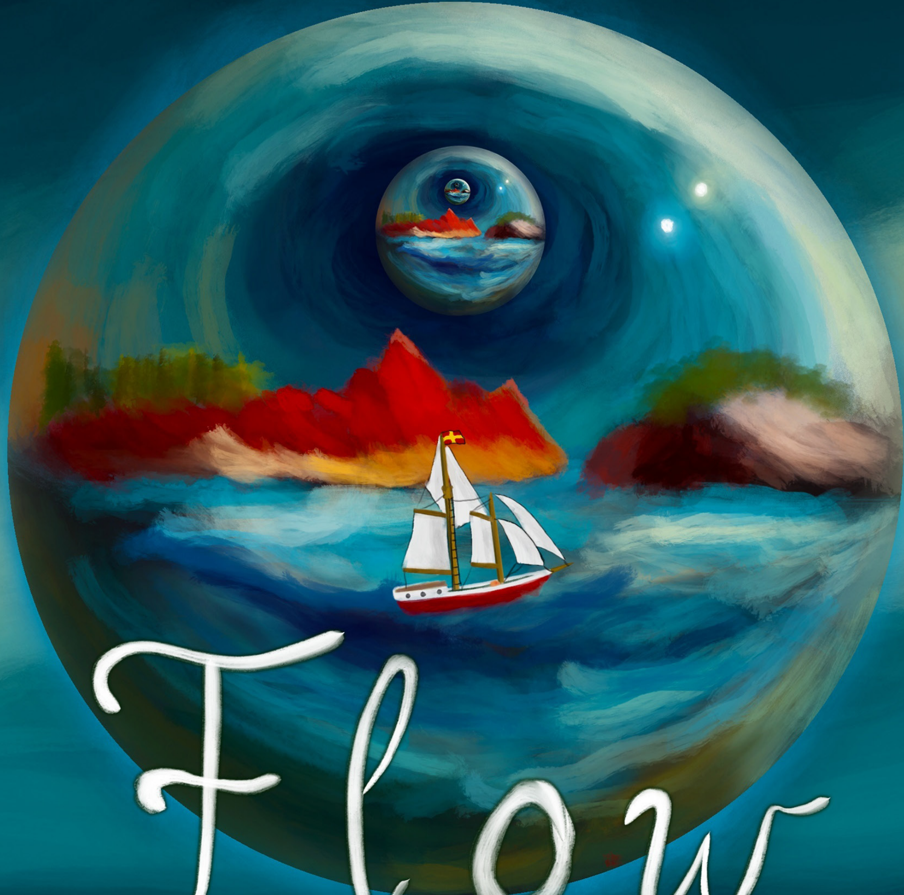
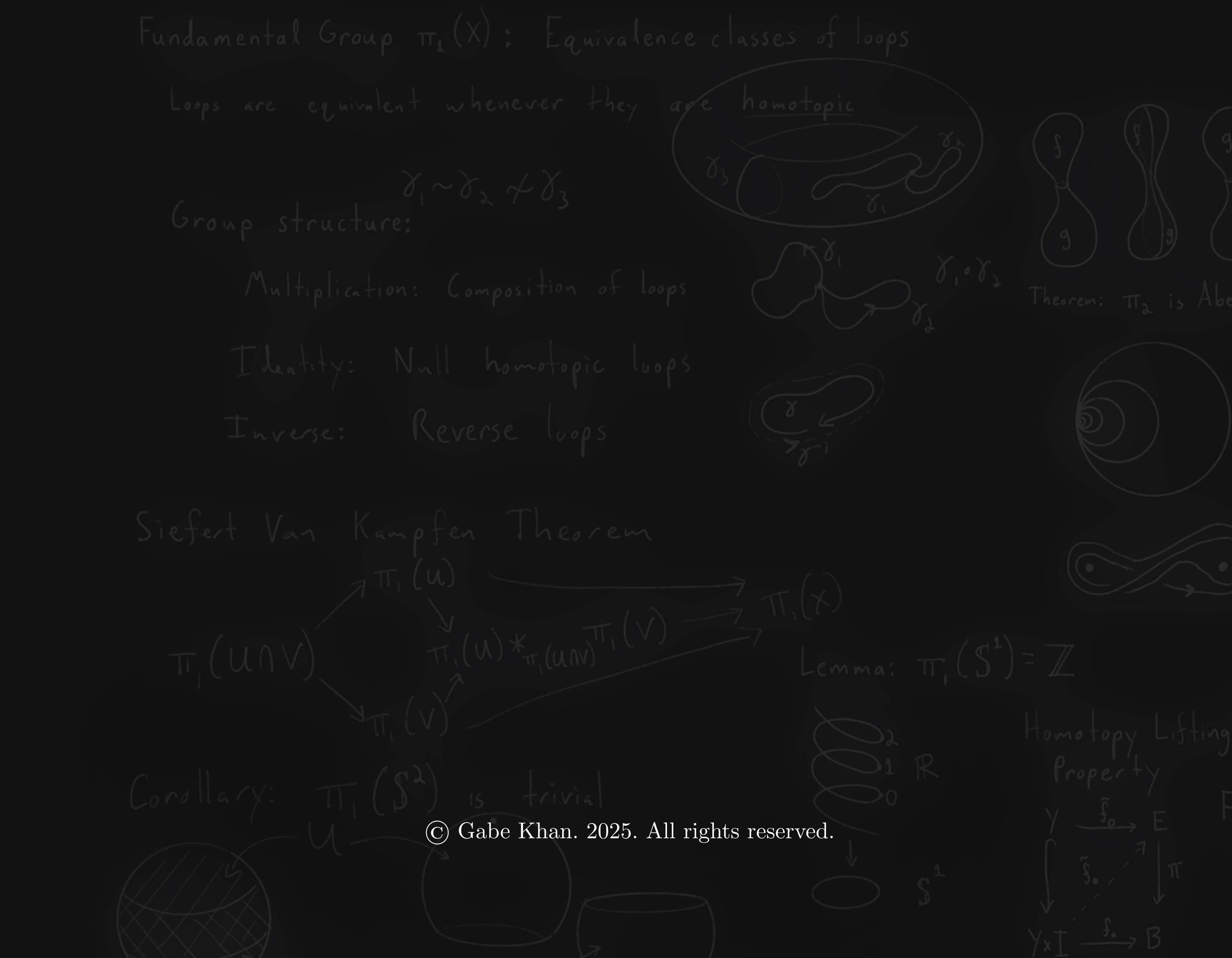
Gabe Khan



Flow

A story of heat and geometry

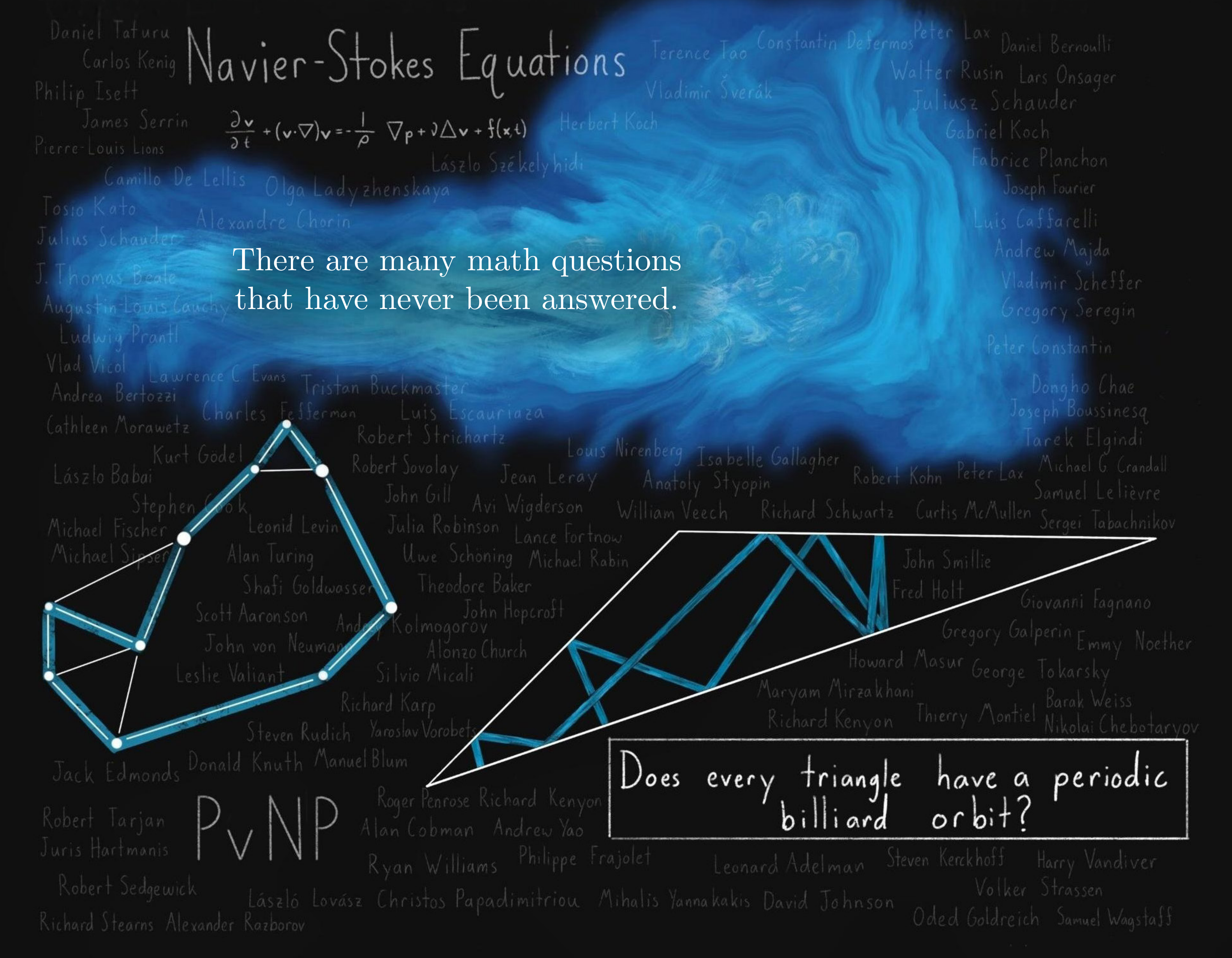
To Remi


Fundamental Group $\pi_1(X)$: Equivalence classes of loops
Loops are equivalent whenever they are homotopic
$\gamma_1 \sim \gamma_2 \not\sim \gamma_3$
Group structure:
Multiplication: Composition of loops
Identity: Null homotopic loops
Inverse: Reverse loops
Theorem: π_2 is Abe
Siefert Van Kampfen Theorem
Lemma: $\pi_1(S^1) = \mathbb{Z}$
Homotopy Lifting Property
Corollary: $\pi_1(S^2)$ is trivial

