THE PROBLEM OF RANDOM INTERVALS ON A LINE

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1. *Statement of the problem.* Suppose that events occur at random points on a line from $t = -\infty$ to $+\infty$, the probability of an event occurring between t and $t+dt$ being λdt. If we select any interval of the line, say the interval $[0, y]$, there will be a finite probability that it contains $0, 1, 2, \ldots, r, \ldots$ events; in fact, it is not difficult to show that these probabilities form a Poisson distribution, the probability that the interval contains r events being $\lambda^r y^r e^{-\lambda y}/r!$ (see e.g. (1)). Consider the case when each event consists of an interval of length α (an event being characterized by its first point). What is the probability that the covered portion of the interval $[0, y]$ lies between x and $x+dx$?

A related problem*, that of n given intervals at random points on the circumference of a circle, has been discussed by W. L. Stevens(2). The methods used in the present paper, however, are basically different from those of Stevens, since the concept of random events on an infinite line leads naturally to the use of continuous processes. Some of the results given by Stevens will be deduced in the course of the work.

The problem has also been discussed recently in a paper by H. E. Robbins(3). He is concerned with the moments of the resulting distribution, whereas we are more concerned with its explicit formulation. The problem of the moments will be dealt with in § 11.

2. *The basic equation.* Let† $T(x, y)$ be the probability that the covered portion of the interval $[0, y]$ is less than or equal to x; and let $W(x, y)\,dx$ be the probability that the covered portion of the interval lies between x and $x+dx$. Then $W(x, y)$ is the differential coefficient of $T(x, y)$ with respect to x. Both $W(x, y)$ and $T(x, y)$ will be zero if $x < 0$ or $y - x < 0$.

It is clear that there will be a finite probability of the interval being completely uncovered or completely covered. Hence the function $T(x, y)$ has a discontinuous jump at $x = 0$ and $x = y$. Similarly, there will be a finite probability of the interval consisting of exactly r non-overlapping events (including no overlapping at the end-points), where r has the values $1, 2, 3, \ldots, n$, n being the integral part of y/α. Hence $T(x, y)$ has a discontinuous jump at each of the points $x = \alpha, 2\alpha, \ldots, n\alpha$. At any discontinuity of $T(x, y)$ the function $W(x, y)$ is not defined in the pure mathematical sense. However, $W(x, y)$ is more convenient to deal with than $T(x, y)$, and we therefore use the δ-function notation; if $T(x, y)$ has a jump of magnitude K at $x = k$ we say that $W(x, y)$ has a δ-function singularity at $x = k$ and contains the term $K\delta(x-k)$. Thus $W(x, y)$ will have δ-function singularities at $x = 0, \alpha, 2\alpha, \ldots, n\alpha, y$.

* I am indebted to H. Jeffreys for drawing my attention to this aspect of the problem, and for a number of helpful suggestions.

† We are here considering the problem described at the beginning of § 1. The case of n given events will be dealt with in § 7.

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Let $w(x,y,\xi)\,dx\,d\xi$ be the probability that the covered portion of the interval $[0,y]$ lies between x and $x+dx$, and that the last event (i.e. that immediately preceding the right-hand end of the interval) occurred between ξ and $\xi+d\xi$ from the right-hand end of the interval.

Consider first the case $\alpha \leqslant x$; then at least one event must have occurred in $[0,y]$, and the possible values of ξ range from 0 to y. Divide $[0,y]$ into two subintervals $[0,y-\xi]$ and $[y-\xi,y]$. Then if $\xi \leqslant \alpha$, $w(x,y,\xi)\,dx\,d\xi$ corresponds exactly to the following two independent events:

(*a*) $w(x-\xi,y-\xi,0)\,dx\,d\xi$ in the interval $[0,y-\xi]$.

(*b*) No event in $[y-\xi,y]$.

Therefore and by a similar argument

$$\left.\begin{aligned} w(x,y,\xi) &= e^{-\lambda\xi}\,w(x-\xi,y-\xi,0) \quad (\xi\leqslant\alpha)\\ w(x,y,\xi) &= e^{-\lambda\xi}\,w(x-\alpha,y-\xi,0) \quad (\alpha\leqslant\xi)\end{aligned}\right\}. \tag{1}$$

By dividing $[0,y]$ into $[0,y-d\xi]$ and $[y-d\xi,y]$ we can show that

$$w(x,y,0)\,dx\,d\xi = \lambda\,d\xi\,W(x,y)\,dx. \tag{2}$$

Using (1) and (2) we obtain

$$W(x,y) = \int_0^y w(x,y,\xi)\,d\xi = \int_0^\alpha \lambda e^{-\lambda\xi}\,W(x-\xi,y-\xi)\,d\xi + \int_\alpha^{y-x+\alpha}\lambda e^{-\lambda\xi}\,W(x-\alpha,y-\xi)\,d\xi \tag{3}$$

for $\alpha\leqslant x$. (The upper limit of the second integral is taken as $y-x+\alpha$, since

$$W(x-\alpha,y-\xi)=0 \text{ if } y-\xi<x-\alpha.)$$

When $x<\alpha$ the covered portions must occur within the regions $[0,x]$ and $[y-x,y]$. Values of ξ can now be divided into two categories (*a*) $0\leqslant\xi\leqslant x$, (*b*) $y<\xi\leqslant y+\alpha$. Case (*a*) gives rise to a term $\int_0^x \lambda e^{-\lambda\xi}\,W(x-\xi,y-\xi)\,d\xi$ as above. Case (*b*) may be further subdivided into the following:

(i) An event in $[x-\alpha,x+dx-\alpha]$ and no event in $[x+dx-\alpha,y]$. The probability is $\lambda e^{-\lambda(y-x+\alpha)}\,dx$.

(ii) The finite probability that the line is completely covered by an event occurring in $[-\alpha,0]$ and no events occurring in $[0,y]$. This is only relevant when $y<\alpha$, and involves $\delta(y-x)$. (The case in which the line is completely covered but events also occur in $[0,y]$ has been covered by (*a*).) It gives rise to a term $\delta(y-x)\,e^{-\lambda x}\,[1-e^{-\lambda(\alpha-x)}]$. We must also add a term $e^{-\lambda(y-x+\alpha)}\,\delta(x)$ to represent the finite probability that the line is completely uncovered.

