irreducible element of* $S(\mathfrak{n}^-) \otimes S(\mathfrak{n}^+)$.

First we remark that the action of $\mathfrak{n}^+$ on $S(\mathfrak{n}^-)$ is independent of the twisting parameter and is always by derivations. Secondly in what follows we shall always view $S(-4\omega)^-$ as a subspace of $S(\mathfrak{n}^-)$. Then the weights are translated by $-\omega$ and in particular it has lowest weight $\mu := w_0\omega - \omega$, where w_0 denotes the unique longest element of W. Let b_μ denote the corresponding element of $S(4\omega)^+$. It is standard that the term $a_\mu b_{-\mu}$ occurs in $y^0_{-4\omega}$ with a nonzero coefficient in the obvious well-defined sense. Suppose $y^0_{-4\omega}$ factors nontrivially. Then an easy weight space analysis shows that a_μ factors nontrivially as a product of weight vectors $a_{\mu_1}, a_{\mu_2} \in S(\mathfrak{n}^-)$.

For each $\beta \in \pi$, let $p_{-\beta}$ denote the unique up to scalars vector of weight $-\beta$ in $S(\mathfrak{n}^-)$ and $x_\beta \in \mathfrak{g}$ (resp. $s_\beta \in W$) the corresponding root vector (resp. reflection). Recall that if $x_\beta a = 0 : a \in S(\mathfrak{n}^-)$, $\forall\ a\ \beta \in \pi$ then a is a scalar. Consequently if $x^r_\beta a = 1$ for some $r \geq 0$, then $a = p^r_{-\beta}$, up to a nonzero scalar. (All this is of course well-known).

Fix $\beta \in \pi$. By nilpotence and because $S(\mathfrak{n}^-)$ is an integral domain, we can find integers $r_1, r_2 \geq 0$ such that $x^{r_1+1}_\beta a_{\mu_1} = x^{r_2+1}_\beta a_{\mu_2} = 0$. Then $x^{r+1}_\beta a_\mu = 0 : r = r_1 + r_2$ whilst up to a nonzero scalar

$$x^r_\beta a_\mu = (x^{r_1}_\beta a_{\mu_1})(x^{r_2} a_{\mu_2}) \neq 0 \ .$$

Thus $x^r_\beta a_\mu$ is an extreme weight vector of $S(-4\omega)^-$, hence of weight $s_\beta w_0\omega - \omega$, and factors nontrivially unless $a_{\mu_i} = p^{r_i}_{-\beta}$ for some $i \in \{1,2\}$. By a standard induction procedure on decreasing length of Weyl group elements we conclude from this that $S(-4\omega)^-$ admits an extreme weight of the form $p^{s_1}_{-\beta_1} p^{s_2}_{-\beta_2} : s_1, s_2$

integers > 0, $\beta_1, \beta_2 \in \pi$. Thus $s_1\beta_1 + s_2\beta_2 = \omega - w\omega$, for some $w \in W$. Then necessarily $\beta_1 \neq \beta_2$ for otherwise $\beta_1 = \beta_2 = \alpha$ and $s_1+s_2 = 1$ which is impossible. Yet applying x_{β_1}, x_{β_2} we conclude that both $p_{-\beta_1}, p_{-\beta_2} \in S(-4\omega)^-$. This again implies $\beta_1 = \alpha = \beta_2$. This contradiction proves the lemma.

Remark. The $y^0_{-4\mu} : \mu \in P^+(\pi)$ are linearly independent by 6.1(ii). It follows that the y^0_{-4w} are algebraically independent.

6.4. Take $\lambda, \mu \in P^+(\pi)$. With respect to $\succ$, let $\mu \cup \lambda$ denote the unique smallest element of $P^+(\pi)$ greater than both μ, λ. Given $\lambda, \mu, \lambda', \mu' \in P^+(\pi)$ such that $\mu + \lambda = \mu' + \lambda'$ one easily checks that $\mu \cup \mu' + \lambda \cap \lambda' = \mu + \lambda$. Of course similar considerations apply to $-4P^+(\pi)$. For all μ, $\nu \in -4P^+(\pi)$ with $\nu \leq \mu$ set $S^\nu(\mu) := y^0_{\nu-\mu}(S(\mu)^- \otimes S(\mu)^+)$. Given $\nu \in -4P^+(\pi)$, set $-4P^+_\nu(\pi) = \{\mu \in -4P^+(\pi) \mid \mu \geq \nu\}$.

PROPOSITION. *For all $\nu \in -4P^+(\pi)$ one has*

(i) $S^\nu(\mu) \cap S^\nu(\mu') = S^\nu(\mu \cap \mu')$, $\forall\ \mu, \mu' \in -4P^+_\nu(\pi)$.

(ii) $S^\nu(\mu) \cap \left(\sum_{j=1}^t S^\nu(\lambda_j)\right) = \sum_{j=1}^t (S^\nu(\mu) \cap S^\nu(\lambda_j))$, $\forall\ \mu, \lambda_1, \lambda_2, \dots, \lambda_t \in -4P^+_\nu(\pi)$, $\forall\, t \in \mathbb{N}^+$.

The inclusions $\supset$ are immediate from 6.2. (i) Since $S(\mathfrak{n}^-) \otimes S(\mathfrak{n}^+)$ is a unique factorization domain it follows by 6.3 that any element in the left hand side of (i) takes the form $y^0_{\mu\cup\mu'}d$ with $d \in S(\mathfrak{n}^-) \otimes S(\mathfrak{n}^+)$. We claim that d is locally finite with respect to the diagonal action of $U(\mathfrak{g})$ in $S(\mathfrak{n}^-) \otimes S(\mathfrak{n}^+)$ where the action in each factor is defined relative to the twisting parameter $\mu \cap \mu'$. By the Leibnitz formula for twisted derivations (twisting parameters add) and using that $y^0_{\mu\cup\mu'}$ is invariant for the twist defined by $\mu \cup \mu'$ and is nonzero divisor, we obtain $d \in S(\mu \cap \mu') := \{a \in S(\mathfrak{n}^-) \otimes S(\mathfrak{n}^+) \mid \mathit{dim}\ U(\mathfrak{g})a < \infty$ for the diagonal action of $\mathfrak{g}$ defined by the twisting parameter $\mu \cap \mu'\}$. It remains to show that $S(\mu) = S(\mu)^- \otimes S(\mu)^+$, $\forall \mu \in -4P^+(\pi)$. The inclusion $\supset$ is clear. Since the action of $U(\mathfrak{n}^+)$ on $S(\mathfrak{n}^-)$ is locally finite and independent of the twist, we can assume that every finite dimensional $U(\mathfrak{n}^+)$ submodule $E \subset S(\mu)$ belongs to some $F \otimes S(\mathfrak{n}^+)$ with F finite dimensional. Let $\psi : E \longrightarrow F \otimes S(\mathfrak{n}^+)$ denote the embedding which results. From the standard isomorphism $\psi \mapsto T_\psi$ of $Hom_{U(\mathfrak{n}^+)}(E,\ F \otimes S(\mathfrak{n}^+))$ onto $Hom_{U(\mathfrak{n}^+)}(E \otimes F^*,\ S(\mathfrak{n}^+))$ we conclude that $Im\ \psi \subset (F \otimes Im\ T_\psi)$. Yet $Im\ T_\psi$ is a finite dimensional $U(\mathfrak{n}^+)$ submodule of $S(\mathfrak{n}^+)$, hence contained in $S(\mu)^+$. A similar argument with + replaced by $-$ completes the proof of (i).

(ii) Given α a positive root we can write $\alpha = \sum k_i\alpha_i : k_i \in \mathbb{N}$, $a_i \in \pi$ and we set $o(\alpha) = \sum k_i$. Give the polynomial algebra $S(\mathfrak{n}^+)$ (resp. $S(\mathfrak{n}^-)$) a grading by declaring the generator x_α (resp. $x_{-\alpha}$) corresponding to the positive root α, to have degree $o(\alpha)$. Then $S(\mu)^-$, $S(\mu)^+$ and $S(\mu) = S(\mu)^- \otimes S(\mu)^+$ are graded subspaces of $S(\mathfrak{n}^-)$, $S(\mathfrak{n}^+)$ and $S(\mathfrak{n}^-) \otimes S(\mathfrak{n}^+)$ respectively. Given $0 \neq a \in S(\mathfrak{n}^-) \otimes S(\mathfrak{n}^+)$ we denote by $gr\ a$ the *lowest* degree component of a and define $\ell\ deg\ a$ to be the degree of $gr\ a$. Observe that $gr\ y_\mu^0 : \mu \in -4P^+(\pi)$ is a nonzero scalar which we may assume equal to 1 and so $\ell\ deg\ y_\mu^0 = 0$.

Take $a \in S^\nu(\mu) \cap \sum_{j=1}^t S^\nu(\lambda_j)$ nonzero. We can write $a = \sum_{j=1}^t a_j$. Then possibly dropping some terms (of higher degree) we have, for some $I \subset \{1, 2, \ldots, t\}$, that one of the following hold.

Case 1. $gr\ a = \sum_{j \in I} gr\ a_j$.

Case 2. $\sum_{j \in I} gr\ a_j = 0$.

We shall prove by induction on $\ell\ deg$, which is permissible because all the elements lie in the finite dimensional space $S(\nu)$, and by induction on t, that these imply that we can choose $a_j \in S^\nu(\mu)$. This will prove the assertion.