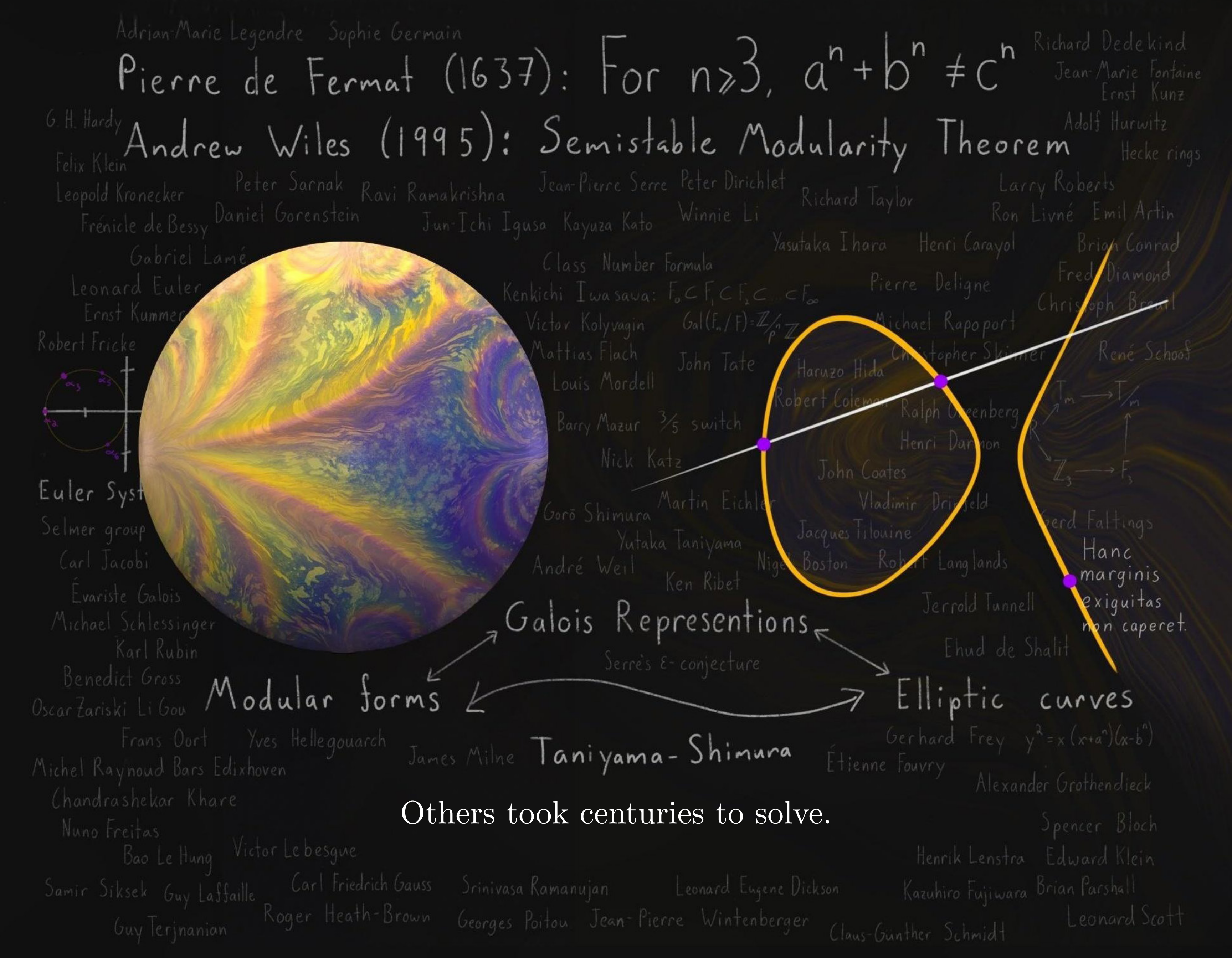
Flow

A story of heat and geometry

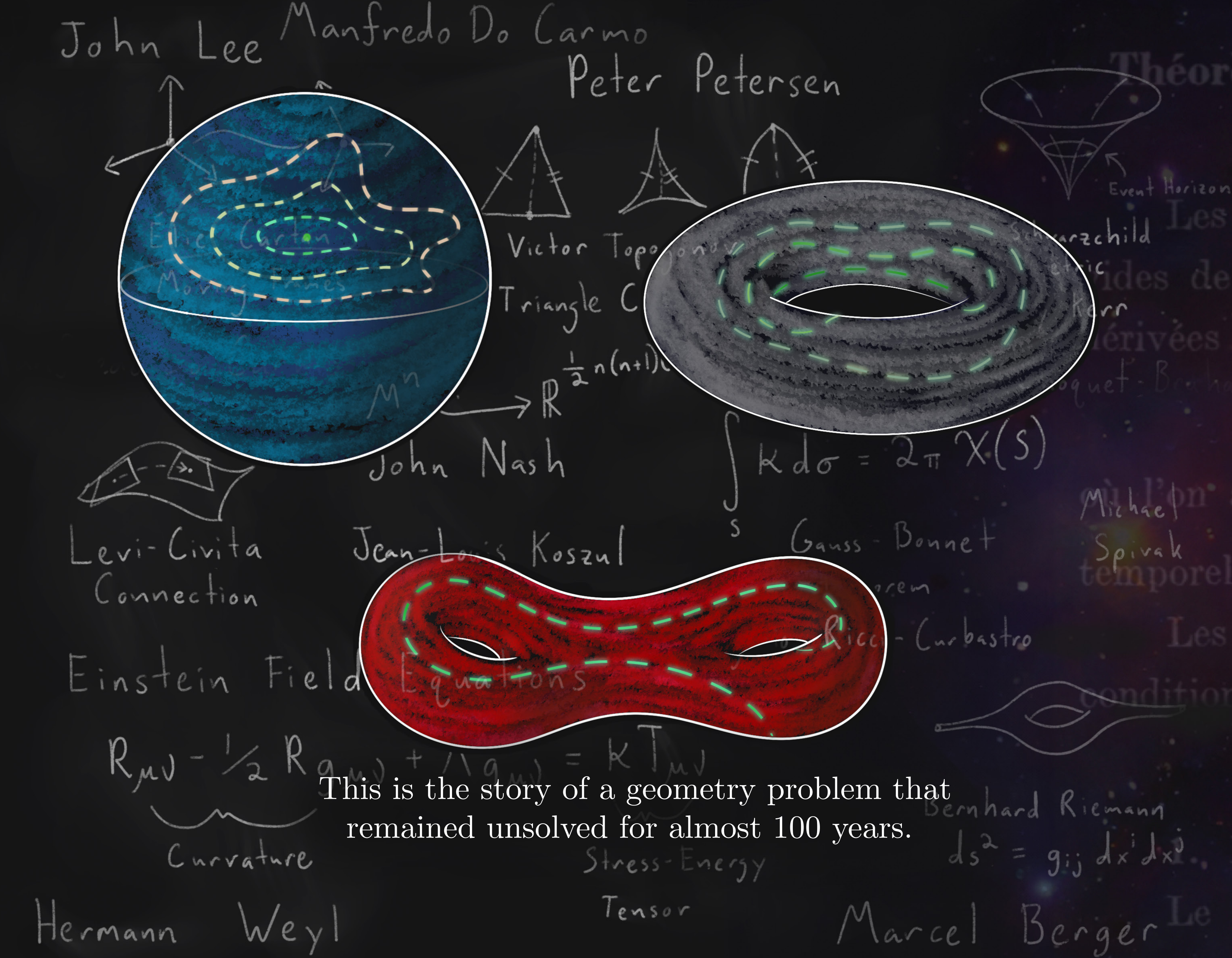
Gabe Khan



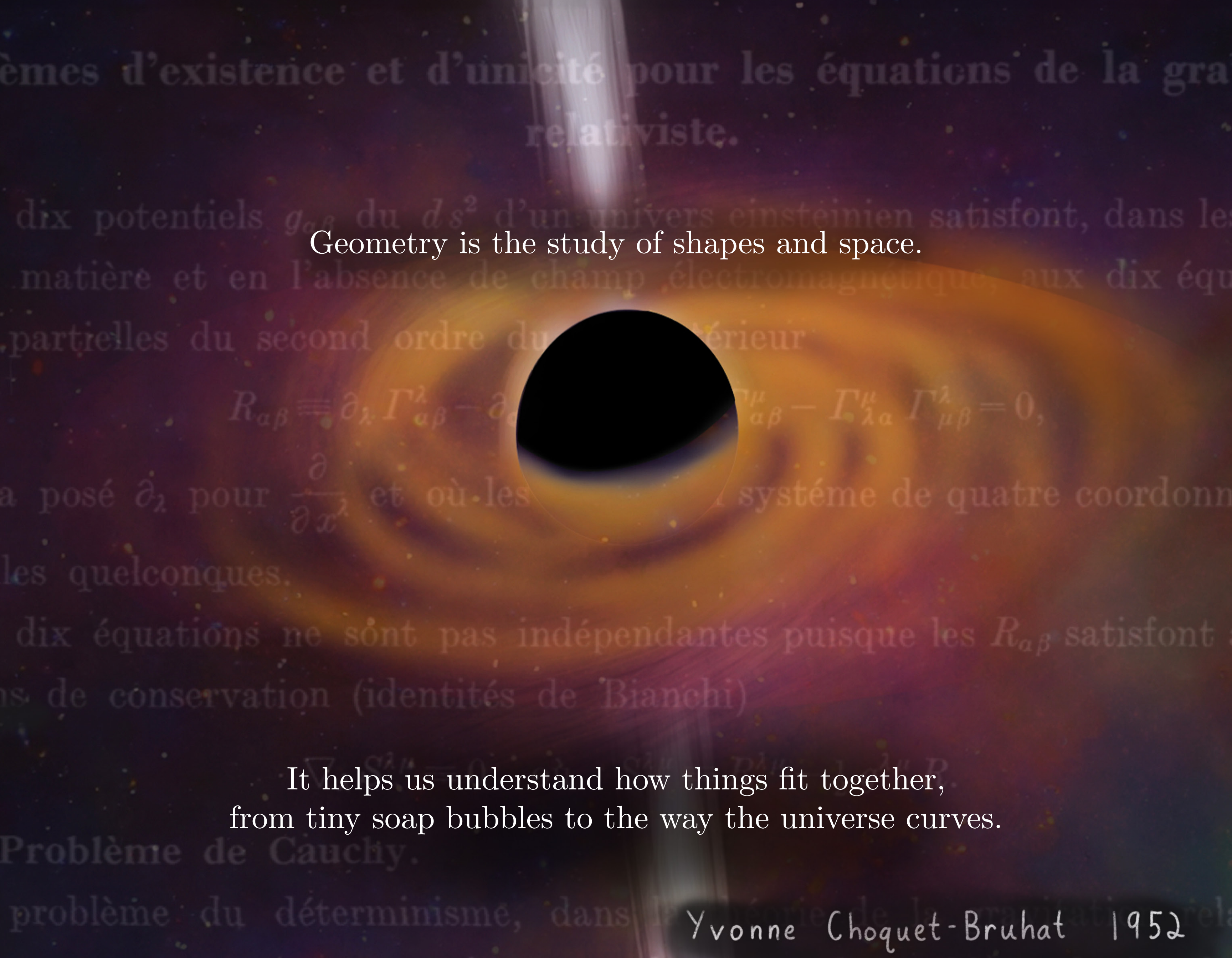
There are many math questions that have never been answered.



Others took centuries to solve.



This is the story of a geometry problem that remained unsolved for almost 100 years.



Geometry is the study of shapes and space.

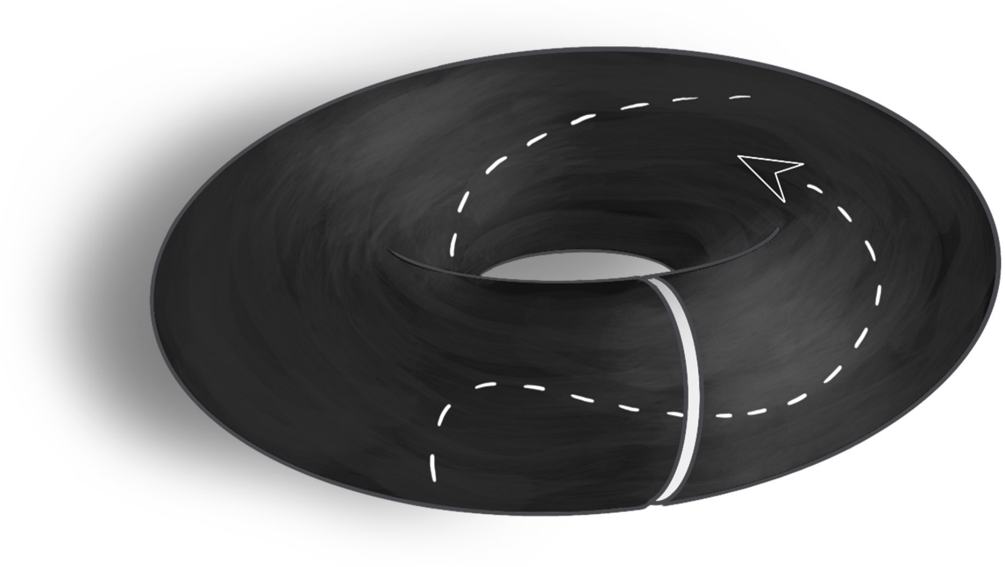
It helps us understand how things fit together,
from tiny soap bubbles to the way the universe curves.

Thousands of years ago, people discovered
that Earth is shaped like a sphere.

But our planet isn't perfectly round.
Its spinning squashes it slightly and
the land forms mountains and valleys.

So mathematicians study how
surfaces bend and stretch.

Here is a surface that is very
different from the earth.



We can bend this surface into a coffee mug,
but not into a sphere!

A mathematician is a device for turning coffee into theorems.
-Alfréd Rényi

A comathematician is a device for turning cotheorems into ffee.
-Anonymous

Some surfaces can be bent into spheres. Others cannot.
How can we tell the difference?

SOLVTIO PROBLEMATIS AD GEOMETRIAM SITVS PERTINENTIS.

