SYMMETRIES, RATIONAL SPHERE MAPS, AND COMPLETE POLYNOMIAL SEQUENCES

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Abstract. We consider the set S of possible target dimensions for rational sphere maps whose Hermitian-invariant group is the unitary group. In each source dimension, we show that S is co-finite by applying a classical theorem of Ron Graham on complete polynomial sequences. We establish several results, some computer assisted, finding the largest exceptional value. We close by posing a purely number-theoretic question about these exceptional values.

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1. Introduction

The unit sphere in $\mathbb{C}^n$ is certainly one of the most symmetric CR manifolds, as automorphisms of the unit ball extend to be CR automorphisms of the sphere. With Ming Xiao in [5] and [6], the author introduced and studied the notion of the Hermitian invariant group associated with a proper holomorphic mapping between balls, and the author further developed this matter in [2] and [3]. One of the results from [5] characterizes those rational sphere maps f with source dimension n whose Hermitian invariant group Γ_f is the unitary group $\mathbf{U}(n)$. Such a map must be either a tensor product $z^{\otimes m}$ for $m \geq 2$, or an orthogonal sum of such maps (with at least two summands) and of degree at least 2. To be explicit, we write the tensor product map as

$$z^{\otimes m} = H_m(z) = (\dots, c_\alpha z^\alpha, \dots) \tag{1}$$

where $|c_\alpha|^2 = \binom{m}{\alpha}$ are the multinomial coefficients. The ordering of the components will not concern us here.

The situation when $f = H_0$ is a constant map is not interesting. When $f = H_1$, it is an automorphism and the group Γ_f is the full automorphism group $Aut(\mathbb{B}_n)$ of the unit ball. An orthogonal sum that is of degree 1 is in fact spherically equivalent to the map $z \mapsto (z, 0)$. It was proved in [5] that intermediate groups between the unitary group $\mathbf{U}(n)$ and $Aut(\mathbb{B}_n)$ do not arise. Hence it is natural to analyze those maps f for which $\Gamma_f = \mathbf{U}(n)$. In Section 8 we briefly discuss the situation for subgroups of $\mathbf{U}(n)$ that contain the torus $\mathbf{U}(1) \times \cdots \times \mathbf{U}(1)$.

The main issue in this paper concerns the possible (minimum) target dimensions of non-trivial orthogonal sums of tensor powers that are of degree at least 2. We can write

$$f = \oplus \sum_{j=0}^{d} \lambda_j H_j \tag{2}$$

for complex numbers λ_j with $\sum_{j=0}^d |\lambda_j|^2 = 1$. The symbol $\oplus$ indicates that each H_j maps into a subspace of the target that is orthogonal to the other images.

In studying the possible target dimensions of such maps, a surprising connection with combinatorial number theory arises. In each source dimension n at least 3, there is a set of gaps in the possible target dimensions. The collection of possible target dimensions depends on whether one allows the term H_0. When one includes this term, and the source dimension is at most 3, this collection turns out be be something well understood. Because of a lemma about spherical equivalence, however, CR geometry suggests disallowing the term H_0 from (2), and further restrictions on the collection of target dimensions arise. Using a classical result of Ron Graham ([7]), we establish the following result.

Theorem 1.1. *Let S be the collection of minimal target dimensions of polynomial sphere maps in source dimension n whose Hermitian-invariant group Γ_f is the unitary group $\mathbf{U}(n)$. Then S is a co-finite subset of the set of integers N with $N > n$. For $n \geq 3$, the complement is non-empty.*

We call the elements of the complement *exceptional values*. They depend in a very complicated way on both the source dimension and whether we assume $f(0) = 0$. To illustrate this theorem, consider source dimension 3. The possible minimal target dimension for a polynomial sphere map for which $\Gamma_f = \mathbf{U}(3)$ is any positive integer **not** in the set $\{2, 5, 8, 12, 23, 33\}$. If we make the natural assumption that $f(0) = 0$ (see Section 1), then additional exceptional values arise; the set of them remains finite, and the maximum exceptional value is 50. The set of these values is $\{1, 2, 4, 5, 7, 8, 11, 12, 14, 17, 20, 22, 23, 26, 29, 32, 33, 35, 50\}$.

