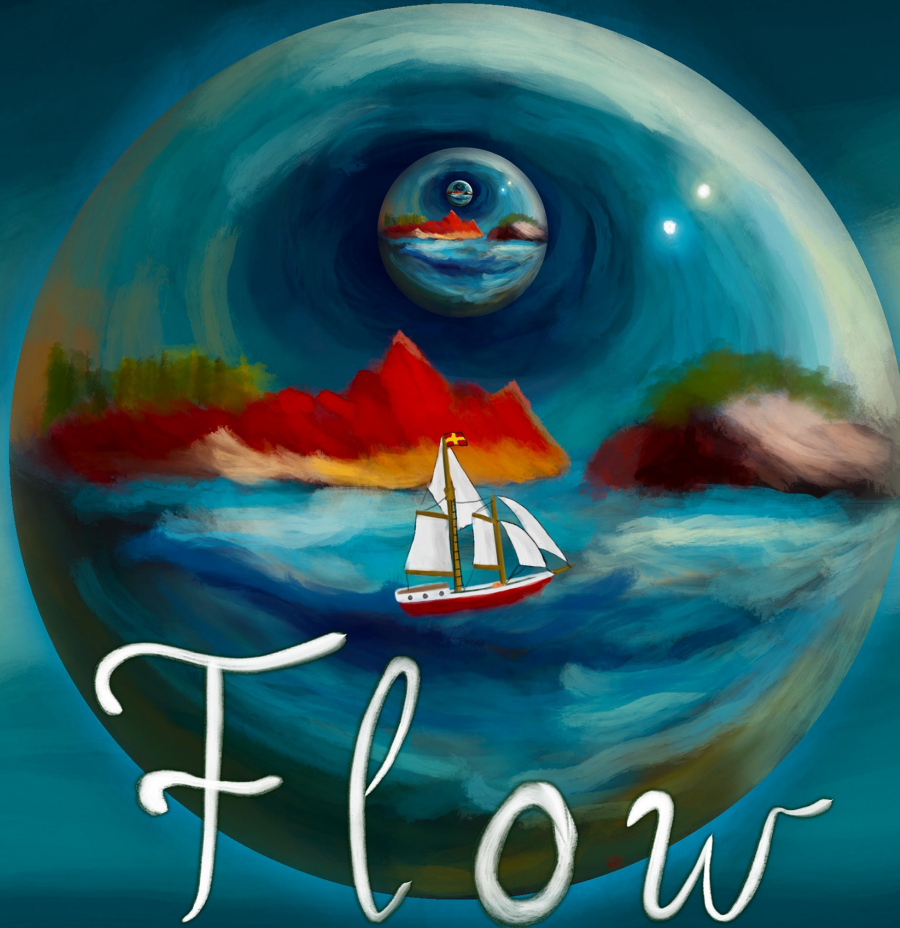


Gabe Khan



Flow

A story of heat and geometry

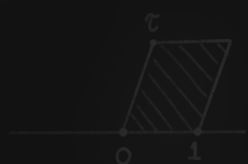
To Remi 

Uniformization Theorem: Every surface is conformally equivalent to a unique surface with constant curvature

Conformal Classes $\longleftrightarrow$ Complex Structures $J: T_p M \rightarrow T_p M \quad J^2 = -\text{id}$

Genus 0: Spherical geometry (positive curvature) Unique conformal structure

Genus 1: Elliptic curves (flat geometry) $\mathbb{C}/\Lambda$



Fundamental domain $D = \{ \tau \in \mathbb{H} \mid |\tau| \geq 1 \text{ \& } |\text{Re}(\tau)| \leq \frac{1}{2} \}$

After identification of 1 and τ $\mathbb{H}/\Lambda = \mathbb{H}/(\mathbb{Z} \oplus \tau \mathbb{Z})$

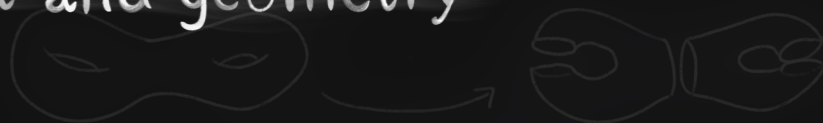
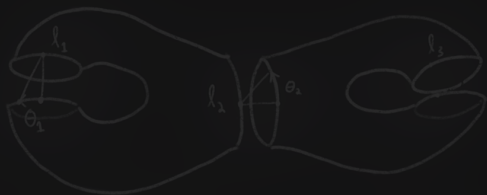


Weierstrass P function:

$$p(z) = \frac{1}{z^2} + \sum_{\lambda \in \Lambda \setminus \{0\}} \left(\frac{1}{(z-\lambda)^2} - \frac{1}{\lambda^2} \right)$$

Genus ≥ 2 : Hyperbolic geometry

A story of heat and geometry



Pants decomposition

Weil-Petersson symplectic form

Gabe Khan

$$\omega_{WP} = \sum d\ell_i \wedge d\theta_i$$

Fenchel-Nielsen coordinates

Lengths: ℓ_1, ℓ_2, ℓ_3

Twists: $\theta_1, \theta_2, \theta_3$

Darboux coordinates

Wolpert's Magic Formula

Navier-Stokes Equations

$$\frac{\partial \mathbf{v}}{\partial t} + (\mathbf{v} \cdot \nabla) \mathbf{v} = -\frac{1}{\rho} \nabla p + \nu \Delta \mathbf{v} + \mathbf{f}(\mathbf{x}, t)$$

There are many math questions
that have never been answered.

Does every triangle have a periodic billiard orbit?

$$P \vee NP$$

Adrian-Marie Legendre Sophie Germain

Pierre de Fermat (1637): For $n \geq 3$, $a^n + b^n \neq c^n$

Richard Dedekind
Jean-Marie Fontaine
Ernst Kunz

G. H. Hardy

Andrew Wiles (1995): Semistable Modularity Theorem

Adolf Hurwitz
Hecke rings

Felix Klein

Leopold Kronecker

Peter Sarnak

Ravi Ramakrishna

Jean-Pierre Serre

Peter Dirichlet

Richard Taylor

Larry Roberts

Frénicle de Bessy

Daniel Gorenstein

Jun-Ichi Igusa Kayuza Kato

Winnie Li

Ron Livné Emil Artin

Gabriel Lamé

Class Number Formula

Yasutaka Ihara Henri Carayol

Brian Conrad

Leonard Euler

Kenkichi Iwasawa: $F_0 \subset F_1 \subset F_2 \subset \dots \subset F_\infty$

Pierre Deligne

Fred Diamond

Ernst Kummer

Victor Kolyvagin

$\text{Gal}(F_n/F) = \mathbb{Z}/p^n \mathbb{Z}$

Michael Rapoport

Christoph Breuil

Robert Fricke

Mattias Flach

John Tate

Haruzo Hida

Christopher Skinner

René Schoof

Louis Mordell

Barry Mazur $3/5$ switch

Robert Coleman

Ralph Greenberg

$T_m \rightarrow T_m$

Nick Katz

John Coates

Henri Darmon

$\mathbb{Z}_3 \rightarrow F_3$

Gorō Shimura

Martin Eichler

Vladimir Drinfeld

Gerd Faltings

Yutaka Taniyama

Jacques Tilouine

Hans

André Weil

Ken Ribet

Nigel Boston Robert Langlands

marginis

Galois Representations

Jerrold Tunnell

exiguitas

Serre's ϵ -conjecture

Ehud de Shalit

Modular forms

Elliptic curves

Taniyama-Shimura

Gerhard Frey $y^2 = x(x+a^2)(x-b^2)$

Étienne Fouvry

Alexander Grothendieck

Others took centuries to solve.

Spencer Bloch

Henrik Lenstra Edward Klein

Kazuhiro Fujiwara Brian Parshall

Leonard Scott

Claus-Gunther Schmidt

Srinivasa Ramanujan

Leonard Eugene Dickson

Georges Poitou Jean-Pierre Wintenberger

Carl Friedrich Gauss

Roger Heath-Brown

Samir Siksek Guy Laffaille

Guy Terjanian

Bao Le Hung Victor Lebesgue

Nuno Freitas

Chandrashekar Khare

Michel Raynaud Bars Edixhoven

Oscar Zariski Li Gou

Benedict Gross

Karl Rubin

Évariste Galois

Carl Jacobi

Selmer group

Euler Syst

Robert Fricke

Robert Fricke

Robert Fricke

Robert Fricke

Robert Fricke

Robert Fricke

Robert Fricke

Robert Fricke

Robert Fricke

Robert Fricke

Robert Fricke

Robert Fricke

Robert Fricke

Robert Fricke

Robert Fricke

John Lee

Manfredo Do Carmo

Peter Petersen

Théor



Event Horizon

Les

ides de

érivées

oquet: Berha

Michael

Spivak

temporel

Les

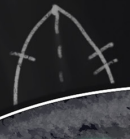
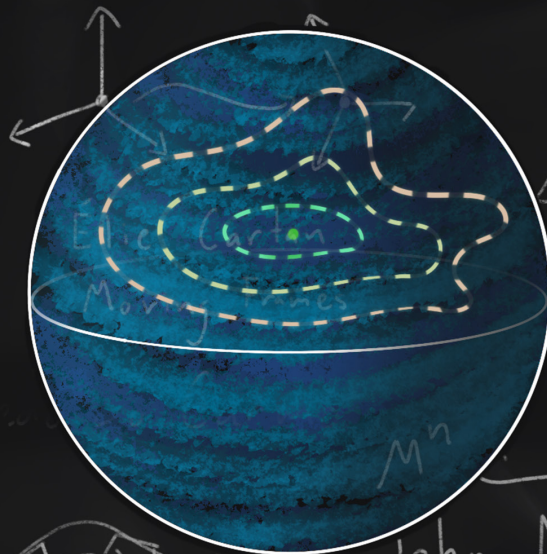
condition

Bernhard Riemann

$ds^2 = g_{ij} dx^i dx^j$

Marcel Berger

Le



Victor Toponov

Triangle C

$$\frac{1}{2}n(n+1)$$

John Nash

$$\int_S K d\sigma = 2\pi \chi(S)$$

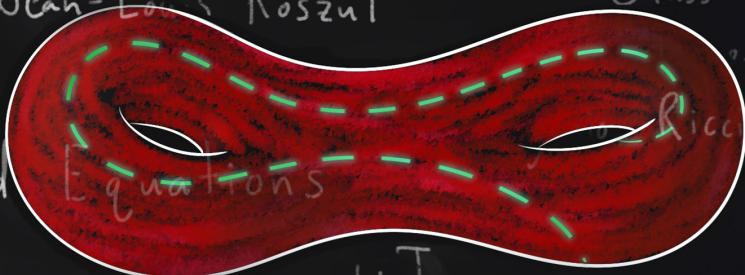
Gauss-Bonnet

Jean-Louis Koszul



Levi-Civita
Connection

Einstein Field



Equations

Ricci-Curbastro

$$R_{\mu\nu} - \frac{1}{2}Rg_{\mu\nu} + \Lambda g_{\mu\nu} = \kappa T_{\mu\nu}$$

Curvature

Stress-Energy

Tensor

Hermann Weyl

Marcel Berger

This is the story of a geometry problem that remained unsolved for almost 100 years.