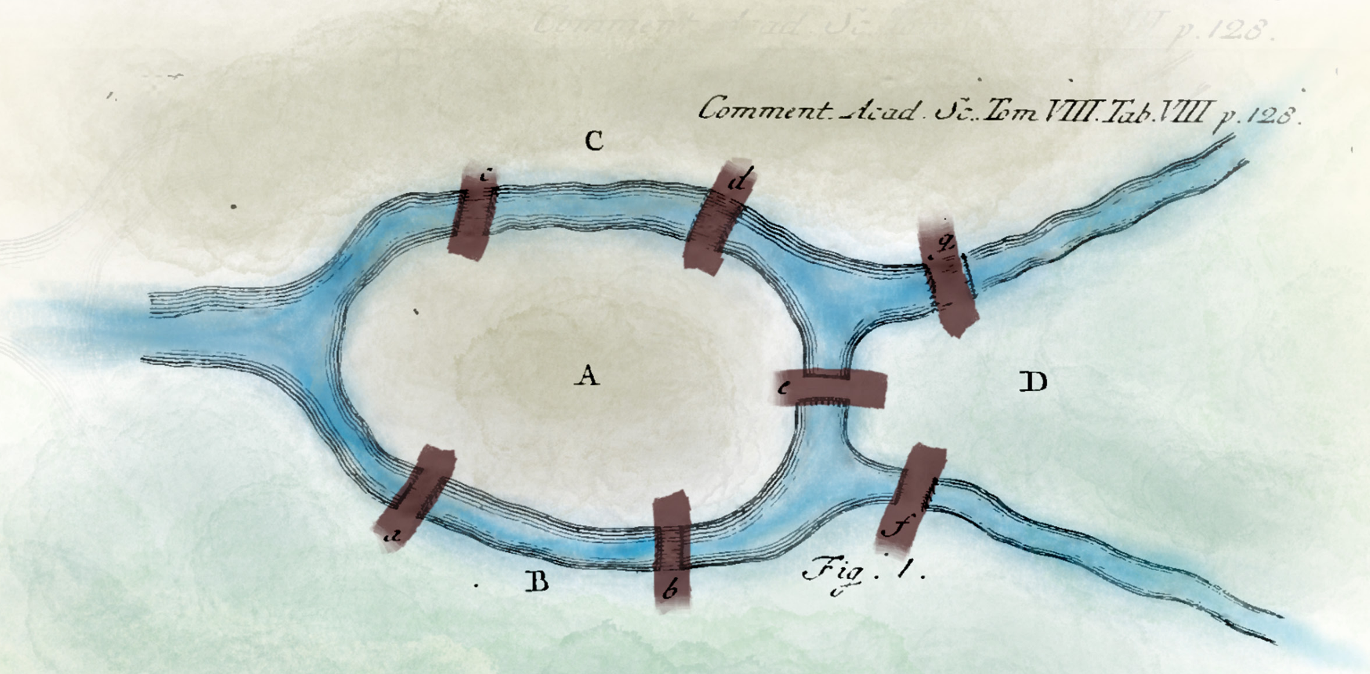
AVCTORE

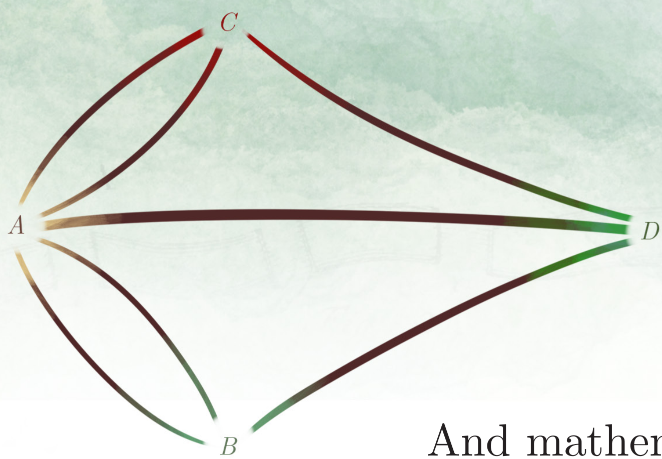
Leonh. Eulero.

§. 1.

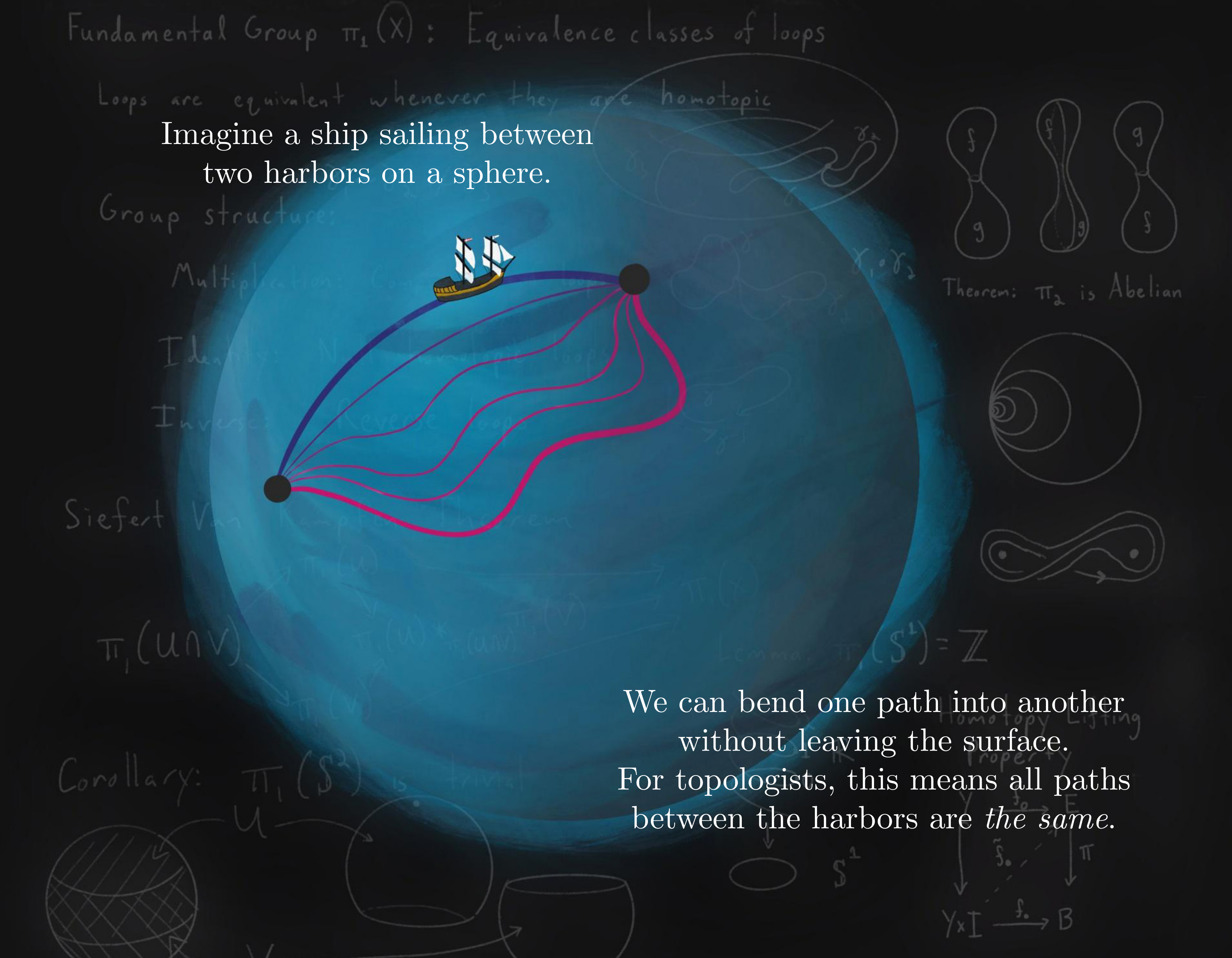
Tabula VIII.

The mathematics which studies how surfaces and other shapes bend is called *topology*.





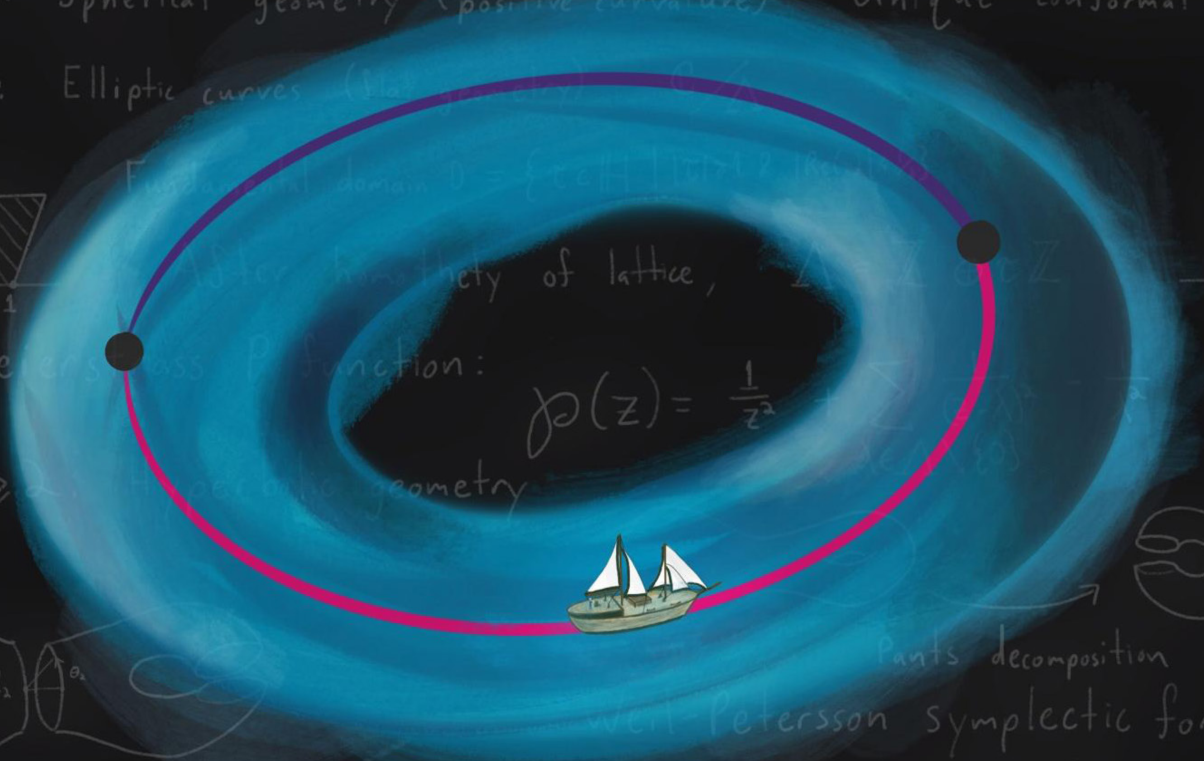
And mathematicians who study topology are called *topologists.*



Imagine a ship sailing between two harbors on a sphere.

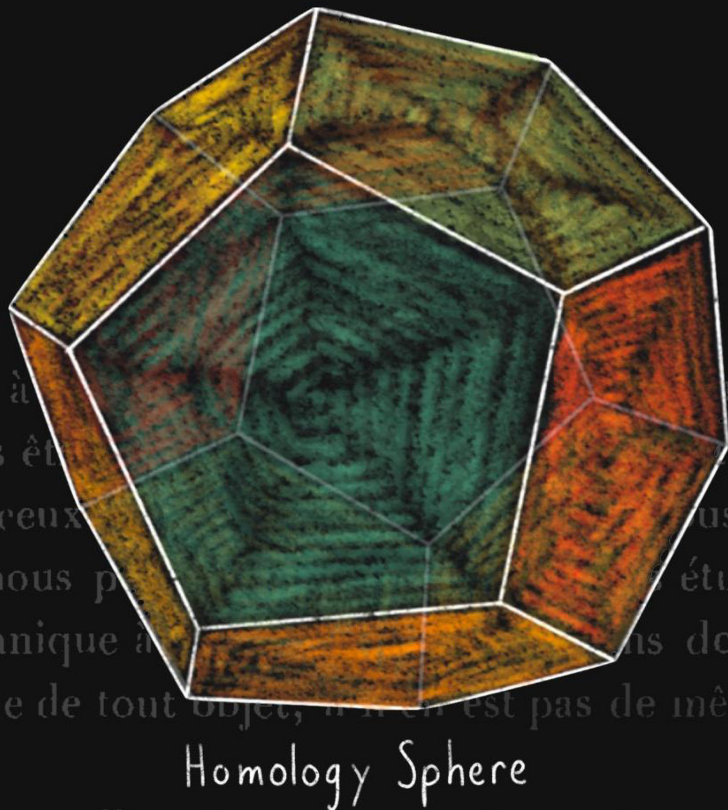
We can bend one path into another without leaving the surface. For topologists, this means all paths between the harbors are *the same.*

But on a planet shaped like a donut, a route that loops around the hole one way cannot bend into one that loops another way. So the topologies of these worlds are different!