Consider case 1. It is clear that $gr\ a \in S(\mu) \cap \sum_{j=1}^t S(\lambda_j)$ and so by 6.1(iii) we can write $gr\ a = \sum b_j : b_j \in S(\mu) \cap S(\lambda_j) = S(\mu \cap \lambda_j)$, by 6.1(ii). Set $c_j = y_{\nu-(\mu\cap\lambda_j)} b_j$ which by 6.1(i) and 6.2(i) lies in $S^\nu(\mu) \cap S^\nu(\lambda_j)$. Then $gr\ c_j = b_j$. Set $c = \sum c_j$. It follows easily that $\ell\ deg(a - c) > \ell\ deg\ a$. Yet $a - c \in S^\nu(\mu) \cap \sum_{j=1}^t S^\nu(\lambda_j)$ and so the assertion follows by the induction hypothesis.

Consider case 2. This is just case 1 with t reduced by 1. It therefore means that we can find $c_j \in S^\nu(\lambda_j)$ such that $\sum c_j = 0$ and $gr\ a_j = gr\ c_j, fa\ j \in I$. Then $\ell\ deg(a_j - c_j) > \ell\ deg\ a_j, \forall\ a\ j \in I$ whilst $a = \sum (a_j - c_j)$. Consequently we return to cases 1 or 2 with $\ell\ deg$ increased, which holds by the induction hypothesis. This completes the proof of (ii).

Remarks. Notice that the argument of (ii) gives a second proof of (i). We remark that the isomorphism T constructed in (i) is valid for any Hopf algebra, in particular, U^+. Combined with 4.7(ii) and 6.7 this leads (by the analysis of (i)) to a more elegant proof of 4.9.

6.5. We need the result corresponding to 6.4 in the noncommutative ring $G(U)$. For this we use specialization at $q = 1$. Set $A = k[q, q^{-1}]$. Here we might define $\hat{G}^-$ to be the A subring generated by the elements $y_i t_i : i = 1, 2, \ldots, \ell$ and set

$$V^- = \hat{G}^- \otimes_A A/\langle q - 1 \rangle \ .$$

As in (JL, 4.10) it follows easily that V^- is isomorphic to $U(\mathfrak{n}^-)$. However this is the wrong thing to do! The reason is that the induced action of $U(\mathfrak{g})$ (obtained from 4.7(*) through the identification $\hat{U} \otimes_A A/\langle q - 1 \rangle \cong U(\mathfrak{g}) \otimes_k k[T]$, notation JL, 4.10 is not well-defined because of the denominator in 4.6(***). Only the action of $U(\mathfrak{b}^-)$ is defined (via 4.7(*)) and it is easy to see that this is just the adjoint action. Nevertheless we shall need this specialization later and we call it the U-specialization of G^- at $q = 1$.

6.6. It is convenient to replace A by the localization A_0 of $k[q]$ at the prime ideal $\langle q-1\rangle$. (Actually only a small number of roots of unity must be avoided). Identify G^- with $G^-\otimes K\tau(\lambda)$ with the action defined by 4.7(*). Call λ the twisting parameter. Note that the action of U^+ on G^- in 4.7(*) is just the adjoint action, so is *independent* of the twisting parameter λ and moreover increases weights. We may therefore define the A_0 submodule $\tilde{G}^-$ of G^- inductively as follows. View G^- as a module for the adjoint action of T and extend the order relation (3.1) on $Q(\pi)$ to a total order. Set $\tilde{G}_0^- = A_0$. Fix $\xi \in Q^-(\pi)$, let G_ξ^- denote the ξ weight subspace of G^-, assume that $\tilde{G}_\eta^-$ is defined for all $\eta > \xi$ and set

$$\tilde{G}_\xi^- = \{a \in G_\xi^- \,|\, (ad\ x_i)a \in \tilde{G}_{\xi+\alpha_i}^-, \forall\ i\} \quad ,$$

$$\tilde{G}^- = \sum \tilde{G}_\xi^- \quad .$$

By say (JL, 2.2, 3.1) we conclude that $\tilde{G}^-$ is an A_0-subring of G^-. It is obviously stable for the adjoint action of U^+ and T. To show that it is stable for the adjoint action of U^- we may use induction on the order. Thus suppose $(ad\ y_i)\ \tilde{G}_\eta^- \subset \tilde{G}_{\eta-\alpha_i}^-$ for all $\eta > \xi$. Then

$$(adx_j)(ady_i)\tilde{G}_\xi^- \subset \left(\frac{q^{2(\xi,\alpha_i)} - q^{-2(\xi,\alpha_i)}}{q^{2d_i} - q^{-2d_i}}\right)\tilde{G}_\xi^- + (ad\ y_i)\tilde{G}_{\xi+\alpha_i}^- \subset \tilde{G}_\xi^-$$

by the induction hypothesis and noting that the offending pole at $q = 1$ cancels. From this we conclude that $(ady_i)\tilde{G}_\xi^- \subset \tilde{G}_{\xi-\alpha_i}^-$, as required. This result also applies to the twisted action of 4.7(*) as this is obtained from the adjoint action by just modifying what happens to 1. Here one quickly checks that $(ady_i).1 \in \tilde{G}_{-\alpha_i}^-$, for all i, where again the offending pole cancels.

From the above we see that $\hat{y}_i = (q-q^{-1})y_it_i \in \tilde{G}^-$, for all i; but $y_it_i \notin \tilde{G}^-$. Set

$$S^- = \tilde{G}^- \otimes_{A_0} A_0/\langle q-1\rangle \quad .$$

We call this the S-specialization of G^- at $q = 1$. A further computation which we omit shows that the images of the $\tilde{y}_i$ in S^- commute! The reader may therefore anticipate that $S^- = S(\mathfrak{n}^-)$ as an algebra. Actually this follows quite easily from the theory of $\mathcal{O}$ rings developed in [8].

Let $\mathcal{O}$ denote the Bernstein-Gelfand-Gelfand $\mathcal{O}$ category for $U(\mathfrak{g})$. A ring S on which $\mathfrak{g}$ acts by derivations such that S belongs as a $U(\mathfrak{g})$ module to $\mathcal{O}$ is called an $\mathcal{O}$ ring. Suppose S is isomorphic to $\delta M(0)$ as a module and that the trivial one-dimensional submodule $L(0)$ of $\delta M(0)$ contains an identity 1 for S.

We claim that S is isomorphic to $S(\mathfrak{n}^-)$ as a ring. Indeed by a general result of Dixmier ([2], 3.3.2) any minimal prime P of S is a submodule and so is either zero or contains the unique simple submodule $L(0)$ of $\delta M(0)$, hence the identity of S. We conclude that S is a prime $\mathcal{O}$ ring and then the required assertion follows from ([8], 5.6(ii)—see also 3.5, 3.6 for conventions). The theory used in [8] is quite deep and in particular uses Formanek's theorem for p.i. rings. However since $(\delta M(0))^{\mathfrak{n}^+}$ is one dimensional the much simpler arguments given in ([8], Sect. 3) also apply.

It remains to show that S^- has the properties claim above. Here we can take the twisting parameter λ to be 0, because the definition of S^- is independent of λ. Then the action of U on G^- is just the adjoint action and so it follows that the resulting action of $\mathfrak{g}$ on S^- is by derivations. That $H^0(\mathfrak{n}^+, S^-)$ reduces to scalars will follow exactly as for the corresponding assertion for G^- noted in 4.7. Obviously each $\tilde{G}^-_\xi$ is A_0 torsion-free. We prove it is finitely generated by induction on the order relation. For this it is enough to recall that $\tilde{G}^-_0 = A_0$ by definition and to note that by 4.7 for $\xi < 0$, the map $a \mapsto \{x_i a\}_{i=1}^{\ell}$ is an A_0 module injection of $\tilde{G}^-_\xi$ into $\bigoplus_{i=1}^{\ell} \tilde{G}^-_{\xi+\alpha_i}$. Since A_0 is principal, we conclude that $\tilde{G}^-_\xi$ is free and $dim_k S^-_\xi = rank\ \tilde{G}^-_\xi = dim_K G^-_\xi$. Hence with respect to formal characters we have

$$ch\ S^- = ch\ G^- = ch\ \delta M(0)\ .$$

As in 4.7, this and the previous observation suffice to prove that $S^- \cong \delta M(0)$. Again clearly the zero weight space of S^- is spanned by the identity. Had we taken the twisted action with twisting parameter λ we would have concluded that $S^- \cong \delta M(\lambda)$. Summarizing

Proposition. *S^- is isomorphic to $S(\mathfrak{n}^-)$ as a ring and to $\delta M(\lambda)$ as a $\mathfrak{g}$ module.*

Remark. Of course if $\lambda \neq 0$, then only $\mathfrak{n}^+$ must act by derivations, whilst $\mathfrak{b}^-$ acts by inhomogeneous derivations.

6.7. It is now completely obvious that $K(\lambda)^- : \lambda \in -4P^+(\pi)$ passes to $S(\lambda)^-$ under S-specialization. More precisely

$$(K(\lambda)^- \cap \tilde{G}^-) \otimes_{A_0} A_0/\langle q-1\rangle$$

is a nonzero finite dimensional $\mathfrak{g}$ submodule of $\delta M(-1/4\lambda)$ hence necessarily isomorphic to $L(-1/4\lambda)$, so equal to $S(\lambda)^-$. To show that y_μ passes to y^0_μ needs one further (amusing) observation.

Fix $\lambda \in -4P^+(\pi)$ and recall that $(ad\ U)\tau(\lambda) = K(\lambda)^- \otimes K\tau(\lambda) \otimes K(\lambda)^+$. Hence we can identify $(ad\ U)\tau(\lambda)$ with $K(\lambda)^- \otimes K(\lambda)^+$. Now of course $(ad\ U)\tau(\lambda)$ admits the adjoint action of U whilst $K(\lambda)^- \otimes K(\lambda)^+$ identifies with $L(-\lambda/4) \otimes L(-\lambda/4)^*$

and so admits a diagonal action of U. We claim that these two actions coincide. More precisely

LEMMA. *The above identification gives a U module isomorphism of* $(ad\ U)\tau(\lambda)$ *onto* $K(\lambda)^- \otimes K(\lambda)^+$.