Let p be a polynomial in one variable with rational coefficients that maps the integers to the integers. The problem described above reduces to asking whether the sequence $\{p(1), p(2), \dots\}$ is what is called *complete*. In this context the term means the following:

Definition 1.1. The polynomial sequence $\{p(1), p(2), \dots\}$ is called *complete* if the set S of sums of distinct elements of the sequence is co-finite. In other words, there is an integer x_0 such that $y \geq x_0$ implies

$$y = \sum_j c[j]p(j),$$

where each c_j is either 0 or 1, and the sum is finite.

Let S denote the set of sums of distinct values of a polynomial p, which we assume has rational coefficients. In the paper [7] from 1964, R. Graham gave a necessary and sufficient condition on p for S to be co-finite. This condition is simple to state. Write the polynomial in terms of the basis of falling factorials:

$$p(x) = \sum_{j=0}^d a_j \binom{x+j}{j}.$$

Assume that each coefficient a_j is reduced to lowest terms. Then S is co-finite if and only if the greatest common divisor of the collection of numerators is 1 and the coefficient of the highest order term is positive. In particular, the polynomial $x \mapsto \binom{x+n-1}{n-1}$ is complete. On the other hand, the polynomial x^2+x is **not** complete.

See for example [1] and its references for additional (and newer) results about complete sets of natural numbers. See also [10] for conditions on sequences guaranteeing completeness. The paper [4] from 1974 gives a computer assisted proof that 12758 is the largest exceptional value when the polynomial is x^3. The author has verified this number and proved similar statements using Mathematica.

We formulate the main questions as follows.

Main questions. Suppose that f is a sphere map of the form (2) with degree at least 2. What (minimal) target dimensions are possible for f? How does this collection change when we assume that $\lambda_0 = 0$?

Although it is a combinatorial nightmare to determine precisely which gaps in target dimension can arise and how things depend on the degree, Theorem 1.1 shows that the number of such gaps is finite. Our methods can be used to find or estimate the largest exceptional value, although we do so in only a few special cases. We close the paper with a natural question along these lines. For linear polynomials, we determine S exactly. Although this result is surely known, we include a proof.

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2. A lemma about spherical equivalence

The purpose of this section is to prove a simple result, known to many people in CR Geometry, that will enable us to assume that the map preserves the origin.

The following lemma holds for any proper mapping of the given form, without assuming rationality. In our context, the map f is a monomial mapping, and the map G is also a monomial mapping with $G(0) = 0$.

Lemma 2.1. *Let $f = g \oplus a$ be a proper mapping between balls, with $g(0) = 0$. Then there is a target automorphism ϕ such that*

$$F = \phi \circ f = (\frac{g}{\sqrt{1 - \|a\|^2}}) \oplus 0 = G \oplus 0$$

is a proper mapping between balls with $F(0) = 0$. In particular, f is spherically equivalent to a map $G \oplus 0$ with $G(0) = 0$.

Proof. Recall that automorphisms of the ball are linear fractional transformations of the form

$$\phi_{\mathbf{a}}(z) = U \ \frac{\mathbf{a} - L_{\mathbf{a}} z}{1 - \langle z, \mathbf{a} \rangle} \tag{3}$$

where U is unitary, $\|\mathbf{a}\| < \mathbf{1}$, and $L_{\mathbf{a}}$ is a certain linear map. The linear map $L_{\mathbf{a}}$ satisfies

$$L_{\mathbf{a}}(z) = \frac{\langle z, \mathbf{a} \rangle \mathbf{a}}{s+1} + sz,$$

where $|s|^2 = 1 - \|\mathbf{a}\|^2$. Given f as in the statement of the lemma, we define $\mathbf{a}$ to be $(0, a)$. Then $\langle f, a \rangle = \|\mathbf{a}\|^{\mathbf{2}}$. Letting U be the identity, and computing $\phi_{\mathbf{a}} \circ f$ gives the conclusion of the lemma. $\square$

This lemma will be used in the following simple setting. Suppose λ is a constant and $f = \lambda \oplus g$ is a polynomial sphere map with $g(0) = 0$. Then f is spherically equivalent to a polynomial sphere map preserving the origin. Hence, when we consider an orthogonal sum of tensor products as in

$$\oplus \sum_{j=0}^{d} \lambda_j H_j,$$

the term $\lambda_0 H_0$ can be eliminated if we study maps of the form

$$\oplus \sum_{j=1}^{d} \mu_j H_j$$

and we identify $0 \oplus G$ with G. This identification lowers the target dimension. We remark that the target group defined in [5] provides a criterion for understanding this identification. Let I denote the identity operator on $\mathbb{C}^N$. The unitary map $U = e^{i\theta} \oplus I$ in $N+1$ variables has the invariance property $U \circ (0 \oplus G) = 0 \oplus G$ and thus the target group contains a group isomorphic to the unit circle.