Henri Poincaré introduced this idea in 1895, and used topology to study the geometry of surfaces.

Henri thought about three-dimensional space too. In 1904, he posed a question:



If every loop in a closed space can be shrunk to a point, does the space have the same topology as a 3-sphere?

Question. *If a compact three-dimensional manifold M^3 has the property that every simple closed curve within the manifold can be deformed continuously to a point, does it follow that M^3 is homeomorphic to the sphere S^3?*

The Poincaré Conjecture
JOHN MILNOR

CINQUIÈME COMPLÉMENT À L'ANALYSIS SITUS.

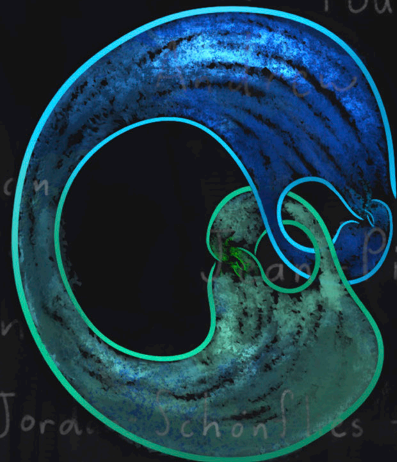
Hopf Fibration

This question became known as the *Poincaré Conjecture*.

A <u>conjecture</u> is a mathematical idea that has not been proven.

Emmy Noether
R H Bing
Walther von Dyck
Frank Adams
Shiing-Shen Chern
Daniel Quillen
H. Blaine Lawson
Poul Heegaard
Oswald Veblen
Casson

John Milnor
Exotic 7-Sphere

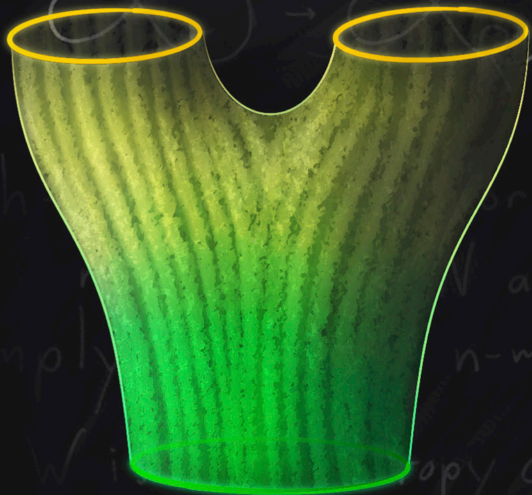


J W Alexander
Horned Sphere

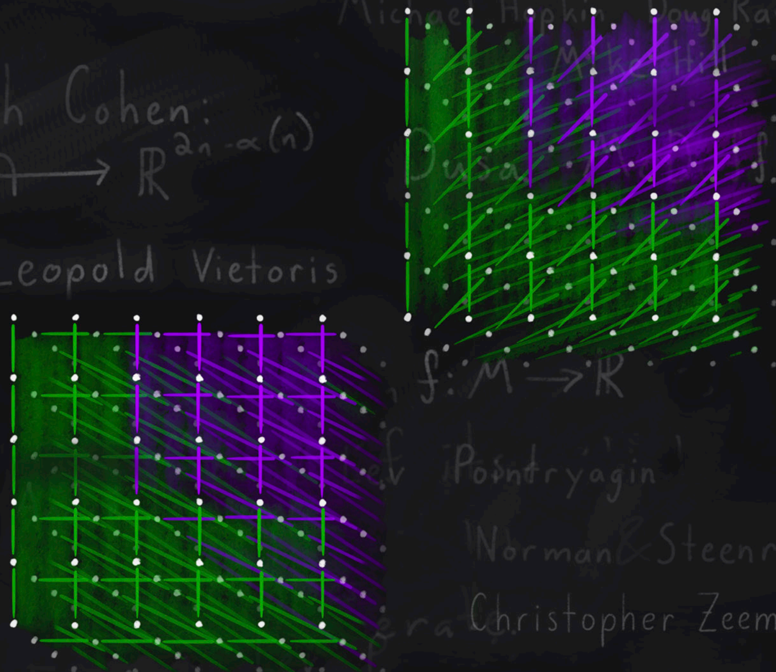
fails in dimension 3.
Allen Hatcher
Maurice Fréchet
Laurent Siebenmann
Camille Jordan
Michael Anderson
Felix Hausdorff
Marston Morse
J H C Whitehead
Heinz Hopf
Mary Rudin
Pavel Uryson
Andrey Kolmogorov

Over the next 80 years, many people thought about topology. They discovered many strange and fantastical spaces.

And they invented tools to study these spaces.
But despite all their efforts, no one knew how
to answer Henri's question.



Stephen Smale
Cobordism Theory



The perfidious lemma of Dehn
Was every topologist's bane
'Til Christos D. Pap-
akyriakop-
oulos proved it without any strain.
- Milnor

BULLETIN (New Series) OF THE
AMERICAN MATHEMATICAL SOCIETY
Volume 6, Number 3, May 1982

THREE DIMENSIONAL MANIFOLDS, KLEINIAN GROUPS AND HYPERBOLIC GEOMETRY

BY WILLIAM P. THURSTON

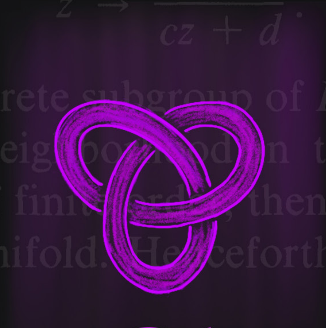
1. A conjectural picture of 3-manifolds. A major thrust of mathematics in the late 19th century, in which Poincaré had a large role, was the uniformization theory for Riemann surfaces: that every conformal structure on a closed oriented surface is represented by a Riemannian metric of constant curvature. For the typical case of negative Euler characteristic (genus greater than 1) this gives a hyperbolic structure: any small neighborhood in the surface is isometric to a neighborhood in the hyperbolic plane, and the surface itself is the quotient of the hyperbolic plane by a discrete group of motions. The exceptional cases, the sphere and the torus, have spherical and Euclidean structures.

Three-manifolds are greatly more complicated than surfaces, and I think it is fair to say that until recently there was little reason to expect any analogous theory for manifolds of dimension 3 (or more)—except perhaps for the fact that so many 3-manifolds are beautiful. The situation has changed, so that I feel fairly confident in proposing the

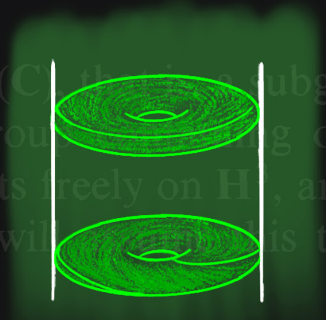
1.1. CONJECTURE. *The interior of every compact 3-manifold has a canonical decomposition into pieces which have geometric structures.*

Then, in 1982, Bill Thurston found a new way to think about three-dimensional geometry.