Hence we have

$$\begin{aligned} W(x,y) = \int_0^x \lambda e^{-\lambda\xi}\,W(x-\xi,y-\xi)\,d\xi + e^{-\lambda(y-x+\alpha)}\,[\lambda+\delta(x)]&\\ +\,\delta(y-x)\,e^{-\lambda x}\,[1-e^{-\lambda(\alpha-x)}] \quad (x<\alpha).& \end{aligned} \tag{4}$$

Equations (3) and (4) are the basic equations which completely determine the form of $W(x,y)$.

3. *Formal solution of the equations.* The basic equations assume a more tractable form if we change variables. Write $y-x=z$ and put $W(x,x+z)\equiv f(x,z)$. Then (3) and (4) become

$$f(x,z)=\int_0^\alpha \lambda e^{-\lambda\xi}f(x-\xi,z)\,d\xi+\int_\alpha^{z+\alpha}\lambda e^{-\lambda\xi}f(x-\alpha,z+\alpha-\xi)\,d\xi \quad (\alpha\leqslant x) \tag{5}$$

and

$$f(x,z)=\int_0^x \lambda e^{-\lambda\xi}f(x-\xi,z)\,d\xi+e^{-\lambda(z+\alpha)}[\lambda+\delta(x)]+e^{-\lambda z}\delta(z)[e^{-\lambda x}-e^{-\lambda\alpha}] \quad (x<\alpha). \tag{6}$$

The coefficient $e^{-\lambda z}$ of $\delta(z)$ in (6) has been inserted for convenience and makes no difference to the value of the term.

The second integral in (5) can be written as

$$e^{-\lambda\alpha}\int_0^z \lambda e^{-\lambda\eta}f(x-\alpha,z-\eta)\,d\eta=e^{-\lambda(\alpha+z)}\int_0^z \lambda e^{\lambda\zeta}f(x-\alpha,\zeta)\,d\zeta.$$

Thus
$$f(x,z)=\int_0^\alpha \lambda e^{-\lambda\xi}f(x-\xi,z)\,d\xi+e^{-\lambda(\alpha+z)}\int_0^z \lambda e^{\lambda\zeta}f(x-\alpha,\zeta)\,d\zeta \quad (\alpha\leqslant x). \tag{7}$$

Equations (6) and (7) are amenable to treatment by Laplace transforms in x. We use the notation of van der Pol; if $f(x)$ has Laplace transform $F(p)$ we write $f(x)\doteqdot F(p)$. Let $f(x,z)\doteqdot F(p,z)$. As mentioned in §2 $f(x,z)$ does not exist in the mathematical sense at certain points, and Laplace transforms are only strictly applicable to

$$\int_0^x f(x,z)\,dx\doteqdot F(p,z)/p.$$

If this integral has a discontinuity of magnitude K at $x=k$ the corresponding Laplace transform contains a term Ke^{-kp}; thus, if for convenience we work with $F(p,z)$, we obtain a term Kpe^{-kp}, which is then to be interpreted as $K\delta(x-k)$. The following results can easily be deduced from Laplace transform theory:

$$\delta(x)\doteqdot p,$$

$$\left.\begin{array}{ll} f(x-\alpha,z) & (x\geqslant\alpha)\\ 0 & (x<\alpha)\end{array}\right\}\doteqdot e^{-\alpha p}F(p,z),$$

$$\int_0^x \lambda e^{-\lambda\xi}f(x-\xi,z)\,d\xi\doteqdot\frac{\lambda}{p+\lambda}F(p,z),$$

$$\left.\begin{array}{ll} \int_0^z \lambda e^{\lambda\zeta}f(x-\alpha,\zeta)\,d\zeta & (x\geqslant\alpha)\\ 0 & (x<\alpha)\end{array}\right\}\doteqdot\int_0^z \lambda e^{\lambda\zeta}e^{-\alpha p}F(p,\zeta)\,d\zeta,$$

$$\left.\begin{array}{ll} \int_\alpha^x \lambda e^{-\lambda\xi}f(x-\xi,z)\,d\xi & (x\geqslant\alpha)\\ 0 & (x<\alpha)\end{array}\right\}\doteqdot\frac{\lambda}{p+\lambda}e^{-\alpha(p+\lambda)}F(p,z),$$

$$\left.\begin{array}{ll} e^{-\lambda x}-e^{-\lambda\alpha} & (x<\alpha)\\ 0 & (x\geqslant\alpha)\end{array}\right\}\doteqdot\frac{p(1-e^{-\lambda\alpha})-\lambda e^{-\lambda\alpha}(1-e^{-\alpha p})}{p+\lambda}.$$

Applying these results to (6) and (7) we obtain

$$F(p,z)=\frac{\lambda}{p+\lambda}F(p,z)[1-e^{-\alpha(p+\lambda)}]+e^{-\lambda(\alpha+z)}\int_0^z \lambda e^{\lambda\zeta}e^{-\alpha p}F(p,\zeta)\,d\zeta$$
$$+\lambda e^{-\lambda(\alpha+z)}[1-e^{-\alpha p}]+pe^{-\lambda(\alpha+z)}+\delta(z)\,e^{-\lambda z}\,\frac{p(1-e^{-\lambda\alpha})-\lambda e^{-\lambda\alpha}(1-e^{-\alpha p})}{p+\lambda}. \tag{8}$$

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If we write $F(p,z)\,e^{\lambda z} = G(p,z)$, (8) simplifies to the form

$$G(p,z) = A + B\int_0^z G(p,\zeta)\,d\zeta + C\delta(z), \tag{9}$$

where $$A = \frac{e^{-\lambda\alpha}(p+\lambda)(p+\lambda-\lambda e^{-\alpha p})}{p+\lambda e^{-\alpha(p+\lambda)}}, \quad B = \frac{\lambda e^{-\alpha(p+\lambda)}(p+\lambda)}{p+\lambda e^{-\alpha(p+\lambda)}},$$

and $$C = \frac{p(1-e^{-\lambda\alpha}) - \lambda e^{-\lambda\alpha}(1-e^{-\alpha p})}{p+\lambda e^{-\alpha(p+\lambda)}}.$$

Equation (9) can be solved in the ordinary way for $G(p,z)$ (by making use of the properties of δ-functions) and we obtain

$$G(p,z) = C\delta(z) + (A+BC)\,e^{Bz}. \tag{10}$$

This solution will be found to be valid on substituting in (9).