Recall that the comultiplication Δ applied to y_i gives $y_i \otimes t_i^{-1} + t_i \otimes y_i$. Then from the remark in 4.8 or directly from 4.7(*) we obtain

$$\begin{aligned}\sigma(y_i)(1 \otimes 1) &= y_i 1 \otimes t_i^{-1} 1 + t_i 1 \otimes y_i 1 \\ &= q^{1/4(\lambda,\alpha_i)}(ad\ y_i)\tau(\lambda) \otimes q^{-1/4(\lambda,\alpha_i)} 1 \quad + \quad 0 \\ &= (ad\ y_i)\tau(\lambda) \ \text{ in } \ (ad\ U)\tau(\lambda) \ .\end{aligned}$$

In other words the offending factor $q^{1/4(\lambda,\alpha_i)}$ cancels! A similar assertion holds for x_i. This proves the lemma.

Remark 1. In passing from $(ad\ U)\tau(\lambda)$ to $K(\lambda)^- \otimes K(\lambda)^+$ we drop the factor $\tau(\lambda)$. In this sense y_λ passes to y_λ^0, that is if we take a representative of $y_\lambda \in K(\lambda)^- \otimes K(\lambda)^+$ then its image in $S(\lambda)^- \otimes S(\lambda)^+$ is invariant by the lemma and hence a multiple of y_λ^0. Clearing denominators we can assume it to be a nonzero multiple.

Remark 2. It is perhaps worth noting that the above analysis can be used to considerably simplify the construction of the centre $Z(U)$ given in (JL, Sect. 8). Take $\lambda \in -R^+(\pi)$. Then $L(-1/4\lambda)$ is finite dimensional and we have isomorphisms $(ad\ U)\tau(\lambda) \xrightarrow{\sim} K(\lambda)^- \otimes K(\lambda)^+ \xrightarrow{\sim} L(-1/4\lambda) \otimes L(-1/4\lambda)^* \xrightarrow{\sim} End_K L(-1/4\lambda)$ of U modules. By 4.5 we can further identify $(ad\ U)\tau(\lambda)$ with a U submodule of $F(U)$. This recovers 3.5 but notice we do not now need either the Rosso form or complete reducibility. Yet the identity on $End_K L(-1/4\lambda)$ defines a (trivial) one dimensional submodule which through the above isomorphisms gives a nonzero element $z_\lambda \in Z(U)$. This can be used in (JL, 8.6) and obliviates the need for (JL, Sect. 6) and complete reducibility (JL, 5.12). One cannot immediately get that

$$Z(U) = \bigoplus_{\lambda \in -R^+(\pi)} K z_\lambda$$

from 4.10, because we used the existence of a simple Verma module in 4.7 which in turn requires via 8.2(i) some knowledge of the centre. Nevertheless this can be provided by the $z_\lambda : \lambda \in -R^+(\pi)$. Consequently we could also avoid the Weyl character formula analysis in (JL, 8.6). In any case the tenacious reader will observe that by these means the centre can be constructed in a remarkably simple and almost computational-free fashion.

6.8. We can now prove the required analogue of 6.4. Fix $\nu \in -4P^+(\pi)$ and recall that $-4P^+_\nu(\pi) = \{\mu \in -4P^+(\pi) \mid \mu \geq \nu\}$. Fix $\nu \in -4P^+(\pi)$ and set $R^\nu(\mu) = y_{\nu-\mu}(ad\ U)\tau(\mu)$, $\forall\ a\ \mu \in -4P^+_\nu(\pi)$.

THEOREM. *For all $\nu \in -4P^+(\pi)$ one has*

(i) $R^\nu(\mu) \cap R^\nu(\mu') = R^\nu(\mu \cap \mu'), \forall\ \mu, \mu' \in -4P^+_\nu(\pi)$.

(ii) $R^\nu(\mu) \cap \sum_{j=1}^t R^\nu(\lambda_j) = \sum_{j=1}^t (R^\nu(\mu) \cap R^\nu(\lambda_j)),$

$$\forall\ \mu, \quad \lambda_1, \lambda_2, \ldots, \lambda_t \in -4P^+_\nu(\pi), \quad \forall\ t \in \mathbb{N}\ .$$

The inclusions $\supset$ are immediate from 4.13. Thus we only have to show that dimensions coincide. This follows from 6.4 and specialization. Here we remark that intersection with $\tilde{G}^- \otimes \tilde{G}^+$ which is a free A_0 module, must give free A_0 submodules.

Remark. The perspicacious reader will notice that we can also prove this result directly via the remarks in 6.1 and 6.4.

6.9. We can now establish the result searched for in 5.5. Let $\check{J}_+$ denote the ideal of $G(\check{U})$ generated by the augmentation ideal of $Y(\check{U})$.

COROLLARY. *For all $\mu \in P^+(\pi)$ one has*

$$[G(\check{U})/\check{J}_+ : L(\mu)] = dim\ L(\mu)_0\ .$$

Fix an isomorphism class E of U modules and let $G(\check{U})_E$ denote the isotypical component of $G(\check{U})$ considered as a U module for the adjoint action. Fix $\nu \in -4P^+(\pi)$ and set

$$G^\nu(\check{U}) = \sum_{\mu \in -4P^+_\nu(\pi)} (ad\ U)\tau(\mu)\ .$$

Set $G^\nu(\check{U})_E = G(\check{U})_E \cap G^\nu(\check{U})$. Since $G(\check{U}) = \lim_{\longrightarrow} G^\nu(\check{U})$,it is enough to show that $[G^\nu(\check{U})/\check{J}_+ \cap G^\nu(\check{U}) : E] = dim\ E_0$ for all ν sufficiently large.

We have

$$G^\nu(\check{U})/\check{J}_+ \cap G^\nu(\check{U}) = \left(\bigoplus_{\mu \geq \nu} (ad\ U)\tau(\mu)\right) \Big/ \sum_{\mu > \eta \geq \nu} R^\eta(\mu)\ .$$

It is clear that 6.8 extends in the obvious fashion to a corresponding assertion for each isomorphism class. Recalling that the y_μ are nonzero divisors, distributativity allows one to compute by the usual set theoretic rules the multiplicity of E in the right hand side above to be $[(ad\ U)\tau(\nu) : E]$. Indeed it is immediate that the multiplicity of E in the denominator of the right hand side takes the form $\sum c_\xi[(ad\ U)\tau(\xi) : E]$. We claim that $c_\xi = 1$ if $\xi > \nu$ and zero otherwise. This will establish the above estimate. First observe that the $R^\nu(\mu)$ for different η lie in the distinct direct summands $F(\eta) = (ad\ U)\tau(\eta)$. For η fixed we can write

$$\sum_{\mu>\eta} R^\eta_{(\mu)} = \sum_{i\in I_\eta} y_{w_i} F(\eta - w_i)$$

where $I_\eta \subset \{1, 2, \ldots, \ell\}$ is maximal with the property that $\eta - w_i \in -4P^+_\nu$. By 6.8 and because the y_{w_i} are nonzero divisors the right hand side has dimension

$$\sum_{s=1}^{r} \sum_{\substack{i,i_2,\ldots,i_s\in I_\eta \\ (distinct)}} (-1)^{s-1} dim\ F(\eta - w_{i_1} - w_{i_2} - \cdots - w_{i_s})$$

where $r = |I_\eta|$. Now a given term in the right hand side, say $dim\ F(\xi)$ with $\xi = \eta - w_{i_1} - w_{i_2} - \cdots - w_{i_s}$, occurs in a similar development of the $R^{\eta'}(\mu)$. Without loss of generality we can assume that η is maximal in $-4P^+_\nu(\pi)$ with the property that such a term appears. Then η is unique because if η_1, η_2 are two such choices then so is $\eta_1 \cup \eta_2 \leq \nu$. Then $\eta' = \eta - w_{j_1} - w_{j_2} - \cdots - w_{j_t}$: with the j indices being all possible subcollections of the i indices. It follows that the overall contribution to such a term is the sum

$$(-1)^{s-1} + s(-1)^{s-2} + \cdots + \binom{s}{1} = \sum_{t=1}^{s} (-1)^{t-1} \binom{s}{t} = 1 \quad ,$$

as required. By 5.2 and 3.5 this gives the required assertion.

6.10. The following sections are not needed for our separation theorem; but are required to determine Verma module annihilators and particularly to set up Duflo's theorem as discussed in 1.6. First we start with a

CONJECTURE. *For all $\lambda \in P(\pi)$, $\mu \in -R^+(\pi)$ one has*

$$Ann_{F(U)} M(\lambda) \cap (ad\ U)\tau(\mu) = 0 \ .$$

A priori this seems very surprising, because $(ad\ U)\tau(\mu)$ already contains a central element z_μ which by analogy with the enveloping algebra situation we would expect to annihilate some Verma module. Actually by (JL, 8.6) we can easily compute how z_μ acts on $M(\lambda)$. Let $(ch\ L(\xi), \eta)$ denote the polynomial in q, q^{-1} obtained from $ch\ L(\xi)$ by replacing e^ν by $q^{(\nu,\eta)}$. Then by (JL, 8.6) we find that z_μ acts on $M(\lambda)$ by the polynomial

$$\varphi(z_\mu)(\lambda) = (ch\ L(-1/4\mu), -4(\lambda + \rho)) \ .$$

This takes the value $dim\ L(-1/4\mu)$ at $q=1$ and so in particular is nonzero.