Sol



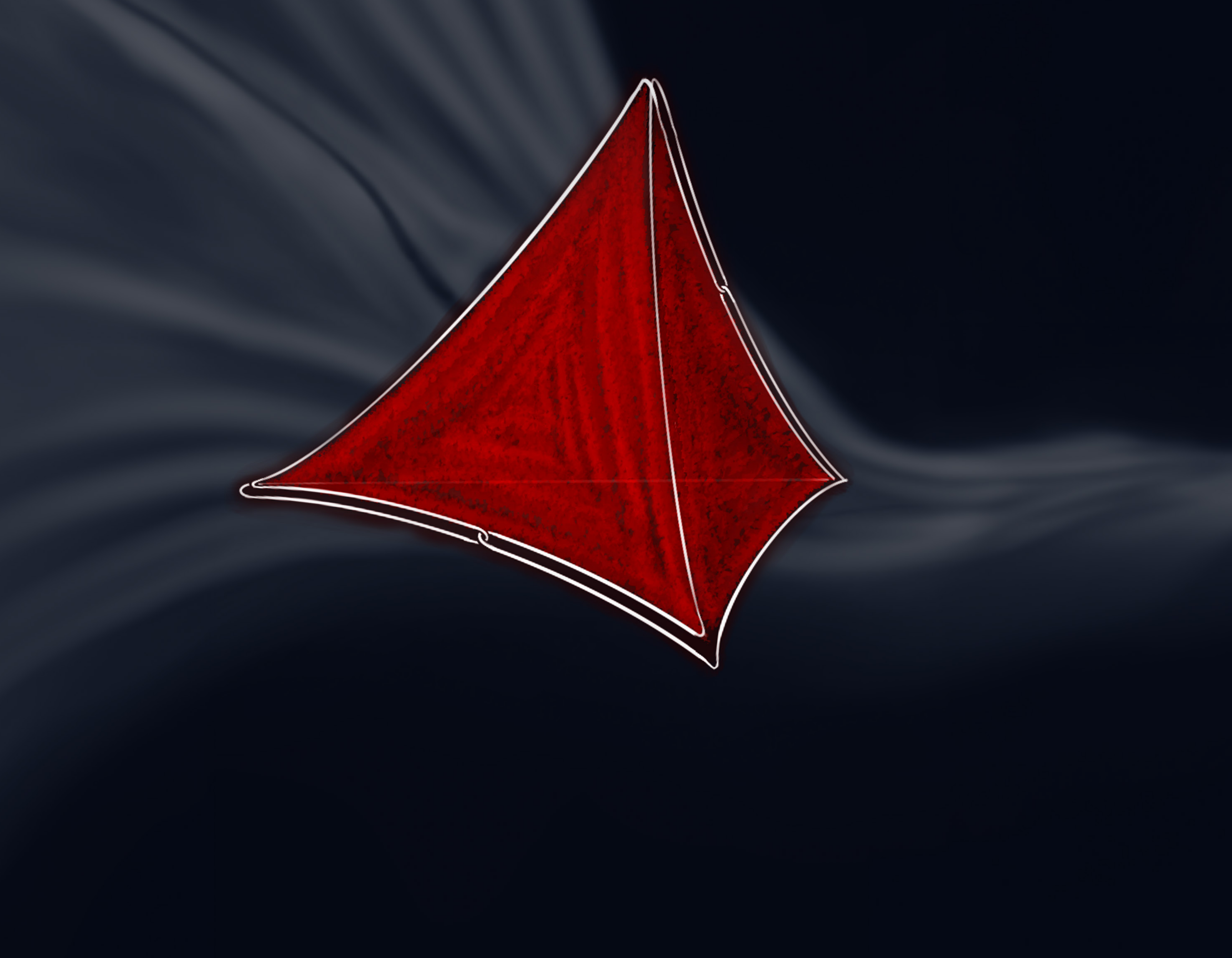
$\widetilde{\mathbb{SL}_2}$

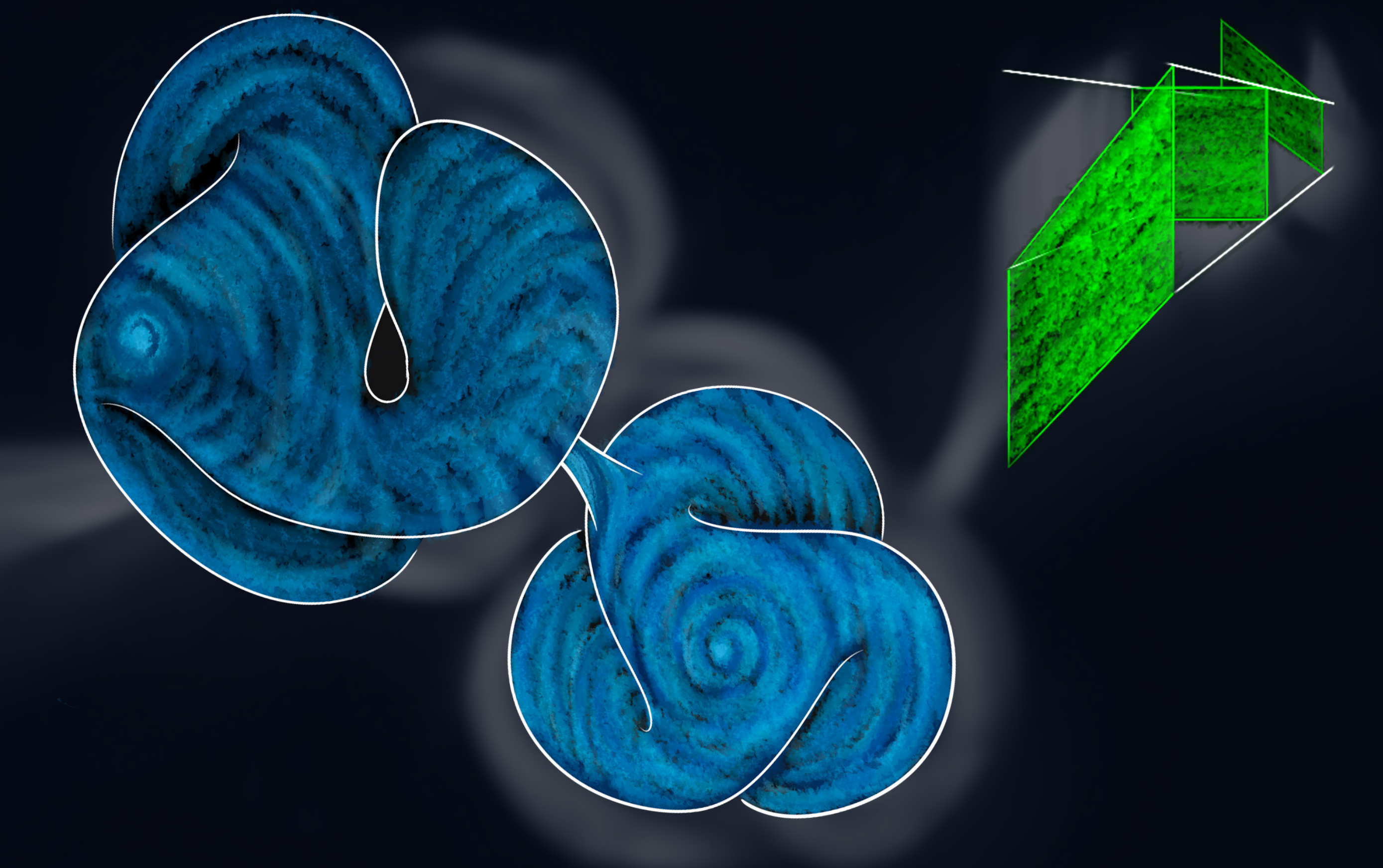


Nil

He described eight special kinds of geometry.

And given any three-dimensional space,





he envisioned a way to cut it so that each
piece had one of these geometries.

This idea became known as the *Geometrization Conjecture*.

The Symplectic Camel

It suggested there was a hidden structure to three-dimensional space.

ein besonderes Max Dehn unserer Art Wolfgang Haken innerhalb der noch Dietmar Salamon Kreise der ungefähre Verlauf George Francis

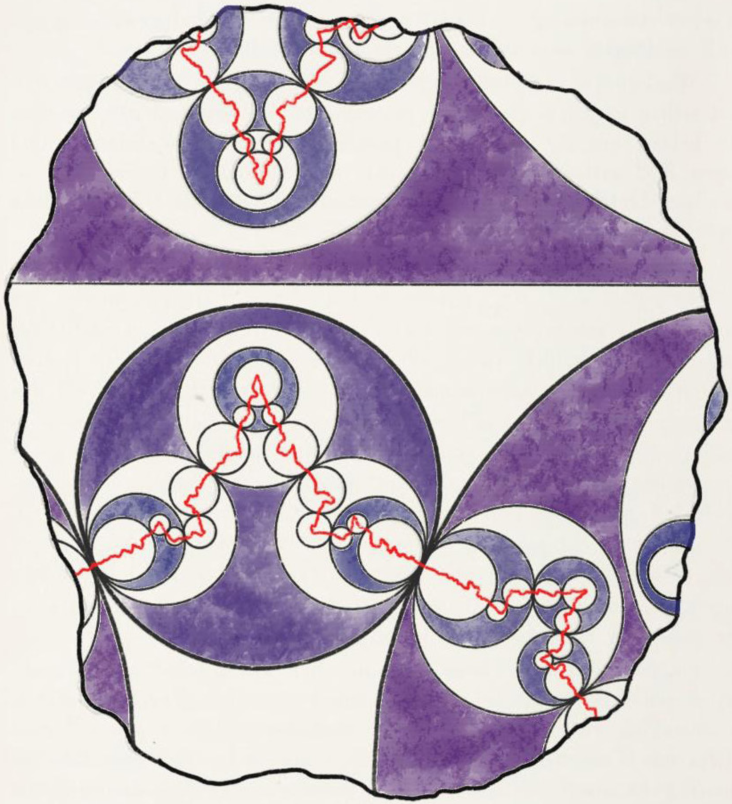


Fig. 145.

Limit set of a quasi-Fuchsian group

Figur 145 werden Grigory Margulis folgenden Paragraphen Kaoru Ono - holt anknüpfen.

Einige weitergehende Simon Donaldson Gestalt der Dusa McDuff können Katrin Wehrheim einer mehr begrifflichen Überlegung machen.

Zu Abigail Thompson Überlegung James Cannon hier Mikhael Gromov nehmen; denn Lipman Bers gelingt Claude LeBrun geometrischen „Anschauung" Vitaly Kapovich bei der Grenzcurve Helmut Hofer Verhältnisse zu erfassen.

Wir beginnen Eduard Zehnder, eine symbolische J. Hyam Rubinstein aller Robert Riley Grenzcurve einzuführen.

Ein *nicht-parabolischer* Grenzpunkt Sun-Yung Alice Chang auch den Spiegelungsprocess fortführen mögen, stets Clifford Taubes bestimmten Paul Yang frei Joan Birman Kreisen. Um vom Ausgangsviereck aus immer näher an den fraglichen Grenzpunkt Aleksandr Aleksandrov man somit Nigel Hitchen bestimmte, unendliche Reihe von Vierecken:

$$1,\quad \overline{V_{i_1}},\quad \overline{V_{i_1}V_{i_2}},\quad \overline{V_{i_1}V_{i_2}V_{i_3}},\quad \overline{V_{i_1}V_{i_2}V_{i_3}V_{i_4}},\ \ldots$$

Robert Gompf wobei die Neil Trudinger Zahlen Gerard Besson 1, 2, 3, 4 sind, und Kenji Fukaya zwei auf einander Gilles Courtois Sylvain Gallot ander gleich sind. Michael Anderson *fraglichen Grenzpunkt symbolisch durch*:

(1) $$[i_1,\ i_2,\ i_3,\ i_4,\ \ldots]$$

darstellen können, Boris Delaunay *dieser* Anton Petrunin *einen bestimmten Grenzpunkt.*

Chiu-Chu Melissa Liu Grenzpunkte Svetlana Katok dieser Ansatz mit Leichtigkeit. Alexander Givental ist nur der, Stephanie Alexander *parabolischen Grenzpunkte immer zwei Arten der Annäherung haben.* So lässt sich z. B. die William Meeks Ausgangsdreiecks Vladimir Arnold [1, 2, 1, 2, ...] als durch [2, 1, 2, 1, ...] darstellen.

Übrigens Michael Kapovich hervor, *dass* Hidehiko Yamabe *Grenzpunkte* Troels Jørgensen *darstellende* Takao Yamaguchi *haben*. Jim Simons ε der Abstand Emmanuel Breuillard von einander, George Mostow im Spiegelungsprocess André Haefliger dass die Durchmesser sämtlicher noch offenen Kreise $< \varepsilon$ sind. Sicher Curtis McMullen an beide Punkte in Danny Calegari Kreisen liegen Takashi Shioya verschiedene Ketten (1) liefern. Was die Matthew Gursky Punkte Sergei Ivanov wird Yuri Burago Überlegung leicht ergänzen.

Charles Conley an die Darstellung (1) folgende Einteilung aller Grenzpunkte: Christina Sormani *entweder* Cornelia Drutu *oder nach einer* … *so heisse der* … *Falle*

And if Bill’s idea was true, it would show that Henri’s idea was true too!

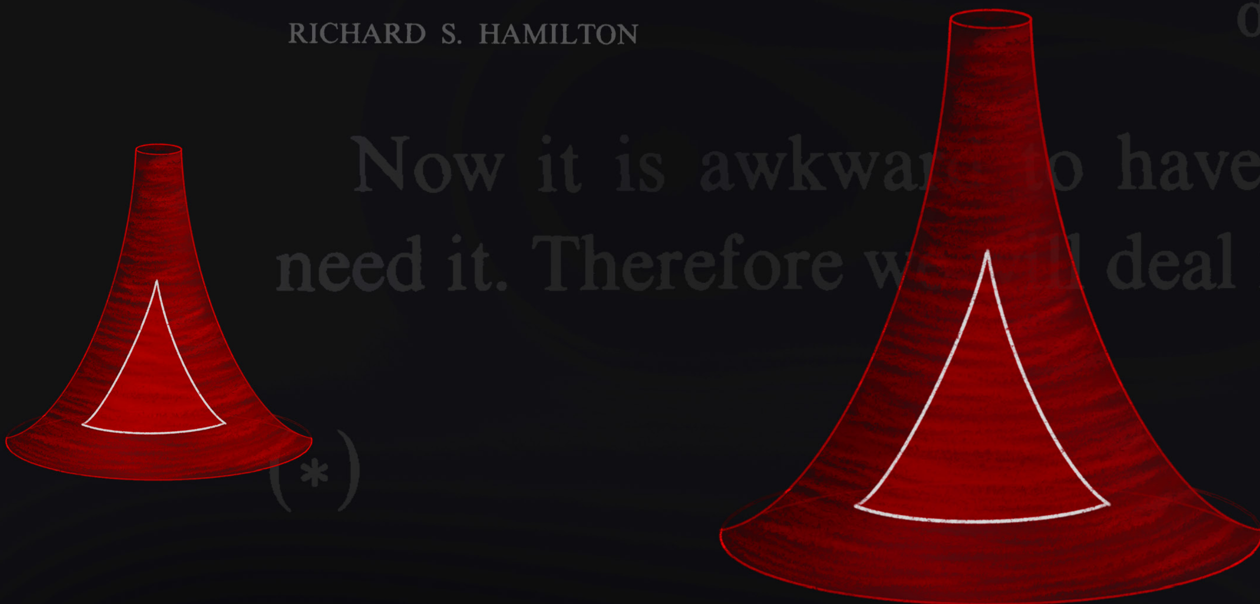
Thierry Aubin *aperiodisch.* Die periodischen Grenzpunkte sind identisch mit Andreas Floer der Substitutionen von Γ; Mladen Bestvina *teilen wir die* Gopal Prasad *Grenzpunkte in die drei weiteren Arten der*

Robert Fricke and Felix Klein (1897)

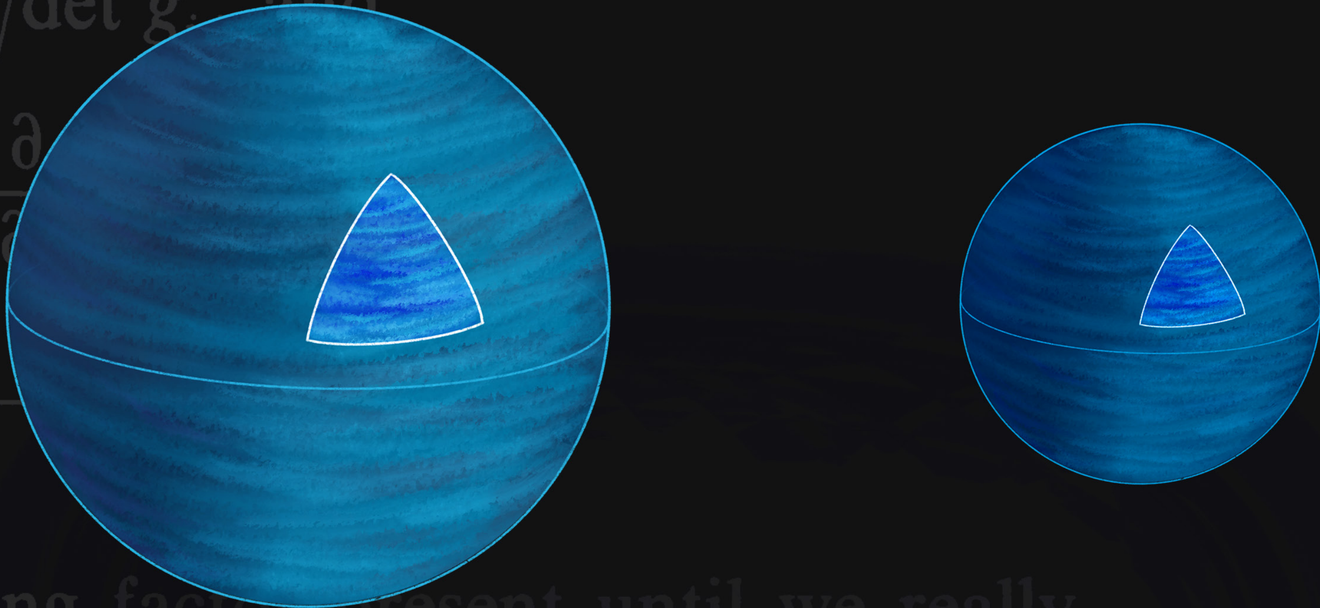
Around the same time, Richard Hamilton had another idea. He found a way to change the shape of space using the way it curves.