Suppose that f is an orthogonal sum of tensor products as in (2). By Lemma 2.1, it is spherically equivalent to a map of the form $G \oplus 0$, where $G(0) = 0$. We say that a sphere map has *minimal target dimension* N (or, *is minimal*) if the image of the map lies in a subspace of dimension N but not in any smaller dimensional space. Thus the map G is minimal, while the map f is not.

3. Graham's theorem

Let us first remark that it is often useful to express a polynomial $p(x)$ in terms of falling factorials rather than in terms of powers of x. Since the expressions

$$\binom{x}{r} = \frac{x(x-1)\cdots(x-r+1)}{r!}$$

are polynomials of different degrees in x, we can write, for any polynomial p, the expression

$$p(x) = \sum_{j=0}^{d} \alpha_j \binom{x}{j}. \tag{4}$$

It is a standard fact, for example, that a polynomial maps $\mathbb{Z}$ to $\mathbb{Z}$ if and only all these coefficients are integers. The dominant term for large x is $\binom{x}{d}$.

The theorem of Graham states the following:

Theorem 3.1. *Let p be a polynomial p of degree d with rational coefficients each reduced to lowest terms. Let P be the set of values $\{p(1), p(2), \dots\}$ and $S = S(P)$ the set of sums of distinct elements of P. Then S is co-finite if and only if the coefficient of $\binom{x}{d}$ is positive and the collection of numerators (of the coefficients) are relatively prime.*

The necessity of the condition on relative primality is obvious, but less us nonetheless give a simple example. The set S for the polynomial $p(x) = x^2 + x + 1$ is co-finite, but S is **not** co-finite for $q(x) = x^2 + x$, because the values of $q(n)$ for integers n is always even. This simple example shows that more than the highest order term matters in deciding completeness.

In our context, the only polynomials that arise are precisely those of the form $p(x) = \binom{x}{n-2}$ where n is the source dimension. For such polynomials the set $S(P)$ is co-finite, an immediate consequence of the theorem of Graham. In order to better understand the set of exceptional values, we will provide proofs in special cases. The first result and subsequent example are elementary but illuminating.

Proposition 3.1. *Let m and b be relatively prime with $m \geq 2$ and $0 < b < m$. Let $T_N = Nm + b$ for a natural number $N \geq 0$. Let S denote the set of sums of distinct elements of the T_N. Then the complement of S is finite. Furthermore, the largest number that is not in S is*

$$B = m(b-1) + \frac{m^2(m-1)}{2}. \tag{5}$$

Proof. First we make a simple remark. If $N \in S$, then $N + sm \in S$ for any non-negative integer s. To see why, it suffices to show this statement for $s = 1$. To verify for $s = 1$, replace the largest term $mt + b$ that arose in the sum with $m(t+1) + b$. As a consequence of the remark, if we obtain m consecutive numbers in S, then all larger numbers are in S. The idea of the proof is then to show that the number B in (5) is the largest exceptional value that is a multiple of m and show that each of the numbers $B+1, B+2, \ldots, B+m-1$ is in S.

Given a number K, for $1 \leq r \leq m$ we write

$$K = ma + r.$$

We then consider numbers $\sum c_j(mj + b)$ where each c_j is 0 or 1, and try to write

$$K = ma + r = \sum_{j=0}^{L-1} c_j(mj+b) = m\sum_{j=0}^{L-1} jc_j + b\sum_{j=0}^{L-1} c_j \tag{6}$$

When $r < m$, the smallest member of S congruent to r arises by satisfying formula (6) when all the $c_j = 1$ and L is minimal. To do so, we must choose L (the number of non-zero c_j) such that Lb is congruent to r mod (m). Since m and b are relatively prime, we can always make this choice. Thus $L = b^{-1}r$ modulo (m).

