On a perturbed Hofstadter Q-recursion

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Abstract

The Hofstadter Q-sequence is a prominent example of nested recurrence. Despite decades of study, it is not even known whether $Q(n)$ is defined for all n. Mantovanelli introduced a parity-perturbed variant $\widetilde{Q}$, obtained by adding $(-1)^n$ to the recursion, which surprisingly replaces the chaotic behaviour of Q by an exact dyadic self-similarity. In this paper we prove that $\widetilde{Q}$ is well-defined for all n and satisfies

$$\left|\frac{\widetilde{Q}(n)}{n} - \frac{1}{2}\right| = O\left(\frac{1}{\sqrt{\log n}}\right).$$

The proof exploits the self-similar structure of the sequence, where alternating arches arise whose frequency combinatorics are governed by the Catalan numbers. A complementary analysis of the arch amplitudes, conditional on two minimal conjectural properties, refines the asymptotic formula to

$$\limsup_{n\to\infty}\left|\frac{\widetilde{Q}(n)}{n} - \frac{1}{2}\right|\sqrt{\log_2 n} \;=\; \frac{1}{3\sqrt{2\pi}}.$$

Numerical experiments suggest the conjecture $Q(n) - \widetilde{Q}(n) = O(n/\sqrt{\log n})$, indicating that $\widetilde{Q}$ may serve as a tractable proxy for Q. This experimental direction will be investigated elsewhere.

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1 Introduction

The Hofstadter Q-sequence

The Hofstadter Q-sequence [8],

$$Q(n) = Q(n-Q(n-1)) + Q(n-Q(n-2)), \qquad Q(1) = Q(2) = 1 \tag{1}$$

(OEIS A005185), is one of the most celebrated examples of a nested recursion. Its defining feature is *self-reference*. The indices at which Q reads its own past depend on the sequence itself, creating a feedback loop that produces chaotic behaviour. Despite decades of study [10, 12, 3, 2, 6, 1] and recent progress on related Hofstadter-type sequences via automata and numeration systems [5, 11, 4, 14], virtually nothing has been proved rigorously about Q itself. It is not known whether $Q(n)$ is defined for all n, nor whether $Q(n)/n$ converges. The ratios $Q(n)/n$ fluctuate chaotically around $1/2$ (Figure 1, red).

The parity perturbation

In 2026, Mantovanelli [13] introduced the perturbed recursion

$$\widetilde{Q}(n) = \widetilde{Q}(n-\widetilde{Q}(n-1)) + \widetilde{Q}(n-\widetilde{Q}(n-2)) + (-1)^n, \qquad \widetilde{Q}(1) = \widetilde{Q}(2) = 1 \tag{2}$$

(OEIS A394051). The perturbation $(-1)^n$ has a powerful regularising effect. While Q is chaotic, $\widetilde{Q}$ exhibits exact dyadic self-similarity (Figure 1, blue). Mantovanelli observed the self-similarity numerically and conjectured $\widetilde{Q}(n)/n \to 1/2$. The present article proves this convergence, establishes the rate $O(1/\sqrt{\log n})$, and identifies (conditionally) the exact envelope constant. In particular, $\widetilde{Q}$ is well-defined for all n and satisfies $1 \le \widetilde{Q}(n) \le n$ (Corollary 4.14), in contrast with the original sequence Q where even this is unknown.

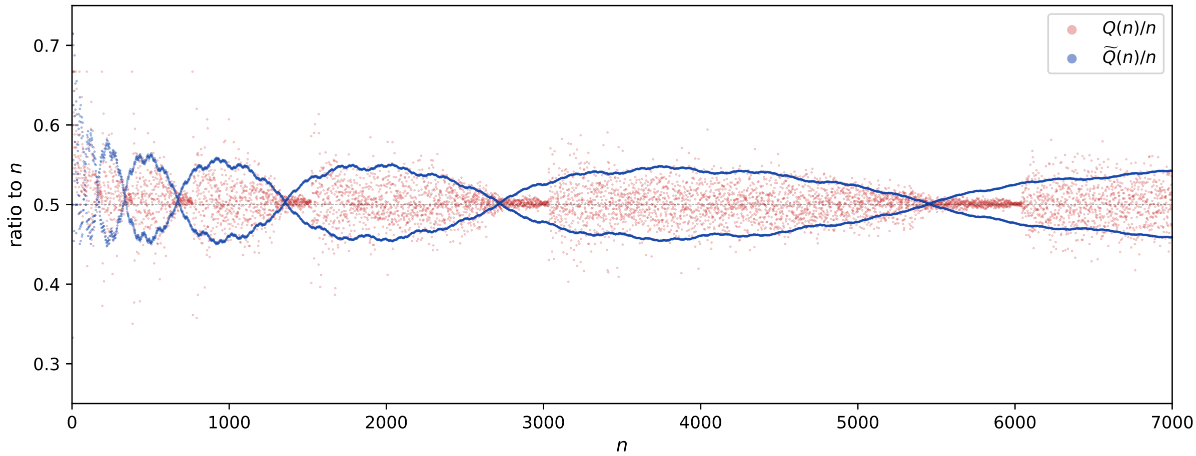

Figure 1: $Q(n)/n$ (red) and $\widetilde{Q}(n)/n$ (blue), $1 \le n \le 7000$.

The parity split

All values of $\widetilde{Q}$ are odd (by induction on n, since $\widetilde{Q}(n)$ is a sum of two past values plus $(-1)^n$, and odd + odd ± 1 is odd). Setting $R(n) := (\widetilde{Q}(n)+1)/2$ and separating by parity of the index gives two subsequences

$$A(m) := R(2m-1), \qquad B(m) := R(2m), \tag{3}$$

and the difference and symmetric modes

$$\sigma(m) := A(m) + B(m) - m, \qquad \delta(m) := B(m) - A(m). \tag{4}$$

Then

$$\widetilde{Q}(2m) = m - 1 + \sigma(m) + \delta(m), \qquad \widetilde{Q}(2m-1) = m - 1 + \sigma(m) - \delta(m). \tag{5}$$

Accordingly, Theorem 1.1 below follows once we prove

$$A(m) = \frac{m}{2} + o(m), \qquad B(m) = \frac{m}{2} + o(m),$$

or equivalently $\sigma(m) = o(m)$ and $\delta(m) = o(m)$. Throughout, $\Delta f(m) := f(m+1) - f(m)$ denotes the forward difference of a sequence f, and for any binary word W of length n,

$$h_W(t) := \#\{0 \le k < t : W[k] = 0\} - \#\{0 \le k < t : W[k] = 1\} \qquad (0 \le t \le n)$$

denotes the *prefix height function.* The arch decomposition controls δ, while the 1-Lipschitz analysis (Theorem 4.8) controls σ. The function δ oscillates between positive and negative values, and its zeros define a natural decomposition into alternating *arches* (Section 2).

Main result

Theorem 1.1. (a) (Unconditional.) $|\widetilde{Q}(n)/n - 1/2| = O(1/\sqrt{\log n})$.

(b) (Conditional on Observations 9.4 and 9.10.) *The amplitude increments* $W_r := V^+(r) - V^+(r-1)$ *satisfy* $W_r = \binom{2r+1}{r}$ *and*

$$\limsup_{n\to\infty} \left| \frac{\widetilde{Q}(n)}{n} - \frac{1}{2} \right| \sqrt{\log_2 n} \;=\; \frac{1}{3\sqrt{2\pi}}.$$

Part (b) depends on two unproved properties of the arch amplitudes: a record-claim property for the negative-arch height (Observation 9.4) and a run-count identification (Observation 9.10), both verified computationally for $r \le 6$ but not yet proved in general. These properties and their consequences are described in Section 9.

Proof strategy

The key observation is that the recursion (2) can be rewritten as a deterministic machine that reads two binary tapes alternately (Section 3). Within each arch, the machine produces a binary word P_r (the *step word*) that encodes the successive differences ΔA and ΔB. The step word at level $r+1$ is built from the step word at level r by an explicit interleaving operation, giving the sequence a self-similar structure.

This self-similarity is the engine of the proof. At each level, the interleaving operation adds a fixed positive contribution to the height of the new step word, which always exceeds the maximum negative contribution inherited from the previous level. By induction, every step word is a non-negative excursion, the differences ΔA and ΔB take only the values 0 and 1, and the step words are anti-palindromes (Section 4).

With these structural properties in hand, we count how often each integer v is visited by A and by B on the dyadic blocks $[4^k, 4^{k+1})$. The interleaving structure translates into a recursion on these frequency counts, which turns out to have a clean geometric solution (Section 7). The imbalance between the A-visits and the B-visits on each block is controlled by the central binomial coefficients $\binom{2k+2}{k+1}$, which grow as $O(4^k/\sqrt{k})$. Summing over blocks, the total number of m with $A(m) \le x$ is $2x + O(x/\sqrt{\log x})$. A standard monotone-inversion argument then recovers $A(m) = m/2 + O(m/\sqrt{\log m})$, and similarly for B, completing the proof.

The convergence rate $O(1/\sqrt{\log n})$ matches that of the Conway–Mallows sequence [10], but the mechanism is different. Conway's sequence is controlled by a single binary tree with central binomial coefficients $\binom{L}{\lfloor L/2 \rfloor}$. The Mantovanelli sequence is controlled by a family of Catalan-type coefficients $\binom{2r+1}{r}$. The article also develops a complementary analysis of the arch amplitudes (Sections 8–9), which identifies the exact envelope constant $1/(3\sqrt{2\pi})$ in part (b), and discusses perspectives on the original Hofstadter Q-sequence (Sections 11–12).

2 The arch decomposition

2.1 Exact recursions for A and B

Substituting $\widetilde{Q}(n) = 2R(n) - 1$ into (2) and separating odd and even indices yields two coupled recursions. On odd indices, $A(m)$ is determined by a "reading head" l_m that looks up a past value of B. On even indices, $B(m)$ is determined by a reading head j_m that looks up a past value of A. The exact recursions are

$$A(m) = B\big(m - B(m-1)\big) + B\big(m - A(m-1)\big) - 1, \qquad (R_A)$$

$$B(m) = A\big(m+1 - A(m)\big) + A\big(m+1 - B(m-1)\big). \qquad (R_B)$$

These identities are exact and hold for all $m \geq 2$. The global monotonicity and 1-Lipschitz property

$$\Delta A(m), \Delta B(m) \in \{0, 1\} \qquad (m \geq 1)$$

are proved later, after the arch skeleton has been established (Theorem 4.8). At this stage we use (R_A)–(R_B) only as exact recursions. When the 1-Lipschitz property holds, δ is a ± 1 lattice path.

2.2 First values

Table 1 gives the first values of A, B, and δ.

m	$A(m)$	$B(m)$	$\delta(m)$	ΔA	ΔB
1	1	1	0		
2	1	2	1	0	1
3	2	2	0	1	0
4	3	3	0	1	1
5	3	4	1	0	1
6	3	5	2	0	1
7	4	5	1	1	0
8	4	6	2	0	1
9	5	6	1	1	0
10	6	6	0	1	0
11	7	6	−1	1	0
12	8	7	−1	1	1

Table 1: First values of $A(m)$, $B(m)$, $\delta(m)$, and the step sequences ΔA, ΔB.

2.3 Zeros of δ and the arch structure

The zeros of δ partition $\mathbb{N}$ into alternating *positive arches* (where $\delta \geq 0$) and *negative arches* (where $\delta \leq 0$). Write $u_0 < v_0 < u_1 < v_1 < u_2 < \cdots$ for the successive zeros, where $[u_r, v_r]$ is the r-th positive arch and $[v_r, u_{r+1}]$ is the r-th negative arch. Figure 2 shows the first few arches.

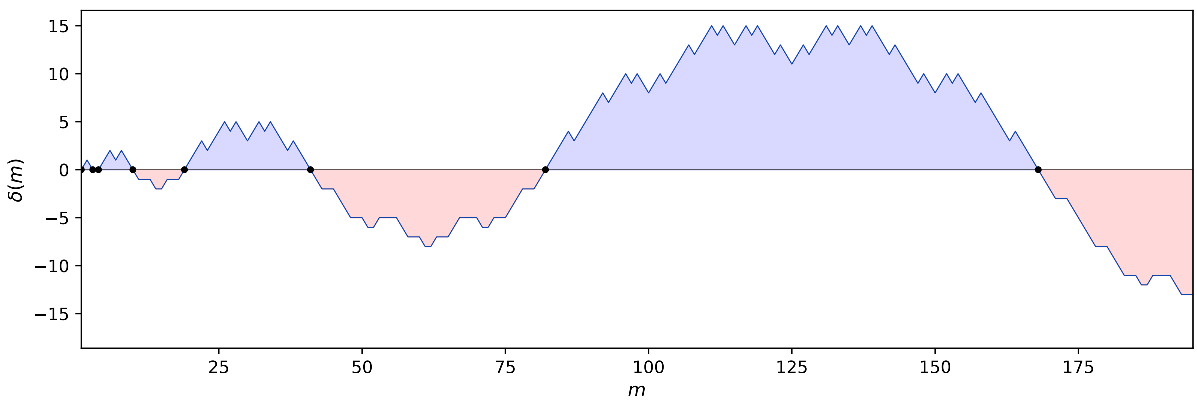

Figure 2: $\delta(m)$ for $1 \leq m \leq 195$. Positive arches (blue) alternate with negative arches (red). Black dots mark the zeros u_r and v_r.

Define the amplitudes

$$V^+(r) := \max_{u_r \leq m \leq v_r} \delta(m), \qquad V^-(r) := \max_{v_r \leq m \leq u_{r+1}} \big(-\delta(m)\big). \tag{6}$$

2.4 The arch skeleton

The zeros and amplitudes satisfy explicit closed forms. Define $a_0 := 3$ and $a_{r+1} := 4a_r - 1$, giving $a_r = (2 \cdot 4^{r+1} + 1)/3$. The arch skeleton hypothesis (H1) collects the inductive target at each level.

Definition 2.1. The hypothesis (H1) at level r has four components.

(H1a) *Boundary values.* $A(u_r) = B(u_r) = a_r$ and $A(v_r) = B(v_r) = b_r$, where $b_r := 2a_r$.

(H1b) *Skeleton geometry.* $u_r = 2a_r - r - 2$ and $v_r = u_r + 2a_r = 4a_r - r - 2$.

(H1c) *Local data.*

$$B(u_r{+}1) = a_r{+}1, \quad A(v_r{-}1) = b_r{-}1, \quad B(v_r{-}1) = b_r,$$

and at the next zero

$$A(u_{r+1}{-}1) = a_{r+1}, \quad B(u_{r+1}{-}1) = a_{r+1}{-}1.$$

(H1d) *Excursion positivity (H1sign).* $\delta(m) \geq 0$ for all $u_r \leq m \leq v_r$, with $\delta(m) \geq 1$ for $u_r < m < v_r$.