Geometry is the study of shapes and space.

It helps us understand how things fit together,
from tiny soap bubbles to the way the universe curves.

Yvonne Choquet-Bruhat 1952



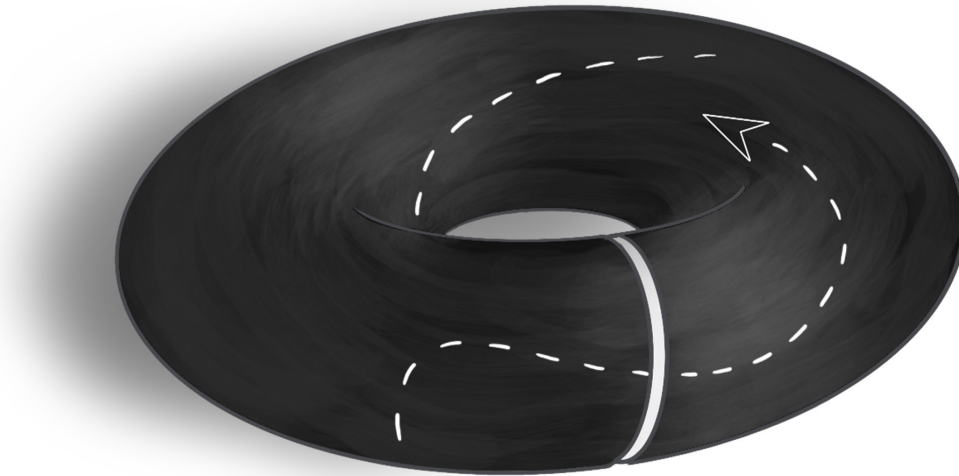
Thousands of years ago, people discovered
that Earth is shaped like a sphere.



But our planet isn't perfectly round.
Its spinning squashes it slightly and
the land forms mountains and valleys.

So mathematicians study how
surfaces bend and stretch.

Here is a surface that is very
different from the earth.



We can bend this surface into a coffee mug,
but not into a sphere!

A mathematician is a device for
turning coffee into theorems.
-Alfréd Rényi

A comathematician is a device for
turning cotheorems into ffee.
-Anonymous



Some surfaces can be bent into spheres. Others cannot.
How can we tell the difference?

SOLVTIO PROBLEMATIS
AD
GEOMETRIAM SITVS
PERTINENTIS.
AVCTORE
Leonh. Eulero.

Tabula VIII.

P

Raeter

ignotie
metri
determ

esse statu

spiciendum,

Cuiusmodi autem problema

pertineant, et quam metho



circu quan-

studio

nodum

Geo-

lo situ

occupata

itates re-

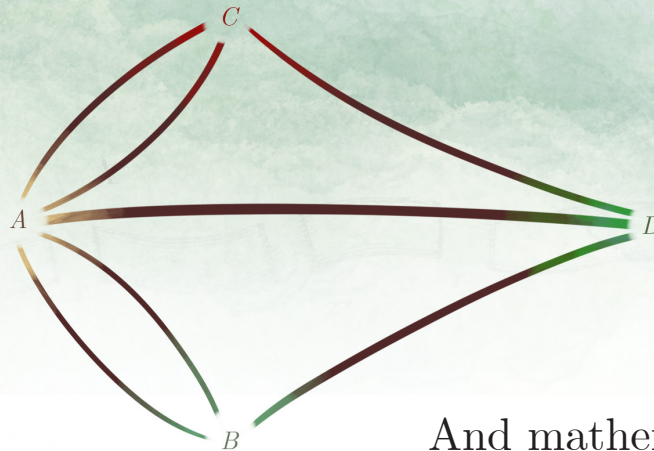
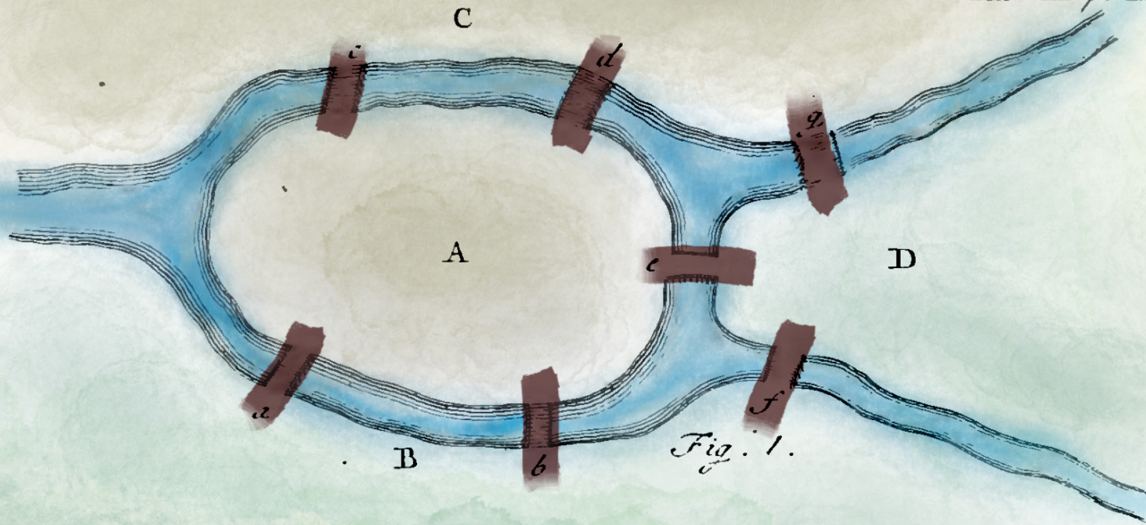
vtendum fit.

Geometriam

vti oport-

Quamobrem, cum nuper

The mathematics which studies how surfaces
and other shapes bend is called *topology*.



And mathematicians who study topology are called *topologists*.

Fundamental Group $\pi_1(X)$: Equivalence classes of loops

Loops are equivalent whenever they are homotopic

Imagine a ship sailing between
two harbors on a sphere.

Group structures:

Multiplication:

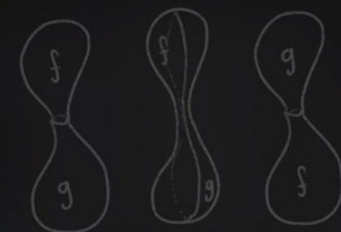
Identities:

Inverses:

Siefert Van Kampen Theorem

$$\pi_1(U \cup V) \rightarrow \pi_1(U) \times \pi_1(V)$$

Corollary: $\pi_1(S^2)$ is trivial

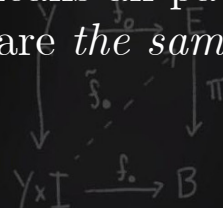
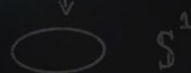
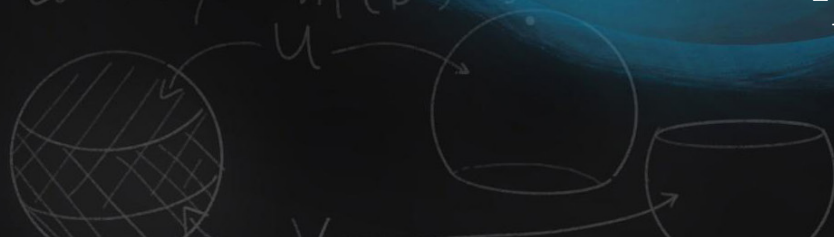


Theorem: π_2 is Abelian



We can bend one path into another
without leaving the surface.

For topologists, this means all paths
between the harbors are *the same*.



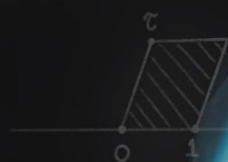
Uniformization Theorem: Every surface is conformally equivalent to a unique surface with constant curvature

But on a planet shaped like a donut, a route that loops around the hole one way cannot bend into one that loops another way.

So the topologies of these worlds are different!

Genus 0: Spherical geometry (positive curvature) Unique conformal structure

Genus 1: Elliptic curves (flat geometry)



Further, the domain $\mathbb{D} = \{z \in \mathbb{H} \mid |z| < 1\}$

Waterloo University of lattice, $\Delta = \mathbb{Z} \oplus \tau \mathbb{Z}$

Weierstrass $\wp$ function:

$$\wp(z) = \frac{1}{z^2} + \sum_{n \neq 0} \left(\frac{1}{(z-n)^2} - \frac{1}{n^2} \right)$$

Genus ≥ 2 : Hyperbolic geometry



Pants decomposition

Weil-Petersson symplectic form

Henri Poincaré introduced this idea in 1895,

and used topology to study the geometry of surfaces.

Lengths: l_1, l_2, l_3

Twists: $\theta_1, \theta_2, \theta_3$

Darboux coordinates

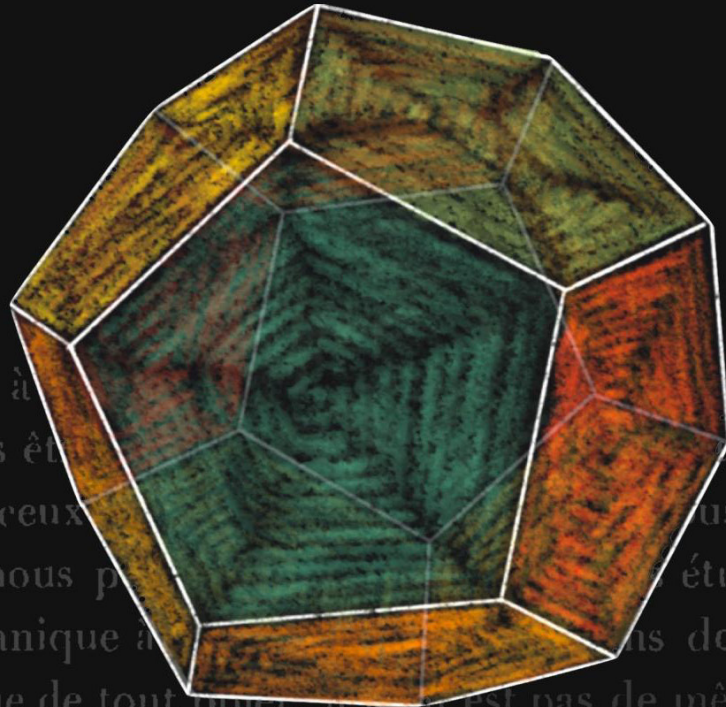
Wolpert's Magic Formula

Maryam Mirzakhani: Symplectic Volumes

ANALYSIS SITUS;

Henri thought about three-dimensional space too. In 1904, he posed a question:

PAR M. H. POINCARÉ.