An instant corollary of 3.5 and the above conjecture would be that

$$(*) \qquad [F(U)/Ann_{F(U)}M(\lambda) : L(\mu)] \geq dim\ L(\mu)_0 \ ,$$

$\forall\ \lambda \in P(\pi),\ \mu \in P^+(\pi)$.

The importance of (*) arises in the following way. Let $M,\ N$ be U modules. Then $Hom_K(M,N)$ is a U bimodule in a standard way. Using the antipode σ of U on the second factor, it then becomes a $U \otimes U$ module and hence a U module for the diagonal action. Set

$$F(M,N) = \{\xi \in Hom_K(M,N) \mid dim\ U\xi < \infty\} \ .$$

Unlike the enveloping algebra case, this won't be a $U \otimes U$ submodule of $Hom_K(M,N)$; but it is an $F(U) \otimes F(U)$ submodule. Moreover if $M = N$ the action of $F(U)$ on M gives a homomorphism $F(U) \longrightarrow F(M,N)$ with kernel $Ann_{F(U)}M$. Now take $M = M(\lambda) : \lambda \in P(\pi)$. As in 5.2 the classical reasoning (combine 8.3 and 8.5(i)) gives

$$[F(M(\lambda), M(\lambda)) : L(\mu)] \leq dim\ L(\mu)_0\ ,\ \forall\ \mu \in P^+(\pi) \ .$$

This forces equality in (*) and by (*) we have an isomorphism

$$F(U)/Ann_{F(U)}M(\lambda) \xrightarrow{\sim} F(M(\lambda), M(\lambda)) \ .$$

In the following we shall establish a weak version of our conjecture which will be enough to establish (*).

6.11. Set $\mathbb{H}' = (ad\ U(\mathfrak{g}))U(\mathfrak{n}^-)$, $H' = (ad\ U(\mathfrak{g}))S(\mathfrak{n}^-)$. The latter is just Kostant's space of harmonic elements of $S(\mathfrak{g})$. It is a deep consequence of his work ([2], 8.4) that $\mathbb{H}'$ does not meet and is even a complementary subspace to the annihilator of any Verma module for $U(\mathfrak{g})$.

Fix $\mu \in -R^+(\pi)$ and recall $F(\mu) = (ad\ U)\tau(\mu)$. Let $\hat{U}$ denote the A subring of U generated by the $x_i, y_i : i = 1, 2, \ldots, \ell$. Recall that

$$\hat{U} \otimes_A A/\langle q-1 \rangle \cong U(\mathfrak{g}) \otimes_k k[T]$$

and further specialize by sending the now central elements t_i to the identity. Under this process $F(\mu)$ (more precisely $\hat{F}(\mu) := \hat{U} \cap (ad\ U)\tau(\mu)$) specializes to an $ad\ U(\mathfrak{g})$ stable subspace $F(\mu)'$ of $U(\mathfrak{g})$ which by 3.5 is isomorphic to an image of $End_k L(-1/4\mu)$ viewed as $\mathfrak{g}$ module under the diagonal action. By the freeness assertion of (JL, 5.10(i)), to prove our conjecture it suffices to show that $dim\ F(\mu)' = dim\ F(\mu)$ and that $F(\mu)' \subset \mathbb{H}'$.

Consider the subspace $L(\mu)^- = K(\mu)^- \otimes_K K\tau(\mu)$ of $F(\mu)$ defined in 4.8, 4.9. We recall that $K(\mu)^- \subset G^-$ consists of zero degree elements of $F(U)$ (or of $G(U)$). We have already analyzed in some detail how $K(\mu)^-$ specializes in 6.6. However this was its S-specialization. Here we are taking its U-specialization $T(\mu)^-$. This lies in $U(\mathfrak{n}^-)$ and is stable under the adjoint action of $\mathfrak{b}^-$; but does not admit an $\mathfrak{n}^+$ action. In particular (see remark in 6.18) one does not have $gr\ T(\mu)^- = S(\mu)^-$ with respect to the canonical filtration of $U(\mathfrak{n}^-)$. Nevertheless weight space considerations do force them to coincide in some limiting sense for large μ.

6.12. To prove our two assertions above and hence our conjecture it is enough to show that

$$(*) \qquad dim\ ad\ U(\mathfrak{g})T(\mu)^- = dim\ F(\mu)' \ .$$

Let us show that (*) holds in some limiting sense.

The inclusion relation implied by 4.12(i) gives that $T(\mu)^- \subset T(\lambda)^-$ whenever $\mu, \lambda \in -R^+(\pi)$ and $\mu > \lambda$. Regarding $-R^+(\pi)$ as a directed set, we may form the corresponding direct limit and it is immediate from 4.7 that

$$(**) \qquad \varinjlim T(\mu)^- = U(\mathfrak{n}^-) \ .$$

Fix $\nu \in P^+(\pi)$. By Kostant ([2], 8.3.9(ii)) one has

$$[\mathbb{H}' : L(\nu)] = dim\ L(\nu)_0 \ .$$

Then by (**) we conclude that

$$[ad\ U(\mathfrak{g})T(\mu)^- : L(\nu)] \leq dim\ L(\nu)_0$$

with equality if $-\mu$ is sufficiently large. Hence

$$[F(\mu)' \cap \mathbb{H}' : L(\nu)] \leq dim\ L(\nu)_0$$

with equality if $-\mu$ is sufficiently large. As discussed above, this further implies (*) of 6.10. We have shown the

THEOREM. *Fix $\lambda \in P(\pi)$. Then for all $\nu \in P^+(\pi)$ one has*

$$[F(U)/Ann_{F(U)}M(\lambda) : L(\nu)] = dim\ L(\nu)_0 \ .$$

Equivalently the action of $F(U)$ on $M(\lambda)$ gives an isomorphism

$$F(U)/Ann_{F(U)}M(\lambda) \xrightarrow{\sim} F(M(\lambda), M(\lambda))$$

of $F(U)$ bimodules.

6.13. In this section we show that $Z(U)$ specializes to the centre $Z(\mathfrak{g})$ of $U(\mathfrak{g})$. This is ultimately not needed either for the separation theorem nor for Verma module annihilators; but its truth removes what would otherwise seem to be a paradox if not a contradiction. Of course the statement itself is paradoxical in that $Z(U)$ is not necessarily a polynomial algebra whereas $Z(\mathfrak{g})$ is. The point is that specialization does not preserve linearity because of the possibility of dividing by a power of $(q-1)$. Indeed $Z(U)$ is spanned by $z_\mu : \mu \in -R^+(\pi)$ which we saw in 6.9 all specialize to scalars!

Recall that $x_i, y_i \in \hat{U}$ (notation, JL 4.10) and so is $(t_i^2 - t_i^{-2})/(q^{2d_i} - q^{-2d_i})$. For our analysis it is convenient to further adjoin the elements

$$h_i := \frac{t_i - 1}{q^{d_i} - 1} \quad .$$

Since

$$\frac{h_i(1 + t_i^{-1})(t_i + t_i^{-1})}{(1 + q^{-d_i})(q^{d_i} + q^{-d_i})} = \frac{t_i^2 - t_i^{-2}}{q^{2d_i} - q^{-2d_i}} \quad ,$$

this ultimately makes no difference to specialization at $q = 1$. It is also convenient to replace A by A_0. With this slight change we now let $\hat{U}^0$ denote the A_0 subring of U^0 generated by T and the h_i, and $\hat{U}$ the A_0 subring of U generated by $\hat{U}^+, \hat{U}^-, \hat{U}^0$, where $\hat{U}^+$ (resp. $\hat{U}^-$) is the A_0 subring of U generated by the x_i (resp. y_i) $i := 1, 2, \ldots, \ell$. Including the divided powers makes no difference as we are specializing at $q = 1$.

Take $i \in \{1, 2, \ldots, \ell\}$ and view t_i as defining the function $\mu \mapsto q^{(\alpha_i, \mu)}$ on $P(\pi)$. In this sense specialization at $q = 1$ also sends each t_i to 1. Set $\alpha_i^\vee = d_i^{-1}\alpha_i$. Then h_i is replaced by the function $\mu \longrightarrow (q^{(\alpha_i, \mu)} - 1)/(q^{d_i} - 1)$ which becomes the function $\mu \mapsto (\alpha_i^\vee, \mu)$ at $q = 1$, that is h_i specializes to the coroot $\alpha_i^\vee$. Identify the Cartan subalgebra $\mathfrak{h}$ of $\mathfrak{g}$ with the linear span of the $\alpha_i^\vee : i = 1, 2, \ldots, \ell$.

Given $a \in KT$, let $a_\mu(q)$ denote the element of K obtained by replacing each t_i by $q^{(a_i, \mu)}$ and by $a(q)$ the function on $P(\pi)$ sending $\mu \mapsto a_\mu(q)$. Given $\hat{a} \in \hat{U}^0$, let $\hat{a}(1)$ denote its image in $\hat{U} \otimes_{A_0} A_0/\langle q-1 \rangle$, that is its specialization at $q = 1$. This can be viewed as an element of $S(\mathfrak{h})$.