The hypothesis (H1) is not an assumption but an *inductive target*: it is proved for all $r \geq 0$ in Section 4 (Theorems 4.7 and 4.8). Until that proof is complete, the results of Section 3 are stated under (H1) as a standing hypothesis. Once Section 4 is reached, all these results become unconditional.

Table 2 collects the skeleton data for the first levels.

r	a_r	u_r	v_r	$2a_r$	$V^+(r)$	$\lvert V^-(r)\rvert$
0	3	4	10	6	2	2
1	11	19	41	22	5	8
2	43	82	168	86	15	28
3	171	337	679	342	50	98

Table 2: Skeleton data for the first four arch levels. The arch lengths $2a_r$ grow as $\Theta(4^r)$ and the amplitudes $V^+(r)$ as $\Theta(4^r/\sqrt{r})$. The depth satisfies $|V^-(r)| \leq 2V^+(r) - 2$ unconditionally (Proposition 3.13), with equality under the record claim (Observation 9.4).

Once Theorem 4.8 is established, one has $\sigma(m) = O(\log m)$, so the convergence $\widetilde{Q}(n)/n \to 1/2$ reduces to showing $\delta(m) = o(m)$.

Proposition 2.2. *Granting the frequency law* (Theorem 7.1), $|\widetilde{Q}(n)/n - 1/2| = O(1/\sqrt{\log n})$.

Proof. By (5), $\widetilde{Q}(2m) = m - 1 + \sigma(m) + \delta(m)$ and $\widetilde{Q}(2m{-}1) = m - 1 + \sigma(m) - \delta(m)$. The 1-Lipschitz property (Theorem 4.8) gives $\sigma(m) = O(\log m)$. The counting function $N_A(x) := \#\{m \geq 1 : A(m) \leq x\} = \sum_{v=1}^{x} F_A(v)$, where $F_A(v) := \#\{m : A(m) = v\}$ is the visit multiplicity, satisfies, by the exact mass formula and the frequency law (Section 7), $N_A(4^K{-}1) = 2(4^K{-}1) + O(K) + \frac{1}{2}\sum_{k<K} D_k$, where D_k is the weighted A-versus-B asymmetry on the dyadic block $[4^k, 4^{k+1})$. Since $D_k = \binom{2k+2}{k+1} = O(4^k/\sqrt{k})$, $\sum_{k<K} D_k = O(4^K/\sqrt{K})$, giving $N_A(x) = 2x + O(x/\sqrt{\log x})$. Monotone inversion yields $A(m) = m/2 + O(m/\sqrt{\log m})$. The same holds for B, so $\delta(m) = B(m) - A(m) = O(m/\sqrt{\log m})$. Combining gives $\widetilde{Q}(n)/n = 1/2 + O(1/\sqrt{\log n})$. □

Remark 2.3. This argument is a preview of the full proof of Theorem 1.1(a), given in Section 7.

The convergence is thus unconditional once the arch skeleton (Section 4), the 1-Lipschitz property (Section 4), and the frequency law (Section 7) are established. The convergence rate $|\widetilde{Q}(n)/n{-}1/2| = O(1/\sqrt{\log n})$ follows unconditionally from the frequency law. The exact envelope constant, conditional on Observations 9.4 and 9.10, is derived in Sections 8 and 9.

2.5 Palindromicity of the excursion

Figure 3 shows the excursion δ on the positive arch $r = 2$ (length $2a_2 = 86$). The symmetry $\delta(u_r + t) = \delta(v_r - t)$ is exact. The excursion is a non-negative ± 1 lattice path (a Dyck-type path) with a Cantor-like distribution of maxima.

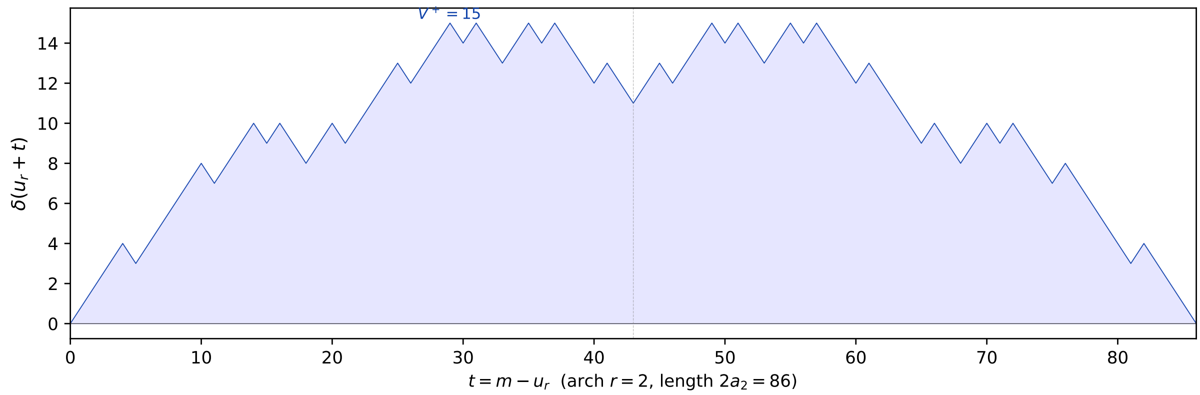

Figure 3: The excursion $\delta(u_r + t)$ on the positive arch $r = 2$ (length 86). The palindromic symmetry around the midpoint is exact.

At a larger scale, the fractal self-similarity is clear. Figure 4 shows the arch $r = 3$ (length 342), where the sub-arch structure of level $r = 2$ is visible inside the larger envelope.

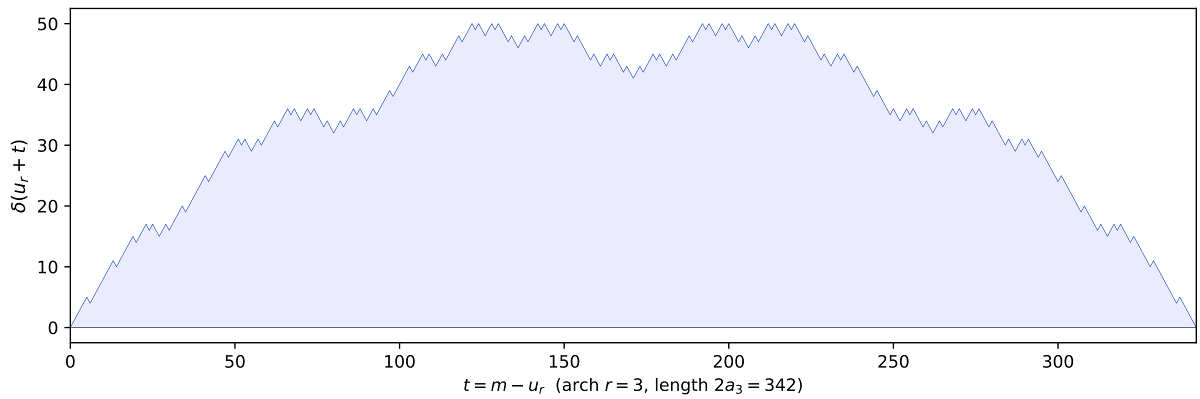

Figure 4: The excursion on the positive arch $r = 3$ (length $2a_3 = 342$). The sub-structure of level $r = 2$ is visible within the larger envelope.

The palindromic symmetry of the excursion is a general property of balanced anti-palindromic words.

Theorem 2.4. *Let W be a binary word of length $2a$, balanced (a zeros and a ones) and anti-palindromic ($W[k] + W[2a-1-k] = 1$). Then $h_W(2a - t) = h_W(t)$ for all $0 \le t \le 2a$.*

Proof. Write $i_t = \#\{0 \le k < t : W[k] = 0\}$, so $h_W(t) = 2i_t - t$. Anti-palindromicity sends the zeros of $W[2a-t : 2a)$ to the ones of $W[0 : t)$, so the suffix contains exactly $t - i_t$ zeros. Since W has a zeros total, $i_{2a-t} = a-(t-i_t)$, giving $h_W(2a-t) = 2(a+i_t-t)-(2a-t) = 2i_t-t = h_W(t)$. ☐

Applied to the step words P_r, this gives the following. Write $\tau_r := \min\{t : h_{P_r}(t) = V^+(r)\}$ for the time of the first maximum of the excursion.

Corollary 2.5. *The height function h_{P_r} is palindromic: $h_{P_r}(2a_r - t) = h_{P_r}(t)$ for all t. The set of times achieving $V^+(r)$ is symmetric about a_r, and $\tau_r \le a_r$.*

Proof. Since P_r is balanced and anti-palindromic (Theorem 3.11), Theorem 2.4 applies with $a = a_r$. The symmetry of the maximum set and the bound $\tau_r \le a_r$ follow immediately. □

3 The interleave transducer

The recursions (R_A)–(R_B) can be recast as a deterministic two-tape interleaving machine. This is the structural discovery that makes the analysis possible.

Convention. Throughout, positions in binary words are 0-indexed. Head counters i_t, j_t, a_t, b_t denote the number of bits consumed after t outputs (prefix lengths).

Notation. The symbols a_r and b_r denote the arch skeleton parameters (Definition 2.1). In Sections 3–5 and 8–9, the same letters a_t and b_t (with subscript t, not r) denote the transducer head positions after t outputs. Likewise, c_t denotes the automaton state, while $c_{r,\ell}$ denotes the run-length count (Proposition 8.3). The amplitude increment is W_r (Section 8). The truncated prefix $P_r[0{:}B)$ is denoted Ω_r (Proposition 9.8).

3.1 The Interleave operator

Definition 3.1. For binary words X, Y and initial state $b \in \{0, 1\}$, $\mathrm{Interleave}(X, Y, b)$ is the word obtained by alternately reading from X (when the state is 0) or Y (when the state is 1), updating the state to the value of the bit just emitted, until both tapes are exhausted. If the designated tape is empty but the other is not, the non-empty tape is read instead.

Example 3.2. Let $X = [0, 1, 0]$ and $Y = [1, 1]$, with initial state $b = 0$. The machine reads $X[0] = 0$ (state stays 0), then $X[1] = 1$ (state becomes 1), then $Y[0] = 1$ (state stays 1), then $Y[1] = 1$ (state stays 1, but Y is now exhausted), then $X[2] = 0$ (forced, since Y is empty). Output: $\mathrm{Interleave}(X, Y, 0) = [0, 1, 1, 1, 0]$.

The operator satisfies a fundamental additivity property.

Lemma 3.3. *If $Z = \mathrm{Interleave}(X, Y, b)$ and i_t, j_t denote the number of bits consumed from X and Y after t outputs, then*

$$h_Z(t) = h_X(i_t) + h_Y(j_t), \tag{7}$$

where $h_W(s) := \#\{0 \le k < s : W[k] = 0\} - \#\{0 \le k < s : W[k] = 1\}$ is the prefix height function.

Proof. The prefix $Z[0 : t)$ consists of exactly $X[0 : i_t)$ and $Y[0 : j_t)$ interleaved. The counts of 0's and 1's are additive over disjoint subsequences. □

The inverse operation is given by extraction operators.

Lemma 3.4. *For any binary word W beginning with 0 and ending with 1, define $E_0(W)$ as the subsequence consisting of the first bit and all bits preceded by a 0, and $E_1(W)$ as the subsequence of all bits preceded by a 1. Then $W = \mathrm{Interleave}(E_0(W), E_1(W), 0)$.*

Proof. Run the Interleave machine on tapes $E_0(W)$ and $E_1(W)$ with initial state 0. At each step, the state equals the previously emitted bit, so the machine reads from E_0 after a 0 and from E_1 after a 1. By construction of E_0 and E_1, this reproduces W bit by bit. □

3.2 Step words

We write $X \circ Y$ for the concatenation of binary words X and Y. Define the *step words*

$$P_r := (\Delta A(m))_{m=u_r}^{v_r-1}, \qquad N_r := (\Delta B(x))_{x=v_r}^{u_{r+1}-1}.$$

Thus P_r records the successive increments of A on the positive arch, and N_r those of B on the negative arch. After the global 1-Lipschitz property is established (Theorem 4.8 below), these words also encode the ± 1 steps of the excursion δ, since on a positive arch $\Delta A + \Delta B = 1$ and therefore $\delta(m+1) - \delta(m) = 1 - 2\,\Delta A(m)$.

Example 3.5. At level $r = 0$, the values of ΔA on $[4, 9]$ are $P_0 = [0, 0, 1, 0, 1, 1]$. The corresponding excursion heights (via $1 - 2\,\Delta A$) are $(0, 1, 2, 1, 2, 1, 0)$.

3.3 Laws 1 and 2

The step word at level $r+1$ is determined by the previous negative-arch data through the following interleaving law, which we call *Law 2* (negative-to-positive).

Proposition 3.6. *Assume* (H1) *at levels* $\leq r$ *and the entry condition* $\Delta A(u_{r+1}-1) = 0$*. Then the step word of the positive arch at level* $r+1$ *satisfies*

$$P_{r+1} = \text{Interleave}\big(S_0,\, S_1,\, 0\big), \qquad S_0 := [0, 0] \circ N_r \circ [1], \quad S_1 := [0] \circ N_r. \tag{8}$$

Proof. On the positive arch $[u_{r+1}, v_{r+1}]$, the recursion (R_A) determines $\Delta A(m)$ from the reading heads $l_m := m - A(m-1)$ and $j_m := m - B(m-1)$. The multiplexer rule is: if $\Delta A(m-1) = 0$ (the previous step was upward), the head l advances and head j stays, so the output is $\Delta A(m) = \Delta B(l_m)$. If $\Delta A(m-1) = 1$ (downward), heads swap roles and $\Delta A(m) = \Delta B(j_m)$. This is exactly the Interleave operation with state $b_m = \Delta A(m-1)$.

The initial state is $b_{\text{init}} = \Delta A(u_{r+1}-1) = 0$ by hypothesis. The landing identities at $m = u_{r+1}$ are

$$l_{u_{r+1}} = u_{r+1} - A(u_{r+1}-1) = u_{r+1} - a_{r+1} = v_r - 2,$$

and

$$j_{u_{r+1}} = u_{r+1} - B(u_{r+1}-1) = u_{r+1} - (a_{r+1}-1) = v_r - 1.$$

The local boundary data gives $\Delta B(v_r-2) = \Delta B(v_r-1) = 0$. Hence the tape scanned by head l is $[0, 0] \circ N_r \circ [1]$ (the two initial zeros from the boundary, followed by N_r, followed by the junction bit 1 at u_{r+1}), while the tape scanned by head j is $[0] \circ N_r$.