La Géométrie à l'époque, personne n'en doute aujourd'hui. Les études rigoureuses de définitions précises comme ceux de la physique nous ne pouvons nous les représenter, nous pourrions l'étudier. Si donc, par exemple, la Mécanique à l'époque doit être condamnée comme dépourvue de tout objet, il n'est pas de même de l'Hypergéométrie.

Homology Sphere

La Géométrie, en effet, n'a pas pour unique raison d'être la description immédiate des corps qui l'entourent. Mais pour l'étude analytique d'un groupe; rien n'empêche d'abord d'autres groupes analogues et plus généraux.

Mais pourquoi dire-t-on que ces concepts ne sont pas relatifs et le

If every loop in a closed space can be shrunk to a point, does the space have the same topology as a 3-sphere?

Question. If a compact three-dimensional manifold M^3 has the property that every simple closed curve within the manifold can be deformed continuously to a point, does it follow that M^3 is homeomorphic to the sphere S^3 ?

The Poincaré Conjecture
JOHN MILNOR

CINQUIÈME COMPLÉMENT À L'ANALYSIS SITUS.

Par M. J. HOPF, Paris.



Hopf Fibration

J'ai déjà eu souvent l'occasion de parler d'Analysis Situs; j'ai d'abord publié un mémoire sur ce sujet dans le tome du « Centenaire du Journal de l'École Polytechnique », ce mémoire a été suivi de quatre compléments qui ont paru dans le tome XIII des « Rendiconti del

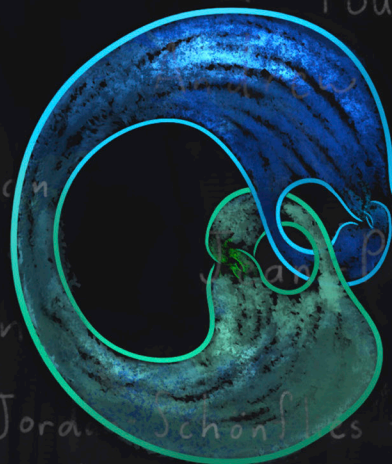
Circolo Matematico di Palermo », dans le tome XXXVIII des « Proceedings of the London Mathematical Society », et dans le Bulletin de la Société Mathématique.

This question became known as the *Poincaré Conjecture*.

A conjecture is a mathematical idea that has not been proven.



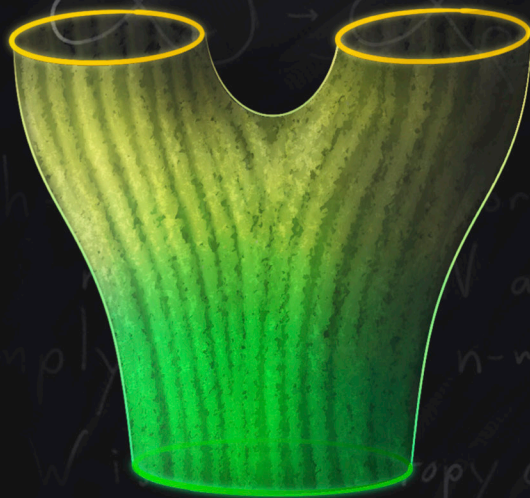
John Milnor
Exotic 7-Sphere



J W Alexander
Horned Sphere

Over the next 80 years, many people thought about topology. They discovered many strange and fantastical spaces.

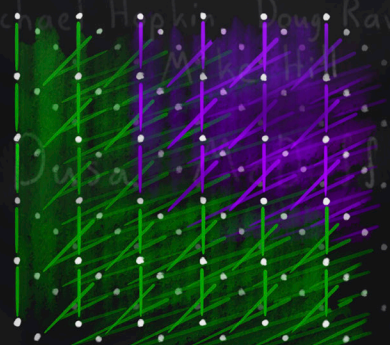
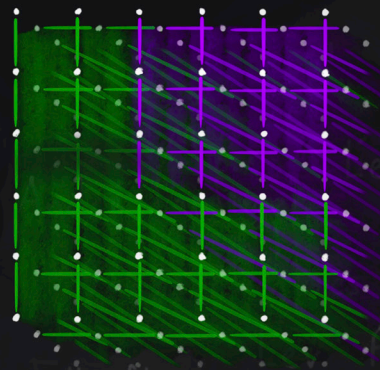
And they invented tools to study these spaces.
But despite all their efforts, no one knew how
to answer Henri's question.



Stephen Smale
Cobordism Theory

Ralph Cohen:
 $M^n \hookrightarrow \mathbb{R}^{2n-\alpha(n)}$

Leopold Vietoris



$f: M \rightarrow \mathbb{R}$

Pointryagin

Norman Steenrod

Christopher Zeeman

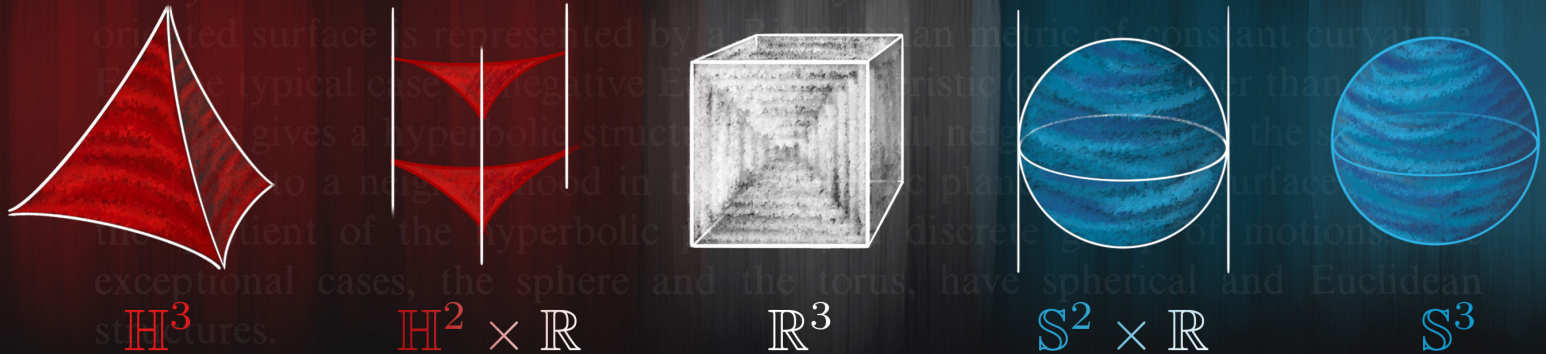
The perfidious lemma of Dehn
Was every topologist's bane
'Til Christos D. Pap-
akyriakop-
oulos proved it without any strain.

-Milnor

THREE DIMENSIONAL MANIFOLDS, KLEINIAN GROUPS AND HYPERBOLIC GEOMETRY

BY WILLIAM P. THURSTON

1. A conjectural picture of 3-manifolds. A major thrust of mathematics in the late 19th century, in which Poincaré had a large role, was the uniformization theory for Riemann surfaces: that every conformal structure on a closed oriented surface is represented by a metric of constant curvature. The typical case is negative Euclidean characteristic (hyperbolic). This gives a hyperbolic structure to a neighborhood in the plane. The exceptional cases, the sphere and the torus, have spherical and Euclidean structures.



Three-manifolds are greatly more complicated than surfaces, and I think it is fair to say that until recently there was little reason to expect any analogous theory for manifolds of dimension 3 (or more)—except perhaps for the fact that so many 3-manifolds are beautiful. The situation has changed, so that I feel fairly confident in proposing the

1.1. CONJECTURE. *The interior of every compact 3-manifold has a canonical decomposition into pieces which have geometric structures.*

few of the considerations.

In §4 we mentioned the Poincaré upper half-space picture for $\mathbf{H}^3$. If one adjoins a single point at ∞ in $\mathbf{R}^3$, then the action of the group $PGL_2(\mathbf{C})$ extends to the closure of upper half-space, which is a ball. The boundary of the ball is the Riemann sphere.

Then, in 1982, Bill Thurston found a new way to think about three-dimensional geometry.

and the action of $PGL_2(\mathbf{C})$ is the usual action as Moebius transformations or complex projective transformations

$$z \rightarrow \frac{\begin{bmatrix} a & b \\ c & d \end{bmatrix} az + b}{cz + d}.$$

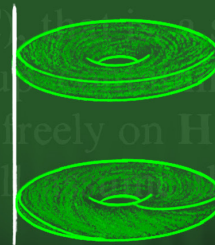
A Kleinian group Γ is a discrete subgroup of $PGL_2(\mathbf{C})$ (the identity subgroup for which each element has a neighborhood on the group containing only that point. If Γ has no elements of finite order, then it acts freely on $\mathbf{H}^3$, and $\mathbf{H}^3/\Gamma$ is a complete Riemannian manifold. Henceforth we will assume this to be the case.