When $r = 0$ we put $L = m$. Otherwise we choose the smallest such L. Then, given r, the following number is in S:

$$m\sum_{j=0}^{L-1} j + Lb. \tag{7}$$

It follows by the remark above that adding nonnegative multiples of m to the value in (7) keeps us in S. Thus we can describe S precisely. For each residue class mod (m) we obtain the smallest member of S by this procedure. Then we add multiples of m to obtain S.

We next establish (5). We note that the expression in (5) is a multiple of m. We try to write Km as a sum of distinct numbers of the form $jm + b$. Thus, for c_j equal to 0 or 1 we have

$$N = Km = \sum_{j=0}^{L-1} c_j(jm+b) = m\sum_{j=0}^{L-1} c_j j + b\sum_{j=0}^{L-1} c_j. \tag{8}$$

By (8), and because b is relatively prime to m, the number of terms (the number of non-zero c_j) must be a multiple of m. Therefore, using all $c_j = 1$, we obtain

$$N \geq \sum_{j=0}^{m-1}(mj+b) = m\frac{(m-1)m}{2} + mb > \frac{m^2(m-1)}{2} + m(b-1) = B. \tag{9}$$

Thus B is an exceptional value. Observe that the number $m\frac{(m-1)m}{2} + mb$ is in S. Each number strictly between B and $B+m$ is in S because each such number is at least as large as the smallest number in its congruence class that is obtained by the above procedure. $\square$

Example 3.1. Put $m = 11$ and $b = 3$. We write 301 as a combination of the numbers $11k+3$. Since 301 is congruent to 4 mod (11), and $5*3$ is congruent to 4, we require 5 terms. The smallest member of S that is congruent to 4 is therefore

$$3 + 14 + 25 + 36 + 47 = 125.$$

Replacing 47 with something of the form $k*(11)+3$ yields

$$301 = 3 + 14 + 25 + 36 + ((20)*(11) + 3)\,.$$

On the other hand, 299 is not in S. Note that 299 is congruent to 2, and $b^{-1}2 = 8$. The smallest member of S that is congruent to 2 is thus the sum of the first 8 terms, namely $3+14+25+36+47+58+69+80 = 332$.

The largest number not in S is 627. To illustrate the last part of the proof, we show that $628 \in S$. Since 628 is congruent to 1 mod (11), and $3^{-1} = 4$, the minimum value in S congruent to 4 is $3+14+25+36 = 78$. Thus we have

$$628 = 3 + 14 + 25 + 36 + (50)*(11) = 3 + 14 + 25 + (53)*(11) + 3 \in S.$$

For polynomials of higher degree it is more difficult to find the maximum exceptional value. We consider the polynomial $p(x) = x^2 + x + 1$.

Proposition 3.2. *Put* $p(x) = x^2+x+1$. *Then* 106 *is* **not** *in* $S(p)$, *but if* $y \geq 107$, *then* $y \in S(p)$.

Proof. The tedious verification that 106 fails is easily done on Mathematica. We give a simple sketch. The allowable values are

$$\{3, 7, 13, 21, 31, 43, 57, 73, 91, \dots\}.$$

We try to write 106 as a distinct sum of these numbers. If we use 91, then we would need to be able to write 15 as such a sum, and doing so is obviously impossible. If the largest number used is 73, we would need to write 33 as such a sum, and doing so is obviously impossible. If the largest number used is 57 or 43, we again fail. If the largest number used is 31, then the sum is at most 75, and again we fail.

We next use Mathematica to check that $x \in S$ for $107 \leq x \leq 421$.

Suppose that we know $107 \leq x < y$ implies $x \in S$. We wish to show that $y \in S$. Since the sequence $p(n)$ is increasing there is a unique N such that

$$p(N) < y \leq p(N+1).$$

If $y = p(N+1)$, then $y \in S$, and hence we may assume that $p(N) < y < p(N+1)$. We consider the number $y - p(N-3)$. If we can show the following two things, then y must be in S:

- $y - p(N-3) \geq 107$
- $y - p(N-3) < p(N-3)$

The first item holds by our assumption that y is the smallest possible exceptional value at least 107, and the second item holds because this inequality precludes using $p(N-3)$ in the sum. Since

$$y - p(N-3) = y - p(N) + p(N) - p(N-3) > p(N) - p(N-3) = 6N - 6,$$

the first item holds if $6N-6 \geq 107$ and hence holds if $N \geq 19$. Since $y < p(N+1)$, the second item holds if $p(N+1) < 2p(N-3)$. Plugging in the values, we see that the second item holds if

$$(N+1)^2 + N + 1 + 1 < 2\left((N-3)^2 + (N-3) + 1\right),$$

which is equivalent to $0 < N^2 - 13N + 11$ and hence holds if $N \geq 13$. Therefore, if $N \geq 19$, then $y \in S$. Since $p(20) = 421$ and we have verified the result from 107 up to 421, we conclude the result for all y at least 107. $\square$

The proof of this proposition generalizes significantly. We find an interval of good values using a computer. We imagine the next smallest exceptional value. We show that it cannot exist if we choose N sufficiently large.

4. Binomial coefficients and source dimensions 1 and 2

We return to the situation for polynomial sphere maps. We require another well-known piece of information. The dimension of the space of homogeneous polynomials of degree m in n variables is $\binom{m+n-1}{n-1} = \binom{m+n-1}{m}$. As a result, the target dimension of the map H_m from (1) equals this number. Furthermore, the target dimension of the map in (2) is

$$\sum_{j=0}^{d} c_j \binom{j+n-1}{n-1}, \tag{10}$$

where each c_j is either 0 or 1.

The case of source dimension 1 is completely trivial. Since each map H_j has target dimension 1, and we are taking orthogonal sums, each target dimension from 1 to the degree d is possible, and each such number arises. Thus the set of values $N(1,d)$ is the set of possible numbers of summands, and hence is the set $\{1, 2, \ldots, d\}$. This result holds whether or not we allow H_0.

The case of source dimension 2 is also somewhat trivial. When the source dimension is 2, the target dimension of the map H_m is $m+1$. Since we are assuming that the degree is at least 2, the minimum possible N is 3. The set of possible values of N for maps of degree at most d is the set $\{3, 4, \ldots, \frac{(d+1)(d+2)}{2}\}$. If we assume that f is of degree d, then the minimum possible N is $d+1$.

We conclude the following simple result.

Proposition 4.1. *Assume the source dimension is* 2. *Let f be an orthogonal sum, of degree d with $d \geq 2$. The set of possible target dimensions for f is the integer interval $[d+1, \binom{(d+1)(d+2)}{2}]$. If we disallow the constant map H_0, then we must exclude $d+2$ and $\binom{(d+1)(d+2)}{2} - 1$ from this interval. If we allow the degree to be arbitrary (but at least 2), then for each $N \geq 3$ there is a polynomial map p with group $\Gamma_p = \mathbf{U}(2)$ and target dimension N.*

Thus all target dimensions at least 3 are possible for maps with group $\mathbf{U}(2)$.

5. Source dimension 3

The case of source dimension 3 is particularly interesting. The crucial point is that the target dimension of H_m becomes the triangular number $\binom{m+2}{2}$. Numbers that are sums of distinct triangular numbers have been studied. We have the following result. The six numbers in the exceptional set are sequence A053614 in the On-Line Encyclopedia of Integer Sequences. [9]

Theorem 5.1. *Suppose that N is a sum of distinct triangular numbers. Then N cannot be in the set*

$$\{2, 5, 8, 12, 23, 33\}$$

but all other numbers are possible.

The following related result is perhaps new, and hence we will provide a proof. The finiteness follows immediately from the theorem of Graham, but we provide an alternative proof because we want to show that 50 is the largest exceptional value. Recall by Lemma 2.1 that we are dropping one dimension by excluding the number 1 as a valid triangular number.

Theorem 5.2. *Suppose that N is a sum of distinct triangular numbers, but that 1 is not included in the list of triangular numbers. Then N cannot be in the set*

$$S' = \{1, 2, 4, 5, 7, 8, 11, 12, 14, 17, 20, 22, 23, 26, 29, 32, 33, 35, 50\}$$

but all other numbers are possible.