Over the $2a_{r+1}$ steps of the arch, both tapes are exhausted: $j_{v_{r+1}} - j_{u_{r+1}} = |N_r| + 1$ and $l_{v_{r+1}} - l_{u_{r+1}} = |N_r| + 3$. Therefore the recursion on $[u_{r+1}, v_{r+1}]$ coincides exactly with the interleave machine, and its output word is precisely P_{r+1}. □

Conversely, the negative-arch step word is built from the previous positive-arch data by *Law 1* (positive-to-negative).

Proposition 3.7. *Under* (H1) *at level* r*,*

$$N_r = \text{Interleave}\big(P_r[2:],\; P_r[:-1],\; 1\big), \tag{9}$$

where $P_r[2:]$ *drops the first two bits and* $P_r[:-1]$ *drops the last bit.*

Proof. On the negative arch, the parity-checkmate analysis (Theorem 4.8, negative-arch case) yields four facts. Set $d := \Delta B(x-1)$, $k_x := x+1-A(x)$, $j_x := x+1-B(x-1)$.

(i) NAND gate. $\Delta A(x) = 1 - d \cdot \Delta A(j_x)$.

(ii) Dual selector. $\Delta B(x) = \Delta A(j_x)$ if $d = 0$, $\Delta B(x) = \Delta A(k_x)$ if $d = 1$.

(iii) Pointer inertia. When $d = 1$, the j-head is parked on a 1 (since j did not advance at the previous step), so $\Delta A(j_x) = 1$.

(iv) Initial positions. $j_{v_r} = k_{v_r} = u_r+1$ and $d_0 = 0$.

The dual selector shows that in state $d = 0$ the machine reads from the j-tape, and in state $d = 1$ from the k-tape. After the first output, define the conjugacy invariant

$$d_t = c_t, \quad k_t = u_r + b_t, \quad j_t = u_r + a_t + 2 - c_t,$$

where (a_t, b_t, c_t) are the read positions and state of Interleave$(P_r[2:], P_r[:-1], 1)$ after t outputs. We verify that this invariant is preserved at each step.

Case $c_t = 0$ ($\equiv d_t = 0$). The recursion reads from the j-tape: $\Delta B(x) = \Delta A(j_x)$. In the interleave machine, state $c_t = 0$ reads from the a-tape $P_r[2:]$, outputting $P_r[a_t{+}2]$. The invariant gives $j_t = u_r + a_t + 2$, so $\Delta A(j_t) = \Delta A(u_r + a_t + 2) = P_r[a_t{+}2]$. Both produce the same output. If the output is 0, then $c_{t+1} = 0$, $a_{t+1} = a_t + 1$, $b_{t+1} = b_t$, and $j_{t+1} = u_r + a_{t+1} + 2 - 0 = j_t + 1$, $k_{t+1} = k_t$, matching the recursion update. If the output is 1, then $c_{t+1} = 1$, $a_{t+1} = a_t + 1$, $b_{t+1} = b_t$, and $j_{t+1} = u_r + a_{t+1} + 2 - 1 = j_t$ (parked), $k_{t+1} = k_t$, again matching.

Case $c_t = 1$ ($\equiv d_t = 1$). The recursion reads from the k-tape: $\Delta B(x) = \Delta A(k_x)$. In the interleave machine, state $c_t = 1$ reads from the b-tape $P_r[:-1]$, outputting $P_r[b_t]$. The invariant gives $k_t = u_r + b_t$, so $\Delta A(k_t) = P_r[b_t]$. Both produce the same output. The pointer inertia (iii) says $\Delta A(j_t) = 1$, so the NAND gate (i) gives $\Delta A(x) = 1 - 1 \cdot 1 = 0$, hence $k_{t+1} = k_t + 1 - 0 = k_t + 1 = u_r + b_t + 1$. If the output is 0, then $c_{t+1} = 0$, $b_{t+1} = b_t + 1$, $k_{t+1} = u_r + b_{t+1}$, and $j_{t+1} = u_r + a_t + 2$, consistent. If the output is 1, then $c_{t+1} = 1$, $b_{t+1} = b_t + 1$, $k_{t+1} = u_r + b_{t+1}$, and $j_{t+1} = j_t$ (parked), consistent.

In all four sub-cases the invariant is preserved. Therefore the recursion and the interleave machine produce the same output word, namely $N_r = \text{Interleave}(P_r[2:], P_r[:-1], 1)$. □

Together, Laws 1 and 2 form a closed dynamical system on step words: $P_r \to N_r \to P_{r+1} \to N_{r+1} \to \cdots$.

3.4 Worked example: $P_0 \to N_0 \to P_1$

Example 3.8. Starting from $P_0 = [0, 0, 1, 0, 1, 1]$:

Law 1 gives $N_0 = \text{Interleave}(P_0[2:], P_0[:-1], 1) = \text{Interleave}([1, 0, 1, 1], [0, 0, 1, 0, 1], 1)$. The machine starts in state 1, reads $[0, 0, 1, 0, 1]$ first bit: output 0, state $\to 0$. Then reads $[1, 0, 1, 1]$ first bit: output 1, state $\to 1$. Continuing:

$$N_0 = [0, 1, 0, 0, 1, 1, 0, 1, 1].$$

Law 2 gives $P_1 = \text{Interleave}(S_0, S_1, 0)$ with

$$S_0 = [0, 0] \circ N_0 \circ [1] = [0, 0, 0, 1, 0, 0, 1, 1, 0, 1, 1, 1],$$
$$S_1 = [0] \circ N_0 = [0, 0, 1, 0, 0, 1, 1, 0, 1, 1].$$

The machine starts in state 0, reads from S_0. The complete output is

$$P_1 = [0, 0, 0, 1, 0, 0, 0, 1, 0, 1, 1, 0, 0, 1, 0, 1, 1, 1, 0, 1, 1, 1],$$

a word of length $22 = 2a_1$. This matches the excursion heights

$$(0, 1, 2, 3, 2, 3, 4, 5, 4, 5, 4, 3, 4, 5, 4, 5, 4, 3, 2, 3, 2, 1, 0)$$

on the arch $[19, 41]$.

3.5 Consequences

The interleaving laws have immediate structural consequences. The step words are balanced.

Corollary 3.9. *P_r contains a_r zeros and a_r ones. Hence $\delta(v_r) = \delta(u_r) = 0$.*

Proof. Since N_r has $|N_r| = 4a_r - 3$ bits with $2a_r - 2$ zeros and $2a_r - 1$ ones, the tape $S_0 = [0, 0] \circ N_r \circ [1]$ contains $2a_r$ zeros and $2a_r$ ones, while $S_1 = [0] \circ N_r$ contains $2a_r - 1$ of each. The total output P_{r+1} has $4a_r - 1 = a_{r+1}$ zeros and a_{r+1} ones. □

The interleaving also transports the initial run lengths across levels.

Corollary 3.10. *Let $\lambda_P(r)$ denote the length of the initial zero-run of P_r, and $\lambda_N(r)$ that of N_r. Then*

$$\lambda_P(r) = r + 2, \qquad \lambda_N(r) = r + 1.$$

Proof. For Law 1 with initial state 1, the first read is from $P_r[: -1]$, which begins with $\lambda_P(r)$ zeros. The first output is 0 (state switches to 0), then $P_r[2 :]$ begins with $\lambda_P(r){-}2$ zeros. Total initial run of N_r: $1 + (\lambda_P(r){-}2) = \lambda_P(r){-}1$. For Law 2 with initial state 0, S_0 begins with $\lambda_N(r){+}2$ zeros, all emitted before any state change. Hence $\lambda_P(r{+}1) = \lambda_N(r) + 2$. With $\lambda_P(0) = 2$ (direct check), induction gives the result. □

Thus P_r begins with 0^{r+2} (upward steps) and, by anti-palindromicity, ends with 1^{r+2} (downward steps). The anti-palindromicity itself propagates through the interleaving laws via a reverse-complement covariance.

Theorem 3.11. *Under* (H1)*, P_r is anti-palindromic for all $r \geq 0$, that is, $P_r[k]{+}P_r[2a_r{-}1{-}k] = 1$ for all k.*

Proof. By induction on r, using the reverse-complement covariance of the Interleave operator. A binary word W is anti-palindromic (AP) if $W[k] + W[|W|{-}1{-}k] = 1$ for all k. We write $\overline{W}$ for the reverse-complement of W: $\overline{W}[k] = 1 - W[|W|{-}1{-}k]$. Then W is AP iff $\overline{W} = W$.

Key identity. If U is AP of length n, beginning with 0 and ending with 1, then $Z :=$ Interleave$([0] \circ U \circ [1], U, 0)$ is AP.

To see this, apply the extraction operators (Lemma 3.4). By construction, $E_0(Z) = [0]{\circ}U{\circ}[1]$ and $E_1(Z) = U$. Since U is AP, $\overline{U} = U$. We claim $\overline{Z} = Z$. Reading $\overline{Z}$ from left to right, each bit of $\overline{Z}$ is the complement of the corresponding bit of Z read from right to left. The extraction $E_0(\overline{Z})$ reverses and complements $E_0(Z) = [0] \circ U \circ [1]$, giving $[0] \circ \overline{U} \circ [1] = [0] \circ U \circ [1] = E_0(Z)$ (using $\overline{U} = U$ and the fact that complementing then reversing $[0] \circ U \circ [1]$ returns $[0] \circ \overline{U} \circ [1]$). Similarly $E_1(\overline{Z}) = \overline{U} = U = E_1(Z)$. Since a word is uniquely determined by its extractions (Lemma 3.4), $\overline{Z} = Z$, so Z is AP.

For the step (ii)⇒(i): if $[0]{\circ}N_{r-1}$ is AP, then $P_r =$ Interleave$([0]{\circ}([0]{\circ}N_{r-1}){\circ}[1], [0]{\circ}N_{r-1}, 0)$ is AP by the key identity with $U = [0] \circ N_{r-1}$.

For the step (i)⇒(ii): if P_r is AP, write $P_r = [0] \circ U \circ [1]$ with U AP (since P_r begins with 0 and ends with 1, and the AP property of P_r implies that of U). Then $[0] \circ N_r =$ Interleave$([0] \circ U \circ [1], U, 0)$ is AP by the key identity applied to U.

The base case $P_0 = [0, 0, 1, 0, 1, 1]$ is AP by direct check. □

The run-length transport provides exact anchor points where $\widetilde{Q}(n)/n = 1/2$.

Proposition 3.12. *For every $r \geq 1$,*

$$\widetilde{Q}(2a_r) = a_r. \tag{10}$$

These are exact points where $\widetilde{Q}(n)/n = 1/2$.

Proof. By Corollary 3.10, N_{r-1} begins with r zeros, so B is constant on $[v_{r-1}, v_{r-1}{+}r{-}1]$. Since $a_r - v_{r-1} = r$ (from the closed forms), $B(a_r) = B(v_{r-1}) = b_{r-1}$. The relation $a_r = 2b_{r-1}{-}1$ gives $B(a_r) = (a_r{+}1)/2$. Hence $\widetilde{Q}(2a_r) = 2B(a_r) - 1 = a_r$. □

The height-additivity of the Interleave operator yields a decomposition of the negative-arch excursion into two independent contributions from the parent positive arch.

Proposition 3.13. *Let (a_x, b_x) denote the Law 1 reading positions. Write $H_{N_r} := h_{N_r}$ for the prefix height of the negative-arch step word. Then*

$$H_{N_r}(x) = h_{P_r}(a_x{+}2) + h_{P_r}(b_x) - 2. \tag{11}$$

At the depth maximum, both heads simultaneously visit maxima of the parent positive-arch excursion. This alignment is verified numerically for $r \le 6$. Under the record claim (Observation 9.4), both heads reach the maximum of h_{P_r} simultaneously, giving the exact relation $|V^-(r)| = 2V^+(r) - 2$.

Proof. The identity (11) follows from Lemma 3.3 applied to Law 1. The tape $P_r[: -1]$ (read by the b-head) starts with the full positive-arch data, contributing $h_{P_r}(b_x)$, while the tape $P_r[2 :]$ (read by the a-head) starts at offset 2, contributing $h_{P_r}(a_x+2)$. The constant -2 accounts for the missing prefix: the two initial zeros of P_r are skipped by the a-tape.

For the depth relation, the upper bound $\max_x H_{N_r}(x) \le 2V^+(r) - 2$ follows immediately from (11), since each term is at most $V^+(r)$ and the constant is -2. The equality $\max H_{N_r} = 2V^+(r) - 2$ requires that both heads reach the maximum of h_{P_r} simultaneously. This is verified for $r \le 6$ and follows from the first-maximum identity (Section 9), which is conditional on Observations 9.4 and 9.10. □

4 Unconditional proof of the arch skeleton

We now prove (H1) for all r by strong induction. The key idea is that the Interleave operator adds the heights of its two input tapes, and the padding bits inject a capital of $+3$ that exceeds twice the maximum debt -1.

Remark 4.1. Several results of Section 3 (balance, run-length transport, anti-palindromicity) are stated under (H1). In the inductive proof below, they are applied to the word $\widetilde{P}_r$ of Lemma 4.2, which is constructed from levels $< r$ alone. The proofs of these results depend only on the algebraic structure of the Interleave operator and on the properties of N_{r-1}, both of which are available under $(\text{H1})_{<r}$. No result at level r or beyond is used, so there is no circularity.

The proof proceeds in six steps: (1) construction of $\widetilde{P}_r$ (§4.1), (2) depth bound on the negative arch (§4.2), (3) interior positivity via fractal inheritance (§4.3), (4) late-tail closure (§4.4), (5) the main induction (§4.5), (6) local data, the stall rule, and the 1-Lipschitz property (§4.6).

4.1 Causal transduction

The following lemma encodes the recursion within each arch.