4. *Interpretation of the solution.* In order to interpret (10) in terms of x we expand the exponential as a power series,

$$1 + Bz + \frac{B^2z^2}{2!} + \dots + \frac{B^rz^r}{r!} + \dots.$$

We then have to interpret terms of the form AB^r, CB^r. For this we must expand the denominators in ascending powers of $e^{-\alpha p}$. Any term multiplied by $e^{-r\alpha p}$ gives zero if $x < r\alpha$. As a typical example AB^r is expanded in the following form:

$$\lambda^r e^{-(r+1)\lambda\alpha} e^{-r\alpha p}(p+\lambda-\lambda e^{-\alpha p})\left(1+\frac{\lambda}{p}\right)^{r+1}$$
$$\times\left[1 - \frac{(r+1)\lambda}{p}e^{-\alpha(p+\lambda)} + \frac{(r+1)(r+2)}{2!}\left(\frac{\lambda}{p}\right)^2 e^{-2\alpha(p+\lambda)} + \dots\right].$$

Clearly all terms of this are zero unless $x \geqslant r\alpha$. The enumeration of all terms in the general case is very complicated. When $r\alpha \leqslant x \leqslant (r+1)\alpha$ only terms up to AB^r and CB^r in the expansion of the exponential need be considered.

5. *The δ-function terms.* By examining the terms in the expansion of (10) it is quite easy to sort out the δ-function singularities which were mentioned in § 2. First, the coefficient of $\delta(z)$, C, is the probability that the whole interval is covered; this will be dealt with separately in the next section. All δ-function terms in x will be given by terms of the form $\phi(z)\,pe^{-n\alpha p}$ in the expansion of (10). B and C do not give rise to such terms but A does; hence the only terms of $F(p,z)$ that need be considered in this connexion are of the form $AB^rz^r e^{-\lambda z}/r!$.

The zero order term is $pe^{-\lambda(\alpha+z)}$; this corresponds to the probability $e^{-\lambda(y+\alpha)}$ of the interval being completely uncovered.

The rth order term is $p\lambda^r z^r e^{-(r+1)\lambda\alpha} e^{-r\alpha p} e^{-\lambda z}/r!$ giving rise to a term $\delta(x-r\alpha)$. Hence the probability that exactly r non-overlapping events occur in the interval (including no overlapping at the end-points) is

$$\lambda^r(y-r\alpha)^r e^{-\lambda(y+\alpha)}/r! \quad (y \geqslant r\alpha). \tag{11}$$

This must be made up of the following independent probabilities:

(*a*) Exactly r events occur in $[0, y-\alpha]$; probability $\lambda^r(y-\alpha)^r e^{-\lambda(y-\alpha)}/r!$.

(*b*) No event occurs in $[-\alpha, 0]$ or $[y-\alpha, y]$; probability $e^{-2\lambda\alpha}$.

(*c*) Given r events occurring in $[0, y-\alpha]$, no two overlap.

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By examining (11) we deduce that the probability of (c) is

$$\left(\frac{y-r\alpha}{y-\alpha}\right)^r. \tag{12}$$

From (12) we can deduce a result obtained by Stevens. Given r intervals of length α at random points on a circle what is the probability that no two overlap? Let y be the length of the circumference of the circle. Choose the beginning of any one interval as origin measuring in an anticlockwise direction round the circumference. Then no two intervals overlap if, and only if, (a) all $(r-1)$ remaining intervals occur in $[\alpha, y-\alpha]$, and (b) no two of these intervals overlap.

Using (12), we obtain for the required probability

$$\left(\frac{y-2\alpha}{y}\right)^{r-1}\left(\frac{y-r\alpha}{y-2\alpha}\right)^{r-1} = \left(\frac{y-r\alpha}{y}\right)^{r-1}. \tag{13}$$

6. *Probability that total interval is covered.* We conclude from (10) that the probability of the total interval being covered is given by

$$\frac{p(1-e^{-\lambda\alpha})-\lambda e^{-\lambda\alpha}(1-e^{-\alpha p})}{p+\lambda e^{-\alpha(p+\lambda)}}. \tag{14}$$

It is easy to deduce this result from first principles by a method similar to that used in §§2 and 3. For if $z(y)$ is the probability that the total interval is covered we can deduce the equations

$$\left.\begin{aligned} z(y) &= \int_0^\alpha \lambda e^{-\lambda\xi} z(y-\xi)\,d\xi && (y \geqslant \alpha),\\ \text{and}\qquad z(y) &= \int_0^y \lambda e^{-\lambda\xi} z(y-\xi)\,d\xi + e^{-\lambda y} - e^{-\lambda\alpha} && (y \leqslant \alpha),\end{aligned}\right\} \tag{15}$$

which lead to the same result as (14).

Expanding (14) we obtain

$$\left[(1-e^{-\lambda\alpha}) - \frac{\lambda}{p}e^{-\lambda\alpha}(1-e^{-\alpha p})\right]\sum_{r=0}^{\infty}(-1)^r\left(\frac{\lambda}{p}\right)^r e^{-r\alpha(p+\lambda)}.$$

Collecting together the terms in $e^{-r\alpha p}$ we find after a little simplification that they are

$$(-1)^{r+1}e^{-(r+1)\lambda\alpha}\left[\left(\frac{\lambda}{p}\right)^r + \left(\frac{\lambda}{p}\right)^{r+1}\right].$$

Hence we can write down $z(y)$ in a simple explicit form

$$\begin{aligned} z(y) = 1 - e^{-\lambda\alpha}(1+\lambda y) + e^{-2\lambda\alpha}\left[\lambda(y-\alpha) + \frac{\lambda^2(y-\alpha)^2}{2!}\right] - \ldots \\ + (-1)^{r+1}e^{-(r+1)\lambda\alpha}\left[\frac{\lambda^r(y-r\alpha)^r}{r!} + \frac{\lambda^{r+1}(y-r\alpha)^{r+1}}{(r+1)!}\right] + \ldots, \end{aligned} \tag{16}$$

all terms beyond $y-n\alpha$ being ignored if $n\alpha < y \leqslant (n+1)\alpha$. The expression for $z(y)$ assumes a more empirical form if we choose new variables, $\lambda\alpha = \beta$, $y/\alpha = \nu$, $z(y) = \zeta(\nu)$.