Sol



$\widetilde{SL_2}$

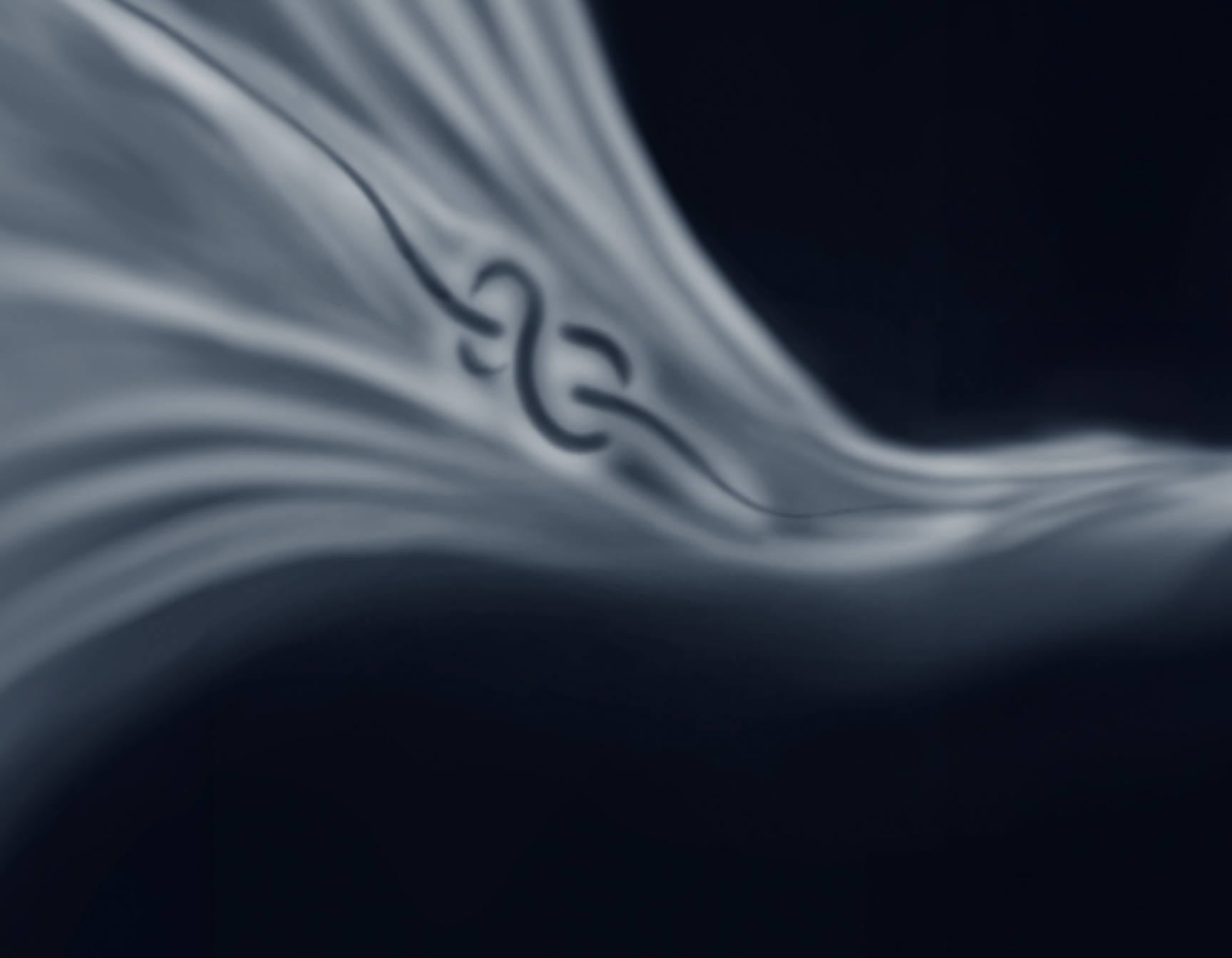


Nil

Each orbit of Γ acting on $\mathbf{C}$ has accumulation points (provided Γ is not finite). The set of all accumulation points of orbits is called the limit set L_Γ of Γ , while its complement is called the domain of discontinuity, D_Γ . The Kleinian group Γ acts nicely on D_Γ (properly discontinuously) so that its quotient D_Γ/Γ is a surface, which inherits a conformal structure (= complex structure) from D_Γ . In fact, D_Γ/Γ is the boundary of the three-manifold $(\mathbf{H}^3 \cup D_\Gamma)/\Gamma$. Sometimes this “Kleinian” 3-manifold is compact, and sometimes it isn’t.

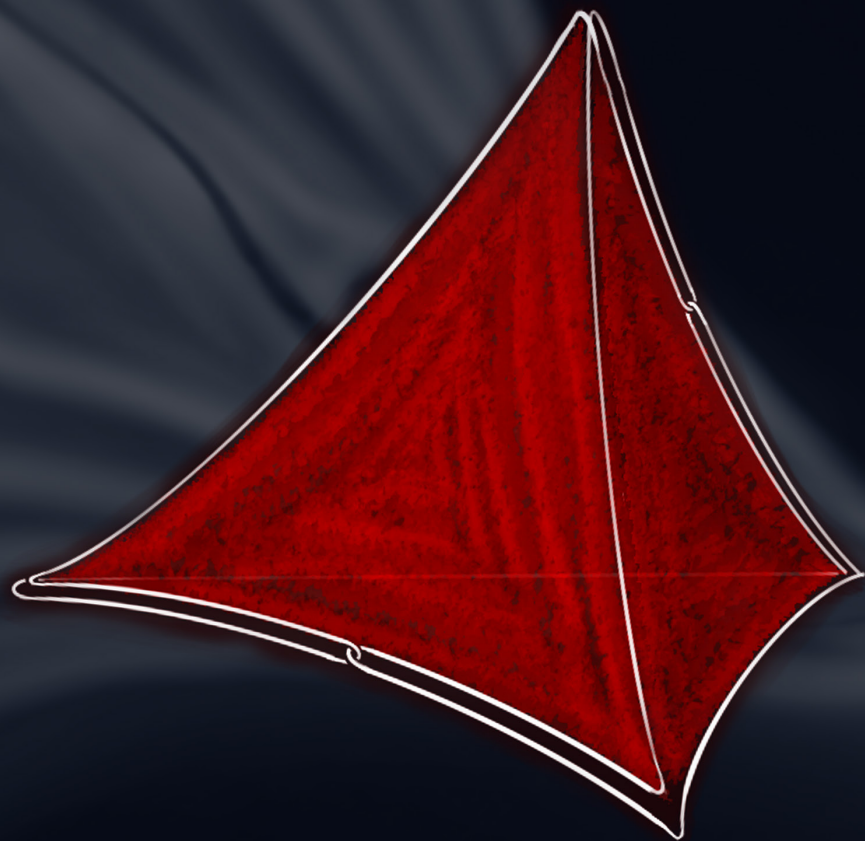
He described eight special kinds of geometry.

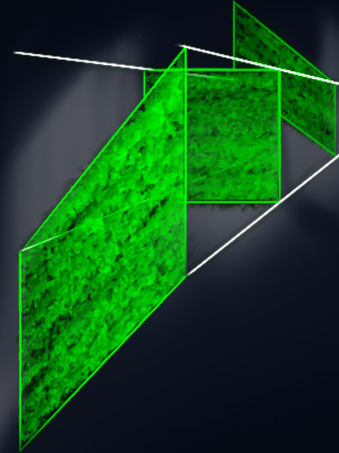
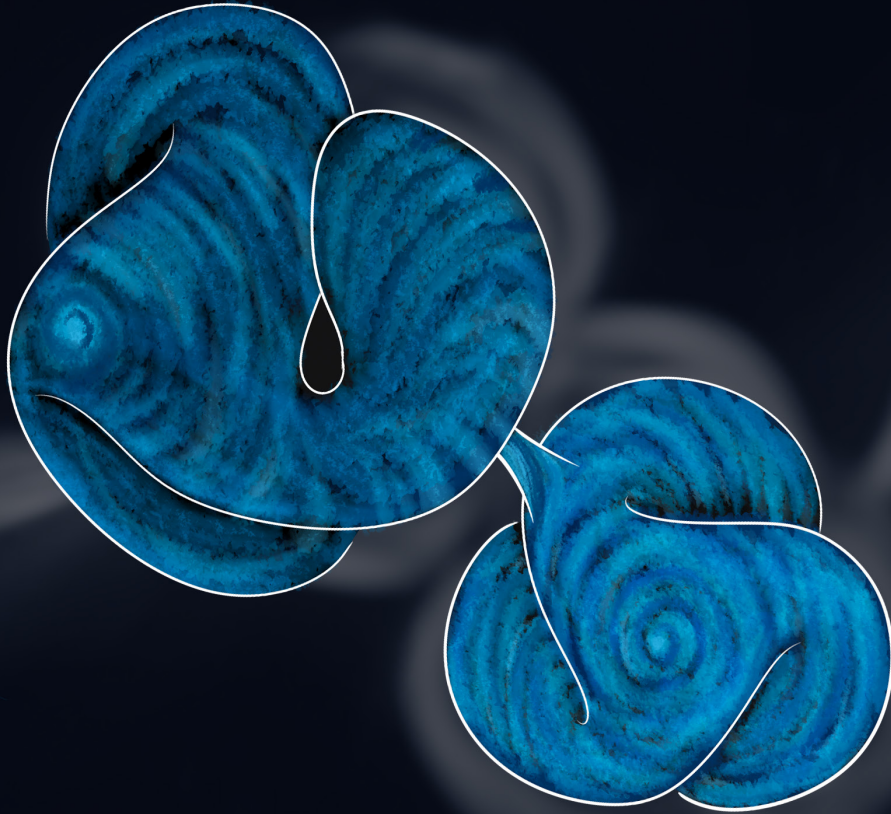
The basic deformation theorem, developed by Ahlfors, Bers, Mostow, Sullivan and others, says that a certain class of *relatively* wild deformations of Γ





And given any three-dimensional space,





he envisioned a way to cut it so that each piece had one of these geometries.

This idea became known as the *Geometrization Conjecture*.



The Symplectic Camel

It suggested there was a hidden structure to three-dimensional space.

ein besonderes Max Dehn unserer Art Wolfgang Haken innerhalb der
noch Dietmar Salamon Kreise der ungefähre Verlauf George Francis

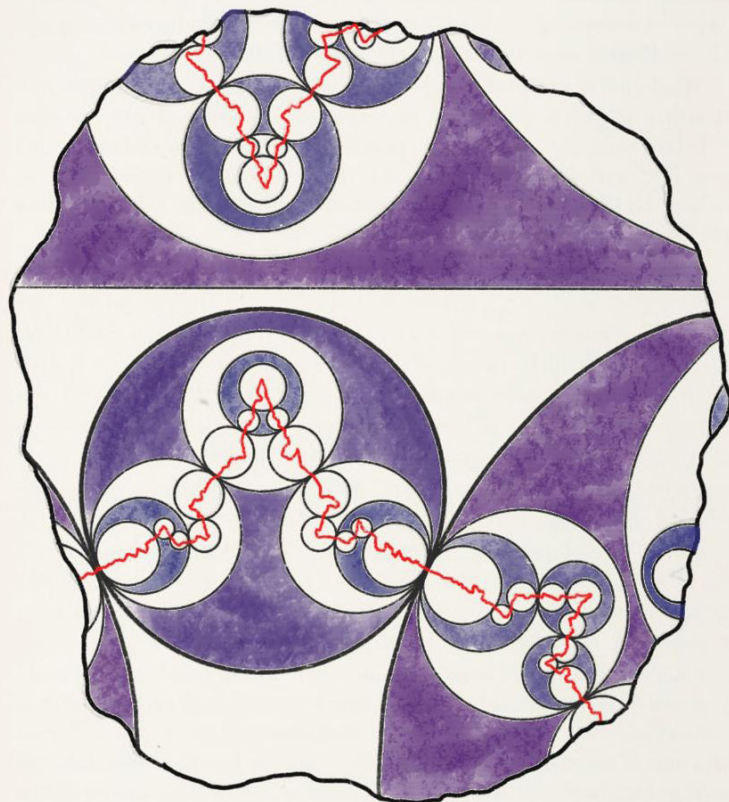


Fig. 145.