Lemma 4.2. *Assuming* $(\text{H1})_{<r}$*, the recursion* (2) *produces on the positive arch at level r the word*

$$\widetilde{P}_r := \text{Interleave}\big([0,0] \circ N_{r-1} \circ [1],\ [0] \circ N_{r-1},\ 0\big),$$

without invoking (H1) *at level r.*

Proof. The recursion (2) is forward-deterministic. The reading heads l_m, j_m satisfy $l_m, j_m < m$ for all m. At $m = u_r$, the initial state and head positions are determined by the boundary data of level $r-1$ (part of $(\text{H1})_{<r}$). During the $2a_r$ steps of the arch, the heads progress monotonically through $[v_{r-1}, u_r]$, the zone containing N_{r-1} and its boundary padding. No value at level r or beyond is ever read. The multiplexer rule coincides with the Interleave operator. □

Under $(\text{H1})_{<r}$, $\widetilde{P}_r$ inherits the properties of P_r because it is produced by the same Interleave operator from the same input data N_{r-1}. The combinatorial arguments of Section 3 (balance, run-length transport, palindromicity) depend only on the algebraic structure of the Interleave operator and on the properties of N_{r-1}, both of which are available under $(\text{H1})_{<r}$. In particular, $\widetilde{P}_r$ has length $2a_r$, is balanced (Corollary 3.9), begins with 0^{r+2} (Corollary 3.10), is anti-palindromic (Theorem 3.11), and therefore ends with 1^{r+2}. Set $S_0 := [0,0] \circ N_{r-1} \circ [1]$ and $S_1 := [0] \circ N_{r-1}$.

4.2 The depth bound

The next two lemmas contain the algebraic core of the induction. The padding bits [0, 0] and [0] inject a height capital of +3 into the Interleave producing $\widetilde{P}_r$. The depth bound shows that the inherited negative-arch height never drops below −1, so two copies contribute at most −2. The net balance $3 - 1 - 1 = 1 > 0$ is the reason every excursion stays positive.

Lemma 4.3. *Assume $h_{P_{r-1}}(s) \geq 1$ for all interior points $1 \leq s \leq 2a_{r-1} - 1$. Then $H_{N_{r-1}}(x) \geq -1$ for all $0 \leq x \leq |N_{r-1}|$.*

Proof. By Lemma 3.3 applied to Law 1,

$$H_{N_{r-1}}(x) = h_{P_{r-1}}(a_x{+}2) + h_{P_{r-1}}(b_x) - 2,$$

where (a_x, b_x) are the reading positions of the Law 1 Interleave.

Law 1 starts in state 1, so the first read is on tape $b = P_{r-1}[: -1]$ (length $2a_{r-1} - 1$). Thus $b_x \geq 1$ for $x \geq 1$, and $b_x \leq 2a_{r-1} - 1$. Since b_x is always an interior point of P_{r-1}, the inductive hypothesis gives $h_{P_{r-1}}(b_x) \geq 1$.

For the fast head, $a_x + 2 \geq 2$ and $a_x + 2 \leq 2a_{r-1}$. At the extreme $a_x + 2 = 2a_{r-1}$ (tape exhausted), $h_{P_{r-1}}(2a_{r-1}) = 0$ by balance. For all other values, $h_{P_{r-1}}(a_x + 2) \geq 1$. In every case, $h_{P_{r-1}}(a_x + 2) \geq 0$.

Combining gives $H_{N_{r-1}}(x) \geq 0 + 1 - 2 = -1$. □

4.3 Fractal inheritance

Lemma 4.4. *Under the hypotheses of Lemma 4.3, $h_{\widetilde{P}_r}(t) \geq 1$ for all $1 \leq t \leq t^*$, where t^* is the last time before S_0 is fully consumed.*

Proof. For $t \leq r{+}2$, the machine reads the prefix 0^{r+2} entirely from S_0, giving $h(t) = t \geq 1$.

For $r{+}2 < t \leq t^*$, Lemma 3.3 gives

$$\begin{aligned} h_{\widetilde{P}_r}(t) &= h_{S_0}(i_t) + h_{S_1}(j_t) \\ &= \big[2 + H_{N_{r-1}}(i_t{-}2)\big] + \big[1 + H_{N_{r-1}}(j_t{-}1)\big] \\ &= 3 + H_{N_{r-1}}(i_t{-}2) + H_{N_{r-1}}(j_t{-}1). \end{aligned}$$

By Lemma 4.3, each $H_{N_{r-1}}$ term is at least −1. Hence $h_{\widetilde{P}_r}(t) \geq 3 - 1 - 1 = 1$. □

4.4 Late-tail closure

It remains to handle the final steps after S_0 is exhausted.

The final segment of each arch identifies with the beginning of the next.

Lemma 4.5. *When S_0 is fully consumed in the Interleave producing $\widetilde{P}_r$, the unread suffix of S_1 is exactly 1^{r+1}.*

Proof. By Lemma 3.4, $E_0(\widetilde{P}_r) = S_0$ and $E_1(\widetilde{P}_r) = S_1$. The word $\widetilde{P}_r$ ends with the block $0, 1^{r+2}$ (since it is AP with prefix 0^{r+2}, the last bit before the terminal block of ones is a 0).

The first 1 of the terminal block 1^{r+2} is preceded by 0, so it belongs to $E_0 = S_0$. This is the padding [1], the last bit of S_0. The remaining $r{+}1$ ones are each preceded by 1, so they belong to $E_1 = S_1$. Since the block is terminal, these are the last $r{+}1$ bits of S_1.

At the instant S_0 is exhausted, the machine has just emitted the padding bit [1] (the last symbol of S_0), so its state is 1. From this point on, the machine reads exclusively from S_1, which contains only ones. Each output is therefore 1 (state stays 1), and the excursion decreases by 1 at each step. □

The late tail therefore descends linearly to zero.

Corollary 4.6. *At the instant t^* when S_0 is exhausted, $h_{\widetilde{P}_r}(t^*) = r{+}1$. The excursion then descends linearly as $r{+}1, r, \dots, 1, 0$, remaining ≥ 1 at all interior points.*

Proof. By Lemma 4.5, the unread suffix of S_1 is 1^{r+1}, each contributing -1 to the height. Since $h_{\widetilde{P}_r}$ reaches $r{+}2$ just before the padding bit [1] of S_0 and drops by 1 at that bit, $h(t^*) = r{+}1$. The $r{+}1$ remaining ones bring the height to 0. □

4.5 Main induction

Theorem 4.7. *For all $r \geq 0$, the positive-arch excursion is strictly positive on the interior: the word $\widetilde{P}_r$ satisfies $h_{\widetilde{P}_r}(t) \geq 1$ for $1 \leq t \leq 2a_r - 1$. Moreover $|\widetilde{P}_r| = 2a_r$ (balanced), $v_r = u_r + 2a_r$, and $u_r = 2a_r - r - 2$.*

Proof. By strong induction on r.

Base cases $r = 0, 1, 2$. Verified by direct computation of (2). The excursion heights are $(0,1,2,1,2,1,0)$ for $r=0$ and $(0,1,2,3,2,3,4,5,4,5,4,3,4,5,4,5,4,3,2,3,2,1,0)$ for $r=1$. For $r=2$, $h \geq 1$ on all interior points of $[82, 168]$ (Figure 3). The formulas $u_r = 2a_r - r - 2$ and $v_r = u_r + 2a_r$ are checked at each base level.

Inductive step. Assume the theorem at all levels $< r$. By Lemma 4.2, the step word at level r is $\widetilde{P}_r$, built from levels $< r$ alone. Since $\widetilde{P}_r$ is produced by the same Interleave operator from the same input data N_{r-1}, the combinatorial consequences (balance, run-length transport, palindromicity) hold for $\widetilde{P}_r$ under $(\mathrm{H1})_{<r}$. By Lemma 4.4, $h \geq 1$ before S_0-exhaustion. By Corollary 4.6, $h \geq 1$ in the late tail. This proves interior positivity.

Balance (Corollary 3.9) gives $|\widetilde{P}_r| = 2a_r$, so $v_r = u_r + 2a_r$. For $u_r = 2a_r - r - 2$, note that $u_r = v_{r-1} + |N_{r-1}|$. By the inductive hypothesis, $v_{r-1} = 4a_{r-1} - (r{-}1) - 2$. The negative arch has $|N_{r-1}| = 4a_{r-1} - 3$ bits. Hence $u_r = (4a_{r-1} - r - 1) + (4a_{r-1} - 3) = 8a_{r-1} - r - 4$. Since $a_r = 4a_{r-1} - 1$, one checks $2a_r - r - 2 = 8a_{r-1} - r - 4$. □

4.6 Local zero data, the stall rule, and the 1-Lipschitz property

The geometric skeleton (Theorem 4.7) gives the arch lengths and the positivity of the excursion. The following theorem completes (H1) by establishing the boundary values (H1a), the local data (H1c), and the global 1-Lipschitz property.

Theorem 4.8. *For all $m \geq 1$,*

$$\Delta A(m), \Delta B(m) \in \{0, 1\}.$$

Moreover, for every $r \geq 0$,

$$A(u_r) = B(u_r) = a_r, \qquad A(v_r) = B(v_r) = b_r,$$

and the local data of Definition 2.1(H1c) hold.

Proof. By outer induction on r, simultaneously with the geometric skeleton. The base cases $r = 0, 1, 2$ are verified directly. Assume all three properties hold for levels $< r$.

The negative arch $[v_{r-1}, u_r]$ (parity checkmate). We prove, by inner induction on $x \in [v_{r-1}, u_r]$, that $\Delta A(x), \Delta B(x) \in \{0,1\}$ and $\sigma(x) + \Delta A(x) = r{+}2$.

Set

$$k_x := x{+}1{-}A(x), \qquad j_x := x{+}1{-}B(x{-}1).$$

These are the two reading heads naturally appearing in (R_B). By the outer induction and the already proved steps on earlier indices, both landing points lie in the previous positive arch, where $\sigma = r{+}1$ and $\Delta A, \Delta B \in \{0,1\}$ are known.

Set $d := \Delta B(x{-}1) \in \{0,1\}$. Summing $A(x{+}1) + B(x)$ via (R_A) and (R_B) yields the master parity equation

$$2\sigma(x) + \Delta A(x) = 2r + 3 + d\,\Delta A(j_x). \tag{12}$$

Since the right-hand side is either $2r + 3$ or $2r + 4$, parity forces $\Delta A(x) \in \{0,1\}$ and $\sigma(x) + \Delta A(x) = r + 2$.

Next, from (R_B) at x and $x + 1$,

$$B(x) = A(k_x) + A(j_x + d), \qquad B(x+1) = A(k_{x+1}) + A(j_x + 1).$$

Hence

$$\Delta B(x) = [A(k_{x+1}) - A(k_x)] + [A(j_x + 1) - A(j_x + d)].$$

Since

$$k_{x+1} - k_x = 1 - \Delta A(x) \in \{0,1\}, \qquad (j_x + 1) - (j_x + d) = 1 - d \in \{0,1\},$$

both brackets lie in $\{0,1\}$ by the outer induction. Thus $0 \le \Delta B(x) \le 2$. If $\Delta B(x) = 2$, then necessarily $\Delta A(x) = 0$ and $d = 0$, so (12) gives

$$2\sigma(x) = 2r + 3,$$

impossible by parity. Therefore $\Delta B(x) \in \{0,1\}$.

The positive arch $[u_r, v_r]$ *(zipper induction).* We prove, by inner induction on $m \in [u_r, v_r{-}1]$, the joint statement

$$\Delta A(m) \in \{0,1\}, \qquad \sigma(m{+}1) = r{+}2, \qquad \Delta B(m) \in \{0,1\}.$$

Base case $m = u_r$. From the skeleton, $A(u_r) = B(u_r) = a_r$. Via (R_A) at $u_r{+}1$ with the identity $u_r{+}1{-}a_r = v_{r-1}{-}1$, we get $A(u_r{+}1) = 2B(v_{r-1}{-}1){-}1 = 2b_{r-1}{-}1 = a_r$, so $\Delta A(u_r) = 0$. From the boundary data at u_r, one has $\Delta B(u_r) = 1$. The σ-constancy at $u_r{+}1$ follows from the bracket analysis below, with all landing points in the zone where the negative-arch analysis applies.

The step $\Delta A(m)$ *is Boolean.* Define $j_m := m - B(m{-}1)$ and $l_m := m - A(m{-}1)$. From (R_A),

$$\Delta A(m) = [B(j_{m+1}) - B(j_m)] + [B(l_{m+1}) - B(l_m)].$$

Using $\Delta A(m{-}1) + \Delta B(m{-}1) = 1$ (which follows from $\sigma(m) = \sigma(m{-}1) = r{+}2$), one gets

$$j_{m+1} - j_m = \Delta A(m{-}1), \qquad l_{m+1} - l_m = 1 - \Delta A(m{-}1).$$

Hence: if $\Delta A(m{-}1) = 0$, then $\Delta A(m) = \Delta B(l_m)$. If $\Delta A(m{-}1) = 1$, then $\Delta A(m) = \Delta B(j_m)$. In both cases, $\Delta A(m)$ copies a value $\Delta B(y)$ at some $y < m$. Since $\Delta B(y) \in \{0,1\}$ is already known at all earlier indices, it follows that $\Delta A(m) \in \{0,1\}$.

The drift is constant: $\sigma(m{+}1) = r{+}2$. Write

$$\sigma(m{+}1) = [B(j) + A(j{+}1)] + [B(l) + A(k)] - 1 - (m{+}1),$$

where $j := (m{+}1) - B(m)$, $l := (m{+}1) - A(m)$, $k := (m{+}2) - A(m{+}1)$.

For the first bracket, all landing points lie either in the negative arch (where the parity-checkmate gives the compensation identity $\sigma + \Delta A = r{+}2$) or in the already treated part of the current positive arch (where the inner induction gives $\sigma = r{+}2$). Hence $B(j) + A(j{+}1) = j + (r{+}2)$.

For the second bracket there are two cases. If $\Delta A(m) = 0$, then $k = l{+}1$, and the same compensation/induction argument gives $B(l) + A(l{+}1) = l + (r{+}2)$. If $\Delta A(m) = 1$, then $k = l$. Set $e := \Delta B(m{-}1) \in \{0,1\}$. Summing $A(m{+}1){+}B(m)$ in two ways gives $\sigma(l){+}\sigma(j){+}e{\cdot}\Delta A(j) = 2r + 4$. If $e \cdot \Delta A(j) = 0$, then $\sigma(l) + \sigma(j) = 2r{+}4$, forcing $\sigma(l) = \sigma(j) = r{+}2$. If $e \cdot \Delta A(j) = 1$, then $\sigma(l) + \sigma(j) = 2r{+}3$, and both values are at most $r{+}2$, so one of them equals $r{+}1$. This case is excluded by the stall rule (T_r) below (Proposition 4.12), which guarantees $e \cdot \Delta A(j) = 0$ whenever $\Delta A(m) = 1$. In every case, $\sigma(l) = r{+}2$, and therefore $B(l) + A(l) = l + (r{+}2)$.

Combining the two brackets and using $j + l = (m{+}1) + 1 - \sigma(m) = m - r$, we obtain $\sigma(m{+}1) = r{+}2$.