LEMMA. *Suppose $\hat{a} \in \hat{U}^0$ and $\hat{a}(1) = 0$. Then $(q-1)^{-1}\hat{a} \in \hat{U}^0$.*

This is proved by the usual trick. We can write

$$\hat{a} = \sum \hat{a}_i t_\ell^i \quad : \quad \hat{a}_i \in \hat{U}_{\hat{\ell}}^0$$

where in $\hat{U}_{\hat{\ell}}^0$ the t_ℓ variable no longer appears (though h_ℓ may still appear). By the hypothesis

$$\hat{a} = \hat{a} - \hat{a}(1) = \sum (\hat{a}_i - \hat{a}_i(1))t_\ell^i + \sum \hat{a}_i(1)(t_\ell^i - 1) \ .$$

In the first term we note that each $\hat{a}_i - \hat{a}_i(1)$ specializes to zero, so we can continue by eliminating the $t_{\ell-1}$ variable. In the second term we can replace the common factor $t_\ell - 1$ by $h_\ell(q^{d_i} - 1)$ and then divide by $q - 1$. This proves the lemma.

6.14. For each $m \in \mathbb{N}$, let $S_m(\mathfrak{h})$ denote the space of homogeneous polynomials of degree m on $\mathfrak{h}^*$ and let $\hat{U}_m^0$ denote the subset of all $a \in A_0T$ such that $(q-1)^{-m}a \in \hat{U}^0$ and let $\theta_m(a)$ denote the specialization of $(q-1)^{-m}a$ at $q = 1$.

Lemma.

(i) Take $a \in \hat{U}_m^0$. Then $\theta_m(a)$ is the element of $S_m(\mathfrak{h})$ given by

$$\mu \mapsto \frac{1}{m!} \frac{d^m a_\mu(q)}{dq^m} \Big|_{q=1}$$

(ii) The map θ_m is a surjection of $\hat{U}_m^0$ onto $S_m(\mathfrak{h})$.

(i) We can write $a = (q-1)^m b$ for some $b \in \hat{U}^0$. Then

$$(*) \qquad a_\mu(q) = (q-1)^m b_\mu(q)$$

and $b_\mu(1)$ is defined (by cancelling the $(q-1)^m$ factor). Moreover $\theta_m(a) = b(1)$ and the latter is just the function $\mu \mapsto b_\mu(1)$ on $P(\pi)$. Yet differentiating (*) we obtain

$$\frac{1}{m!} \frac{d^m a_\mu(q)}{dq^m} \Big|_{q=1} = b_\mu(1)$$

as required.

(ii) Take $\xi \in -R^+(\pi)$. Then for each integer $j \geq 0$, $\tau(j\xi)$ is represented by the function $\mu \mapsto q^{j(\xi,\mu)}$. Its m^{th} derivative at $q = 1$ takes the value $(j\xi)^m = j^m\xi^m$. From the nonvanishing of the determinant of the matrix with entries $j^n : j, n \in \{0, 1, 2, \ldots, m\}$ we can find rational coefficients $c_j : j = 0, 1, 2, \ldots, m$ such that

$$\sum c_j (j\xi)^n = \begin{cases} 0 & : n < m, \\ m!\ \xi^m & : n = m \end{cases}$$

Set $a = \sum c_j \tau(j\xi)$. By the first equation, $a(q)$ has a zero of order m at $q-1$ so $(q-1)^{-m}a \in \hat{U}^0$ by 6.12. By the second equation and (i), this element specializes to ξ^m. Thus $\xi^m \in \theta_m$ for all $\xi \in -R^+(\pi)$. Obviously $Im\ \theta_m$ is a linear space and it is an elementary fact that all such functions span $S_m(\mathfrak{h})$.

Remark. Notice we have proved the slightly stronger fact that it is enough to take $\tau(-R^+(\pi))$ instead of $T = \tau(Q(\pi))$. Notice that we could even have taken smaller subsets, for example $\tau(-rR^+(\pi))$: r any integer > 0 would do.

6.15. Take $\lambda \in -R^+(\pi)$ and set

$$\hat{\tau}(\lambda) = \sum_{w \in W} \tau(w\lambda) q^{(\rho, w\lambda)} \quad .$$

Define

$$\hat{U}^{00} = \left(\sum_{\lambda \in -R^+(\pi)} K\hat{\tau}(\lambda) \right) \cap \hat{U}^0$$

and $\hat{U}^{00}_m = \hat{U}^{00} \cap \hat{U}^0_m$.

Define (as usual) a translated action of W on $\mathfrak{h}^*$ by $w \cdot \mu = w(\mu + \rho) - \rho$, and set

$$S(\mathfrak{h})^{W\cdot} = \{ b \in S(\mathfrak{h}) \mid b(w.\mu) = b(\mu),\ \forall\ w \in W \} \ .$$

LEMMA. $\sum_{m \in \mathbb{N}} \theta_m(\hat{U}^{00}_m) = S(\mathfrak{h})^{W\cdot}$

One checks that

$$\mu \mapsto \left. \frac{d^m}{dq^m} \hat{\tau}(\lambda)(\mu) q^{-(\rho,\lambda)} \right|_{q=1} = (\lambda^m, \sum_{w \in W} w.(\rho + \mu))$$

and so defines an element of $S(\mathfrak{h})^{W\cdot}_m$. Moreover all invariant polynomials so obtain. The assertion is then an easy consequence of 6.13.

6.16. We showed in (JL, 8.6) that for each $\lambda \in -R^+(\pi)$ there exists $z'_\lambda \in Z(U)$ such that $\varphi(z'_\lambda) = \hat{\tau}(\lambda)$. (The z'_λ are an appropriate linear combination of the z_λ defined in 4.13, see JL, 8.6, eq.(6)). It would seem from 6.15 and Harish-Chandra's ([2], 7.4) description of the centre $Z(\mathfrak{g})$ of $U(\mathfrak{g})$ that this would imply that $Z(U)$ specializes to $Z(\mathfrak{g})$. However there is one more snag which is resolved by the following

LEMMA. *Given $z \in Z(U)$ such that $\varphi(z) \in \hat{U}^0$. Then $z \in Z(U) \cap \hat{U}$.*

Fix $\mu \in \mathbb{N}\pi$ and let $\hat{U}^+_\mu$ (resp. $\hat{U}^-_{-\mu}$) denote the A_0 submodule of $\hat{U}^+$ (resp. $\hat{U}^-$) of vectors of T-weight μ (resp. $-\mu$). As noted in (JL, 4.10) these are free A_0 modules. Take $\lambda \in -P^+(\pi)$ and let e_λ denote the canonical highest weight vector of the Verma module $\hat{N}(\lambda)$ for $\hat{U}$ (as defined in say JL, 5.10) of highest weight λ. If $x_{\mu,i} \in \hat{U}^+_\mu$, $y_{-\mu,j} \in \hat{U}_\mu$, then we can write $x_{\mu,i}y_{-\mu,j}e_\lambda$ in the form $m^\lambda_{\mu;i,j}e_\lambda$ for some $m^\lambda_{\mu,i,j} \in A_0$. We claim that there exist bases $\{x_{\mu,i}\}$, $\{y_{-\mu,j}\}$ of the free A_0 modules $\hat{U}^+_\mu$ and $\hat{U}^-_{-\mu}$ such that the matrix m^λ_μ with entries $\{m^\lambda_{\mu;i,j}\}$ is invertible in A_0, for all $\lambda \in -P^+(\pi)$. This is assured if m^λ_μ evaluated at $q = 1$ is invertible in k, for all $\lambda \in -P^+(\pi)$. Then we see that our claim is an immediate consequence of the fact that $\hat{U}^+_\mu$ (resp. $\hat{U}^-_{-\mu}$) specializes to the corresponding weight space of $U(\mathfrak{n}^+)$ (resp. $U(\mathfrak{n}^-)$) and that the contravariant form on the simple (8.2(i)) Verma module $M(\lambda)$ for $U(\mathfrak{g})$ is nondegenerate.

Now we can write

$$(*) \qquad z = \sum_{i,j} \sum_{\mu \in \mathbb{N}\pi} y_{-\mu,i}\chi^{i,j}_\mu x_{\mu,j}$$

for some $\chi^{i,j}_\mu \in U^0$. Then $\varphi(z) = \chi_0$ and we must show that $\chi_0 \in \hat{U}^0$ implies $\chi^{ij}_\mu \in \hat{U}^0$ for all $\mu > 0$, all i,j. Lift the order relation (3.1) in $Q(\pi)$ to a total order. Fix $\mu \in Q^+(\pi)$ and assume that the assertion has been proved for all $\eta < \mu$.

Consider $zy_{-\mu,k}e_\lambda$. On the one hand this equals $y_{-\mu,k}ze_\lambda = \chi_0(\lambda)y_{-\mu,k}e_\lambda$. On the other hand it may be computed using (*). Comparison of terms and using the induction hypothesis gives

$$y_{-\mu,i}\chi^{i,j}_\mu x_{\mu,j}y_{-\mu,k}e_\lambda \in \hat{N}(\lambda) \quad .$$

By our choice of bases, we deduce that

$$y_{-\mu,i}\chi^{i,j}_\mu(\lambda)e_\lambda \in \hat{N}(\lambda) \quad .$$

Since $\hat{N}(\lambda)$ is a free $\hat{U}^-$ module it follows that $\chi^{i,j}_\mu(\lambda) \in A_0$. Since this holds for all λ in Zariski dense set $-P^+(\pi)$ we conclude from 6.13 that $\chi^{i,j}_\mu \in \hat{U}^0$, as required.

Remark. The analysis here is similar in spirit to ([1], Prop. 2.2); see also ([1], 3.9).