Then
$$\begin{aligned} \zeta(\nu) = 1 - e^{-\beta}(1+\beta\nu) + e^{-2\beta}\left[\beta(\nu-1) + \frac{\beta^2}{2!}(\nu-1)^2\right] - \ldots \\ + (-1)^{r+1}e^{-(r+1)\beta}\left[\frac{\beta^r}{r!}(\nu-r)^r + \frac{\beta^{r+1}}{(r+1)!}(\nu-r)^{r+1}\right] + \ldots. \end{aligned} \tag{17}$$

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7. *The case of n given events. Derivation of Stevens's solution.* We can divide the general solution $W(x, y)$ into mutually exclusive categories in which exactly n events occur in $[-\alpha, y]$. The probability that exactly n events occur in $[-\alpha, y]$ is

$$\lambda^n (y+\alpha)^n e^{-\lambda(y+\alpha)}/n!.$$

Let $W_n(x, y)\,dx$ be the probability that, given n events occurring in $[-\alpha, y]$, the covered portion of $[0, y]$ lies between x and $x+dx$. Then

$$W(x,y) = \sum_{n=0}^{\infty} \frac{\lambda^n (y+\alpha)^n}{n!} e^{-\lambda(y+\alpha)} W_n(x,y). \tag{18}$$

Hence, if we expand $W(x, y)\, e^{\lambda(y+\alpha)}$ in ascending powers of λ, the coefficient of λ^n will be a function of x and y only, and completely determines $W_n(x, y)$. The complete solution $W_n(x, y)$ is very complicated, but the method can fairly easily be applied to $z(y)$ to determine $z_n(y)$, the probability that given n events in $[-\alpha, y]$ the interval $[0, y]$ is completely covered.

We have to expand $z(y)\, e^{\lambda(y+\alpha)}$ in ascending powers of λ. It is convenient to go back to the Laplace transform of equation (14). If $z(y) \doteqdot Z(p)$, then

$$z(y)\, e^{\lambda(y+\alpha)} \doteqdot \frac{pe^{\lambda\alpha}}{p-\lambda} Z(p-\lambda) = \frac{pe^{\lambda\alpha}}{p-\lambda} - \frac{p}{p-\lambda}\, \frac{p}{p-\lambda+\lambda e^{-\alpha p}}. \tag{19}$$

The first term of (19), $pe^{\lambda\alpha}/(p-\lambda) \doteqdot e^{\lambda(y+\alpha)}$, and the coefficient of λ^n is $(y+\alpha)^n/n!$.

The second term is $$-\sum_{s=1}^{\infty} \left(\frac{\lambda}{p}\right)^s \sum_{t=1}^{\infty} \left(\frac{\lambda}{p}\right)^t (1-e^{-\alpha p})^t.$$

The term involving λ^n is

$$-\left(\frac{\lambda}{p}\right)^n [1 + (1-e^{-\alpha p}) + \ldots + (1-e^{-\alpha p})^n] = -\left(\frac{\lambda}{p}\right)^n e^{\alpha p}[1-(1-e^{-\alpha p})^{n+1}]. \tag{20}$$

Taking the Laplace transform of (20) and dividing the coefficient of λ^n by $(y+\alpha)^n/n!$ we deduce that

$$z_n(y) = 1 - {}^{n+1}C_1 \left(\frac{y}{y+\alpha}\right)^n + {}^{n+1}C_2 \left(\frac{y-\alpha}{y+\alpha}\right)^n + \ldots + (-1)^s\, {}^{n+1}C_s \left(\frac{y-\overline{s-1}\alpha}{y+\alpha}\right)^n + \ldots, \tag{21}$$

the series terminating with s equal to the integral part of $(y+\alpha)/\alpha$ if this is not greater than $n+1$, otherwise with $s = n+1$. In the latter case $z_n(y) \equiv 0$, as can easily be seen from the form of the Laplace transform of $(y+\alpha)^n z_n(y)$.

Now consider n intervals at random points on the circumference of a circle. Choose the end of one interval arbitrarily as origin, and measure distances anticlockwise round the circumference. Let y be the length of the circumference of the circle. Then the probability that the circle is completely covered is the probability that given $(n-1)$ events occurring in $[-\alpha, y-\alpha]$ the interval $[0, y-\alpha]$ is completely covered. This is $z_{n-1}(y-\alpha)$, which is the solution given by Stevens.

A problem somewhat similar to that discussed in this section has been dealt with in a recent paper by Votaw (4) by methods substantially different from those used here.

8. *The distribution of gaps.* The method of the preceding sections can be applied to determine the probability distribution of the number of gaps in a given interval of line $[0, y]$, and hence to deduce Stevens's results for a circle. Let $u_r(y)$ be the probability

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that the interval $[0, y]$ contains r gaps (including gaps at the beginning and end of the interval). Then $u_r(y) = 0$ when $y \leqslant (r-1)\alpha (r \geqslant 1)$, $u_0(y)$ is identical with $z(y)$ of § 6, and $u_1(y)$ includes the case when the interval is completely uncovered. Let $u_r(y, \xi)\, d\xi$ be the probability that the interval $[0, y]$ contains r gaps, and that the last event occurred between ξ and $\xi + d\xi$ from the end of the interval. By analogy with equation (1) we deduce that

$$\left.\begin{aligned} u_r(y,\xi) &= \lambda e^{-\lambda\xi} u_r(y-\xi)\, d\xi \quad (\xi \leqslant \alpha) \\ \text{and} \qquad u_r(y,\xi) &= \lambda e^{-\lambda\xi} u_{r-1}(y-\xi)\, d\xi \quad (\alpha < \xi \leqslant y) \end{aligned}\right\} \quad (y \geqslant \alpha). \tag{22}$$

When $r \geqslant 2$ values of ξ between 0 and y cover all possible cases; but when $r = 1$ a term must be added to take account of values of $\xi > y$. Hence we have

$$\left.\begin{aligned} u_r(y) &= \int_0^\alpha \lambda e^{-\lambda\xi} u_r(y-\xi)\, d\xi + \int_\alpha^y \lambda e^{-\lambda\xi} u_{r-1}(y-\xi)\, d\xi \quad (r \geqslant 2) \\ u_1(y) &= \int_0^\alpha \lambda e^{-\lambda\xi} u_1(y-\xi)\, d\xi + \int_\alpha^y \lambda e^{-\lambda\xi} u_0(y-\xi)\, d\xi + e^{-\lambda y} \end{aligned}\right\} \quad (y \geqslant \alpha). \tag{23}$$

When $y < \alpha$ only $u_1(y)$ need be taken into account, and we have

$$u_1(y) = \int_0^y \lambda e^{-\lambda\xi} u_1(y-\xi)\, d\xi + e^{-\lambda\alpha} \quad (y < \alpha), \tag{24}$$

but it is also convenient from the point of view of Laplace transforms to write

$$u_r(y) = \int_0^y \lambda e^{-\lambda\xi} u_r(y-\xi)\, d\xi \quad (r \geqslant 2, y < \alpha).$$