The step $\Delta B(m)$ is Boolean. From $\sigma(m) = \sigma(m{+}1) = r{+}2$ it follows that $\Delta A(m) + \Delta B(m) = 1$. Since $\Delta A(m) \in \{0, 1\}$, we conclude $\Delta B(m) = 1 - \Delta A(m) \in \{0, 1\}$.

Boundary values. Since P_r ends with 1^{r+2} (Corollary 3.10), $\Delta A(v_r{-}1) = 1$ and $\Delta B(v_r{-}1) = 0$, giving $A(v_r{-}1) = b_r{-}1$ and $B(v_r{-}1) = b_r$. The balance P_r having a_r zeros and a_r ones gives $A(v_r) = B(v_r) = b_r$. Applying the same base-case argument on the next negative arch gives $A(u_{r+1}{-}1) = a_{r+1}$ and $B(u_{r+1}{-}1) = a_{r+1}{-}1$. □

Remark 4.9. The proof of Theorem 4.8 combines two distinct techniques. On the negative arch, the *parity checkmate* uses the master parity equation (12) to rule out $\Delta B = 2$. On the positive arch, the *zipper induction* propagates the step-word structure forward in m using the palindromicity of P_r. Both arguments rely on the entry/exit conditions established in Theorem 4.7.

Remark 4.10. With Theorems 4.7 and 4.8 established, all results stated under (H1) in Section 3 are unconditional. In particular, Laws 1 and 2, palindromicity, run-length transport, and the anchor identity hold for all r. The step words P_r and N_r are confirmed to be binary ($\{0, 1\}$-valued), so they encode both the increments ΔA, ΔB and the ± 1 steps of the excursion δ.

The recursions (R_A)–(R_B) combine into a master identity for the sum $S(m) = A(m) + B(m)$.

Proposition 4.11. *Define $S(m) := A(m) + B(m)$ and $S^*(x) := B(x) + A(x{+}1)$. For m on the positive arch r, with $x_1 := m - B(m{-}1)$ and $x_2 := m - A(m{-}1)$,*

$$S(m) = S^*(x_1) + S^*(x_2) - 1 - \Delta A(m{-}1) \cdot \Delta A(x_2). \tag{13}$$

Under the negative-arch identity $S^*(x) = x + r + 2$ (proved on $[v_{r-1}{-}2, u_r{+}1]$ by the parity checkmate) and $S(m{-}1) = m + r + 1$ (inner induction), this gives

$$S(m) = m + r + 2 - \Delta A(m{-}1) \cdot \Delta A\big(m - A(m{-}1)\big). \tag{14}$$

Proof. By (R_A), $A(m) = B(x_1) + B(x_2) - 1$. By (R_B),

$$B(m) = A(m{+}1{-}A(m)) + A(m{+}1{-}B(m{-}1)) = A(x_2{+}1{-}\Delta A(m{-}1)) + A(x_1{+}1).$$

Set $\varepsilon := \Delta A(m{-}1) \in \{0, 1\}$. Adding the two equations gives

$$\begin{aligned} S(m) &= \big[B(x_1) + A(x_1{+}1)\big] + \big[B(x_2) + A(x_2{+}1{-}\varepsilon)\big] - 1 \\ &= S^*(x_1) + \big[B(x_2) + A(x_2{+}1{-}\varepsilon)\big] - 1. \end{aligned}$$

For the second bracket, write $A(x_2{+}1{-}\varepsilon) = A(x_2{+}1) - \varepsilon \cdot \Delta A(x_2)$ (since $A(x_2{+}1) - A(x_2) = \Delta A(x_2)$ and $(x_2{+}1{-}\varepsilon)$ is either $x_2{+}1$ or x_2 depending on ε). Therefore

$$B(x_2) + A(x_2{+}1{-}\varepsilon) = S^*(x_2) - \varepsilon \cdot \Delta A(x_2).$$

Substituting back gives (13).

For (14), note that $x_1{+}x_2 = 2m{-}S(m{-}1)$. Under $S(m{-}1) = m{+}r{+}1$ and $S^*(x_i) = x_i{+}r{+}2$, the sum $S^*(x_1) + S^*(x_2) - 1 = x_1 + x_2 + 2r + 3 = m + r + 2$. □

The error term in the master identity vanishes by a stall rule: whenever the step word outputs a 1, the landing point in the negative arch reads a 0.

Proposition 4.12. *On the positive arch r, for $u_r < m \le v_r$:*

$$\Delta A(m{-}1) = 1 \implies \Delta A\big(m - A(m{-}1)\big) = 0. \tag{15}$$

Equivalently, $\Delta A(m{-}1) \cdot \Delta A(m{-}A(m{-}1)) = 0$.

Proof. Let $\widetilde{P}_r$ be the word of Lemma 4.2. Write $m = u_r + t + 1$ and $Z(t) := \#\{0 \le k < t : \widetilde{P}_r[k] = 0\}$, $O(t) := t - Z(t)$. Then $\Delta A(m{-}1) = \widetilde{P}_r[t]$. Assume $\widetilde{P}_r[t] = 1$. We must show $\Delta A(x_2) = 0$ where $x_2 := m - A(m{-}1)$.

Since $A(u_r) = a_r$, we have $A(m{-}1) = a_r{+}O(t)$, hence $x_2 = u_r{+}1{-}a_r{+}Z(t) = v_{r-1}{-}1{+}Z(t)$.

Case 1: $Z(t) = a_r$. Then $x_2 = v_{r-1} - 1 + a_r = u_r + 1$. Since $\widetilde{P}_r$ begins with 0^{r+2}, $\Delta A(u_r{+}1) = \widetilde{P}_r[1] = 0$.

Case 2: $Z(t) \le a_r - 1$. Set $x := x_2 - 1 = v_{r-1} - 2 + Z(t)$. Since $\widetilde{P}_r[t] = 1$ and $\widetilde{P}_r$ begins with at least two zeros, $Z(t) \ge 2$, so $v_{r-1} \le x \le u_r - 1$. The negative-arch part of Theorem 4.8 gives $\sigma(y) + \Delta A(y) = r{+}2$ on $[v_{r-1}, u_r]$. Subtracting at $y{+}1$ and y and comparing with $\sigma(y{+}1) - \sigma(y) = \Delta A(y) + \Delta B(y) - 1$ yields

$$\Delta A(y{+}1) = 1 - \Delta B(y) \qquad (v_{r-1} \le y \le u_r{-}1). \tag{16}$$

At $y = x$, this gives $\Delta A(x_2) = 1 - \Delta B(x)$. It remains to show $\Delta B(x) = 1$.

By (16) applied at $y = x$ and the definition of S_0, we have $\Delta B(x) = S_0[Z(t)]$ (since the $Z(t)$-th bit of S_0 records $\Delta B(v_{r-1}{-}2 + Z(t)) = \Delta B(x)$, as $Z(t)$ lies strictly before the final padding bit of S_0). So it suffices to prove $S_0[Z(t)] = 1$.

Recall that in $\widetilde{P}_r = \mathrm{Interleave}(S_0, S_1, 0)$, step s reads from S_0 if and only if $s = 0$ or $\widetilde{P}_r[s{-}1] = 0$. Hence the number of bits consumed from S_0 before step t is $i_t = 1 + Z(t{-}1)$.

If $\widetilde{P}_r[t{-}1] = 0$, then step t reads from S_0 and outputs 1 (since $\widetilde{P}_r[t] = 1$). In this sub-case $Z(t) = Z(t{-}1) + 1 = i_t$, so the bit read is $S_0[Z(t)] = 1$.

If $\widetilde{P}_r[t{-}1] = 1$, then step t reads from S_1, so $Z(t) = Z(t{-}1)$. By $i_t = 1 + Z(t{-}1) = Z(t) + 1$, exactly $Z(t) + 1$ bits of S_0 have been consumed, and the last one read was $S_0[Z(t)]$. Let $k < t$ be the last step reading from S_0. If the output at step k were 0, the next step would also read from S_0, contradicting maximality of k. Hence $S_0[Z(t)] = \widetilde{P}_r[k] = 1$.

In both sub-cases $S_0[Z(t)] = 1$, so $\Delta B(x) = 1$ and $\Delta A(x_2) = 0$. □

Remark 4.13. The stall rule closes the gap in the zipper induction (Theorem 4.8, positive-arch case, $\Delta A(m) = 1$): whenever $e \cdot \Delta A(j)$ could be nonzero, the stall rule forces it to vanish, establishing $\sigma(m{+}1) = r{+}2$ unconditionally. The proof uses only $\widetilde{P}_r$ (Lemma 4.2), the negative-arch identity $\sigma{+}\Delta A = r{+}2$ (already proved in Theorem 4.8), and the initial run 0^{r+2} (Corollary 3.10). No result from Sections 8–9 is needed.

An immediate consequence is that the recursion never breaks down.

Corollary 4.14. *The sequence $\widetilde{Q}$ is well-defined for all $n \ge 1$, and satisfies $1 \le \widetilde{Q}(n) \le n$.*

Proof. The strong induction establishing Theorems 4.7 and 4.8 constructs $\widetilde{Q}(n)$ level by level. At each step, the reading heads $n - \widetilde{Q}(n{-}1)$ and $n - \widetilde{Q}(n{-}2)$ land in the already-constructed past (Lemma 4.2), so the recursion produces a well-defined value.

For the upper bound, the 1-Lipschitz property gives $A(m) \le m$ and $B(m) \le m$ for all $m \ge 1$ (since $A(1) = 1$, $\Delta A \in \{0, 1\}$, and likewise for B). Hence $\widetilde{Q}(2m{-}1) = 2A(m) - 1 \le 2m - 1$ and $\widetilde{Q}(2m) = 2B(m) - 1 \le 2m - 1 < 2m$. In both cases $\widetilde{Q}(n) \le n$.

The lower bound $\widetilde{Q}(n) \ge 1$ follows from $A(m) \ge 1$ and $B(m) \ge 1$ (both hold since $A(1) = B(1) = 1$ and $\Delta A, \Delta B \ge 0$). □

The entry condition assumed in Proposition 3.6 is now confirmed.

Corollary 4.15. *For every $r \ge 0$, $\Delta A(u_{r+1}{-}1) = 0$. Hence Law 2 (Proposition 3.6) is unconditional.*

Proof. By Theorem 4.8, $A(u_{r+1}{-}1) = a_{r+1}$ and $A(u_{r+1}) = a_{r+1}$. Therefore $\Delta A(u_{r+1}{-}1) = A(u_{r+1}) - A(u_{r+1}{-}1) = 0$. □

5 The gap sequence and the topological identity

The arch skeleton provides the step words P_r as balanced binary words of length $2a_r$. We now extract from P_r a finer combinatorial object, the 1-*gap sequence* G_r, whose cumulative excess controls the amplitude $V^+(r)$ through an exact topological identity.

The idea is to pass from the full binary word P_r to a simpler derived object. The excursion height h_{P_r} rises at every 0 and falls at every 1. The amplitude $V^+(r)$ is therefore governed by how the ones are distributed among the zeros. Recording only the spacings between consecutive ones captures exactly this information, in a linearised form that is amenable to induction.

5.1 The 1-gap sequence

Definition 5.1. Let $o_0 < o_1 < \cdots < o_{a_r-1}$ be the positions of the ones in P_r (indexed from 0). The 1-*gap sequence* of level r is

$$G_r := (g_0, g_1, \ldots, g_{a_r-1}),$$

where $g_0 := o_0$ and $g_i := o_i - o_{i-1}$ for $i \geq 1$.

Since P_r has exactly a_r ones (Corollary 3.9), G_r has a_r entries, and $\sum_{i=0}^{a_r-1} g_i = o_{a_r-1} = 2a_r - 1$ (the last one in P_r sits at position $2a_r - 1$, because P_r ends with 1^{r+2}).

Example 5.2. For $r = 0$, $P_0 = [0, 0, 1, 0, 1, 1]$, and $G_0 = (2, 2, 1)$. For $r = 1$, $G_1 = (3, 4, 2, 1, 3, 2, 1, 1, 2, 1, 1)$.

5.2 The topological identity

Define the *cumulative excess*

$$S_r(j) := \sum_{i=0}^{j} (g_i - 2), \qquad 0 \leq j \leq a_r - 1. \tag{17}$$

Theorem 5.3. *For all $r \geq 0$ and $0 \leq j \leq a_r - 1$,*

$$S_r(j) = h_{P_r}(o_j) - 2. \tag{18}$$

Proof. Among the first o_j bits of P_r, exactly j are ones (at positions $o_0, \ldots, o_{j-1}$) and $o_j - j$ are zeros. Therefore $h_{P_r}(o_j) = (o_j - j) - j = o_j - 2j$. Since $o_j = \sum_{i=0}^{j} g_i$,

$$h_{P_r}(o_j) = \sum_{i=0}^{j} g_i - 2j = \sum_{i=0}^{j} (g_i - 2) + 2 = S_r(j) + 2. \qquad \square$$

The amplitude is therefore determined by the maximum of the cumulative excess.

Corollary 5.4. $V^+(r) = 2 + \max_{0 \leq j \leq a_r - 1} S_r(j)$.

Proof. Every local maximum of h_{P_r} is achieved at some o_j (the position immediately before a one). Hence $V^+(r) = \max_j h_{P_r}(o_j) = \max_j (S_r(j) + 2)$. $\square$

Write $j_r^* := \arg\max_j S_r(j)$ for the index at which the cumulative excess first reaches its maximum. Then $V^+(r) = 2 + S_r(j_r^*)$, and the first maximum of the excursion δ on the positive arch is achieved at $\tau_r = o_{j_r^*}$.

Proposition 5.5. *Assuming the first-maximum identity $\tau_r + V^+(r) = 4a_{r-1}$ at level r,*

$$j_r^* = 2a_{r-1} - V^+(r). \tag{19}$$

Proof. From the topological identity (Theorem 5.3), $\tau_r = o_{j_r^*}$, where o_j is the position of the j-th one in P_r. Since $h_{P_r}(o_j) = o_j - 2j$, we obtain $\tau_r = 2j_r^* + V^+(r)$. The first-maximum identity then gives $j_r^* = 2a_{r-1} - V^+(r)$. □

Remark 5.6. This derivation uses only the topological identity (unconditional) and the first-maximum identity (conditional on Observations 9.4 and 9.10). It does not require the staircase recursion or the kernel method.

Combining the index of the first maximum with the singleton count gives a first algebraic constraint.