Proof. Let S denote the set of allowable numbers. We will show that its complement is the set S' in the statement. Suppose $1 \leq m \leq 190$. First, using Mathematica, we consider solving the equation

$$m = \sum_{j=2}^{K} c_j \binom{j+1}{2} \tag{11}$$

where each c_j is 0 or 1. We discover that there are no solutions to (11) when $m \in S'$ and that are solutions for all other m up to 190. Now we suppose that $51 \leq x < y$ and that we have shown that $x \in S$. We want to prove that $y \in S$. We do so as follows. Given y, there is a unique N such that $\binom{N+1}{2} < y \leq \binom{N+2}{2}$. If $y = \binom{N+2}{2}$, then $y \in S$ and we may assume that

$$\binom{N+1}{2} < y < \binom{N+2}{2}.$$

We consider $y - \binom{N-2}{2}$. Then we have

$$y - \binom{N-2}{2} = y - \binom{N+1}{2} + \binom{N+1}{2} - \binom{N-2}{2} > \binom{N+1}{2} - \binom{N-2}{2} = 3N - 3.$$

Thus if $3N - 3 \geq 51$ and we do not require the use of $\binom{N-2}{2}$ in the sum, then $y \in S$. To guarantee that we don't use $\binom{N-2}{2}$, it suffices to show that

$$y - \binom{N-2}{2} < \binom{N-2}{2}. \tag{12}$$

But (12) is equivalent to $y < 2\binom{N-2}{2}$, which will follow if we show that

$$\binom{N+2}{2} \leq 2\binom{N-2}{2} \tag{13}$$

But (13) is equivalent to $0 < N^2 - 13N + 10$,which holds for $N \geq 13$. Hence, if $N \geq 18$, both conditions hold, and $y \in S$. Therefore we need to know that $x \in S$ for $51 \leq x \leq 190 = \binom{20}{2}$, and the conclusion follows. $\square$

Theorem 5.2 leads to the following corollary about rational sphere maps.

Theorem 5.3. *Let f be a polynomial sphere map with source dimension 3 whose Hermitian group Γ_f is the unitary group $\mathbf{U}(3)$. The following hold:*

1. *f is of degree d at least 2.*
2. *f is spherically equivalent to a map of the form*
$$\oplus \sum_{j=1}^{d} \lambda_j H_j \tag{14}$$
where $\sum_{j=1}^{d} |\lambda_j|^2 = 1$.
3. *The possible minimal target dimension N is any positive integer except*
$$\{1, 2, 4, 5, 7, 8, 11, 12, 14, 17, 20, 22, 23, 26, 29, 32, 33, 35, 50\}$$

Proof. The first statement comes from [5]. Furthermore f must be an orthogonal sum of tensor products. By Lemma 2.1, f is spherically equivalent to an orthogonal sum g of tensor powers with $g(0) = 0$. Therefore there are constants λ_j such that
$$g = \oplus \sum_{j=1}^{d} \lambda_j H_j.$$
Since the summands are orthogonal, the minimum target dimension is the sum of the target dimensions of the maps H_j for which $\lambda_j \neq 0$. Since $n = 3$, the target dimension of H_j is the triangular number $\binom{j+2}{2}$. Since the first term in the sum is when $j = 1$ and the degree is d, we see that the target dimension of g is a sum of distinct triangular numbers, where the number 1 is disallowed. The result therefore follows from Theorem 5.2. ☐

Remark 5.1. Instead of considering $y - \binom{N-2}{2}$ in the proof of Theorem 4.2, we could try $y - \binom{N-a}{2}$ for other integers a. We can do better if we choose $y - \binom{N-3}{2}$. We used $y - \binom{N-2}{2}$ because this number works also to prove the result that 33 is the largest exceptional value when we include 1 as a triangular number.

6. Source dimension 4

In source dimension 4 we are considering the sequence $\binom{j+3}{3}$ for $j \geq 1$. If we allow the term H_0 to occur, we would start with $j = 0$. In that case, the sequence of exceptional numbers appears in the on-line encyclopedia [9], and the maximum exceptional number is 558. By Lemma 2.1, however, we wish to study the exceptional values when we disallow the value when $j = 0$. We obtain the following result:

Theorem 6.1. *For each N with $N \geq 898$, there is a rational sphere map f with $f(0) = 0$, with source dimension 4, with minimum target dimension N, and with $\Gamma_f = \mathbf{U}(4)$. There is no such map when $N = 897$.*