THREE-MANIFOLDS WITH POSITIVE RICCI CURVATURE

RICHARD S. HAMILTON



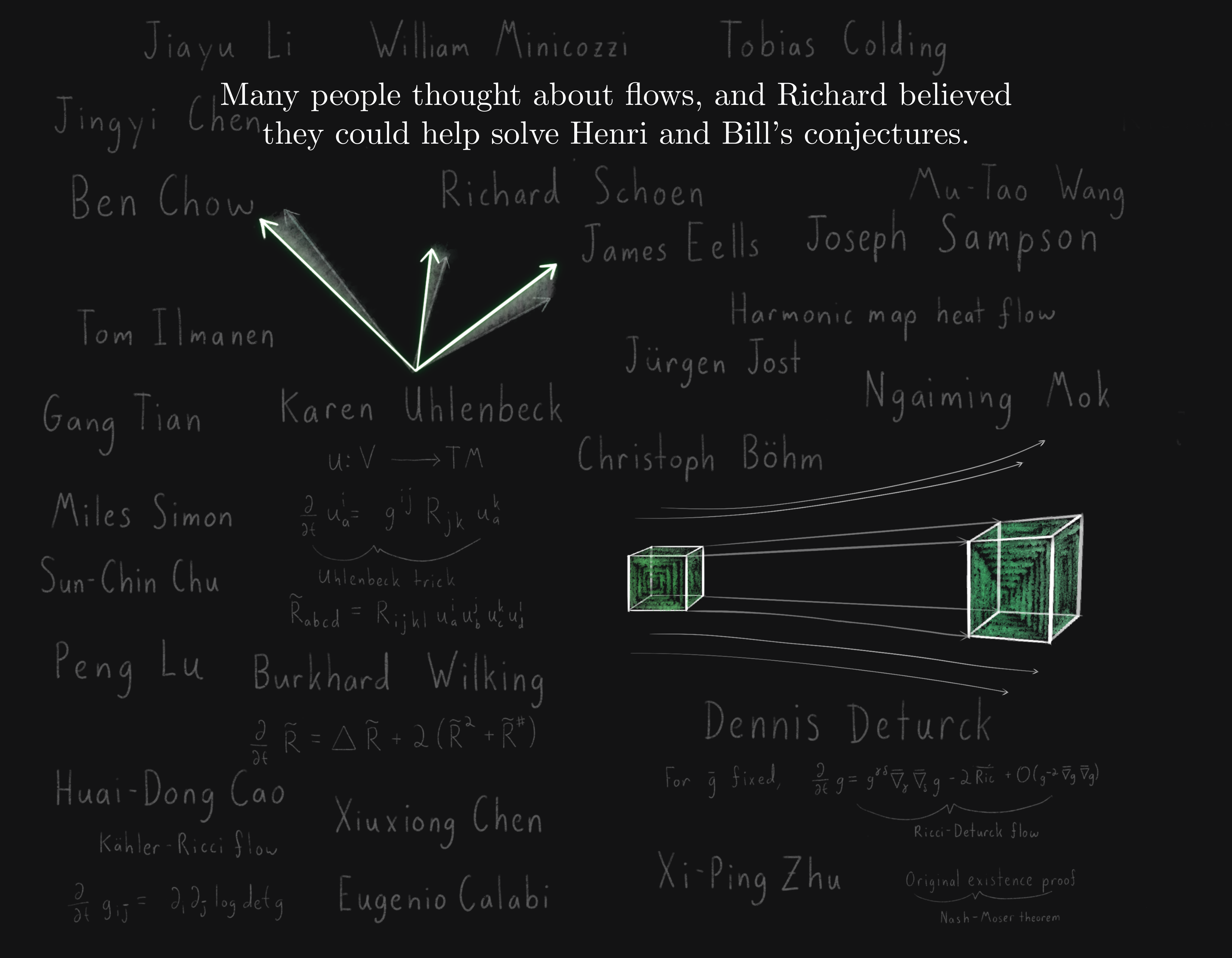
Parts which curve one way get larger as time passes.

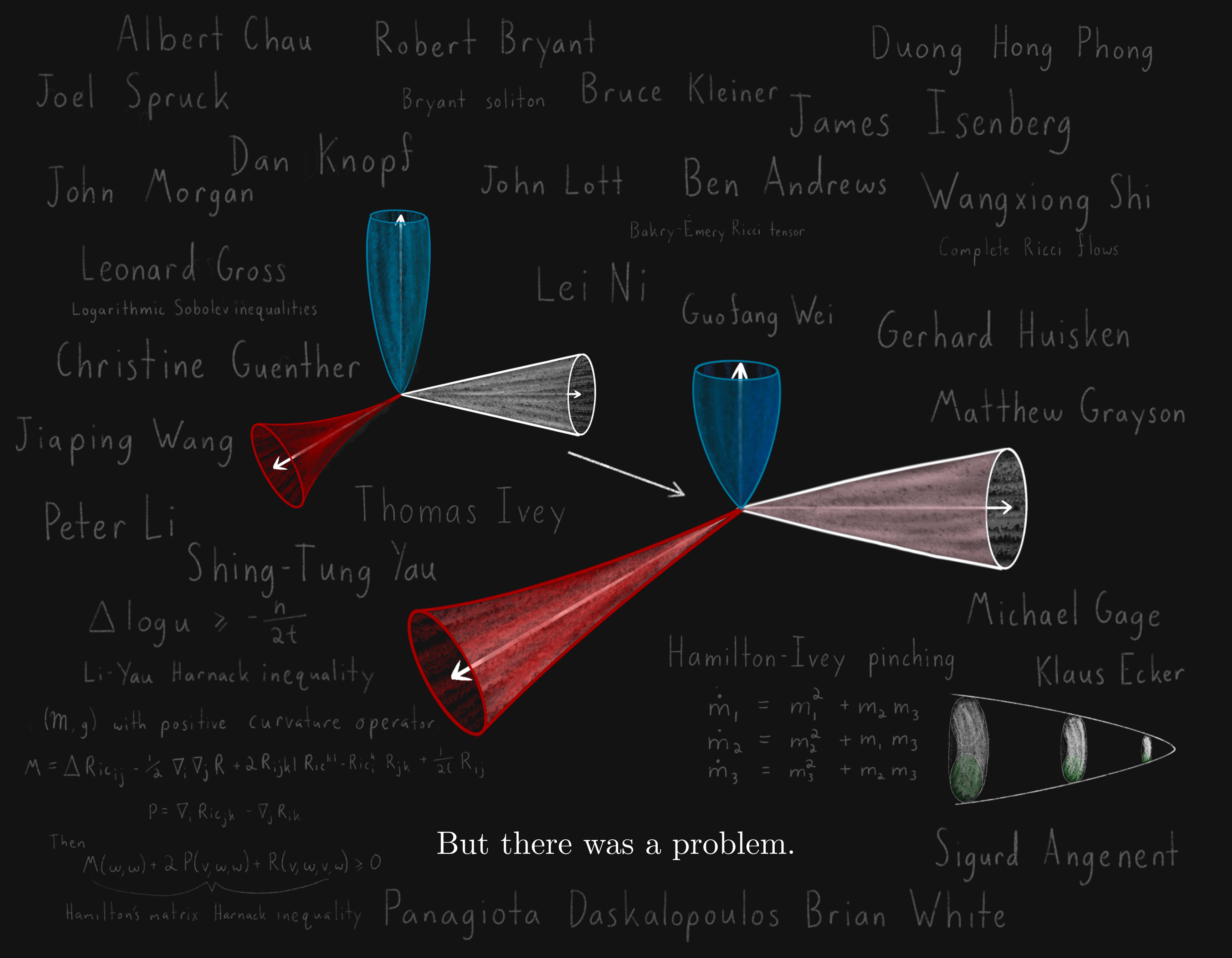


Parts which curve another way get smaller.

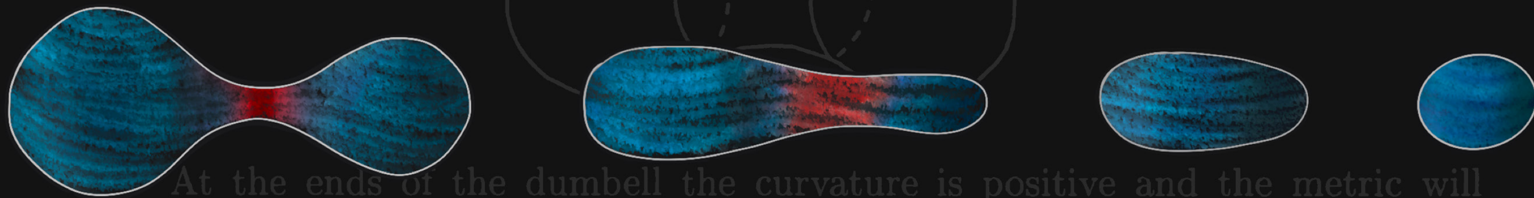
This process is called *Ricci flow*.

Many people thought about flows, and Richard believed they could help solve Henri and Bill's conjectures.

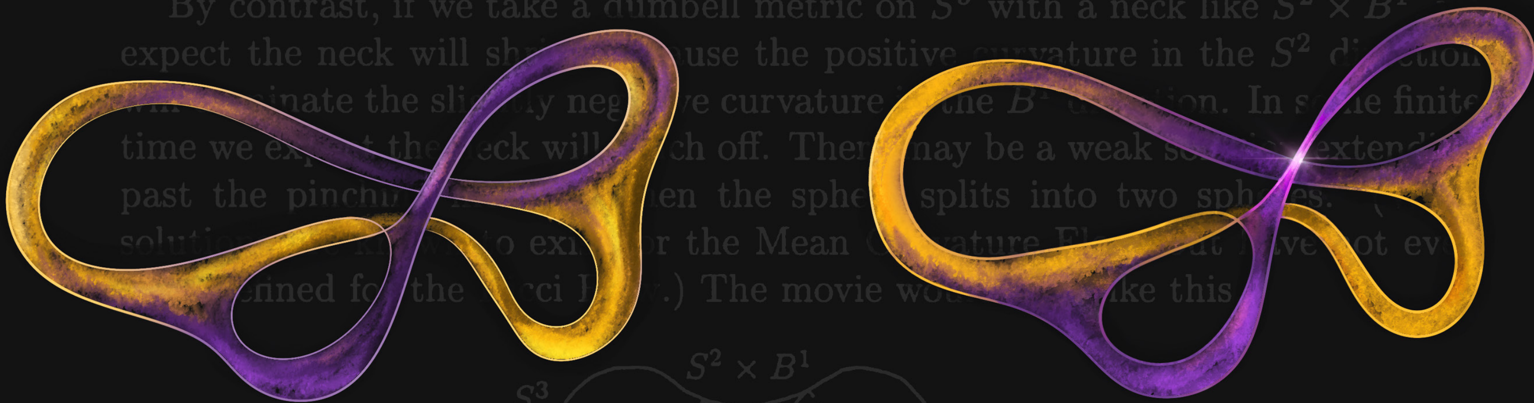




But there was a problem.