Corollary 5.7. *Assuming the first-maximum identity and the singleton scaffold count* $c_{r+1,1} = (4^{r+1}-1)/3$ (Lemma 8.6),

$$c_{r+1,1} - j_r^* = V^+(r) - 1. \tag{20}$$

Proof. Substitute $j_r^* = 2a_{r-1} - V^+(r)$ (Proposition 5.5) and $a_{r-1} = (2 \cdot 4^r+1)/3$ into $c_{r+1,1} - j_r^*$. □

5.3 Mirror symmetry of the gaps

Proposition 5.8. *For all* $r \geq 0$ *and* $0 \leq j \leq a_r - 2$, $g_j = g_{a_r-2-j}$.

Proof. Anti-palindromicity of P_r (Theorem 3.11) gives $o_j + o_{a_r-1-j} = 2a_r - 1$ for all j. Differencing two consecutive instances yields the mirror identity. □

6 Catalan duality

With the arch skeleton established, we derive the convergence rate $|\widetilde{Q}(n)/n - 1/2| = O(1/\sqrt{\log n})$ unconditionally via Route B (frequencies), and the exact envelope constant via Route A (amplitudes), both governed by the Catalan kernel $1 - z + xz^2 = 0$.

Route B (frequencies, unconditional) gives an exact multiplicity law and the Catalan identity

$$D_k = \binom{2k+2}{k+1}.$$

A monotone inversion argument then gives the convergence (Section 7).

Route A (amplitudes, conditional on Observations 9.4 and 9.10) controls the arch amplitudes through the staircase recursion and yields the exact formula

$$W_r = \binom{2r+1}{r},$$

from which $V^+(r) \sim \frac{8}{3\sqrt{\pi}}\, 4^r/\sqrt{r}$ follows (Section 8). This identifies the exact envelope constant $1/(3\sqrt{2\pi})$ in Theorem 1.1(b).

The two routes are connected by a telescoping identity that we establish unconditionally.

6.1 Boundary lag and the median identity

For $v \geq 1$, define the entry times $m_A(v) := \min\{m \geq 1 : A(m) = v\}$, $m_B(v) := \min\{m \geq 1 : B(m) = v\}$, and the lag $E(v) := m_A(v) - m_B(v)$.

Proposition 6.1. *For all* $K \geq 1$,

$$E(4^K) = \sum_{k=0}^{K-1} D_k.$$

Proof. Since $\Delta A, \Delta B \in \{0,1\}$ (Theorem 4.8), $F_A(v) = m_A(v{+}1) - m_A(v)$ and $F_B(v) = m_B(v{+}1) - m_B(v)$. Summing $F_A(v) - F_B(v)$ over $v \in I_k$ gives $E(4^{k+1}) - E(4^k) = \sum_{v \in I_k} (F_A(v) - F_B(v)) = D_k$. The claim follows by telescoping, with $E(1) = 0$. □

The telescoping evaluates cleanly at the dyadic boundaries because of the following arithmetic coincidence.

Proposition 6.2. *For every $r \geq 0$, the dyadic boundary 4^{r+1} is the median of the values taken on the positive arch r:*

$$\frac{3a_r - 1}{2} = 4^{r+1}.$$

Proof. $a_r = (2 \cdot 4^{r+1} + 1)/3$, so $(3a_r - 1)/2 = (2 \cdot 4^{r+1} + 1 - 1)/2 = 4^{r+1}$. □

On any positive arch, the σ-constancy $A(m) + B(m) = m + r + 2$ gives the following. The lag $E(v)$ can be expressed as a sum over the excursion.

Proposition 6.3. *Let $V \in [a_r{+}1, 2a_r{-}1]$ lie in the interior of the positive arch r. Then*

$$E(V) = \delta(m_B(V)) + \delta(m_A(V)),$$

where $\delta(m_B(V))$ is a value on the ascending part of the excursion and $\delta(m_A(V))$ on the descending part.

Proof. At $m = m_B(V)$, $B(m) = V$ and $A(m) = V - \delta(m)$, so $m_B = 2V - \delta(m_B) - r - 2$. At $m = m_A(V)$, $A(m) = V$ and $B(m) = V + \delta(m)$, so $m_A = 2V + \delta(m_A) - r - 2$. Subtracting gives $E(V) = \delta(m_A) + \delta(m_B)$. Since $\delta \geq 0$ on the positive arch and $B \geq A$, $m_B(V) \leq m_A(V)$. The excursion rises then falls (palindromic structure), so $m_B(V)$ lies in the ascent and $m_A(V)$ in the descent. □

Evaluating at the median yields an unconditional lower bound for the amplitudes.

Corollary 6.4. *For all $r \geq 0$,*

$$V^+(r) \geq \frac{1}{2} \sum_{k=0}^{r} D_k = \frac{1}{2} \sum_{k=0}^{r} \binom{2k+2}{k+1}.$$

Numerically, $V^+(r) = 1 + \sum_{k=0}^{r} \binom{2k+1}{k}$ for $r = 0, \ldots, 6$, and the bound is tight up to the additive constant 1.

Proof. By Proposition 6.3 at the median $V = 4^{r+1}$ (Proposition 6.2), $E(4^{r+1}) = \delta(m_B) + \delta(m_A)$. Each summand is bounded by $V^+(r)$, giving $V^+(r) \geq E(4^{r+1})/2$. By Proposition 6.1, $E(4^{r+1}) = \sum_{k=0}^{r} D_k$. □

In both routes, the transducer converts a threshold-s imbalance into a threshold-$(s{+}1)$ imbalance via the same cumulative tail-sum transform. The exact identity $D_k = 2W_k$ (that is, $\binom{2k+2}{k+1} = 2\binom{2k+1}{k}$) quantifies this duality as a binomial identity. Whether it admits a direct dynamical proof (via the palindromic folding of the arch structure) remains open (see Section 12).

7 Route B: frequencies and the geometric law

This section establishes the exact frequency law on dyadic blocks and the Catalan identity for D_k. It uses the distribution of visit multiplicities on dyadic blocks and relies only on the macro-transduction lemma.

7.1 Setup

For $v \in \mathbb{N}$, define the visit multiplicities $F_A(v) := \#\{m : A(m) = v\}$ and $F_B(v) := \#\{m : B(m) = v\}$. Set $I_k := [4^k, 4^{k+1} - 1]$, with $|I_k| = 3 \cdot 4^k$, and

$$a_{k,r} := \#\{v \in I_k : F_A(v) = r\}, \qquad b_{k,r} := \#\{v \in I_k : F_B(v) = r\},$$
$$S_{k,r} := a_{k,r} + b_{k,r}, \qquad \Delta_{k,r} := a_{k,r} - b_{k,r}. \tag{21}$$

Note that $\sum_r S_{k,r} = 2|I_k|$ (each value in I_k is counted once in the a-row and once in the b-row). The notation $S_{k,r}$ should not be confused with the cumulative gap excess $S_r(j)$ of Section 5. The total mass is $M_k := \sum_{v \in I_k}(F_A(v) + F_B(v)) = \sum_r r\, S_{k,r}$.

7.2 The frequency law

Theorem 7.1. *For all $k \geq 1$,*

$$S_{k,r} = 3 \cdot 2^{2k-r+1} \quad (1 \leq r \leq 2k{+}1), \qquad S_{k,2k+2} = 2, \quad S_{k,2k+3} = 1.$$

In particular $\sum_r S_{k,r} = 2|I_k| = 6 \cdot 4^k$.

Table 3 displays the frequency counts for $k = 1, \ldots, 4$. The geometric progression $S_{k,r} = 3 \cdot 2^{2k-r+1}$ occupies columns $1 \leq r \leq 2k{+}1$, followed by the boundary values $S_{k,2k+2} = 2$ and $S_{k,2k+3} = 1$.

k	$r{=}1$	2	3	4	5	6	7	8	9	$\cdots$	Σ
1	12	6	3	2	1						24
2	48	24	12	6	3	2	1				96
3	192	96	48	24	12	6	3	2	1		384
4	768	384	192	96	48	24	12	6	3	2, 1	1536

Table 3: The symmetric frequency counts $S_{k,r}$ for $k = 1, \ldots, 4$. Each row is a geometric progression with ratio 1/2, ending at two boundary values. The row sums equal $2|I_k| = 6 \cdot 4^k$.

7.3 The macro-transduction lemma

For $v \in I_k$, the "plateau word" records $F_A(v)$ in unary as a block $0^{F_A(v)-1}1$. The concatenation W_k^A of these blocks for $v \in I_k$ encodes the frequency profile of A on I_k. Similarly for W_k^B.

Proposition 7.2. *For $s \geq 1$ and $k \geq 1$,*

$$A_{k,\geq s} = \#\{v \in I_k : F_A(v) \geq s\} = \#\{\text{occurrences of } 0^{s-1}1 \text{ in } W_k^A\},$$

and similarly for B.

Proof. Each value $v \in I_k$ contributes the block $0^{F_A(v)-1}1$ to W_k^A. This block contains exactly one occurrence of $0^{s-1}1$ for every $s \leq F_A(v)$, and none for $s > F_A(v)$. Summing over $v \in I_k$ gives $A_{k,\geq s}$. Taking differences recovers $a_{k,r} = A_{k,\geq r} - A_{k,\geq r+1}$. □

The boundary plateau at each arch junction has a precise length.

Lemma 7.3. *For every $k \geq 0$, $F_B(2a_k) = 2k{+}4$.*

Proof. By Theorem 4.8, $B(v_{k+1}) = b_{k+1} = 2a_k$. It suffices to count the length of the constant plateau of B through the index v_{k+1}.

On the positive arch $[u_{k+1}, v_{k+1}]$, the step word is P_{k+1}, which ends with exactly 1^{k+3} (by palindromicity, since it begins with 0^{k+3}). On this arch, $\Delta A(m) + \Delta B(m) = 1$ (Theorem 4.8), so $\Delta B(m) = 0$ for the last $k{+}2$ indices before v_{k+1}.

On the negative arch starting at v_{k+1}, the word N_{k+1} begins with exactly 0^{k+2} (Corollary 3.10), so $\Delta B(m) = 0$ for the first $k{+}1$ indices from v_{k+1} onward.

These two zero-blocks concatenate across the boundary v_{k+1}, giving a contiguous run of $(k{+}2) + (k{+}1) = 2k{+}3$ zeros in ΔB. Hence B equals $2a_k$ on exactly $2k{+}4$ consecutive indices.

The run length is exact: just before the trailing block 1^{k+3} in P_{k+1} one has $\Delta A = 0$, hence $\Delta B = 1$, and just after the initial block 0^{k+2} in N_{k+1} one has $\Delta B = 1$. □

The interleaving laws translate the frequency profile from level k to level $k{+}1$.

Lemma 7.4. *For all $k \geq 1$ and $r \geq 2$,*

$$a_{k+1,r} = A_{k,\geq r-1} + \mathbf{1}_{r=2k+5}, \tag{22}$$
$$b_{k+1,r} = B_{k,\geq r-1} + \mathbf{1}_{r=2k+4}, \tag{23}$$

where $A_{k,\geq s} := \#\{v \in I_k : F_A(v) \geq s\}$ and $B_{k,\geq s} := \#\{v \in I_k : F_B(v) \geq s\}$.

Proof. The main terms follow from Law 2. Each block $0^{s-1}1$ in W_k^A appears as 0^s1 in W_{k+1}^A, incrementing the run length by 1. Thus $a_{k+1,r}$ counts the values with $F_A(v) \geq r - 1$, giving $A_{k,\geq r-1}$. The same argument applies to B.

The boundary term $\mathbf{1}_{r=2k+5}$ in (22) accounts for the unique maximal plateau at $v = a_{k+1}$. By palindromicity (Theorem 3.11) and run-length transport (Corollary 3.10), the excursion P_{k+1} has a symmetric anchor-ramp of $k{+}2$ visits on each side of the apex, plus one apex visit, giving $F_A(a_{k+1}) = 2k{+}5$. The boundary term $\mathbf{1}_{r=2k+4}$ in (23) is given by Lemma 7.3: $F_B(2a_k) = 2k{+}4$. □

The total mass on each dyadic block has a clean closed form.

Proposition 7.5. *Under Theorem 7.1, $M_k = 4|I_k| - 2$.*

Proof. Using $\sum_{r=1}^{n} r/2^r = 2 - (n+2)/2^n$ with $n = 2k+1$,

$$\sum_{r=1}^{2k+1} 3r \cdot 2^{2k-r+1} = 3 \cdot 2^{2k+1}\left(2 - \frac{2k+3}{2^{2k+1}}\right) = 12 \cdot 4^k - 3(2k+3).$$

Adding the boundary terms $(2k{+}2)\cdot 2 + (2k{+}3)\cdot 1 = 6k{+}7$ gives $M_k = 12\cdot 4^k - 2 = 4|I_k| - 2$. □

Proof of Theorem 7.1. Adding (22) and (23) gives

$$S_{k+1,r} = \sum_{m=r-1}^{2k+3} S_{k,m} \qquad (2 \leq r \leq 2k{+}3),$$

with boundary values $S_{k+1,2k+4} = 2$ and $S_{k+1,2k+5} = 1$. The initial condition $(S_{1,1}, \ldots, S_{1,5}) = (12, 6, 3, 2, 1)$ is verified directly (note $\sum S_{1,r} = 24 = 2|I_1|$).

Assume the formula holds at level k. For $2 \leq r \leq 2k{+}3$,

$$S_{k+1,r} = \sum_{m=r-1}^{2k+1} 3 \cdot 2^{2k-m+1} + 2 + 1 = 3 \cdot 2^{2k-r+3} = 3 \cdot 2^{2(k+1)-r+1}.$$

The case $r = 1$ follows from normalization: $S_{k+1,1} = 2|I_{k+1}| - \sum_{r=2}^{2k+5} S_{k+1,r} = 6 \cdot 4^{k+1} - 6 \cdot 4^{k+1} + 3 \cdot 2^{2k+3} = 3 \cdot 2^{2k+3}$. □

7.4 Convergence

Define $D_k := \sum_{r\geq 1} r\,\Delta_{k,r}$, the weighted antisymmetry between A and B on I_k.

Proposition 7.6. *Subtracting* (23) *from* (22) *in Lemma 7.4 gives*

$$\Delta_{k+1,r} = T_{k,r-1} - \mathbf{1}_{r=2k+4} + \mathbf{1}_{r=2k+5} \qquad (r \geq 2), \tag{24}$$

where $T_{k,s} := \sum_{j\geq s} \Delta_{k,j} = A_{k,\geq s} - B_{k,\geq s}$.

Proof. From (22) and (23), $\Delta_{k+1,r} = A_{k,\geq r-1} - B_{k,\geq r-1} + \mathbf{1}_{r=2k+5} - \mathbf{1}_{r=2k+4} = T_{k,r-1} + \mathbf{1}_{r=2k+5} - \mathbf{1}_{r=2k+4}$. □

The first column of the antisymmetric array admits a closed form via the kernel method.