6.17. Combining 6.15, 6.16 (JL, 8.6) and ([2], 7.4) we deduce (as explained above) the

THEOREM. *The centre $Z(U)$ of U specializes at $q = 1$, $t_i = 1, \forall\, i$ to the centre $Z(\mathfrak{g})$ of $U(\mathfrak{g})$. More precisely*

$$(Z(U) \cap \hat{U}) \bigotimes_{A_0 T} A_0 T / \langle q-1, t_i - 1, \forall\, i \rangle = Z(\mathfrak{g}) \quad .$$

6.18. There is an amusing application of the above to the computation of the centre $Z(\mathfrak{n}^-)$ of $U(\mathfrak{n}^-)$. Take $\mu \in -R^+(\pi)$ and consider a lowest weight vector f in $L^-_{(\mu)}$. Recalling 4.8(i), this has weight $-\frac{1}{4} w_0 \mu$. Under U-specialization f becomes an *ad* $U(\mathfrak{n}^-)$ invariant element of $U(\mathfrak{n}^-)$, that is an element of $Z(\mathfrak{n}^-)$, of weight $\frac{1}{4}(\mu - w_0\mu)$. (Here we must take account of the shift by $\frac{1}{4}\mu$ of weights). Replacing U by $\check{U}$ (notation 4.2) we may conclude that the same holds for $\mu \in -4P^+(\pi)$. Thus we have shown the

THEOREM. *For each $\mu \in -P^+(\pi)$, the weight space of $Z(\mathfrak{n}^-)$ of weight $\mu - w_0\mu$ is nonzero.*

Remarks. It can be checked from ([6], Tables I, II) that in nearly all cases the $\omega - w_0\omega : \omega$ fundamental run over the weights of the generators of $Z(\mathfrak{n}^-)$ which is a polynomial algebra. Indeed outside types E_7, E_8 at most the highest root is missing from this set. In these good cases one obtains a much simpler proof of ([6], Thm. 4.12). We also remark ([6], 4.4) that *dim* $Z(\mathfrak{n}^-)_{\mu - w_0\mu} = 1$. Now one can easily arrange distinct elements $\mu, \mu' \in -R^+(\pi)$ to satisfy $\mu - w_0\mu = \mu' - w_0\mu'$. The corresponding lowest weight vectors $f \in L(\mu)^-, f' \in L(\mu')^-$ must have the same U-specialization; yet they cannot have the same S-specialization, because this would contradict 6.1(ii) and 6.7.

7. Separation of Variables.

7.1. Recall that $G(U) = gr_{\mathcal{F}} F(U)$, set $Y = gr_{\mathcal{F}} Z(U)$ and let Y_+ denote the space of homogeneous elements in Y of positive degree. In particular Y_+ is a graded ideal of codimension 1 in Y. Let $Max_1 Y$ denote the set of ideals of codimension 1 in Y. We shall also use $\mathcal{F}$ to denote the induced filtration on $G(U)$.

LEMMA. *For all $Y_\chi \in Max_1 Y$, one has an isomorphism $gr_{\mathcal{F}}(T_< Y_\chi) \xrightarrow{\sim} T_< Y_+$ of graded vector spaces.*

Since $\varepsilon(x_i) = 0$ (augmentation) it follows that *ad* x_i acts by zero on Z and hence by zero on Y. On the other hand in $G(U)$ one has for any Laurent polynomial $p(t) \in U^0$ that

$$\begin{aligned}(ad\ x_i)p(t) &= x_i p(t) t_i - q^{-2d_i} t_i p(t) x_i \\ &= x_i (p(t) - p_i(t)) t_i\end{aligned}$$

where $p_i(t)$ is obtained from $p(t)$ by replacing each factor t_j by $q^{(\alpha_i,\alpha_j)}t_j$. This operation can be viewed as some exponentiated version of differentiation. In any case using the nondegeneracy of the Cartan matrix $(\alpha_i,\alpha_j) : i,j = 1,2,\dots,\ell$ and the action of the $x_i : i = 1,2,\dots,\ell$ it easily follows that the multiplication map

$$KT_< \otimes_K Y \twoheadrightarrow T_<Y$$

is injective and hence an isomorphism.

Set $(T_<Y_\chi)^m = \mathcal{F}^m(G(U)) \cap T_<Y_\chi$ and $(T_<Y_\chi)_m = (T_<Y_\chi)^m/(T_<Y_\chi)^{m-1}$. Note that we may view Y_+ both as an element of Max_1Y and as a graded ideal. Since $Y_\chi \in Max_1Y$ and only the scalars have degree 0 we obtain an isomorphism $gr_\mathcal{F}Y_\chi \xrightarrow{\sim} Y_+$ of graded vector spaces and hence an inclusion $(T_<Y_+)_m \subset (T_<Y_\chi)_m$, for all $m \in \mathbb{N}$. On the other hand each $c \in T_<Y_\chi$ can be written in the form

$$c = \sum_{i=1}^{n} a_ib_i : a_i \in T_< \ , \quad b_i \in Y_\chi$$

with the $gr\ a_i$ linearly independent over K. Set $m = max\{deg\ a_i + deg\ b_i : i = 1,2,\dots,n\}$. Then $c \in (T_<Y_\chi)^m$. By the above isomorphism the products $(gr\ a_i)(gr\ b_i)$ are linearly independent over K and so (possibly dropping some terms of lower degree) we can write

$$gr\ c = \sum_{i=1}^{r} gr\ a_i\ gr\ b_i \in grT_<grY_\chi = T_<Y_+ \ .$$

Moreover by the linear independence, $gr\ c \in (T_<Y_\chi)^m - (T_<Y_+)^{m-1}$. This proves the opposite inequality and establishes the required isomorphism.

7.2. The remaining results of this section are only valid for the simply connected algebra $\check{U}$. We set in particular $\check{Y} = Y(\check{U})$ and $\check{T}_< = \{\tau(-\lambda) : \lambda \in 4P^+(\pi)\}$.

Given $Y_\chi \in Max_1\check{Y}$, set $J_\chi = G(\check{U})Y_\chi$. Let $\check{Y}_+$ denote the augmentation ideal of $\check{Y}$.

Proposition. *For all $Y_\chi \in Max_1\check{Y}$, one has an isomorphism $gr_\mathcal{F}J_\chi \xrightarrow{\sim} G(\check{U})\check{Y}_+$ of graded vector spaces.*

By 4.10 we can write $G(\check{U}) = (ad\ U)\check{T}_<$ and so

$$(*) \qquad J_\chi = Y_\chi(ad\ U)\check{T}_< = (ad\ U)\check{T}_<Y_\chi \ .$$

Set $J_\chi^m = \mathcal{F}^m(G(\check{U})) \cap J_\chi$ and $(\check{T}_<Y_\chi)^m = \mathcal{F}^m(G(\check{U})) \cap \check{T}_<Y_\chi$. Since $\mathcal{F}$ is $ad\ U$

invariant, $(*)$ gives the inclusion

$$(**) \qquad J_\chi^m \supset (ad\ U)(\check{T}_< Y_\chi)^m \ , \quad \forall\ m \in \mathbb{N} \ .$$

It is easy to appreciate that 6.8 is just what we need to get equality in (**) and hence by 7.1 the assertion of the proposition. In more detail a strict inequality in (**) would imply (without loss of generality) that there exist $\nu \in -4P^+(\pi)$, $\mu_i \in -4P^+_\nu(\pi)$, $a_{\mu_i} \in (ad\ U)\tau(\mu_i) : i = 1, 2, \ldots, s$ such that

$$\sum_{i=1}^{s} \left(y_{\nu-\mu_i} - \chi(y_{\nu-\mu_i})\right) a_{\mu_i}$$

does not lie in $(ad\ U)(\check{T}_< Y_\chi)^m$ and yet has degree m. Here we can obviously assume $\tau(\nu)$ has degree $n > m$ and that $deg\ y_{\nu-\mu_i} > 0$ all i. Then the cancellation of the degree n terms gives $\sum y_{\nu-\mu_i} a_{\mu_i} = 0$. Successive application of 6.8 to this identity implies that we can find $\xi \in -4P^+_{\cap \mu_i}(\pi)$ such that $a_{\mu_i} = y_{\mu_i - \xi} b_i$ with $b_i \in (ad\ U)\tau(\xi)$ satisfying $\sum b_i = 0$, or that there will be "shorter" cancellations of a similar nature which we dispense with in a fashion similar to that described below. Thus

$$\begin{aligned} \sum_{i=1}^{s} \left(y_{\nu-\mu_i} - \chi(y_{\nu-\mu_i})\right) a_{\mu_i} &= -\sum_{i=1}^{s} \chi(y_{\nu-\mu_i}) y_{\mu_i-\xi} b_i \\ &= -\sum_{i=1}^{s-1} (\chi(y_{\nu-\mu_i}) y_{\mu_i-\xi} - \chi(y_{\nu-\mu_{i+1}}) y_{\mu_{i+1}-\xi}) \Sigma_{j=1}^{i} b_j \ . \end{aligned}$$

From our degree choices, this clearly belongs to $(ad\ U)(\check{T}_< Y_\chi)^{n-1}$. Successive reduction then gives the required contradiction.

7.3. By local finiteness and complete irreducibility, $G(\check{U})\check{Y}_+$ admits a graded *ad U* stable complement $\acute{H}$.

PROPOSITION. *The map $h \otimes y \mapsto hy$ is an isomorphism of $H \otimes_K \check{Y}$ onto $G(\check{U})$.*

Surjectivity follows as in ([2], 8.2.2). For injectivity consider a sum of the form $\sum h_\gamma \otimes y_\gamma$, $h_\gamma \in H$, $y_\gamma \in \check{Y}$ with the h_γ linearly independent of K. By choice of H and 7.2 it follows that $H \cap J_\chi = 0$, for each $Y_\chi \in Max_1 Y$. Now in J_χ the y_γ are replaced by scalars say $\chi_\gamma \in K$. Thus we obtain

$$\sum h_\gamma y_\gamma = 0 \Longrightarrow \sum h_\gamma \chi_\gamma = 0 \Longrightarrow \chi_\gamma = 0, \quad \forall_\gamma \ .$$

It is enough to choose χ such that $y_\gamma \notin Y_\chi$ for some γ to obtain a contradiction.