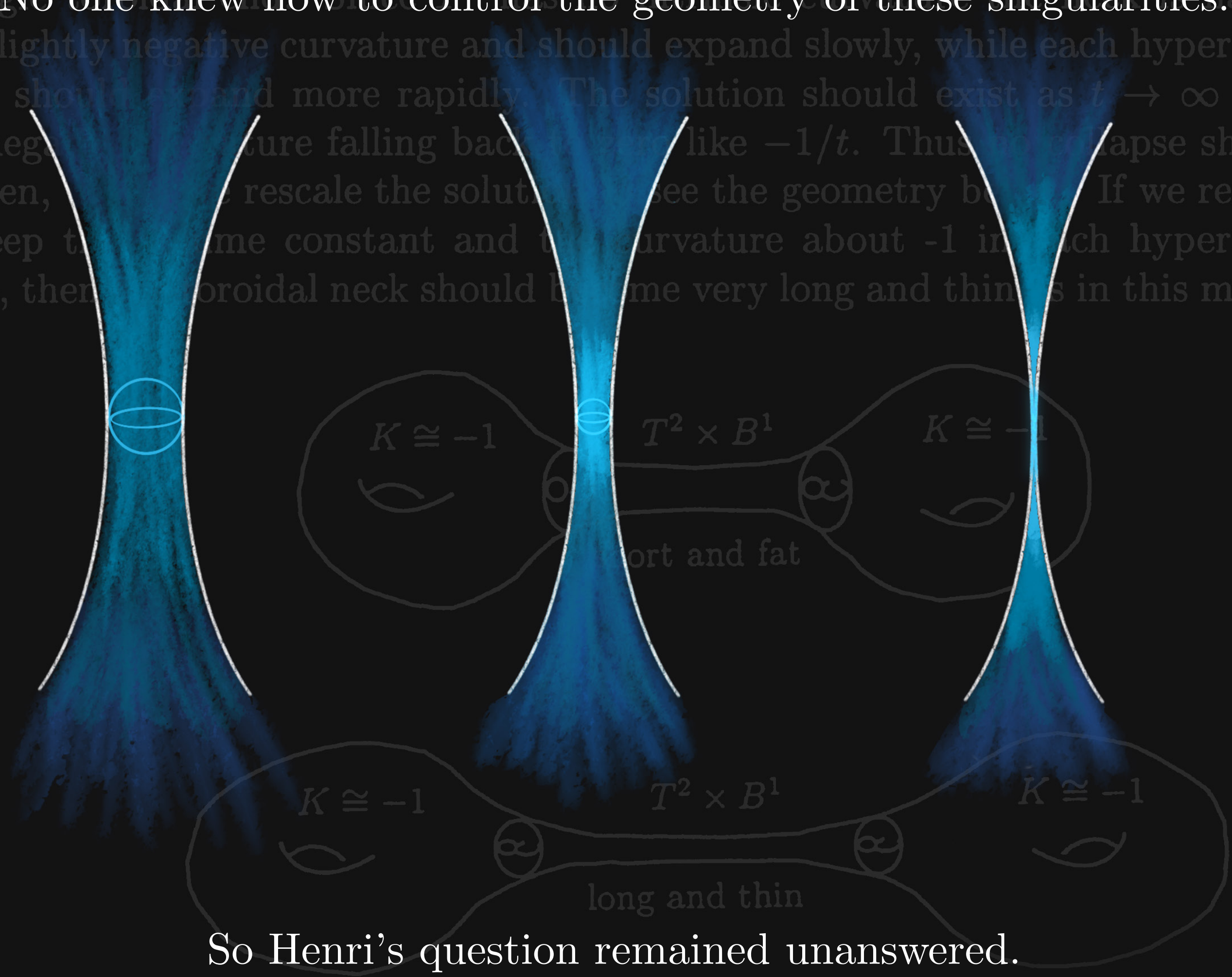


Sometimes Ricci flow would reveal the underlying geometry.



But other times something would go wrong,
and space would tear itself apart.

No one knew how to control the geometry of these singularities.

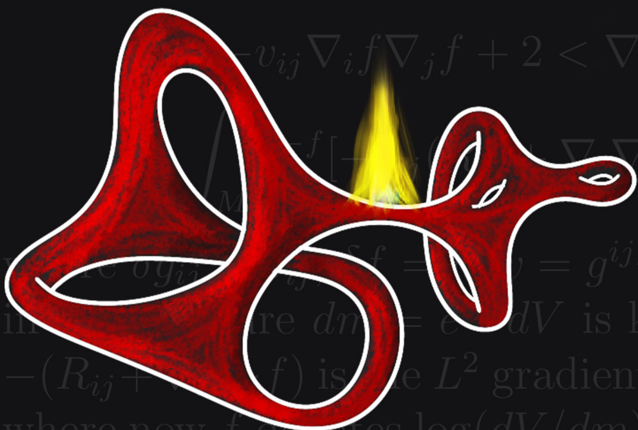


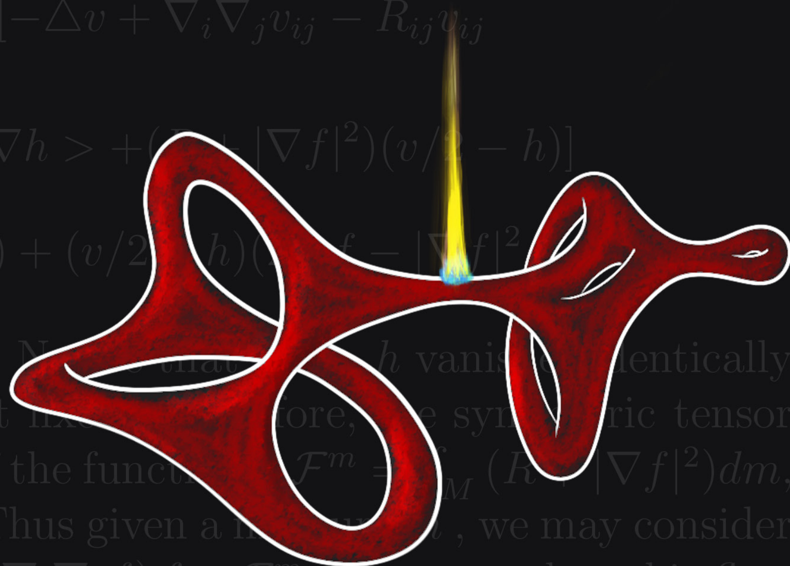
So Henri's question remained unanswered.

The Formation of Singularities in the Ricci Flow

Richard S. Hamilton

Fifteen years later, Grigori Perelman made a crucial breakthrough.



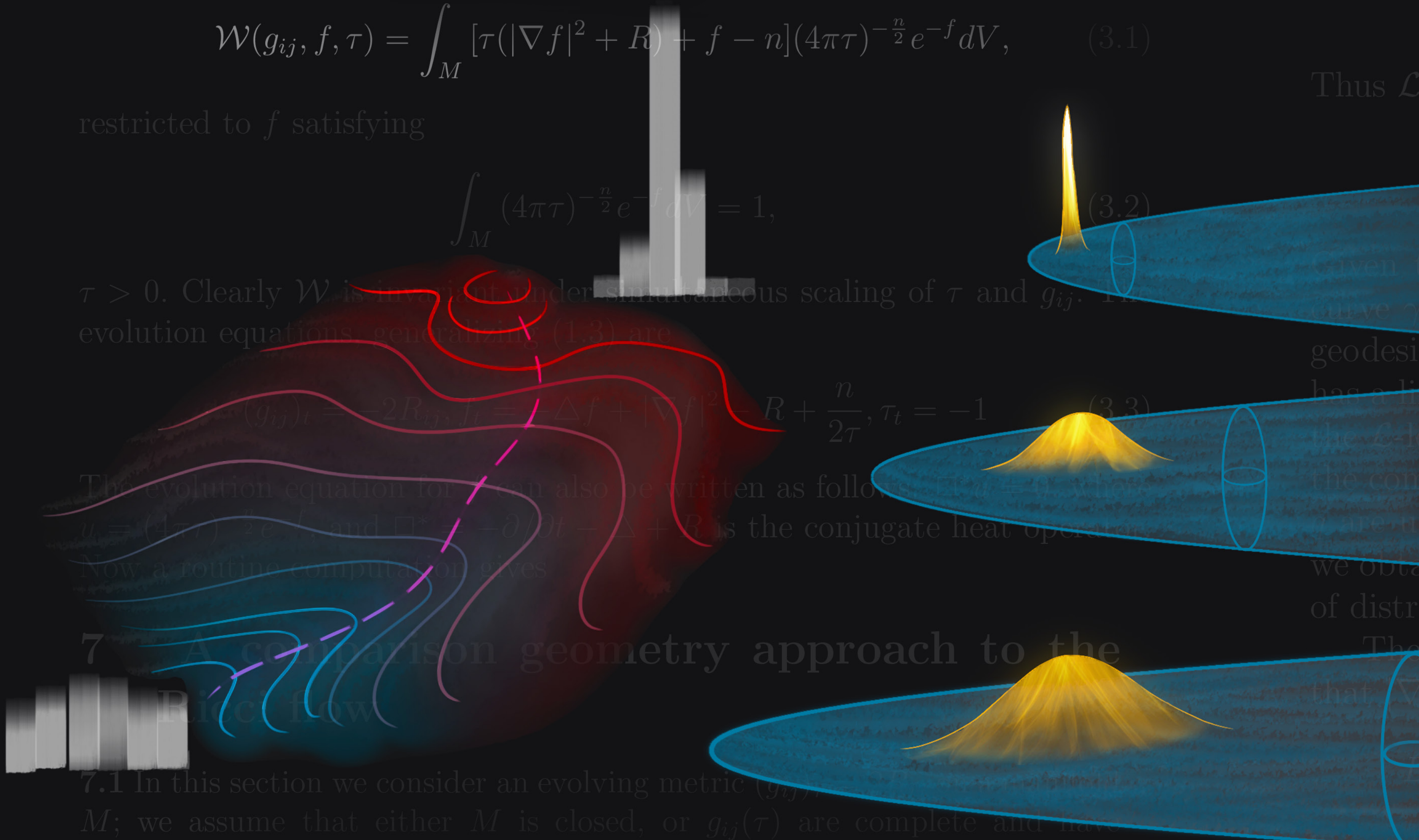


The entropy formula for the Ricci flow and its geometric applications

Grisha Perelman*

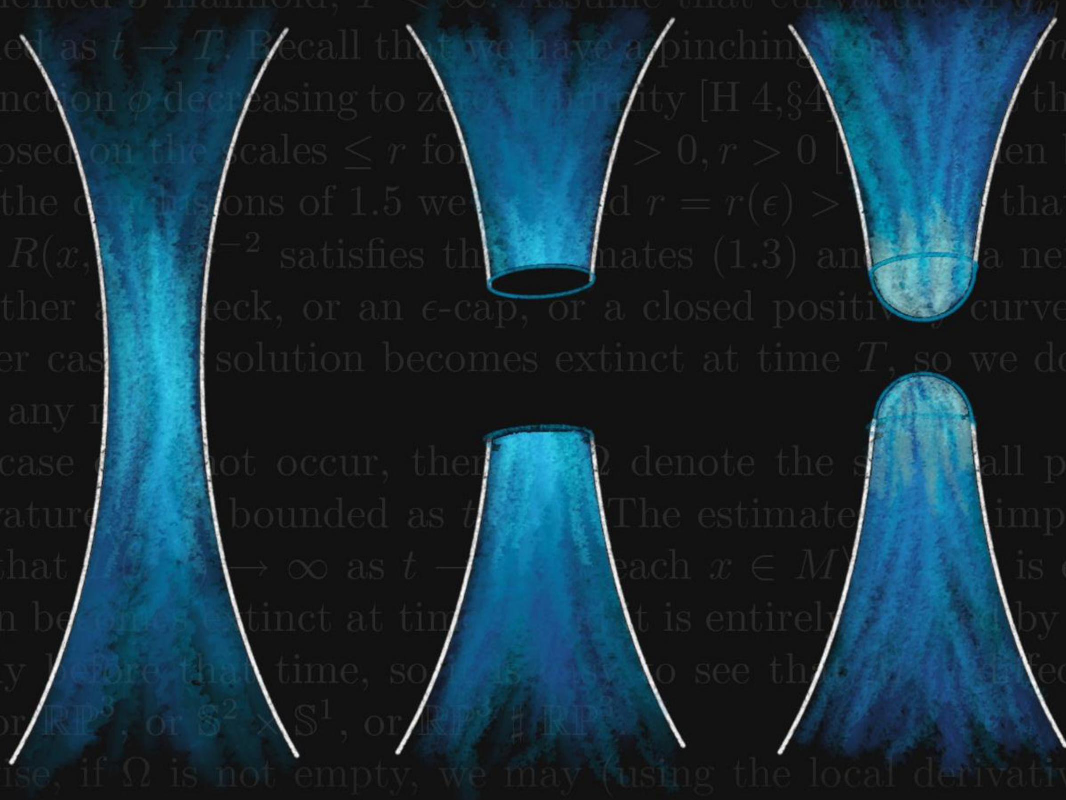
He imagined putting a very hot object in space and letting its heat spread out *backwards* in time.