We now take Laplace transforms in y, and write $u_r(y) \doteqdot U_r(p)$. Using results similar to those quoted in § 3 we obtain

$$U_r(p) = \frac{\lambda}{p+\lambda}[1 - e^{-\alpha(p+\lambda)}]\, U_r(p) + \frac{\lambda e^{-\alpha(p+\lambda)}}{p+\lambda} U_{r-1}(p) \quad (r \geqslant 2),$$

or

$$U_r(p) = \frac{\lambda e^{-\alpha(p+\lambda)}}{p + \lambda e^{-\alpha(p+\lambda)}} U_{r-1}(p) \quad (r \geqslant 2), \tag{25}$$

and $$U_1(p) = \frac{\lambda}{p+\lambda}[1 - e^{-\alpha(p+\lambda)}]\, U_1(p) + \frac{\lambda e^{-\alpha(p+\lambda)}}{p+\lambda} U_0(p) + \frac{p e^{-\lambda(p+\alpha)}}{p+\lambda} + e^{-\lambda\alpha}(1 - e^{-\alpha p}).$$

Substituting the value of $U_0(p)$ from § 6 we obtain, after a little simplification,

$$U_1(p) = \frac{p[1 - U_0(p)]}{p + \lambda e^{-\alpha(p+\lambda)}} = \frac{p(p+\lambda)\, e^{-\lambda\alpha}}{[p + \lambda e^{-\alpha(p+\lambda)}]^2}. \tag{26}$$

Hence, by (25), $$U_r(p) = \frac{\lambda^{r-1} p(p+\lambda)\, e^{-r\lambda\alpha}\, e^{-(r-1)\alpha p}}{[p + \lambda e^{-\alpha(p+\lambda)}]^{r+1}} \quad (r \geqslant 1). \tag{27}$$

We may verify that

$$\sum_{r=1}^{\infty} U_r(p) = U_1(p) \left[\frac{p + \lambda e^{-\alpha(p+\lambda)}}{p}\right] = 1 - U_0(p),$$

by (26). To derive an expression for $u_r(y)$ we expand (27) in the form

$$U_r(p) = \left(1 + \frac{\lambda}{p}\right)\left(\frac{\lambda}{p}\right)^{r-1} e^{-r\lambda\alpha}\, e^{-(r-1)\alpha p} \left[1 - \frac{(r+1)}{1!}\left(\frac{\lambda}{p}\right) e^{-\alpha(p+\lambda)} + \frac{(r+1)(r+2)}{2!}\left(\frac{\lambda}{p}\right)^2 e^{-2\alpha(p+\lambda)} + \dots + \frac{(-1)^s (r+1)\dots(r+s)}{s!}\left(\frac{\lambda}{p}\right)^s e^{-s\alpha(p+\lambda)} + \dots\right],$$

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so that, for $r \geqslant 1$,

$$u_r(y) = e^{-r\lambda\alpha}\left[\frac{\lambda^{r-1}(\overline{y-r-1\alpha})^{r-1}}{(r-1)!}+\frac{\lambda^r(\overline{y-r-1\alpha})^r}{r!}\right]-(r+1)\,e^{-(r+1)\lambda\alpha}$$
$$\times\left[\frac{\lambda^r(y-r\alpha)^r}{r!}+\frac{\lambda^{r+1}(y-r\alpha)^{r+1}}{(r+1)!}\right]+\ldots+(-1)^s\frac{(r+1)\ldots(r+s)}{s!}e^{-(r+s)\lambda\alpha}$$
$$\times\left[\frac{\lambda^{r+s-1}(y-\overline{r+s-1}\alpha)^{r+s-1}}{(r+s-1)!}+\frac{\lambda^{r+s}(y-\overline{r+s-1}\alpha)^{r+s}}{(r+s)!}\right]+\ldots, \tag{28}$$

where all terms beyond $(y-n\alpha)$ are ignored if $n\alpha<y\leqslant(n+1)\alpha$. In terms of the empirical variables of (17), if $u_r(y)$ becomes $\zeta_r(\nu)$,

$$\zeta_r(\nu) = e^{-r\beta}\left[\frac{\beta^{r-1}(\nu-\overline{r-1})^{r-1}}{(r-1)!}+\frac{\beta^r(\nu-\overline{r-1})^r}{r!}\right]-(r+1)\,e^{-(r+1)\beta}$$
$$\times\left[\frac{\beta^r(\nu-r)^r}{r!}+\frac{\beta^{r+1}(\nu-r)^{r+1}}{(r+1)!}\right]+\ldots+(-1)^s\frac{(r+1)\ldots(r+s)}{s!}e^{-(r+s)\beta}$$
$$\times\left[\frac{\beta^{r+s-1}(\nu-\overline{r+s-1})^{r+s-1}}{(r+s-1)!}+\frac{\beta^{r+s}(\nu-\overline{r+s})^{r+s}}{(r+s)!}\right]+\ldots. \tag{29}$$

We can now use the method of § 7 to deal with the case of n given events. If we expand $u_r(y)\,e^{\lambda(y+\alpha)}$ in the form $\sum\limits_{n=0}^{\infty}\frac{\lambda^n(y+\alpha)^n}{n!}u_{rn}(y)$, $u_{rn}(y)$ is the probability that, given n events occurring in $[-\alpha,y]$, the interval $[0,y]$ contains r gaps. We again return to the Laplace transform $U_r(p)$; we have, by (27),

$$u_r(y)\,e^{\lambda(y+\alpha)} \doteqdot \frac{p}{p-\lambda}U_r(p-\lambda)\,e^{\lambda\alpha} = \frac{p^2\lambda^{r-1}\,e^{-(r-1)\alpha p}}{(p-\lambda+\lambda e^{-\alpha p})^{r+1}}$$
$$=\left(\frac{\lambda}{p}\right)^{r-1}e^{-(r-1)\alpha p}\left[1+(r+1)\frac{\lambda}{p}(1-e^{-\alpha p})+\ldots+\frac{(r+1)\ldots(r+s)}{s!}\left(\frac{\lambda}{p}\right)^s(1-e^{-\alpha p})^s+\ldots\right].$$

The term in λ^n is $\qquad \left(\frac{\lambda}{p}\right)^n e^{-(r-1)\alpha p}\frac{(n+1)!}{r!(n-r+1)!}(1-e^{-\alpha p})^{n-r+1},$

and hence

$$u_{rn}(y) = {}^{n+1}C_r\left[\left(\frac{y-\overline{r-1}\alpha}{y+\alpha}\right)^n-{}^{n-r+1}C_1\left(\frac{y-r\alpha}{y+\alpha}\right)^n+\ldots+(-1)^s\,{}^{n-r+1}C_s\right.$$
$$\left.\times\left(\frac{y-\overline{r+s-1}\alpha}{y+\alpha}\right)^n+\ldots\right]. \tag{30}$$

The case of n intervals at random points on the circumference of a circle can be dealt with by an argument similar to that given at the end of § 7. We deduce that for n intervals the probability of r gaps is $u_{rn-1}(y-\alpha)$, which is in agreement with the solution given by Stevens.