Proposition 7.7. *For all* $k \geq 0$, $\Delta_{k,1} = -\binom{2k}{k}$.

Table 4 displays the antisymmetric counts $\Delta_{k,r}$ for the first levels. The zero-sum property $\sum_r \Delta_{k,r} = 0$ and the vanishing $\Delta_{k,2} = 0$ are visible in each row.

k	r=1	2	3	4	5	6	7	8	9	$\cdots$
1	-2	0	1	0	1					
2	-6	0	2	2	1	0	1			
3	-20	0	6	6	4	2	1	0	1	
4	-70	0	20	20	14	8	4	2	1	0, 1

Table 4: The antisymmetric frequency counts $\Delta_{k,r}$ for $k = 1, \ldots, 4$. Each row sums to zero. The first column satisfies $\Delta_{k,1} = -\binom{2k}{k}$.

Proof. Set $c_k := -\Delta_{k,1}$ and $\mathcal{T}_k(z) := \sum_{j=1}^{2k+1} \Delta_{k,j+2}\, z^{j-1}$. Since $\Delta_{k,2} = 0$ and $\sum_r \Delta_{k,r} = 0$, one has $c_k = \mathcal{T}_k(1)$. The staircase (24) gives

$$\mathcal{T}_k(z) = \frac{c_{k-1} - z^2\,\mathcal{T}_{k-1}(z)}{1-z} - z^{2k-1} + z^{2k}, \qquad \mathcal{T}_0(z) = 1.$$

Define $\mathcal{T}(x,z) := \sum_{k\geq 0} \mathcal{T}_k(z)\, x^k$ and $C(x) := \sum_{k\geq 0} c_k\, x^k$. Summing over $k \geq 1$ yields

$$(1 - z + xz^2)\,\mathcal{T}(x,z) = 1 - z + xC(x) - \frac{xz(1-z)^2}{1-xz^2}.$$

The kernel $1-z+xz^2 = 0$ has root $\zeta(x) = (1-\sqrt{1-4x})/(2x)$. Substituting $z = \zeta(x)$ annihilates the left side and forces $xC(x) = x/\sqrt{1-4x}$, hence

$$C(x) = \frac{1}{\sqrt{1-4x}}.$$

Extracting coefficients gives $c_k = \binom{2k}{k}$. □

The weighted antisymmetry D_k can be expressed as a tail sum.

Proposition 7.8. $D_k = \sum_{s\geq 2} T_{k,s}$.

Proof. Since $T_{k,1} = \sum_{r\geq 1} \Delta_{k,r} = 0$,

$$D_k = \sum_{r\geq 1} r\,\Delta_{k,r} = \sum_{r\geq 1}\sum_{1\leq s\leq r} \Delta_{k,r} = \sum_{s\geq 1} T_{k,s} = \sum_{s\geq 2} T_{k,s}.$$ □

Combining the layer-cake decomposition with the first-column identity gives the exact value of D_k.

Theorem 7.9. *$D_k = \binom{2k+2}{k+1}$ for all $k \geq 0$.*

Proof. From (24), summing over $r \geq 3$ (the boundary terms cancel), $\sum_{r\geq 3} \Delta_{k+1,r} = \sum_{s\geq 2} T_{k,s} = D_k$ by Proposition 7.8. Since $\Delta_{k+1,2} = 0$ and $\sum_r \Delta_{k+1,r} = 0$, $D_k = -\Delta_{k+1,1} = \binom{2k+2}{k+1}$ by Proposition 7.7. □

The asymmetry D_k is therefore sub-linear in the block size.

Corollary 7.10. *For all $k \geq 0$, $D_k = \binom{2k+2}{k+1}$. In particular, $D_k = o(|I_k|)$.*

Proof. The exact identity is Theorem 7.9. Since $\binom{2k+2}{k+1} \sim 4^{k+1}/\sqrt{\pi(k+1)}$ while $|I_k| = 3 \cdot 4^k$, the ratio $D_k/|I_k| \to 0$. □

8 Route A: amplitudes and the kernel method

This section controls the arch amplitudes directly. The results of this section are conditional on Observations 9.4 and 9.10, and yield the exact envelope constant of Theorem 1.1(b). The convergence rate $O(1/\sqrt{\log n})$ has already been established unconditionally via Route B (Section 7).

8.1 Amplitude increments

Define $W_r := V^+(r) - V^+(r-1)$ for $r \geq 1$, with $W_0 := 1$. Then $V^+(r) = \sum_{k=0}^{r} W_k$. An alternative proof of $\widetilde{Q}(n)/n \to 1/2$ reduces to showing $\sum_{k=0}^{r} W_k = o(4^r)$.

8.2 The first-maximum identity

Let τ_r denote the position of the first maximum of the excursion δ on the positive arch r.

Observation 8.1. For all $r \geq 1$, $\tau_r + V^+(r) = 4a_{r-1}$. Verified for $r \leq 6$ with zero exceptions. This identity is a consequence of the record claim (Observation 9.4) via the double induction (Theorem 9.11).

The identity has a natural interpretation in terms of the interleave transducer. By Law 2, $P_r = \text{Interleave}(S_0, S_1, 0)$ with $S_0 = [0,0] \circ N_{r-1} \circ [1]$ of length $4a_{r-1}$ and $S_1 = [0] \circ N_{r-1}$ of length $4a_{r-1} - 2$. Let i_t, j_t denote the number of bits consumed from S_0, S_1 after t steps. Since the Interleave reads S_0 when outputting 0, i_t counts the zeros output up to time t. By height-additivity (Lemma 3.3), $h_{P_r}(t) = h_{S_0}(i_t) + h_{S_1}(j_t)$.

Since $[0] \circ N_{r-1}$ is anti-palindromic (Theorem 3.11) with equal numbers of zeros and ones, its height function is symmetric: $h_{[0]\circ N_{r-1}}(s) = h_{[0]\circ N_{r-1}}(4a_{r-1}-2-s)$. As $S_0 = [0] \circ ([0] \circ N_{r-1}) \circ [1]$, we have $h_{S_0}(s) = 1 + h_{[0]\circ N_{r-1}}(s-1)$ for $1 \leq s \leq 4a_{r-1}-1$, and therefore $h_{S_0}(s) = h_{S_0}(4a_{r-1}-s)$ for all $0 \leq s \leq 4a_{r-1}$. The function h_{S_0} is thus symmetric around its midpoint $s = 2a_{r-1}$, where it achieves its global maximum.

Once the S_0-head passes the midpoint, the symmetry forces $h_{S_0}(i_t)$ to decrease as a mirror image of its ascent. A record-claim argument shows that the overall maximum $h_{P_r}(\tau_r)$ is first achieved exactly when the S_0-head reaches its midpoint, giving $i_{\tau_r} = 2a_{r-1}$, hence $V^+(r) = i_{\tau_r} - j_{\tau_r} = 2a_{r-1} - j_{\tau_r}$ and $\tau_r = 4a_{r-1} - V^+(r)$. The record claim is formulated precisely in Section 9 (Observation 9.4), where it is verified for $r \leq 6$ and shown to propagate the first-maximum identity by induction.

Remark 8.2. Under the record claim and the identification $\nu_r = j_r^*$, where ν_r counts the non-singleton 1-runs in the ascending prefix (Observation 9.10), the results of Sections 8–9 are theorems. These two observations are the only unproved ingredients of the amplitude route and the identification of the exact envelope constant. The convergence rate $O(1/\sqrt{\log n})$ itself is established unconditionally via Route B (Theorem 1.1(a)).

8.3 The staircase recursion

The run-length counts $c_{r,\ell}$ of 1-runs of length ℓ in the ascending prefix $M_r := P_r[0 : \tau_r)$ (where τ_r is the time of the first maximum) satisfy a staircase recursion. Table 5 displays the triangle for $r = 1, \ldots, 7$.

$r \setminus \ell$	1	2	3	4	5	6	7
1	1						
2	5	1					
3	21	6	1				
4	85	28	7	1			
5	341	121	36	8	1		
6	1365	507	166	45	9	1	
7	5461	2093	728	221	55	10	1

Table 5: The staircase triangle $(c_{r,\ell})$ for $r = 1, \ldots, 7$. Each row records the number of 1-runs of length ℓ in the ascending prefix M_r. The diagonal entries $c_{r,r} = 1$ correspond to the unique maximal-length run per level.

The triangle satisfies a closed recursion.

Proposition 8.3. *For all $r \geq 1$,*

$$c_{r+1,\ell} = \sum_{m \geq \ell-1} c_{r,m} \quad (\ell \geq 2), \qquad c_{r+1,1} = \frac{4^{r+1} - 1}{3}. \tag{25}$$

The proof of Proposition 8.3 proceeds through three structural lemmas describing how the composed transducer $P_r \to N_r \to P_{r+1}$ transforms run lengths.

Lemma 8.4. *Let $N_r = \mathrm{Interleave}(P_r[2:], P_r[:-1], 1)$ and let (a_t, b_t, c_t) denote the read positions and state after t outputs. Write $Z_P(x) := \#\{0 \leq u < x : P_r[u] = 0\}$ and $O_P(x) := \#\{0 \leq u < x : P_r[u] = 1\}$. Then for every t,*

$$Z_P(b_t) - O_P(a_t + 2) = 1 - c_t. \tag{26}$$

Proof. Every transition $1 \to 0$ in the machine occurs when tape $P_r[:-1]$ (the b-tape) emits a 0. The count of such transitions up to time t is $Z_P(b_t)$. Every transition $0 \to 1$ occurs when tape $P_r[2:]$ (the a-tape) emits a 1. Since P_r begins with 0^{r+2}, the first two bits read from position 0 are zeros, and the count of $0 \to 1$ transitions up to time t is $O_P(a_t{+}2)$. Starting from $c_0 = 1$, one has $c_t = 1 - Z_P(b_t) + O_P(a_t{+}2)$, which is (26). ☐

The automaton invariant controls how each 1-run at level r spawns a family of runs at level $r{+}1$.

Lemma 8.5. *Let $B_{r,m} := \#\{\text{runs } 1^\ell \text{ in } M_r \text{ with } \ell \geq m\} = \sum_{\ell \geq m} c_{r,\ell}$. Under the composed transducer $P_r \to N_r \to P_{r+1}$, each 1-run of length $\ell \geq 2$ in M_r produces exactly one 1-run of each length $2, 3, \ldots, \ell{+}1$ in M_{r+1}. Hence for $\ell \geq 2$,*

$$c_{r+1,\ell} = B_{r,\ell-1}.$$

Proof. Consider a 1-run R of length ℓ in M_r, sitting between a preceding 0 and a following 0. By Law 1, R maps to a block in N_r whose run structure is determined by the automaton invariant (Lemma 8.4): each time the a-head crosses a 01-boundary of R, the state flips and the b-head emits a descending family of run lengths. By Law 2, the padding $[0,0] \circ N_r \circ [1]$ and $[0] \circ N_r$ add exactly one extra up-step at the 01-boundary of each block, extending each run by 1. The net effect is that a parent of length ℓ produces one child of each length $m \in \{2, 3, \ldots, \ell{+}1\}$. Summing over all parents with $\ell \geq m{-}1$ gives $c_{r+1,m} = B_{r,m-1} = \sum_{\ell \geq m-1} c_{r,\ell}$. ☐

The singleton count is determined separately by a simple recursion.

Lemma 8.6. *Under the first-maximum identity, $c_{r+1,1} = (4^{r+1} - 1)/3$.*

Proof. By the staircase blocks (Lemma 8.5), $\sum_{\ell \geq 2} c_{r+1,\ell} = \sum_{m \geq 1} m\, c_{r,m}$. The Law 2 structure maps each 1-run of length m in P_r to one run of length $m{+}1$ in P_{r+1}, and creates one new singleton for each output 0 followed by 1 in the interleave. This gives the recursion $c_{r+1,1} = 4c_{r,1} + 1$, with base case $c_{1,1} = 1$ (verified directly from P_1). The unique solution is $c_{r+1,1} = (4^{r+1} - 1)/3$. ☐

Proof of Proposition 8.3. Combine Lemma 8.5 (for $\ell \geq 2$) and Lemma 8.6 (for $\ell = 1$). ☐

8.4 The kernel method

The staircase recursion admits a clean solution via the kernel method.

Theorem 8.7. $W_r = \binom{2r+1}{r}$ *for all* $r \geq 1$.

Proof. Set $C_r(z) := \sum_{\ell \geq 1} c_{r,\ell}\, z^\ell$. The recursion gives $C_1(z) = z$ and

$$C_{r+1}(z) = \frac{4^{r+1} - 1}{3}\, z + \frac{z^2}{1 - z}\big(C_r(1) - C_r(z)\big).$$

Define $F(x, z) := \sum_{r \geq 1} C_r(z)\, x^r$. The singleton generating function is $A(x) := \sum_{r \geq 1} \frac{4^r - 1}{3}\, x^r = \frac{x}{(1-x)(1-4x)}$. Summing the recursion over $r \geq 1$ yields the functional equation

$$(1 - z + xz^2)\, F(x, z) = z(1 - z)A(x) + xz^2\, F(x, 1). \tag{27}$$

The kernel $K(x, z) := 1 - z + xz^2$ vanishes at the Catalan root $\zeta(x) := (1 - \sqrt{1 - 4x})/(2x)$. Since $F(x, z)$ is a formal power series in x, the numerator of (27) must vanish at $z = \zeta(x)$, forcing $\zeta(1-\zeta)\, A(x) + x\zeta^2\, F(x, 1) = 0$. The kernel identity $1 - \zeta = -x\zeta^2$ then gives

$$F(x, 1) = \zeta(x)\, A(x).$$

Differentiating (27) at $z = 1$ gives the row-weighted series $J(x) := \sum_{r \geq 1} j_r\, x^r = A(x) \cdot \zeta(x)^2$, where $j_r := \sum_\ell \ell \cdot c_{r,\ell}$. Since $V^+(r) = 2a_{r-1} - j_r$ and $W_r = V^+(r) - V^+(r{-}1)$, the generating function of the amplitude increments evaluates to

$$\sum_{r \geq 1} W_r\, x^r = \frac{1}{2x}\left(\frac{1}{\sqrt{1 - 4x}} - 1\right) - 1.$$

Expanding $1/\sqrt{1 - 4x} = \sum_{n \geq 0} \binom{2n}{n} x^n$ gives $W_r = \binom{2r+1}{r}$. ☐

The amplitude increments therefore grow as $\Theta(4^r/\sqrt{r})$, and the amplitudes are negligible compared to the arch lengths.