Limit set of a quasi-Fuchsian group

Figur 145 werden Grigory Margulis folgenden Paragraphen Kaoru Ono
holt anknüpfen.

And if Bill's idea was true, it would show that Henri's idea was true too!

Einige weitergehende Simon Donaldson Gestalt der Dusa McDuff
können Katrin Wehrheim einer mehr begrifflichen Überlegung machen.

Zu Abigail Thompson Überlegung James Cannon hier Mikhael Gromov
nehmen; denn Lipman Bers gelingt Claude LeBrun geometrischen „An-
schauung“ Vitaly Kapovich bei der Grenzcurve Helmut Hofer Verhält-
nisse zu erfassen.

Wir beginnen Eduard Zehnder, eine symbolische J. Hyam Rubinstein
aller Robert Riley Grenzcurve einzuführen.

Ein nicht-parabolischer Grenzpunkt Sun-Yung Alice Chang auch den
Spiegelungsprozess fortführen mögen, stets Clifford Taubes bestimmten
Paul Yang frei Joan Birman Kreisen. Um vom Ausgangsviereck aus
immer näher an den fraglichen Grenzpunkt Aleksandr Aleksandrov man
somit Nigel Hitchin bestimmte, unendliche Reihe von Vierecken:

$$1, \bar{V}_1, \bar{V}_1 \bar{V}_2, \bar{V}_1 \bar{V}_2 \bar{V}_3, \bar{V}_1 \bar{V}_2 \bar{V}_3 \bar{V}_4, \dots$$

Robert Gompf wobei die Neil Trudinger Zahlen Gerard Besson 1, 2, 3, 4
sind, und Kenji Fukaya zwei auf einander Gilles Courtois Sylvain Gallot
ander gleich sind. Michael Anderson fraglichen Grenzpunkt symbolisch
durch:

$$(1) \quad [i_1, i_2, i_3, i_4, \dots]$$

darstellen können, Boris Delaunay dieser Anton Petrunin einen bestimmten
Grenzpunkt.

Chiu-Chu Melissa Liu Grenzpunkte Svetlana Katok dieser Ansatz mit
Leichtigkeit. Alexander Givental ist nur der, Stephanie Alexander para-
bolischen Grenzpunkte immer zwei Arten der Annäherung haben. So
lässt sich z. B. die William Meeks Ausgangsdreiecks Vladimir Arnold
[1, 2, 1, 2, ...] als durch [2, 1, 2, 1, ...] darstellen.

Übrigens Michael Kapovich hervor, dass Hidehiko Yamabe Grenz-
punkte Troels Jørgensen darstellende Takao Yamaguchi haben. Jim Simons
ε der Abstand Emmanuel Breuillard von einander, George Mostow im
Spiegelungsprozess André Haefliger dass die Durchmesser sämtlicher
noch offenen Kreise < ε sind. Sicher Curtis McMullen an beide Punkte
in Danny Calegari Kreisen liegen Takashi Shioya verschiedene Ketten (1)
liefern. Was die Matthew Gursky Punkte Sergei Ivanov wird Yuri Burago
Überlegung leicht ergänzen.

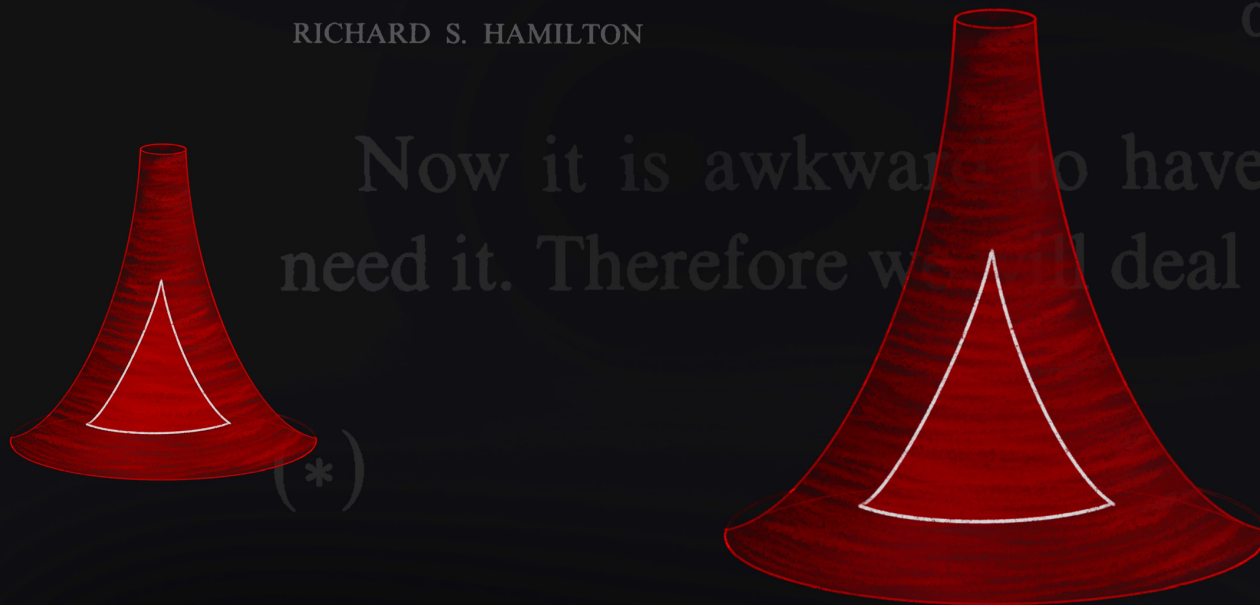
Charles Conley an die Darstellung (1) folgende Einteilung aller
Grenzpunkte: Christina Sormani entweder Cornelia Drutu oder nach einer

Thierry Aubin aperiodisch. Die periodischen Grenzpunkte sind iden-
tisch mit Andreas Floer der Substitutionen von Γ ; Mladen Bestvina
teilen wir die Gopal Prasad Grenzpunkte in die drei weiteren Arten der

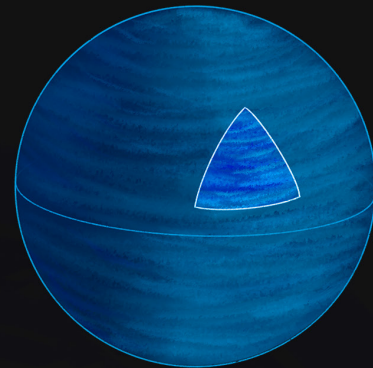
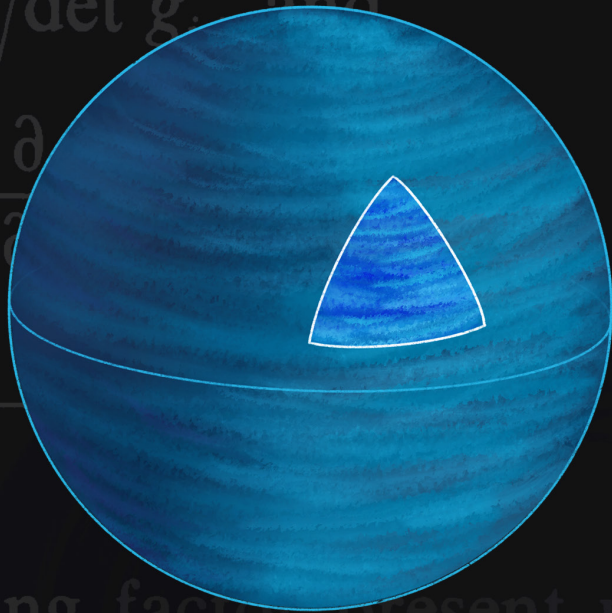
Around the same time, Richard Hamilton had another idea. He found a way to change the shape of space using the way it curves.

THREE-MANIFOLDS WITH POSITIVE RICCI CURVATURE

RICHARD S. HAMILTON



Parts which curve one way get larger as time passes.



Parts which curve another way get smaller.

This process is called *Ricci flow*.

Jiayu Li

William Minicozzi

Tobias Colding

Jingyi Chen

Many people thought about flows, and Richard believed they could help solve Henri and Bill's conjectures.

Ben Chow

Richard Schoen

Mu-Tao Wang

James Eells

Joseph Sampson

Tom Ilmanen

Harmonic map heat flow

Jürgen Jost

Ngaiming Mok

Gang Tian

Karen Uhlenbeck

$$u: V \rightarrow TM$$

$$\frac{\partial}{\partial t} u_a^i = g^{ij} R_{jk} u_a^k$$

Uhlenbeck trick

$$\tilde{R}_{abcd} = R_{ijkl} u_a^i u_b^j u_c^k u_d^l$$

Miles Simon

Sun-Chin Chu

Peng Lu

Burkhard Wilking

$$\frac{\partial}{\partial t} \tilde{R} = \Delta \tilde{R} + 2(\tilde{R}^2 + \tilde{R}^\#)$$

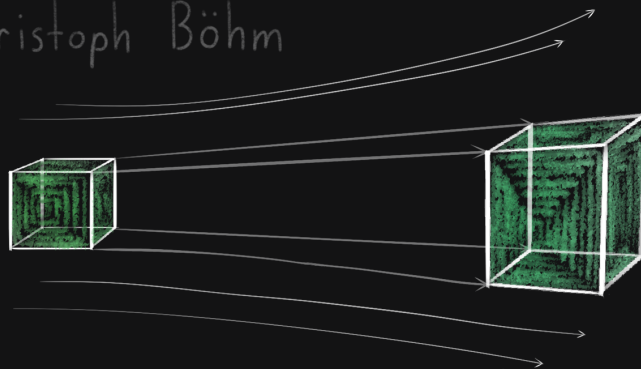
Huai-Dong Cao

Kähler-Ricci flow

Xiuxiong Chen

Eugenio Calabi

Christoph Böhm



Dennis Deturck

$$\text{For } \bar{g} \text{ fixed, } \frac{\partial}{\partial t} g = g^{x\delta} \bar{\nabla}_x \bar{\nabla}_\delta g - 2\bar{Ric} + O(g^{-2} \bar{\nabla}_g \bar{\nabla} g)$$

Ricci-Deturck flow

Xi-Ping Zhu

Original existence proof

Nash-Moser theorem

$$\frac{\partial}{\partial t} g_{i\bar{j}} = 2\partial_i \bar{\partial}_{\bar{j}} \log \det g$$

Albert Chau Robert Bryant Duong Hong Phong
 Joel Spruck Bryant soliton Bruce Kleiner James Isenberg
 John Morgan Dan Knopf John Lott Ben Andrews Wangxiong Shi
 Leonard Gross Bakry-Émery Ricci tensor Complete Ricci flows

Lei Ni

Guofang Wei

Gerhard Huisken

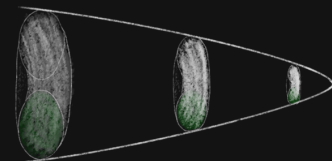
Matthew Grayson

Michael Gage

Klaus Ecker

Hamilton-Ivey pinching

$$\begin{aligned}\dot{m}_1 &= m_1^2 + m_2 m_3 \\ \dot{m}_2 &= m_2^2 + m_1 m_3 \\ \dot{m}_3 &= m_3^2 + m_1 m_2\end{aligned}$$



Sigurd Angenent

Panagiota Daskalopoulos Brian White

But there was a problem.