Remark. Notice that to prove 7.3 we can restrict to a given isomorphism class E. Moreover we only need to show that $gr_{\mathcal{F}} J_\chi = \check{J}_+ := G(\check{U})\check{Y}_+$ holds with respect to that class.

By 5.3(**) and 5.4, we have $[G(\check{U})/gr_{\mathcal{F}} J_\chi : E] \geq dim\ E_0$. Since $gr_{\mathcal{F}} J_\chi \supset \check{J}_+$ we have a surjection

$$G(\check{U})/gr_{\mathcal{F}} J_\chi := \oplus_{m\in\mathbb{N}} \mathcal{F}^m(G(\check{U}))/J_\chi^m \twoheadleftarrow G(\check{U})/\check{J}_+$$

which is an isomorphism if and only if equality holds for each $m \in \mathbb{N}$. By 6.9 and the above we conclude that $gr_{\mathcal{F}} J_\chi = \check{J}_+$ and in particular equality does hold in 7.2(**). This alternative analysis highlights the essence of our proof.

7.4. By 4.11 we can choose an *ad* U stable subspace $\mathbb{H}$ of $\check{U}$ such that $gr\ \mathbb{H} = H$. By 5.3 and the reasoning in ([2], 8.2.4) we obtain the

THEOREM. *The map $h \otimes z \mapsto hz$ is an isomorphism of $\mathbb{H} \otimes_K Z(\check{U})$ onto $F(\check{U})$.*

7.5. A more canonical choice of $\mathbb{H}$ would result if we could prove the following variation of (*) of 3.5. Namely, for all $m \in \mathbb{N}$ that $\mathcal{R}$ restricted to $F_m(\check{U})$ is nondegenerate. Let $(\check{T}_< Y_+)_m$ denote the space of homogeneous elements of degree m in $\check{T}_< Y_+$. By 4.11 we can identify $(ad\ U)(\check{T}_< Y_+)_m$ with an *ad* U stable subspace of $F_m(\check{U})$. Then its orthogonal $\mathbb{H}_m$ in $F_m(\check{U})$ is a complementary *ad* U stable subspace (if 3.5(*) holds). Then we may choose

$$\mathbb{H} = \bigoplus_{m\in\mathbb{N}} \mathbb{H}_m \ .$$

This construction more closely follows that of Kostant as described in 1.4. Again (if the above assumption on $\mathcal{R}$ holds) the sum of the restrictions of $\mathcal{R}$ to each $F_m(\check{U})$ defines (most naturally) a nondegenerate *ad* U invariant bilinear form on $gr_{\mathcal{F}} F(\check{U})$. One can ask if this form has a simpler description than $\mathcal{R}$.

7.6. (Notation 5.4.) It is clear that

$$P_\mu(q) = \sum_{m=0}^{\infty} [\mathbb{H}_m : L(\mu)] q^m$$

for all $\mu \in P^+(\pi)$. In particular from 5.4 we obtain the quantum analogue of Kostant's result ([2], 8.3.9(ii)).

COROLLARY. *For all $\mu \in P^+(\pi)$ one has*

$$[\mathbb{H} : L(\mu)] = dim\ L(\mu)_0 \ .$$

8. Verma Module Annihilators.

8.1. By extending K one may define (JL, 5.3) a universal highest weight module $M(\lambda)$ with any highest weight $\lambda \in \mathfrak{h}^*$. Explicitly we set $B = U^0U^+$ and $M(\lambda) = U \otimes_B K_\lambda$ where K_λ is the one-dimensional B module in which U^+_+ acts by zero and t_i by $q^{(\lambda,\alpha_i)}$. However this is rather unnatural and we prefer to restrict to the case $\lambda \in P(\pi)$ for which no extension of K is needed. The first place where this restriction would make a significant difference would be in conjecture 6.10 which we are unable to prove anyway. Without further notification we shall assume henceforth that $\lambda \in P(\pi)$.

LEMMA. *$Ann_{F(U)}M(\lambda)$ is completely prime.*

Equivalently $F(U)/Ann_{F(U)}M(\lambda)$ is an integral domain. Since the action of *ad* U is locally nilpotent on $F(U)$ it is enough by a standard argument ([9], 8.1) to show that $(F(U)/Ann_{F(U)}M(\lambda))^{U^-}$ is an integral domain. Through the action $F(U)$ on $M(\lambda)$ the latter ring embeds in $(End_K M(\lambda))^{U^-}$ where $End_k M(\lambda)$ is viewed as U^- module for the diagonal action (cf. 6.10).

Let e_λ be a choice of highest weight vector for $M(\lambda)$. Recall that $M(\lambda)$ is freely generated over U^- with generator e_λ. Hence for each $a \in (End_K M(\lambda))^{U^-}$ there is a unique element $\gamma(a) \in U^-$ such that $\gamma(a)e_\lambda = ae_\lambda$. Obviously γ is a linear map. It is clear that $(End_K M(\lambda))^{U^-}$ is a direct sum of its *ad* T weight spaces. Let $a_\mu \in (End_K M(\lambda))^{U^-}$ be an element of weight μ. Then

$$\begin{aligned} a_\mu y_i e_\lambda &= q^{-(\alpha_i,\mu+\lambda-\alpha_i)} t_i a_\mu y_i e_\lambda \\ &= q^{-(\alpha_i,\mu+\lambda)}[y_i a_\mu t_i e_\lambda - (ad\ y_i)a_\mu e_\lambda] \\ &= q^{-(\alpha_i,\mu)} y_i a_\mu e_\lambda, \quad \forall\ i \in \{1,2,\dots,\ell\}\ . \end{aligned}$$

From this we conclude that γ is injective. Furthermore $\gamma(a_\mu a_\nu)e_\lambda = a_\mu a_\nu e_\lambda = a_\mu \gamma(a_\nu)e_\lambda = q^{-(\mu,\nu)}\gamma(a_\nu)a_\mu e_\lambda = q^{-(\mu,\nu)}\gamma(a_\nu)\gamma(a_\mu)e_\lambda$ and so

$$\gamma(a_\mu a_\nu) = q^{-(\mu,\nu)}\gamma(a_\nu)\gamma(a_\mu).$$

A standard argument (cf. JL, 9.2) shows that if $ab = 0$ for some $0 \neq a, b \in (End_K M(\lambda))^{U^-}$ then the highest weight component a_μ (resp. b_ν) of a (resp. b) must satisfy $a_\mu b_\nu = 0$. Then $\gamma(b_\nu)\gamma(a_\mu) = 0$, which contradicts the fact that U^- is an integral domain (JL, 4.10) and the injectivity of γ. This proves the lemma.

8.2. Consider $M(\mu) \in Ob\mathcal{O}_{P(\pi)}$.

LEMMA.

(i) $M(\mu)$ is simple $\iff \mu \in -P^+(\pi)$.

(ii) $M(\mu)$ *is projective* $\Longleftrightarrow \mu \in P^+(\pi)$.

Suppose $(\alpha_i^\vee, \mu) \in \mathbb{N}$. Then by (JL, 5.6) we have an exact sequence

$$0 \longrightarrow M(s_i \cdot \mu) \longrightarrow M(\mu) \longrightarrow M(\mu)/M(s_i \cdot \mu) \longrightarrow 0$$

which is already nonsplit over U^- (because the latter is an integral domain (JL, 4.8) and a Verma module is *a* free rank one U^- module). This gives $\Longrightarrow$ in (i), (ii). The converse assertions follow from (JL, 8.6) and the same reasoning as in the enveloping algebra situation.

8.3. Let E be a simple finite dimensional U module and for each $\nu \in P(\pi)$, let E_ν denote its ν-weight subspace. Recall the semisimplicity of finite dimensional U modules (LJ, 5.12).

LEMMA. *If either* $\nu \in P^+(\pi)$ *or* $\mu \in -P^+(\pi)$ *then* $[F(M(\nu), M(\mu)) : E] = \dim E_{\mu-\nu}$.

For the first assertion, we use, as in the enveloping algebra case, the canonical isomorphisms

$$\begin{aligned} Hom_U(E,\ Hom_K(M(\nu), M(\mu)) &\xrightarrow{\sim} Hom_U(E \otimes_K M(\nu), M(\mu)) \\ &\xrightarrow{\sim} Hom_U(M(\nu), E^* \otimes_K M(\mu)) \ . \end{aligned}$$

Now as in 4.4, $E^* \otimes M(\mu)$ admits a Verma flag with factors $M(\mu+\xi) : \xi \in \Omega(E^*)$. Since ν is dominant $Hom_U(M(\nu), M(\mu+\xi)) = 0$ unless $\nu = \mu + \xi$. Since $M(\nu)$ is projective (8.2(ii)) the left hand side has dimension $\dim E^*_{\nu-\mu} = \dim E_{\mu-\nu}$, as required.