9. *Moments of the distribution of gaps.* The moments of the gap distribution can be conveniently determined by means of Laplace transforms. Let

$$m_\theta(y) = \sum_{r=1}^{\infty}u_r(y)\,e^{r\theta} = K+\sum_{r=1}^{\infty}\frac{\mu^{(k)}(y)\,\theta^k}{k!}, \tag{31}$$

where K is independent of θ, and let

$$m_\theta(y)\doteqdot M_\theta(p) = \sum_1^{\infty}U_r(p)\,e^{r\theta}. \tag{32}$$

Then the coefficient of $\theta^k/k!$ in $M_\theta(p)$ is the Laplace transform of the kth moment $\mu^{(k)}(y)$. Substituting for $U_r(p)$ from (27), we obtain

$$M_\theta(p) = U_1(p)\,e^\theta\left[1+\frac{\lambda e^{\theta-\alpha(p+\lambda)}}{p+\lambda e^{-\alpha(p+\lambda)}}+\ldots+\left(\frac{\lambda e^{\theta-\alpha\overline{p+\lambda}}}{p+\lambda e^{-\alpha\overline{p+\lambda}}}\right)^r+\ldots\right]$$

$$= \frac{U_1(p)\,e^\theta[p+\lambda e^{-\alpha(p+\lambda)}]}{p-\lambda e^{-\alpha(p+\lambda)}(e^\theta-1)} = \frac{U_1(p)}{e^{-\theta}-\lambda e^{-\alpha(p+\lambda)}/[p+\lambda e^{-\alpha(p+\lambda)}]}$$

$$= \frac{(p+\lambda)\,e^{-\lambda\alpha}}{pl}\,\frac{1}{1-l\left(\theta-\dfrac{\theta^2}{2!}+\dfrac{\theta^3}{3!}-\ldots\right)}, \tag{33}$$

where $l = 1+\lambda e^{-\alpha(p+\lambda)}/p$. The coefficient of θ is $e^{-\lambda\alpha}(1+\lambda/p)$; hence

$$\mu^{(1)}(y) = e^{-\lambda\alpha}(1+\lambda y). \tag{34}$$

Similarly, the coefficient of $\theta^2/2!$ is $(1+\lambda/p)\,e^{-\lambda\alpha}(-1+2l)$; hence

$$\mu^{(2)}(y) = e^{-\lambda\alpha}(1+\lambda y)+2e^{-2\lambda\alpha}\left[\lambda(y-\alpha)+\frac{\lambda^2}{2!}(y-\alpha)^2\right]. \tag{35}$$

Higher moments may be determined similarly. The mean-square deviation $\mu^{(2)}-\mu^{(1)2}$ is equal to

$$\lambda y e^{-\lambda\alpha}[1-2\lambda\alpha e^{-\lambda\alpha}]+e^{-\lambda\alpha}[1-e^{-\lambda\alpha}(1+2\lambda\alpha-\lambda^2\alpha^2)]. \tag{36}$$

The moments of the gap distribution for n given intervals on a line (and for n intervals round a circle) can also be conveniently determined by using the Laplace transform of

$$(y+\alpha)^n\,u_{rn}(y)/n!.$$

Let

$$m_{n\theta}(y) \doteqdot M_{n\theta}(p),$$

where

$$m_{n\theta}(y) = \frac{(y+\alpha)^n}{n!}\left[1+\sum_{r=1}^{\infty}u_{rn}(y)\,e^{r\theta}\right] = \frac{(y+\alpha)^n}{n!}\left[K_n+\sum_{k=1}^{\infty}\frac{\mu_n^{(k)}(y)\,\theta^k}{k!}\right]$$

and K_n is independent of θ. Then, by (30),

$$M_{n\theta}(p) = (1/p)^n\,e^{\alpha p}\sum_{r=0}^{\infty}{}^{n+1}C_r\,e^{-r\alpha p}\,e^{r\theta}(1-e^{-\alpha p})^{n-r+1}$$

$$= (1/p)^n\,e^{\alpha p}[1-e^{-\alpha p}+e^{\theta-\alpha p}]^{n+1}$$

$$= (1/p)^n\,e^{\alpha p}[1-e^{-\alpha p}(\theta+\theta^2/2!+\theta^3/3!+\ldots)]^{n+1}. \tag{37}$$

The coefficient of θ is $(n+1)/p^n$; hence

$$\mu_n^{(1)} = (n+1)\,(y/y+\alpha)^n. \tag{38}$$

The coefficient of $\theta^2/2!$ is $\quad 1/p^n[(n+1)+n(n+1)\,e^{-\alpha p}]$;

hence

$$\mu_n^{(2)} = (n+1)\,(y/y+\alpha)^n+n(n+1)\,(y-\alpha/y+\alpha)^n. \tag{39}$$

10. *Approximations for large lengths of line.* For large values of y, expressions such as (28) become extremely cumbersome to deal with. But an alternative expansion is then possible which is convenient for this case. If $f(y)\doteqdot F(p)$, it is well known that $f(y)$ is given by the integral

$$\frac{1}{2\pi i}\int_{c-i\infty}^{c+i\infty}\frac{F(p)}{p}e^{py}\,dp,$$

where the constant c must be appropriately chosen, and hence that $f(y)$ can usually be expanded as a series of residues of $F(p)\,e^{py}/p$. The Laplace transforms with which we are concerned give rise to an infinite series of residues; but for sufficiently large y all but the first term can effectively be neglected.