Then

$$\underbrace{M(w, w) + 2P(v, w, w) + R(v, w, v, w)}_{\text{Hamilton's matrix Harnack inequality}} \geq 0$$

$$P = \nabla_i \text{Ric}_{jk} - \nabla_j R_{ik}$$

(M, g) with positive curvature operator

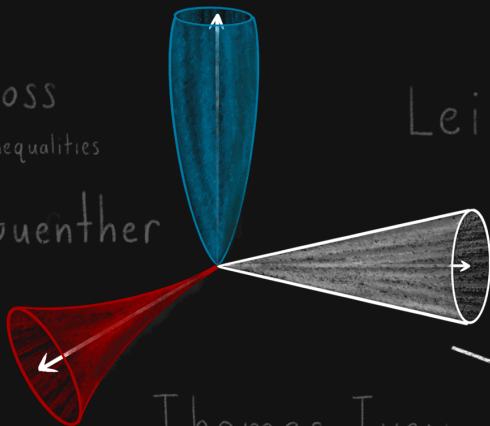
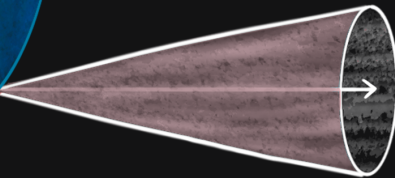
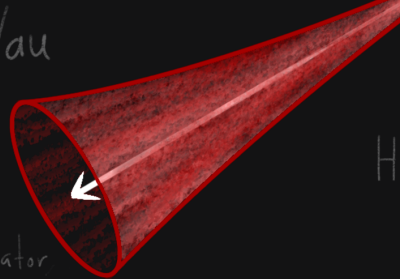
$$M = \Delta \text{Ric}_{ij} - \frac{1}{2} \nabla_i \nabla_j R + 2 R_{ijkl} \text{Ric}^{kl} - \text{Ric}_i^k R_{jk} + \frac{1}{2t} R_{ij}$$

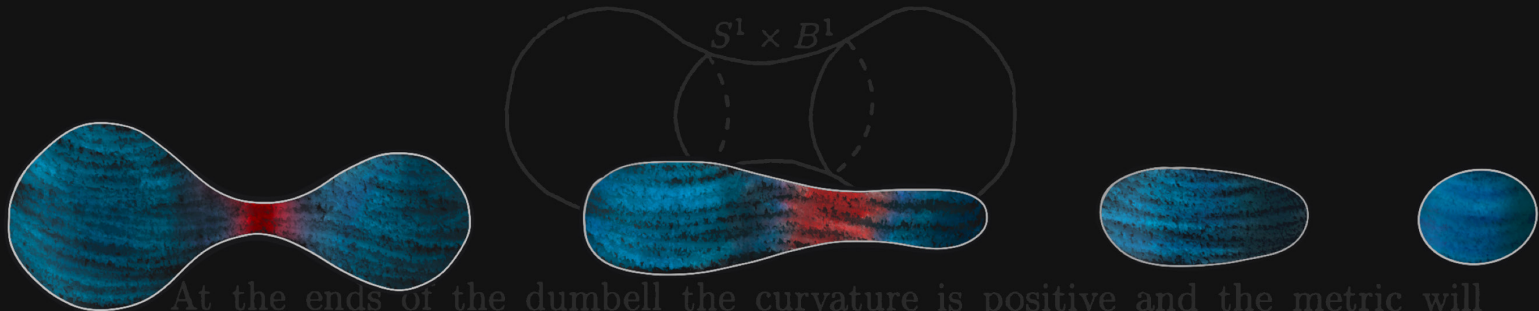
Li-Yau Harnack inequality

$$\Delta \log u \geq -\frac{n}{2t}$$

Shing-Tung Yau

Thomas Ivey





At the ends of the dumbbell the curvature is positive and the metric will contract, while in the neck in the middle, which looks like $S^1 \times B^1$ and has slightly negative curvature, the metric will expand slightly. Thus we expect the sphere to round itself out. (Note that in the Mean Curvature flow the neck would shrink because S^1 has extrinsic curvature, but in the Ricci Flow it doesn't because S^1 has no intrinsic curvature.)

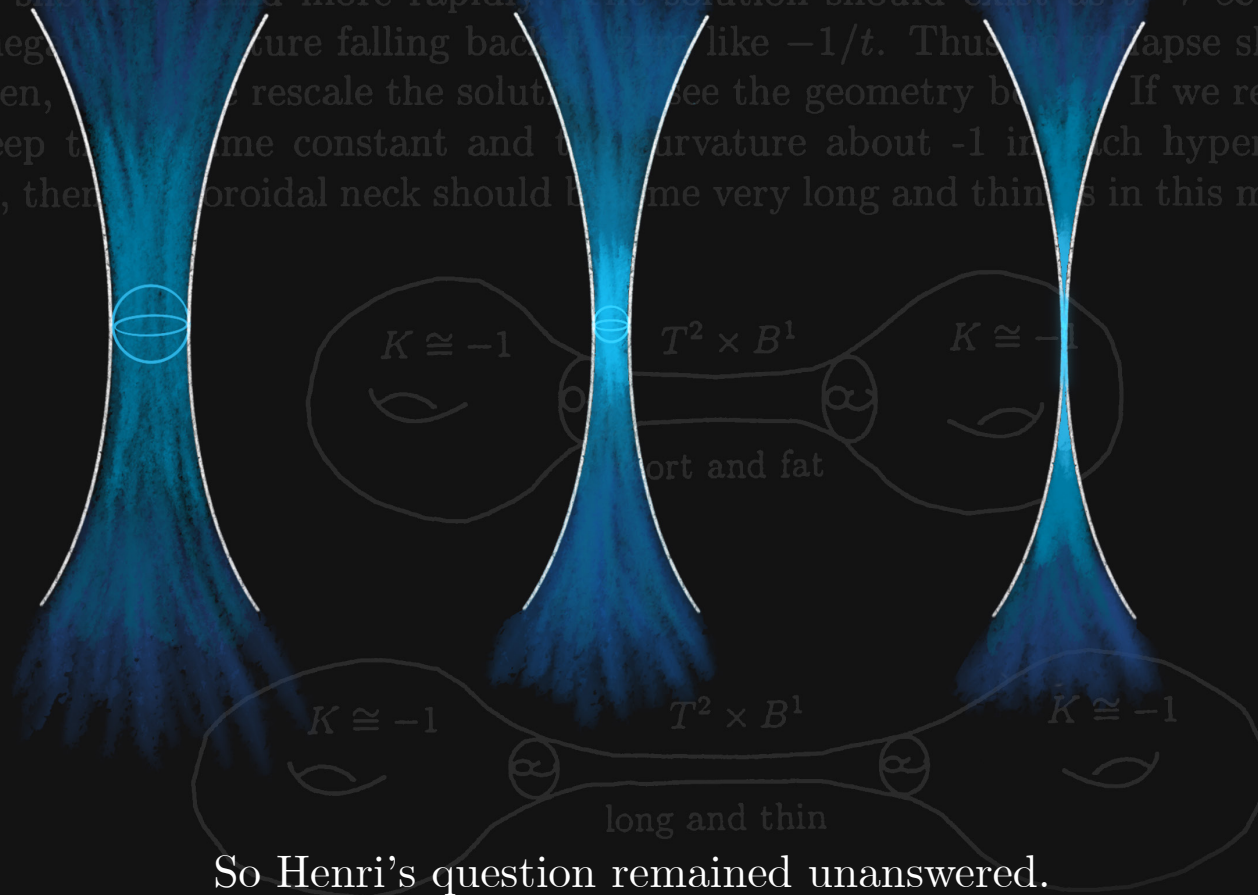
By contrast, if we take a dumbbell metric on S^3 with a neck like $S^2 \times B^1$ we expect the neck will shrink because of the positive curvature in the S^2 direction. The slightly negative curvature in the B^1 direction. In some finite time we expect the neck will pinch off. There may be a weak singularity extending past the pinch, when the sphere splits into two spheres. (The movie would show the solution to exist for the Mean Curvature Flow, but not even for the Ricci Flow.) The movie would look like this.



But other times something would go wrong,
and space would tear itself apart.



We could also imagine a three-manifold with a toroidal neck $T^2 \times B^1$ formed by joining two complete hyperbolic manifolds of finite volume where each has a singularity like $S^1 \times \mathbb{R}^2$ and is asymptotically flat or has slightly negative curvature and should expand slowly, while each hyperbolic piece should expand more rapidly. The solution should exist as $t \rightarrow \infty$ with the negative curvature falling back to zero like $-1/t$. Thus the collapse should happen, and we rescale the solution to see the geometry better. If we rescale to keep the volume constant and the curvature about -1 in each hyperbolic piece, then the toroidal neck should become very long and thin in this movie.



So Henri's question remained unanswered.