Proof. By our discussion thus far, the collection S of minimum source dimension N is any number that can be written
$$\sum_{j \geq 1} c_j \binom{j+3}{j},$$

where each c_j is 0 or 1. By Graham's theorem S is co-finite. We claim that 897 is **not** in S. The values of the sequence are

$$4, 10, 20, 35, 56, 84, 120, 165, 220, 286, 364, 455, 560, 680, 816, \ldots$$

Since 897 is odd, we must use an odd number of terms where the values are odd. The only possibilities are that we include all of $35, 165, 455$ or that we include exactly one of these. If we use all three, then we must have $897-(35+165+455) = 242 \in S$. Trying 120 as a term doesn't work, but if we don't use it, then the sum must be at most $4 + 10 + 20 + 56 + 84 = 174$. We conclude that we must use exactly one of $35, 65, 455$. Similar reasoning, best done using Mathematica, shows that none of these three cases works. Thus 897 is in the complement S'.

Next, we verify on computer that there is no element of S' in the interval $[898, 2600]$. We then prove that this fact implies that there is no element in S' larger than 2600 as well. The idea of the proof is the same as in source dimension 3. We imagine the smallest possible counter-example y. Let us write T_N for $\binom{N+3}{3}$. There is a unique N for which we have $T_N < y < T_{N+1}$. We consider $y - T_{N-4}$. Following the same idea as in the previous proof we require

- $898 \leq T_N - T_{N-4} = 2(1 + N^2)$
- $T_{N+1} \leq 2T_{N+1}$, or $\frac{N^3-21N^2-4N-36}{6} \geq 0$.

Both statements hold if $N \geq 22$. Since $T_{23} = 2600$, and we have verified the result (by computer) in the interval $[898, 2600]$, the conclusion follows. □

7. Remarks on Theorem 1.1

Put $p(j) = \binom{n+j-1}{n-1}$. Then p is a polynomial of degree $n-1$. Our considerations thus far show that Theorem 1.1 follows immediately from knowing that the sequence of values of p is complete. The completeness for this particular polynomial follows immediately from Graham's result in [7], as p consists of a single term.

We summarize what we have done so far. There are restrictions on the possible minimal target dimensions for rational sphere maps with given properties. The restrictions arise as follows. There are $\binom{n+d-1}{d} = \binom{n+d-1}{n-1}$ linearly independent homogeneous polynomials of degree d in n variables. Therefore, for the special case of orthogonal sums of tensor powers, of total degree at most d, the possible minimum target dimension for such sums is given by

$$N = \sum_{j=0}^{d} a_j \binom{n+d-1}{n-1},$$

where each a_j is 0 or 1. Allowing d to vary we see that the collection of such allowable dimensions is precisely the set of numbers of the form

$$\sum_j a_j \binom{n+j-1}{n-1},$$

where again each a_j is 0 or 1. We have studied the special case when $n = 3$, but the ideas are same for higher dimensions. The difference is that the analogue of the triangular numbers is a sparser increasing sequence of natural numbers, but completeness follows immediately from Graham's result.

Consider source dimension 4. We are then considering distinct sums of numbers of the form $\binom{j+3}{3}$. If we allow the number 1 by starting with $j = 0$, then the largest exceptional value is 558. When we start with 4 (thus $j = 1$), then Theorem 6.1 shows that the largest exceptional value is 897.

We next provide an example concerning completeness where dropping the first term changes whether the sequence is complete.

Example 7.1. Put $T_N = 2^N$ for $N \geq 0$. Then, by binary expansion, $S = \mathbb{N}$ (allowing 0 as a possibility), and thus its complement is empty. Put $T_N = 2^N$ for $N \geq 1$. Then the complement of S is infinite, as it includes every odd number. Put $T_N = 3^N$ for $N \geq 0$. Then the complement of S is infinite, as no number of the form $3n + 2$ can be in S.

Remark 7.1. In order that the complement of S be finite, it is necessary that the greatest common divisor of all the elements of S be 1.

Remark 7.2. The proof we gave for Theorem 4.3 can be generalized. The idea is to find a long interval I of elements in S, to seek the next smallest number y **not** in S, and show that there is no such number. For a given polynomial p, for sufficiently large n we will have $p(N) < p(N + 1)$. Assume $p(N) < y < p(N + 1)$. We want $y - p(N-3) \in I$. We also require that $p(N+1) < 2p(N-3)$. If N is large enough to make both hold, then $p(N) < y < p(N + 1)$ implies that $y \in S$. Thus the highest order term seems to be playing the primary role. But the existence of the good interval depends on lower order terms. For example, the polynomial x^2 is complete but the polynomial $x^2 + x$ is not.