As an example consider $z(y)$ of § 6, the probability that the interval $[0, y]$ is covered. It is convenient to use the empirical notation of (17) and to determine $\zeta(\nu)$; from (14) we find that this is given by

$$\sum_{\text{residues}} \frac{q(1-e^{-\beta})-\beta e^{-\beta}(1-e^{-q})}{q+\beta e^{-(\beta+q)}} \frac{e^{q\nu}}{q} \quad (q = \alpha p). \tag{40}$$

We must now investigate solutions of $q+\beta e^{-(\beta+q)} = 0$ or of $qe^q = -\beta e^{-\beta}$. It is easy to see that there are two real negative roots, $-\beta$ and $-\gamma$; and by plotting the curves $|qe^q| =$ constant, $\arg(qe^q) =$ constant, in the complex q plane it can be shown that all the roots lie to the left of $\min(-\beta, -\gamma)$. On examination of (37) it will be seen that the numerator vanishes at $q = 0$, $-\beta$, and there are no poles at these points. Hence $\zeta(\nu)$ is the sum of residues at $q = -\gamma$ and the remaining complex roots of $qe^q = -\beta e^{-\beta}$; and for sufficiently large ν the latter can be neglected. (It may be noted that this approximation is better when $\gamma < \beta$ than when $\gamma > \beta$.) The residue at $q = -\gamma$ is easily evaluated, and hence

$$\zeta(\nu) \sim \frac{e^{-\beta}(\beta-\gamma)}{\gamma(1-\gamma)} e^{-\gamma\nu}. \tag{41}$$

If the remaining complex roots are given by $-\gamma_s$ $(s = 1, 2, 3, \ldots)$ the quantity neglected in (38) is

$$\sum_s \frac{e^{-\beta}(\beta-\gamma_s)}{\gamma_s(1-\gamma_s)} e^{-\gamma_s\nu}. \tag{42}$$

As a second example consider the behaviour of the gap distribution for large y. We determine the limiting form of $m_\theta(y)$; θ can be taken to be as small as we please, and we require a power series expansion in θ. In empirical notation $m_\theta(\nu)$ is given from (33) and (26) by

$$\sum_{\text{residues}} \frac{(q+\beta)\,e^{-\beta}}{[q+\beta e^{-(q+\beta)}]} \frac{e^{\nu q}}{[qe^{-\theta}+\beta e^{-(q+\beta)}\,(e^{-\theta}-1)]}. \tag{43}$$

The zeroes of the denominator are of the type discussed previously. Consider the smallest zero, δ, of

$$qe^q = \beta e^{-\beta}(e^\theta - 1) = \epsilon. \tag{44}$$

When θ is small ϵ is also small, and as a first approximation $\delta \simeq \epsilon$. It is possible to expand δ as a power series in ϵ, for sufficiently small ϵ, and substituting for ϵ in terms of θ from (41) to obtain an expansion as a power series in θ. (The justification for this procedure follows along the usual lines; see, for example, (5).) The important point to notice is that for large ν and small δ only the residue near $q = \delta$ need be considered. Hence

$$m_\theta(\nu) \sim \frac{e^{-\beta}(\beta+\delta)}{\delta+\beta e^{-(\beta+\delta)}} \frac{e^{\nu\delta}}{e^{-\theta}-\beta e^{-(\beta+\delta)}\,(e^{-\theta}-1)}. \tag{45}$$

To determine the behaviour of the distribution for large ν we form the cumulant generating function

$$K_\theta(\nu) = \log[\zeta(\nu)+m_\theta(\nu)] \sim \log m_\theta(\nu) \sim \nu\delta + 0(1), \tag{46}$$

for large ν, by (41). The first three terms in the expansion of δ are $\delta = \epsilon - \epsilon^2 + \frac{3}{2}\epsilon^3$ and, substituting from (44) and using (46), we obtain

$$\kappa_1 \sim \nu\beta e^{-\beta}, \quad \kappa_2 \sim \nu[\beta e^{-\beta} - 2\beta^2 e^{-2\beta}], \tag{47}$$

in agreement with (34) and (36); also

$$\kappa_3 \sim \nu[\beta e^{-\beta} - 6\beta^2 e^{-2\beta} + 9\beta^3 e^{-3\beta}], \tag{48}$$

and, similarly, all higher cumulants are of order ν. Hence, if we take $\sqrt{\nu}$ as unit, the distribution tends to normal about κ_1 with mean square deviation $\beta e^{-\beta} - 2\beta^2 e^{-2\beta}$.

11. *Moments of the $W(x, y)$ distribution.* Equation (10) can be used to determine the moments of the $W(x, y)$ distribution, but a certain amount of algebraic manipulation of Laplace transforms is necessary, since we need the transform of $W(x, y)$ with respect to x, and not of $W(x, x+z)$ as previously. Let us write

$$v(x, y) = W(x, y) - z(y)\,\delta(y - x), \tag{49}$$

and suppose that $$v(x, y) \doteqdot V(p, y).$$

We use the formula from Laplace transform theory (6) that if

$$f(x) \doteqdot F(p),$$

then $$x^r f(x) \doteqdot (-1)^r p \left(\frac{d}{dp}\right)^r \left[\frac{F(p)}{p}\right] \quad (r = 0, 1, 2, \ldots). \tag{50}$$

We have, from (10),

$$F(p, z) = Ce^{-\lambda z}\delta(z) + (A + BC) \sum_{r=0}^{\infty} \frac{(B-\lambda)^r z^r}{r!}. \tag{51}$$

Expanding $z^r = (y-x)^r$, and using (50), we easily see that

$$V(p, y) = p \sum_{r=0}^{\infty} \left[y^r + {}^rC_1 y^{r-1} \frac{d}{dp} + \ldots + {}^rC_s y^{r-s} \left(\frac{d}{dp}\right)^s + \ldots + \left(\frac{d}{dp}\right)^r\right] \frac{(B-\lambda)^r}{r!} \frac{(A+BC)}{p}. \tag{52}$$

It will be convenient, for the purpose of the algebra, to introduce the operators

$$g \equiv \frac{d}{dp} \quad \text{and} \quad h \equiv \frac{1}{p}.$$

Then $$\int_0^x x^k v(x, y)\,dx \doteqdot (-1)^k g^k h V(p, y) \quad (k = 0, 1, 2, \ldots)$$

$$= (-1)^k \sum_{r=0}^{\infty} [y^r g^k + {}^rC_1 y^{r-1} g^{k+1} + \ldots + {}^rC_s y^{r-s} g^{k+s} + \ldots + g^{k+r}] \frac{(B-\lambda)^r}{r!} \frac{(A+BC)}{p}. \tag{53}$$

Now let $$\phi^{(k)}(y) = \int_0^y x^k v(x, y)\,dx,$$

and consider the Laplace transform in y of $\phi^{(k)}(y)$. We may apply the formula (50) to (53) and obtain

$$\phi^{(k)}(y) \doteqdot (-1)^k p \sum_{r=0}^{\infty} (-1)^r [g^r h g^k - {}^rC_1 g^{r-1} h g^{k+1} + \ldots$$
$$+ (-1)^s\, {}^rC_s g^{r-s} h g^{k+s} + \ldots + (-1)^r h g^{k+r}] \frac{(B-\lambda)^r}{r!} \frac{(A+BC)}{p}. \tag{54}$$