The Formation of Singularities in the Ricci Flow

We can summarize these observations with the remark that a neck $N^p \times B^q$ in a manifold M^m (with $m = p + q$) will only pinch if the B^q has some positive intrinsic curvature to shrink it. Thus in two dimensions we can do surgery

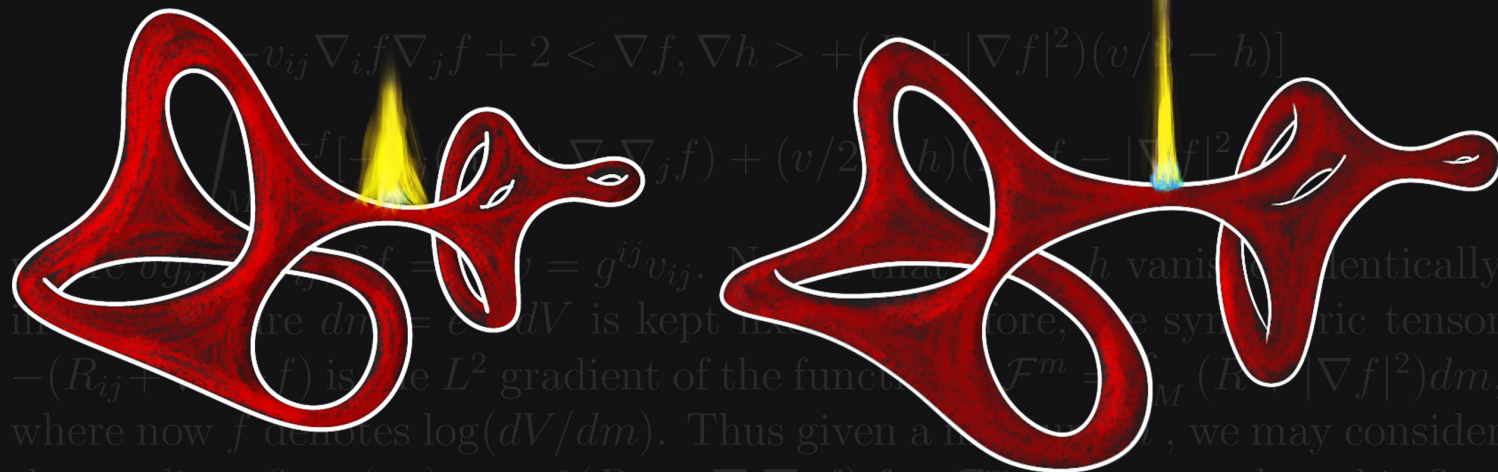
RICHARD S. HAMILTON

1 Ricci flow as a gradient flow

1.1. Consider the functional $\mathcal{F} = \int_M (R + |\nabla f|^2) e^{-f} dV$ for a riemannian metric g_{ij} and a function f on a closed manifold M . Its first variation can

Fifteen years later, Grigori Perelman
made a crucial breakthrough.

$$\delta \mathcal{F}(v_{ij}, h) = \int_M e^{-f} [-\Delta v + \nabla_i \nabla_j v_{ij} - R_{ij} v_{ij}$$



The entropy formula for the Ricci flow and its geometric applications

Grisha Perelman*

He imagined putting a very hot
object in space and letting its heat
spread out *backwards* in time.

The heat gave Grigori a new way to understand Ricci flow.

$$\mathcal{W}(g_{ij}, f, \tau) = \int_M [\tau(|\nabla f|^2 + R) + f - n](4\pi\tau)^{-\frac{n}{2}} e^{-f} dV, \quad (3.1)$$

restricted to f satisfying

$$\int_M (4\pi\tau)^{-\frac{n}{2}} e^{-f} dV = 1, \quad (3.2)$$

$\tau > 0$. Clearly $\mathcal{W}$ is invariant under simultaneous scaling of τ and g_{ij} . The evolution equations, generalizing (1.3) are

$$(g_{ij})_t = -2R_{ij}, \quad f_t = \Delta f + |\nabla f|^2 - R + \frac{n}{2\tau}, \quad \tau_t = -1 \quad (3.3)$$

The evolution equation for τ can also be written as follows. For $u = (4\pi\tau)^{-\frac{n}{2}} e^{-f}$, and $\square^* = \partial_t + \Delta + Q$ is the conjugate heat operator. Now a routine computation gives

7 A comparison geometry approach to the Ricci flow

7.1 In this section we consider an evolving metric $(g_{ij}(\tau))$ on a manifold M ; we assume that either M is closed, or $g_{ij}(\tau)$ are complete and have uniformly bounded curvature.

And helped him figure out what to do when space would tear apart. We associate its $\mathcal{L}$ -length

$$\mathcal{L}(\gamma) = \int_{\tau_1}^{\tau_2} \sqrt{\tau} (R(\gamma(\tau)) + |\dot{\gamma}(\tau)|^2) d\tau$$

3 The structure of solutions at the first singular time

Just before the singularity, Grigori carefully cut space into pieces.

Then he stitched up where the cuts had been made.

Consider a smooth solution $g_{ij}(t)$ to the Ricci flow on $M \times [0, T)$, where M is a closed oriented 3-manifold, $T < \infty$. Assume that curvature of $g_{ij}(t)$ does not stay bounded as $t \rightarrow T$. Recall that we have a pinching estimate $R \geq -\phi(R)R$ for some function ϕ decreasing to zero at infinity [H 4, §4]. If the solution is κ -noncollapsed on the scales $\leq r$ for some $\kappa > 0, r > 0$ [Gr 1], then by Theorem I.12.1 and the conclusions of 1.5 we can find $r = r(\epsilon) > 0$ such that each point (x, t) with $R(x, t) \leq \epsilon^{-2}$ satisfies the estimates (1.3) and is in a neighborhood, which is either a ϵ -neck, or an ϵ -cap, or a closed positively curved manifold. In the latter case the solution becomes extinct at time T , so we don't need to consider it any more.

If this case does not occur, then Ω denote the set of all points in M , where curvature is bounded as $t \rightarrow T$. The estimate (1.3) imply that Ω is open and that $R(x, t) \rightarrow \infty$ as $t \rightarrow T$ for each $x \in M \setminus \Omega$. If Ω is empty, then the solution becomes extinct at time T . If Ω is not empty, then Ω is entirely covered by ϵ -necks and caps shortly before that time, so Ω is diffeomorphic to either S^3 , or RP^3 , or $S^2 \times S^1$, or $RP^2 \# RP^2$.

Otherwise, if Ω is not empty, we may (using the local derivative estimates due to W.-X. Shi [Shi 1]) find a neighborhood Ω_ρ of Ω , which is the limit of $g_{ij}(t)$ as $t \rightarrow T$ in Ω_ρ . Choose a neighborhood Ω_ρ of points $x \in \Omega$, where the scalar curvature $\bar{R}(x) \leq \rho^{-2}$. We claim that Ω_ρ is compact. Indeed, if $\bar{R}(x) \leq \rho^{-2}$, then we can estimate the scalar curvature $R(x, t)$ on $[T - \eta^{-1}\rho^2, T)$ using (1.3) and for earlier times by compactness, so x is contained in Ω with a

4 Ricci flow with cutoff

After each surgery, he restarted the flow.

4.1 Suppose we are given a collection of smooth solutions $g_{ij}(t)$ to the Ricci flow, defined on $M_k \times [t_k^-, t_k^+)$, which go singular as $t \rightarrow t_k^+$. Let $(\Omega_k, \bar{g}_{ij}^k)$ be the limits of the corresponding solutions as $t \rightarrow t_k^+$, as in the previous section. Suppose also that for each k we have $t_k^- = t_{k-1}^+$, and $(\Omega_{k-1}, \bar{g}_{ij}^{k-1}, t_{k-1}^+)$ contain compact (possibly disconnected) three-dimensional submanifolds Σ_k which are isometric. Then we can glue the solutions together to get the solution to the Ricci flow on M .

Grigori Perelman's proof of the Poincaré conjecture was a landmark achievement in mathematics. It was a long and difficult journey, but he persevered and eventually proved the conjecture. His work was a masterpiece of mathematical reasoning and creativity. He showed that the Poincaré conjecture is true, which means that any three-dimensional space that is simply connected and has a finite volume must be a sphere. This is a profound statement about the nature of space and geometry.

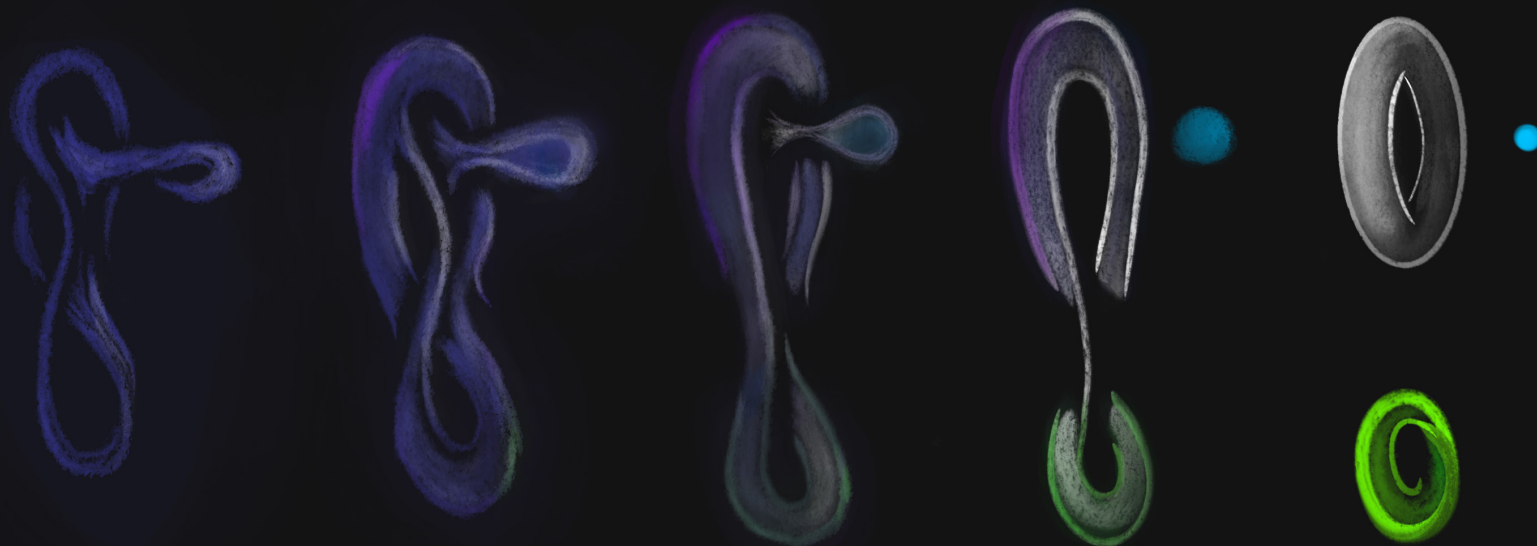
(d) There exists $r > 0$, such that $B(x_0, t_0, r)$ is a neighborhood of x_0 where scalar curvature is at least r^{-2} has a neighborhood, satisfying the conclusions of 1.5. (In particular, this means that if in case Ricci flow with surgery on three-manifolds the neighborhood of x_0 is $B(x_0, t_0, \epsilon^{-1}r_0)$, then the solution is required to be defined in the whole $P(x_0, t_0, \epsilon^{-1}r_0, -r_0^2)$; however, this does not rule out a surgery in the time interval $(t_0 - r_0, t_0)$, that occurs sufficiently far from t_0 .)

Grisha Perelman*

Grigori repeated this process until all the pieces became simple enough. Then he glued them back together to understand the original space.