The heat gave Grigori a new way to understand Ricci flow.



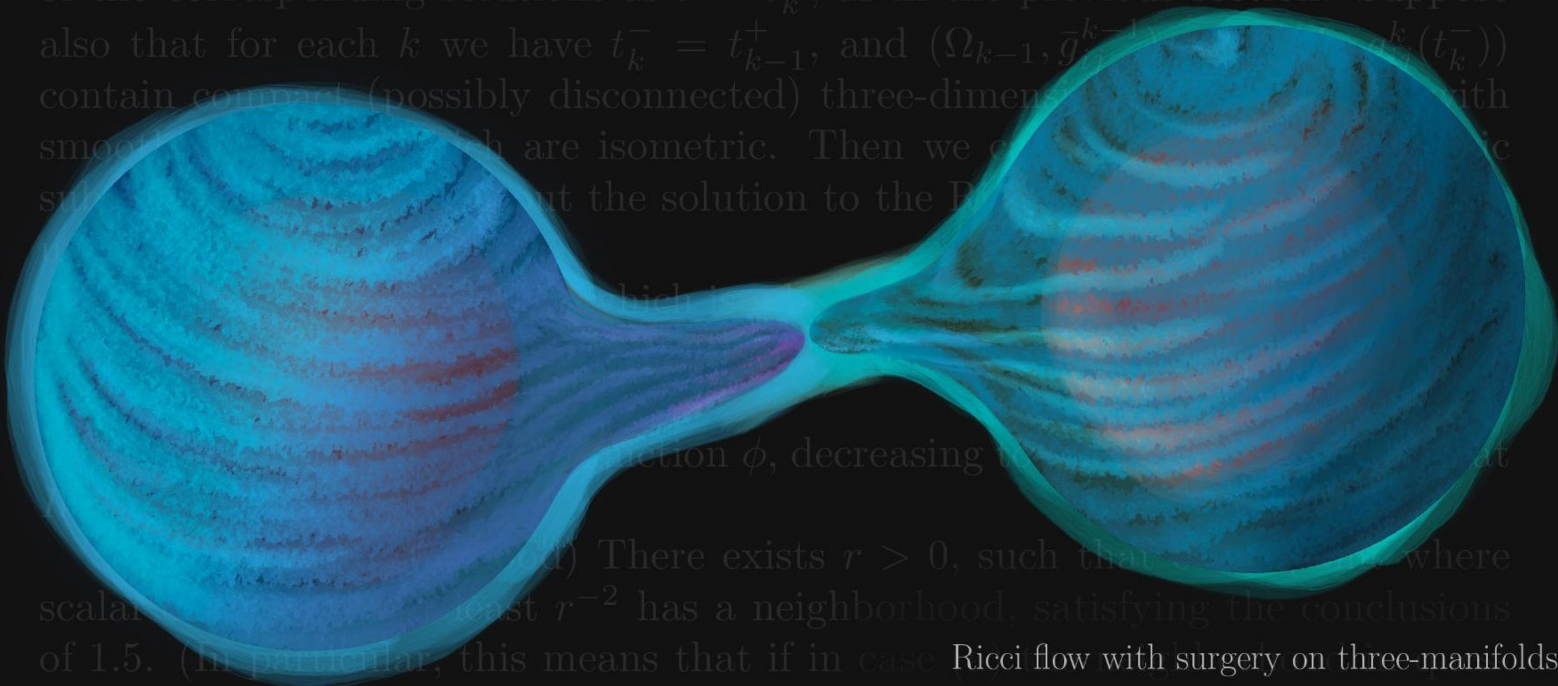
And helped him figure out what to do when space would tear apart.

Just before the singularity, Grigori carefully cut space into pieces.
Then he stitched up where the cuts had been made.



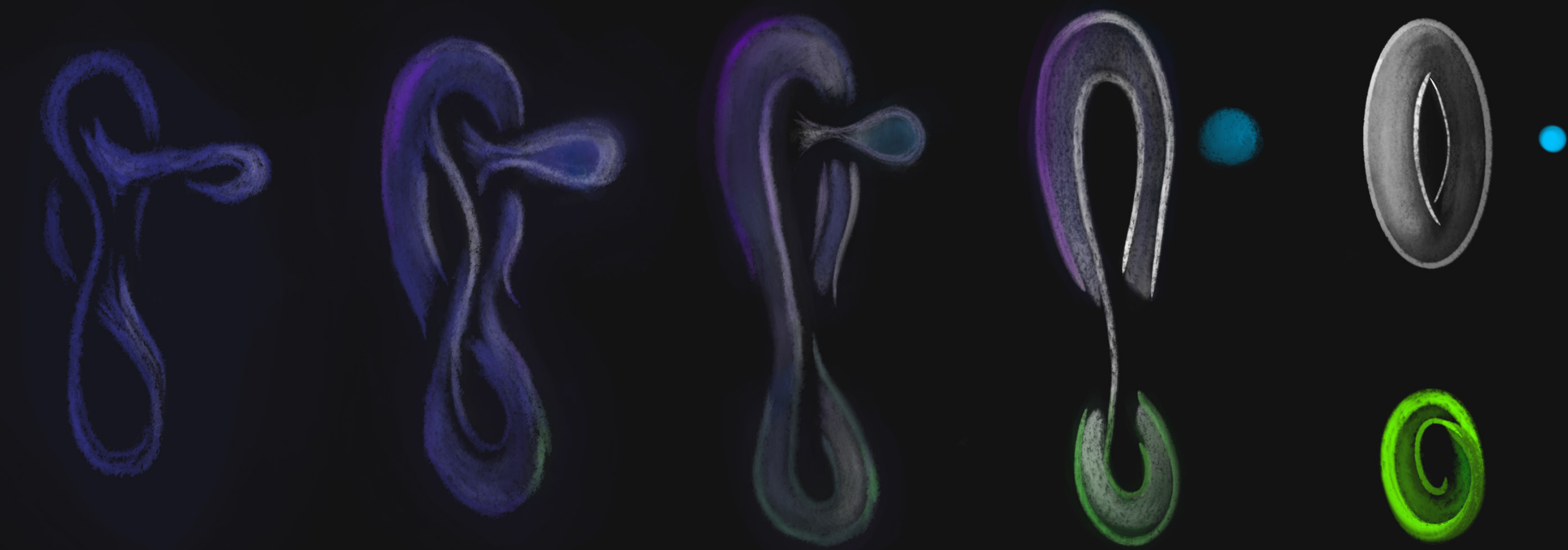
This process is known as surgery, and the heat
helped him understand how to make the cuts.

After each surgery, he restarted the flow.



Grigori repeated this process until all the pieces became simple enough. Then he glued them back together to understand the original space.

Over the course of several years, Grigori developed these ideas and used them to study the structure of three-dimensional space.



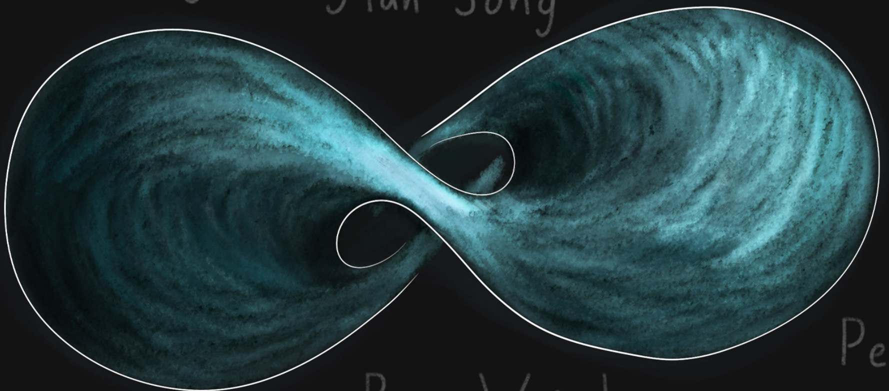
In November of 2002, Grigori finally shared his work to the world. Using Ricci flow, heat and surgery, he solved both the Poincaré and Geometrization conjectures.

Simon Brendle David Glicktenstein Jeffrey Streets

And mathematicians continue to explore new mysteries.

Lu Wang Xiaokui Yang Richard Bamler

Jian Song



Valentino Tosatti

Nataša Šešum Sun Song

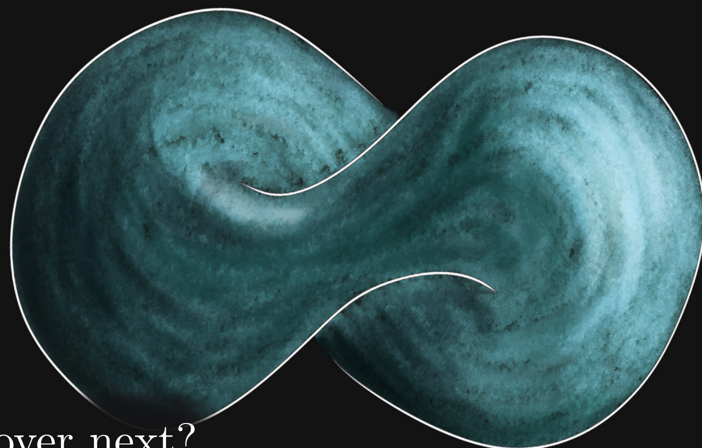
Bing Wang Brett Kotschwar

Peter Topping Xiangwen Zheng

Ben Weinkove

Aaron Naber

Sebastien Picard



Ovidiu Munteanu Robert Haslhofer

Fernando Codá Marques

André Neves Gábor Székelyhidi

What will they discover next?

Esther Cabezas-Rivas Man-Chun Lee

Felix Schulze

Sébastien Boucksom Jason Lotay

The proof of the Poincaré conjecture is one of the most significant achievements in mathematics, marking the culmination of centuries of development in geometry, topology, and partial differential equations.

I was in high school when I first heard about the conjecture and its solution. Although I didn't understand any of the details at the time, I was fascinated by the imagery of curvature flows and surgeries carving geometric spaces. It felt like something out of science fiction, completely unlike the math in my classes.

In this book, I have done my best to tell the history of the Poincaré conjecture and to communicate that mathematics is a collaborative endeavor, built on the contributions of countless mathematicians over generations. For young readers, I hope this book inspires curiosity and sparks an interest in geometry. For readers with more expertise, I hope the illustrations generate further discussion. Math is a beautiful field, and I'm excited to share this story with you.

Technical Notes

My goal was to tell the story accurately, both with respect to the mathematics as well as the history. In order to provide a cohesive narrative, this book focuses on developments throughout the 20th century, although it is worth emphasizing that there were many important discoveries before that time. In addition, there are several anachronisms between the background text and the main story, such as Möbius' name appearing on the page discussing mathematicians who worked on topology after Poincaré.

There are two additional points of clarification that should be made. First, symplectic geometry does not play a role in the Poincaré conjecture. I included the symplectic camel as a reference to Gromov, whose contributions to geometric group theory and metric geometry have been instrumental in shaping modern geometry and play an important role in both Thurston's and Perelman's work. Second, the idea of combining Ricci flow with surgery was originally proposed by Hamilton. However, he was unable to fully control the geometry of three-dimensional singularities or rule out certain singularity models that could not be excised by surgery. Although Perelman did not invent surgery, his insights into the structure of singularities enabled him to successfully carry out the surgical process and complete the proof.

The background text on several pages incorporates excerpts from the papers cited in the references as well as the *Universalis Cosmographia*, which was the first world map to label the Americas. The remaining images were created using Procreate and Photoshop. Many were inspired by existing mathematical figures or techniques for visualizing mathematical concepts, while others drew influence from artworks and posts depicting specific geometric figures, which are cited in the artwork references. The cover art depicts an artistic interpretation of a planet with $\mathbb{S}^2 \times \mathbb{S}^1$ geometry.

Acknowledgments

This book would not have been possible without the support of many people. Thanks to everyone who helped with proofreading and editing, particularly Kori, Mizan, Therese, Karen, and Nicholas. I am especially grateful to David Kramer for his assistance and encouragement. I am also deeply appreciative of the community at Iowa State University. This work was supported in part by a Collaboration grant from the Simons Foundation (849022, G.K.).

Finally, I would like to thank all the teachers who have guided me in my mathematical journey, especially Glen Hall, Fangyang Zheng, and Jun Zhang.

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