In order to simplify (54) we must reduce the operator

$$g^r h - {}^rC_1 g^{r-1} h g + \ldots + (-1)^s\, {}^rC_s g^{r-s} h g^s + \ldots + (-1)^r h g^r. \tag{55}$$

From the formula $$hg - gh = h^2, \tag{56}$$

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it is not difficult to show that $$h^r g - g h^r = r h^{r+1}, \tag{57}$$

$$g^r h = h g^r - r h^2 g^{r-1} + \ldots + (-1)^s r(r-1)\ldots(r-s+1) h^{s+1} g^{r-s} + \ldots + (-1)^r r!\, h^{r+1}, \tag{58}$$

and hence that (55) reduces to the single term $(-1)^r r!\, h^{r+1}$. On substituting in (54) we obtain

$$\phi^{(k)}(y) \doteqdot (-1)^k \sum_{r=0}^{\infty} h^r g^k \frac{(B-\lambda)^r (A+BC)}{p}. \tag{59}$$

We easily see that $$\phi^{(0)}(y) \doteqdot \frac{A+BC}{p+\lambda-B},$$

and verify that $$\phi^{(0)}(y) + z(y) \doteqdot C + \frac{A+BC}{p+\lambda-B} = 1, \tag{60}$$

by (9).

To evaluate $\phi^{(k)}(y)$ from (58) we use the relation

$$h^r g^k = g^k h^r + {}^kC_1 r g^{k-1} h^{r+1} + \ldots + {}^kC_s r(r+1)\ldots(r+s-1) g^{k-s} h^{r+s} + \ldots + r(r+1)\ldots(r+k-1) h^{r+k}, \tag{61}$$

which can easily be deduced from (5). Then, substituting in (59), and summing with respect to r, we have

$$\phi^{(k)}(y) \doteqdot (-1)^k \left[g^k \left(\frac{A+BC}{p+\lambda-B} \right) + \sum_{s=1}^{k} \frac{k!}{(k-s)!} g^{k-s} \frac{(B-\lambda)}{p} \frac{(A+BC)}{(p+\lambda-B)^{s+1}} \right]. \tag{62}$$

The kth moment of the distribution $W(x,y)$, $\mu^{(k)}(y)$, is given by

$$\begin{aligned}\mu^{(k)}(y) &= \phi^{(k)}(y) + y^k z(y) \\ &\doteqdot (-1)^k \left[p g^k \left(\frac{C}{p}\right) + g^k \left(\frac{A+BC}{p+\lambda-B} \right) + \sum_{s=1}^{k} \frac{k!}{(k-s)!} g^{k-s} \frac{(B-\lambda)}{p} \frac{(A+BC)}{(p+\lambda-B)^{s+1}} \right],\end{aligned}$$

and, by (9) and (60), this can be simplified to

$$\mu^{(k)}(y) \doteqdot (1-e^{-\lambda\alpha}) \frac{k!}{p^k} + (-1)^{k+1} \sum_{s=2}^{k} \frac{k!\, \lambda e^{-\lambda\alpha}}{(k-s)!} g^{k-s} \frac{[1-e^{-\alpha(p+\lambda)}][p+\lambda e^{-\alpha(p+\lambda)}]^{s-2}}{p^s (p+\lambda)^{s-1}}. \tag{63}$$

When $k = 1$ only the first term must be taken into account. We thus deduce that

$$\mu^{(1)}(y) = y(1-e^{-\lambda\alpha}), \tag{64}$$

$$\mu^{(2)}(y) = y^2(1-e^{-\lambda\alpha}) - e^{-\lambda\alpha}(y^2 - 2y/\lambda + 2/\lambda^2) + e^{-2\lambda\alpha}[(y-\alpha)^2 - 2(y-\alpha)/\lambda + 2/\lambda^2]. \tag{65}$$

The last term in (65) is only to be taken into account when $y > \alpha$. Higher moments can be evaluated similarly from (63), if required, but they become rather cumbersome.

The moments of the distribution $W_n(x,y)$, for n given events, can be dealt with by the same method as used for the gap distribution, that is by expanding $\mu^{(k)}(y)\, e^{\lambda(y+\alpha)}$ in powers of λ. We easily obtain from (64) and (65),

$$\mu_n^{(1)}(y) = y\left[1 - \left(\frac{y}{y+\alpha}\right)^n\right], \tag{66}$$

$$\mu_n^{(2)}(y) = y^2\left[1 - \left(\frac{y}{y+\alpha}\right)^n\right] - \frac{n}{n+2} \frac{y^{n+2}}{(y+\alpha)^n} + \frac{n}{n+2} \frac{(y-\alpha)^{n+2}}{(y+\alpha)^n}. \tag{67}$$

The last two results are in agreement with Robbins(3), if we replace his p by $\alpha/(y+\alpha)$.

Finally we consider the problem of n intervals on a circle, and the resulting distribution, $W_n'(x,y)\, dx$ representing the probability that the covered portion of the circle of circumference y lies between x and $x+dx$. By an argument similar to that of § 7 we deduce that

$$W_n'(x,y) = W_{n-1}(x-\alpha, y-\alpha),$$

and hence that

$$\mu_n^{(k)\prime}(y) = \int_\alpha^y x^k W_{n-1}(x-\alpha, y-\alpha)\,dx = \int_0^{y-\alpha} (u+\alpha)^k W_{n-1}(u, y-\alpha)\,du$$

$$= \mu_{n-1}^{(k)}(y-\alpha) + {}^kC_1\,\alpha\mu_{n-1}^{(k-1)}(y-\alpha) + \dots + {}^kC_s\,\alpha^s\mu_{n-1}^{(k-s)}(y-\alpha) + \dots + \alpha^k. \quad (68)$$

As particular examples of (68)

$$\mu_n^{(1)\prime}(y) = y\left[1-\left(\frac{y-\alpha}{y}\right)^n\right], \quad (69)$$

$$\mu_n^{(2)\prime}(y) = y^2\left[1-\left(\frac{y-\alpha}{y}\right)^{n-1}\right] + \alpha^2\left(\frac{y-\alpha}{y}\right)^{n-1} - \frac{n-1}{n+1}\frac{(y-\alpha)^{n+1}-(y-2\alpha)^{n+1}}{y^{n-1}}. \quad (70)$$

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