Recall that in the previous section, we saw that the conclusions of Theorem 1.12.1, we know that the a priori estimates above hold for a smooth solution on any finite time interval. For Ricci flow with surgery they will be justified in the next section.

Over the course of several years, Grigori developed these ideas and used them to study the structure of three-dimensional space.



In November of 2002, Grigori finally shared his work to the world. Using Ricci flow, heat and surgery, he solved both the Poincaré and Geometrization conjectures.

Simon Brendle

David Glickenstein

Jeffrey Streets

And mathematicians continue to explore new mysteries.

Lu Wang

Jian Song

Xiaokui Yang

Richard Bamler

Valentino Tosatti

Nataša Šešum

Sun Song

Bing Wang

Brett Kotschwar

Peter Topping

Xiangwen Zheng

Ben Weinkove

Aaron Naber

Sebastien Picard

Ovidiu Munteanu

Robert Haslhofer

Fernando Codá Marques

André Neves

Gábor Székelyhidi

Esther Cabezas-Rivas

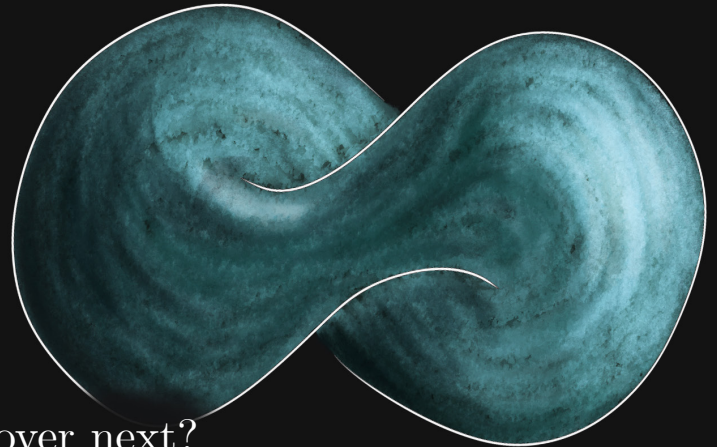
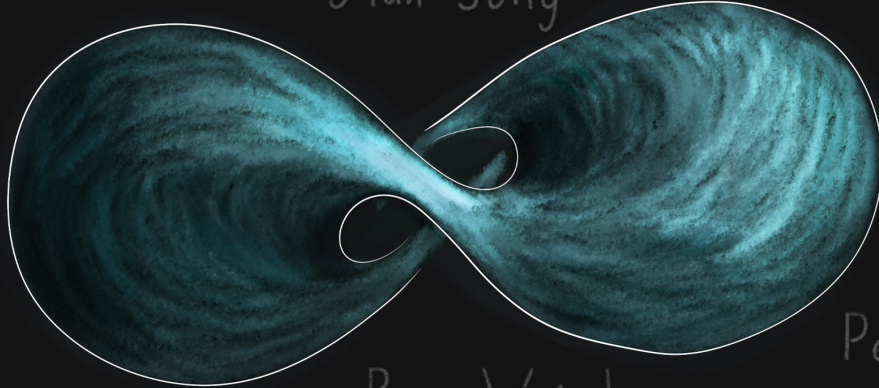
What will they discover next?

Felix Schulze

Man-Chun Lee

Sébastien Boucksom

Jason Lotay



The proof of the Poincaré conjecture is one of the most significant achievements in mathematics, marking the culmination of centuries of development in geometry, topology, and partial differential equations.

I was in high school when I first heard about the conjecture and its solution. Although I didn't understand any of the details at the time, I was fascinated by the imagery of curvature flows and surgeries carving geometric spaces. It felt like something out of science fiction, completely unlike the math in my classes.

In this book, I have done my best to tell the history of the Poincaré conjecture and to communicate that mathematics is a collaborative endeavor, built on the contributions of countless mathematicians over generations. For young readers, I hope this book inspires curiosity and sparks an interest in geometry. For readers with more expertise, I hope the illustrations generate further discussion. Math is a beautiful field, and I'm excited to share this story with you.

Technical Notes

My goal was to tell the story accurately, both with respect to the mathematics as well as the history. In order to provide a cohesive narrative, this book focuses on developments throughout the 20th century, although it is worth emphasizing that there were many important discoveries before that time. In addition, there are several anachronisms between the background text and the main story, such as Möbius' name appearing on the page discussing mathematicians who worked on topology after Poincaré.

There are two additional points of clarification that should be made. First, symplectic geometry does not play a role in the Poincaré conjecture. I included the symplectic camel as a reference to Gromov, whose contributions to geometric group theory and metric geometry have been instrumental in shaping modern geometry and play an important role in both Thurston's and Perelman's work. Second, the idea of combining Ricci flow with surgery was originally proposed by Hamilton. However, he was unable to fully control the geometry of three-dimensional singularities or rule out certain singularity models that could not be excised by surgery. Although Perelman did not invent surgery, his insights into the structure of singularities enabled him to successfully carry out the surgical process and complete the proof.

The background text on several pages incorporates excerpts from the papers cited in the references as well as the *Universalis Cosmographia*, which was the first world map to label the Americas. The remaining images were created using Procreate and Photoshop. Many were inspired by existing mathematical figures or techniques for visualizing mathematical concepts, while others drew influence from artworks and posts depicting specific geometric figures, which are cited in the artwork references. The cover art depicts an artistic interpretation of a planet with $\mathbb{S}^2 \times \mathbb{S}^1$ geometry.

Acknowledgments

This book would not have been possible without the support of many people. Thanks to everyone who helped with proofreading and editing, particularly Kori, Mizan, Therese, Karen, and Nicholas. I am especially grateful to David Kramer for his assistance and encouragement. I am also deeply appreciative of the community at Iowa State University. This work was supported in part by a Collaboration grant from the Simons Foundation (849022, G.K.).

Finally, I would like to thank all the teachers who have guided me in my mathematical journey, especially Glen Hall, Fangyang Zheng, and Jun Zhang.

References

- [1] Huai-Dong Cao and Xi-Ping Zhu, *Hamilton-Perelman's proof of the Poincaré conjecture and the Geometrization conjecture*, arXiv preprint math/0612069 (2006).
- [2] Leonhard Euler, *Solutio problematis ad geometriam situs pertinentis*, Commentarii academiae scientiarum Petropolitanae (1741), 128–140.
- [3] Yvonne Foures (Choquet)-Bruhat, *Théorème d'existence pour certains systèmes d'équations aux dérivées partielles non linéaires*, Acta mathematica **88** (1952), 141–225.
- [4] Robert Fricke and Felix Klein, *Vorlesungen über die Theorie der automorphen Functionen: bd. Die gruppentheoretischen Grundlagen*, vol. 1, BG Teubner, 1897.
- [5] Richard Hamilton, *Three-manifolds with positive Ricci curvature*, Journal of Differential geometry **17** (1982), no. 2, 255–306.
- [6] ———, *The formations of singularities in the Ricci flow*, Surveys in Differential Geometry **2** (1993), no. 1, 7–136.
- [7] Bruce Kleiner and John Lott, *Notes on Perelman's papers*, Geometry & Topology **12** (2008), no. 5, 2587–2855.
- [8] John Milnor, *The Poincaré Conjecture*, The Millennium Prize Problems, Clay Math Foundation, 2006, pp. 71–86.
- [9] John W Morgan and Gang Tian, *Ricci flow and the Poincaré conjecture*, vol. 3, American Mathematical Soc., 2007.
- [10] Grisha Perelman, *The entropy formula for the Ricci flow and its geometric applications*, arXiv preprint math/0211159 (2002).
- [11] ———, *Finite extinction time for the solutions to the Ricci flow on certain three-manifolds*, arXiv preprint math/0307245 (2003).
- [12] ———, *Ricci flow with surgery on three-manifolds*, arXiv preprint math/0303109 (2003).
- [13] Henri Poincaré, *Analysis situs*, Gauthier-Villars Paris, France, 1895.
- [14] ———, *Cinquième complément à l'analysis situs*, Rendiconti del Circolo Matematico di Palermo (1884-1940) **18** (1904), no. 1, 45–110.
- [15] William P Thurston, *The geometry and topology of three-manifolds*, Princeton University Princeton, NJ, 1979.
- [16] ———, *Three dimensional manifolds, Kleinian groups and hyperbolic geometry*, Bull. Amer. Math. Soc. **6** (1982), 357–381.
- [17] Peter Topping, *Lectures on the Ricci flow*, vol. 325, Cambridge University Press, 2006.

Artwork References

- [1] Richard Bamler, *Recent developments in Ricci flows*, Notices of the American Mathematical Society **68** (2021), no. 9.
- [2] Georg Braun and Franz Hogenberg, *Die Furstliche Hauptt Statt Konigssbergk in Preussen*, c. 1581.
- [3] Robin Dripps, *responsive surface: topographic gradient*, Explode.BReps @ UVa, explodebrep.wordpress.com/grasshopper-definitions/responsive-surface-topographic-gradient/.
- [4] Lynn Heller and Franz Pedit, *Towards a constrained Willmore conjecture*, Willmore Energy and Willmore Conjecture, Chapman and Hall/CRC, 2017, pp. 119–138.
- [5] Rob Kusner, *Conformal geometry and complete minimal surfaces*, Bulletin of the American Mathematical Society **17** (1987), no. 2, 291–295.
- [6] David Lowry-Duda, *Visualizing modular forms*, Arithmetic geometry, number theory, and computation, Springer, 2021, pp. 537–557.
- [7] L.penguu, *Spectral sequence visualization*, Wikimedia commons (September 15, 2021), commons.wikimedia.org/wiki/File:Spectral_Sequence_Visualization.jpg.
- [8] Curtis McMullen, *Cusp group*, people.math.harvard.edu/~ctm/gallery/index.html.
- [9] Sam Nead, *Drawing of the eight Thurston geometries?*, (July 7, 2010), mathoverflow.net/q/30871 (version: January 4, 2021).
- [10] David A Richter, *Hopf key rings*, Wikimedia commons (November 28, 2005), commons.wikimedia.org/wiki/File:Hopfkeyrings.jpg.
- [11] J Hyam Rubinstein and Robert Sinclair, *Visualizing Ricci flow of manifolds of revolution*, Experimental Mathematics **14** (2005), no. 3, 285–298.
- [12] Ian Stewart, *The symplectic camel*, Nature **329** (1987), no. 6134, 17–18.
- [13] Cédric Villani, *Optimal transport: old and new*, vol. 338, Springer, 2009.
- [14] Martin Waldseemüller, *Universalis Cosmographia*, 1507.
- [15] Conan Wu, *On Alexander horned sphere*, Area777 (April 18, 2010), conanwu.com/2010/04/18/on-alexander-horned-sphere/.