8. Return to sphere maps

Let $f : \mathbb{B}_n \to \mathbb{B}_N$ be a proper mapping between balls. Let $\mathrm{Aut}(\mathbb{B}_n)$ denote the group of holomorphic automorphisms of the unit ball. The Hermitian-invariant group Γ_f is defined to be the subgroup of $\mathrm{Aut}(\mathbb{B}_n)$ consisting of those automorphisms γ such that there exists a target automorphism ξ for which $f \circ \gamma = \xi \circ f$.

Let us briefly discuss why $\Gamma_f = \mathbf{U}(n)$ when $f = H_m$ for $m \geq 2$. Note that

$$\|H_m\|^2 = \|z\|^{2m} = (\|z\|^2)^m.$$

Hence when U is unitary, we have

$$\|H_m(U \circ z)\|^2 = (\|U \circ z\|^2)^m = \|z\|^{2m} = \|H_m(z)\|^2.$$

Therefore there exists a unitary map in the target such that

$$V \circ H_m \circ U = H_m.$$

Hence Γ_f contains $\mathbf{U}(n)$. See [5] for a proof that this containment is an equality.

Let us consider polynomial sphere maps $p : \mathbb{C}^n \to \mathbb{C}^N$ with $p(0) = 0$. Suppose that N is minimal. If the degree of p is 1, then p is a unitary map, and $N = n$. In this case $\Gamma_p = \mathrm{Aut}(\mathbb{B}_n)$. If the degree is at least 2, then $\Gamma_p \subset \mathbf{U}(n)$. By [5], $\Gamma_p = \mathbf{U}(n)$ if and only if p is an orthogonal sum of tensor products. We have analyzed the possible target dimensions for such maps in this paper. It is natural to ask what happens when we further weaken the amount of symmetry. For example, Γ_p contains the n-torus if and only p is a monomial map. The problem of determining the minimum target dimensions possible for monomial maps has not been completely solved. It is believed that the gaps in minimal target dimension for monomial maps are precisely the same as those for general rational sphere maps. See [8] for information on the gap conjecture in this particular case.

Remark 8.1. Consider groups of the form $G = \mathbf{U}(n_1) \times \mathbf{U}(n_2) \cdots \times \mathbf{U}(n_k)$ where $n = n_1 + \ldots n_k$. Using the ideas in this paper it is possible to consider the possible target dimensions of polynomial sphere maps f with $\Gamma_f = G$. We have done so in this paper when there is just one factor in this product. The generic monomial case is when $n_k = 1$ for each k and the group is thus the torus.

The author's book [2] includes considerable information about Hermitian-invariant groups for rational sphere maps. A summary appears on page 172. That summary includes a small omission, which we correct here. It states that "$\Gamma_f = \mathbf{U}(n)$ if and only if f is a juxtaposition of tensor powers". The requirement that the degree must be at least 2 is not stated. For clarity we provide the precise statement.

Let f be a polynomial sphere map. Then $\Gamma_f = \mathbf{U}(n)$ if and only if,

- $f = H_m = z^{\otimes m}$ for $m \geq 2$, or
- f is of degree at least 2 and f is an orthogonal sum of the H_j for $j \geq 0$.

Let g be a rational sphere map. Then Γ_g is a conjugate of $\mathbf{U}(n)$ (in the group-theoretic sense) if and only if g is spherically equivalent to such a polynomial.

9. An open problem

The author suspects that the following problem is both worthwhile and open.

Let m be a positive integer. Let $S_m(0)$ be the set of numbers that are sums of distinct values of the sequence $\binom{m+j}{j}$ beginning with $j = 0$. Define $S_m(1)$ similarly but begin with $j = 1$. Study the largest exceptional value in each case. (The largest integer not in $S_m(0)$ or $S_m(1)$.) Call these numbers $\lambda_0(m)$ and $\lambda_1(m)$. How does the ratio $\frac{\lambda_0(m)}{\lambda_1(m)}$ behave as $m \to \infty$?

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