For the second assertion we define an algebra homomorphism $\tilde{\Delta} : U \longrightarrow U \otimes U$ through $\tilde{\Delta} = (\sigma^{-1}\kappa \otimes 1)\Delta$ where σ is the antipode and κ the Chevalley antiautomorphism (JL, 4.8). Using triangular decomposition one checks that $B \otimes B \cap \tilde{\Delta}(U) = \tilde{\Delta}(U^0)$ and then as in ([2], 2.2.9) that the multiplication map defines a linear isomorphism of $(B \otimes B) \otimes_{\tilde{\Delta}(U^0)} \tilde{\Delta}(U)$ onto $U \otimes U$. Then as in ([2], 5.5.4, 5.5.8) we obtain isomorphisms

$$\begin{aligned} (M(\mu) \otimes M(\nu))^* &\xrightarrow{\sim} Hom_{B\otimes B}(U \otimes U, K_{-\mu} \otimes K_{-\nu}) \\ &\xrightarrow{\sim} Hom_{\tilde{\Delta}(U^0)}(\tilde{\Delta}(U), K_{\mu-\nu}) \end{aligned}$$

of $U \otimes U$ and $\tilde{\Delta}(U)$ modules respectively. Then as in ([2], 5.5.7) Frobenius reciprocity gives

$$(*) \qquad [(M(\mu) \otimes M(\nu))^* : E] = \dim E_{\mu-\nu} \ .$$

Given $M, N \in Ob\mathcal{O}_{P(\pi)}$, set

$$F(M \otimes N)^* = \{\xi \in (M \otimes N)^* \mid dim\ \tilde{\Delta}(U)a < \infty\} \ .$$

Taking account that δM is viewed as a U module using κ, whereas $(M \otimes N)^*$ is viewed as a $U \otimes U$ module using $\sigma \otimes \sigma$, one checks that the canonical isomorphism

$$Hom_K(N, Hom_K(M, K)) \xrightarrow{\sim} Hom_K(M \otimes N, K)$$

restricts to an isomorphism of $F(N, \delta M)$ onto $F(M \otimes N)^*$. Since $\delta M \cong M$ if M is simple (JL, 5.12) the second assertion of the lemma follows from 8.2(i) and (*).

8.4. Let A be a K-algebra and M an A module. We denote by $d_A(M)$ the Gelfand-Kirillov dimension of M over A. General definitions can be found in [14]. Gelfand-Kirillov dimension is rather well-behaved for enveloping algebras (for example it takes integer values and is exact—i.e., behaves properly on exact sequences). Although we won't need this, recently McConnell [18] has shown that this good behaviour extends to $U_q(\mathfrak{g})$. For present purposes we only need the following two facts which are rather easy and well-known.

Suppose A is an integral domain and L a nonzero left ideal of A. Then

$$(*) \qquad d_A(A/L) \leq d(A) - 1 \ .$$

Suppose A is a bi-algebra. Let E, M be A modules. Via the comultiplication map we may view $E \otimes M$ as an A module.

Suppose $dim_K E < \infty$, then

$$(**) \qquad d_A(E \otimes M) \leq d_A(M) \ .$$

Lemma. *Suppose M, N are simple U modules. Then $F(M, N) = 0$ unless $d_U(M) = d_U(N)$.*

This follows from (**) exactly as in the proof of ([5], 10.13).

8.5. *Fix $\lambda \in P^+(\pi)$. Set*

$$F(x, y) = F(M(x \cdot \lambda), M(y \cdot \lambda)), \quad \forall \ x, y \in W \ .$$

PROPOSITION. *For all $w \in W$,*

(i) The natural maps define an embedding

$$F(M(w \cdot \lambda), M(w \cdot \lambda)) \hookrightarrow F(M(w_0 \cdot \lambda), M(w_0 \cdot \lambda))$$

of $F(U)$ bimodules.

(ii) $Ann_{F(U)} M(\lambda) = Ann_{F(U)} M(w \cdot \lambda)$.

It is immediate from (JL, 6.4) that $d_{F(U)}$ and d_U coincide on modules admitting a weight decomposition. On modules in $\mathcal{O}_{P(\pi)}$ it is also obvious that d_U and d_{U^-} coincide. We denote this common dimension operator by d. From the formal character of U^- one easily checks that $d(U^-) = dim\ \mathfrak{n}^-$, so it is finite.

(i) Recall that $M(w \cdot \lambda)$ is isomorphic to U^- as a U^- module. Its submodule $M(w_0 \cdot \lambda)$ then identifies with a left ideal L of the integral domain U^-. Since $M(w_0 \cdot \lambda)$ is simple (8.2(i)) we conclude that $M(w_0 \cdot \lambda) = Soc\ M(w \cdot \lambda)$. Hence by 8.4 we have $F(N, M(w \cdot \lambda)) = 0$ unless $d(N) \geq d(M(w_0 \cdot \lambda))$. Now take $N = M(w \cdot \lambda)/M(w_0 \cdot \lambda)$. By 8.4(*), $d(N) \leq d(M(w \cdot \lambda)) - 1 = d(M(w_0 \cdot \lambda)) - 1 < d(M(w_0 \cdot \lambda))$. Consequently $F(M(w \cdot \lambda)/M(w_0 \cdot \lambda), M(w \cdot \lambda)) = 0$. This in turn implies that the natural map $F(w, w) \longrightarrow F(w_0, w)$ is injective. Again by 8.4, one has $F(M(w_0 \cdot \lambda), M(w \cdot \lambda)/M(w_0 \cdot \lambda)) = 0$ and so the natural injection $F(w_0, w) \hookrightarrow F(w_0, w_0)$ is an isomorphism. Combined these prove (i).

(ii) From the embedding $M(w_0 \cdot \lambda) \hookrightarrow M(w \cdot \lambda)$ we obtain $Ann_{F(U)} M(w_0 \cdot \lambda) \supset Ann\ M(w \cdot \lambda)$. For the converse, consider $F(w, w)$ as a left $F(U)$ module. Obviously $Ann_{F(U)} F(w, w) \supset Ann\ M(w \cdot \lambda)$ and equality holds because $F(w, w) M(w \cdot \lambda) = M(w \cdot \lambda)$. From this the required inclusion results by (i).

Remark. We apologize here for a slight flaw in the logical order. We used 8.5(i) in the proof of 6.12 as was noted there. The conclusion of 6.12 (which uses Kostant) shows that the embedding in 8.5(i) is an isomorphism. For this last result, one can avoid the use of Kostant by using 8.3 with $\nu \in P^+(\pi)$ and reasoning as in ([3], Sect. 3). Again the results of ([3], Sect. 3) carry over to our present situation.

8.6. It is clear $Ann_{Z(U)} M(\mu)$ is an ideal of codimension 1 in $Z(U)$. By (JL, 8.6) it follows that $Ann_{Z(U)} M(\mu) = Ann_{Z(U)} M(\nu)$ if and only if μ, ν lie in the same W. orbit, that if there exists $\lambda \in P^+(\pi)$ such that $\mu, \nu \in W \cdot \lambda$. We set $Z_{\hat{\lambda}} = Ann_{Z(U)} M(w \cdot \lambda) : w \in W$. Obviously $Ann_{F(U)} M(w \cdot \lambda) \supset F(U) Z_{\hat{\lambda}}$ and we may anticipate that equality holds. This would follow easily from Gelfand-Kirillov dimension estimates had we known $F(U) Z_{\hat{\lambda}}$ to be completely prime. Actually since $gr_{\mathcal{F}} F(U) Z_{\hat{\lambda}} \supset G(U) J_+$, it is enough to show that $G(U) J_+$ is completely prime. We do not know this yet. Despite this we shall prove the

THEOREM. *For all $\mu \in P(\pi)$, one has $Ann_{F(U)} M(\mu) = F(U) Ann_{Z(U)} M(\mu)$.*

By 8.5(ii) it is enough to prove this when $\mu \in -P^+(\pi)$. Then the assertion follows from 8.3 and 6.12.

8.7. It is clear from 7.4, 7.6, 6.12 and 8.6 that the inclusion $gr_{\mathcal{F}}F(U)Z_{\hat{\lambda}} \supset G(U)J_+$ is in fact an equality, though the proof is hardly direct, nor does it imply that $G(U)J_+$ is completely prime.

Recalling the notation 4.13 and 7.1 choose any homomorphism $\chi : Y \longrightarrow K$ such that $\chi(y_\mu) \neq 0, \forall\ \mu \in -R^+(\pi)$ and set $Y_\chi = ker\ \chi$. It turns out that there is a direct rather easy proof that $G(U)Y_\chi$ is completely prime. However this cannot hold for $G(U)Y_+$ or even $G(\check{U})\check{Y}_+$ since one can show that this implies that 7.4 holds for $G(U)$. We expect to come back to these points in a subsequent paper.

Appendix. Table of Notation Symbols used frequently are given below where they are first defined (see also JL, Index of Notation).

1.1. $\mathfrak{g}$, $U(\mathfrak{g})$, $Z(\mathfrak{g})$, $\mathfrak{h}$
1.2. $U_q(\mathfrak{g})$
1.3. U
1.6. Z

2.1. x_i, y_i, t_i, α_i
2.2. $\mathcal{F}$, U^0, U^+, U^-, deg
2.3. ad

3.1. T, π, $P(\pi)$, $Q(\pi)$, $R(\pi)$, τ, $T_\diamond$, $\check{U}$
3.2. φ, $M(\lambda)$, $L(\lambda)$
3.3. $\mathcal{R}$
3.4. $P^+(\pi)$, $R^+(\pi)$, $F_\diamond(U)$

4.2. ω_i, w_0
4.4 $T_<$, $T_<^m$, w_0
4.5. T_m
4.6. G, G^+, G^-, G^0, G_λ, G_m^+, G_m^-, $G(\lambda)_m^-$
4.9. $K(\lambda)^-$, $F(\lambda)$
4.13. z_λ, y_λ
4.14. $\succ$, $\mu \cap \lambda$

5.4. $R_\mu(q)$, $P_\mu(q)$

5.5. $G(U)$, $Y(U)$, Y_+, J_+

6.1. $S(\lambda)^-$
6.2. y_λ^0
6.10. $F(M,N)$
6.11. $\mathbb{H}'$, H'
6.15. $w \cdot \lambda$

7.3. H
7.5. $\mathbb{H}$

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