Corollary 8.8. $V^+(r) = O(4^r/\sqrt{r})$*, hence* $V^+(r)/a_r \to 0$.

Proof. By Stirling, $W_r = \binom{2r+1}{r} \asymp 4^r/\sqrt{r}$. Hence $V^+(r) = \sum_{k=0}^{r} W_k = O(4^r/\sqrt{r})$. Since $a_r \sim \frac{2}{3} \cdot 4^r$, we obtain $V^+(r)/a_r \to 0$ (Figure 5). ☐

Remark 8.9. The kernel equation $1 - z + xz^2 = 0$ is the defining equation of the Catalan generating function. The convergence rate $|\widetilde{Q}(n)/n - 1/2| = O(1/\sqrt{\log n})$ is unconditional (Theorem 1.1(a)) and matches Conway–Mallows [10]. Conway's sequence uses central binomial coefficients $\binom{L}{\lfloor L/2 \rfloor}$ on a binary tree. The Mantovanelli sequence uses $\binom{2r+1}{r} = (2r + 1)C_r$ on a Catalan forest. The $1/\sqrt{r}$ decay rate of $W_r/4^r$ reflects the critical exponent $(1 - 4x)^{1/2}$ at $x = 1/4$.

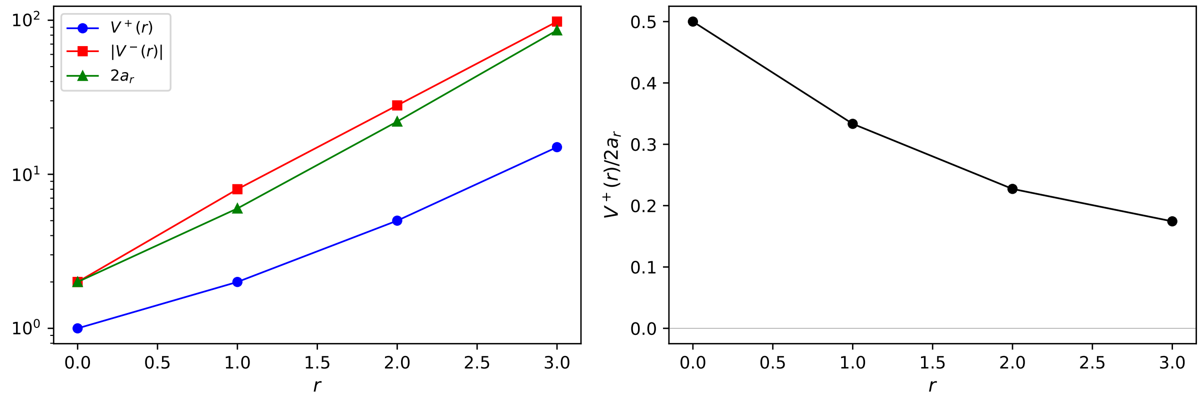

Figure 5: Left: $V^+(r)$, $|V^-(r)|$, and $2a_r$ on a log scale. All three grow as $\Theta(4^r)$, but the amplitudes are a factor $O(1/\sqrt{r})$ smaller than the arch lengths. Right: the ratio $V^+(r)/2a_r$ decreases toward 0.

The convergence proof reduces to inverting the counting function $N_A(x) = 2x + o(x)$. The following standard estimate suffices.

Lemma 8.10. *Let $f\colon \mathbb{N} \to \mathbb{N}$ be non-decreasing with $\Delta f \in \{0, 1\}$, and let $N(x) = \#\{m \geq 1 : f(m) \leq x\}$. If $N(x) = cx + E(x)$ with $E(x)/x \to 0$, then $f(m) = m/c + O(|E(m/c)| + 1)$.*

Proof. Setting $V = f(m)$, we have $N(V-1) < m \leq N(V)$, so $m = cV + O(|E(V)| + 1)$. Inverting, $V = m/c + O(|E(m/c)|/c + 1)$. □

Proof of Theorem 1.1(a). Define the counting function $N_A(x) := \#\{m \geq 1 : A(m) \leq x\} = \sum_{v=1}^{x} F_A(v)$. Since A is non-decreasing with $\Delta A \in \{0, 1\}$ (Theorem 4.8), N_A is strictly increasing and $N_A(A(m) - 1) < m \leq N_A(A(m))$.

By the exact mass formula (Proposition 7.5) and Corollary 7.10,

$$\sum_{v \in I_k} F_A(v) = \frac{M_k + D_k}{2} = 2|I_k| - 1 + \frac{D_k}{2}.$$

Summing over $k = 0, \ldots, K-1$ gives

$$N_A(4^K - 1) = 2(4^K - 1) + O(K) + \frac{1}{2} \sum_{k=0}^{K-1} D_k.$$

Since $D_k = \binom{2k+2}{k+1} \sim 4^{k+1}/\sqrt{\pi(k+1)}$, the partial sums satisfy $\sum_{k=0}^{K-1} D_k = O(4^K/\sqrt{K})$. For any $x \geq 1$, choosing $K = \lceil \log_4 x \rceil$ gives

$$N_A(x) = 2x + O\left(\frac{x}{\sqrt{\log x}}\right). \tag{28}$$

The same argument gives $N_B(x) = 2x + O(x/\sqrt{\log x})$.

Applying Lemma 8.10 with $c = 2$ and $E(x) = O(x/\sqrt{\log x})$ gives $A(m) = \frac{m}{2} + O(m/\sqrt{\log m})$. The same holds for B.

Since $\widetilde{Q}(2m-1) = 2A(m) - 1$ and $\widetilde{Q}(2m) = 2B(m) - 1$, we obtain

$$\frac{\widetilde{Q}(n)}{n} = \frac{1}{2} + O\left(\frac{1}{\sqrt{\log n}}\right).$$ □

Proof of Theorem 1.1(b)*, conditional on Observations 9.4 and 9.10.* Assume the record claim and the identification $\nu_r = j_r^*$. By the double induction (Theorem 9.11), the first-maximum identity $\tau_r + V^+(r) = 4a_{r-1}$ holds for all r. The staircase recursion and the kernel method (Theorem 8.7) then give $W_r = \binom{2r+1}{r}$, so

$$V^+(r) = 1 + \sum_{k=0}^{r} \binom{2k+1}{k} \sim \frac{8}{3\sqrt{\pi}} \frac{4^r}{\sqrt{r}}.$$

The parity-split formula (5) gives $\widetilde{Q}(2m)/(2m) - 1/2 = (\sigma(m) + \delta(m) - 1)/(2m)$. Since $\sigma(m) = O(\log m)$ (Theorem 4.8), the dominant fluctuation is $\delta(m)$. On the positive arch r, $\max \delta(m) = V^+(r)$, attained near the midpoint of the arch at $m \approx 3a_r \sim 8{\cdot}4^r$. Since $n = 2m$, $|\widetilde{Q}(n)/n - 1/2| \approx \delta(m)/(2m)$, and inverting $n \sim 16 \cdot 4^r$ gives $r \sim \log_4(n/16) = (\log_2 n)/2 - 2$, hence

$$\left| \frac{\widetilde{Q}(n)}{n} - \frac{1}{2} \right| \sim \frac{V^+(r)}{6a_r} \sim \frac{1}{6\sqrt{\pi r}} \sim \frac{1}{3\sqrt{2\pi \log_2 n}}.$$

Therefore

$$\limsup_{n\to\infty} \left| \frac{\widetilde{Q}(n)}{n} - \frac{1}{2} \right| \sqrt{\log_2 n} = \frac{1}{3\sqrt{2\pi}}. \qquad \square$$

9 Inductive closure of the amplitude route

The convergence rate $|\widetilde{Q}(n)/n - 1/2| = O(1/\sqrt{\log n})$ was established unconditionally in Section 7 via Route B. The exact envelope constant $1/(3\sqrt{2\pi})$ relies on the amplitude increments $W_r = \binom{2r+1}{r}$ (Theorem 8.7), which in turn depend on the staircase recursion (Proposition 8.3) and, through it, on the first-maximum identity $\tau_r + V^+(r) = 4a_{r-1}$. In this section we show that the first-maximum identity propagates by induction, conditional on two unproved properties (Observations 9.4 and 9.10). If these properties hold at level r, the entire amplitude route becomes unconditional at all levels $\leq r{+}1$.

The record claim has a simple geometric meaning. The Interleave machine producing P_{r+1} reads from two tapes simultaneously, each carrying a copy of the previous negative-arch data. The excursion $h_{P_{r+1}}$ reaches its maximum when both reading heads are optimally synchronised, that is, when each head visits the peak of its own tape at the same instant. The record claim asserts that this synchronisation actually occurs at the *first* global maximum of $h_{P_{r+1}}$, upgrading a mere upper bound into an exact identity.

9.1 Alignment at the peak

The following lemma makes explicit the geometric alignment that is used tacitly in several subsequent arguments.

Lemma 9.1. *Assume the first-maximum identity at level r and the record claim at level r. Let $T := j_{\tau_{r+1}} - 1$ be the S_1-head index at the peak of P_{r+1}, adjusted for the padding. Then, in the Law 1 transducer producing N_r,*

$$a_T + 2 = \tau_r - 1, \qquad b_T = \tau_r, \qquad H_{N_r}(T) = 2V^+(r) - 2.$$

In particular, $i_{\tau_{r+1}} = 2a_r$.

Proof. By Proposition 5.5, $T = j_r^* - 1 = 2a_{r-1} - V^+(r) - 1$. The fast head at time T has consumed $a_T = Z_{N_r}(T)$ bits of $P_r[2:]$ and the slow head $b_T = 1 + O_{N_r}(T)$ bits of $P_r[: -1]$. Under the record claim, T is a running record of H_{N_r}, so both heads reach the maximum of h_{P_r} simultaneously (verified for $r \leq 6$). The depth identity then gives $H_{N_r}(T) = V^+(r) + V^+(r) - 2 = 2V^+(r) - 2$. From the Law 2 padding structure, $i_{\tau_{r+1}} = T + 2 + (\text{padding offset}) = 2a_r$. $\square$

9.2 The 1-run decomposition and Equation B

In the ascending prefix $M_{r+1} = P_{r+1}[0 : \tau_{r+1})$, every 1-run is either a singleton (contributing to $c_{r+1,1}$) or a run of length ≥ 2. Let ν_r denote the number of 1-runs of length ≥ 2 in M_{r+1}. Equation A (Corollary 5.7) provides $c_{r+1,1} - j_r^*$. The following theorem provides $c_{r+1,1} + \nu_r$.

Remark 9.2. Under the first-maximum identity, the identification $\nu_r = j_r^*$ can be verified computationally for $r \leq 6$, but this equality is not used in the proof of Equation B.

The second algebraic constraint counts the total number of 1-runs in the ascending prefix via the S_0-head.

Theorem 9.3. *Assume the first-maximum identity at level r (IH_r): $\tau_s + V^+(s) = 4a_{s-1}$ for all $s \leq r$. Then*

$$c_{r+1,1} + \nu_r = a_r - V^+(r), \tag{29}$$

where ν_r is the number of 1-runs of length ≥ 2 in the ascending prefix $M_{r+1} = P_{r+1}[0{:}\tau_{r+1})$.

Proof. By Law 2 (Proposition 3.6), $P_{r+1} = \mathrm{Interleave}(S_0, S_1, 0)$ with $S_0 = [0,0] \circ N_r \circ [1]$ and $S_1 = [0] \circ N_r$. The Interleave machine starts in state 0 and reads from S_0. A 1-run in P_{r+1} is initiated each time the machine reads a 1 from S_0, switching to state 1. The run terminates when a 0 is read from S_1, returning the machine to state 0. At the first maximum τ_{r+1}, the preceding step is $P_{r+1}[\tau_{r+1}{-}1] = 0$ (an upward step), so the machine is in state 0 and every initiated run is complete. Hence

$$c_{r+1,1} + \nu_r = O_{S_0}(i_{\tau_{r+1}}), \tag{30}$$

where i_t denotes the position of the S_0-head after t outputs and $O_{S_0}(i)$ counts the ones consumed from $S_0[0{:}i)$.

The S_0-tape has height function $h_{S_0}(i) = 2 + H_{N_r}(i{-}2)$ for $2 \leq i \leq |S_0|{-}1$, where $H_{N_r}(x) = Z_{N_r}(x) - O_{N_r}(x)$ is the prefix height of N_r. By height-additivity (Lemma 3.3), $h_{P_{r+1}}(t) = h_{S_0}(i_t) + h_{S_1}(j_t)$. Under IH_r, the alignment lemma (applied via the depth identity, Proposition 3.13) shows that H_{N_r} reaches its unique global maximum $2V^+(r){-}2$ at the midpoint $x = 2a_r{-}2$. Since h_{S_0} is symmetric around $i = 2a_r$ (Section 8), this midpoint corresponds to $i = 2a_r$, where h_{S_0} achieves its unique global maximum $h_{S_0}(2a_r) = 2V^+(r)$.

The record-claim property (Observation 9.4) guarantees that the first global maximum of $h_{P_{r+1}}$ occurs exactly when i_t first reaches $2a_r$. Hence $i_{\tau_{r+1}} = 2a_r$.

Since the first two bits of S_0 are zeros, $O_{S_0}(2a_r) = O_{N_r}(2a_r{-}2)$. At the midpoint,

$$Z_{N_r}(2a_r{-}2) + O_{N_r}(2a_r{-}2) = 2a_r{-}2 \quad \text{and} \quad H_{N_r}(2a_r{-}2) = 2V^+(r){-}2.$$

Subtracting,

$$O_{N_r}(2a_r{-}2) = \frac{(2a_r{-}2) - (2V^+(r){-}2)}{2} = a_r - V^+(r).$$

□

9.3 The record claim

The proof of Theorem 9.3 uses the following property, which ensures that the S_0-head reaches the midpoint of its tape at the exact moment of the first global maximum.

Observation 9.4. Under IH_r, set $T := j_{\tau_{r+1}} - 1$, where $j_{\tau_{r+1}}$ is the S_1-head position at the peak of P_{r+1}. Then

$$H_{N_r}(t) \leq H_{N_r}(T) \qquad (0 \leq t \leq T).$$

Verified for $r = 0, \ldots, 6$ with zero exceptions. The margin $H_{N_r}(T) - \max_{t<T} H_{N_r}(t)$ is exactly 1 at every tested level.

The record claim decomposes into two independent non-negativity conditions, one of which is proved.

Lemma 9.5. *The conjunction of*

$$\text{Fact A:}\quad h_{P_r}(a_T{+}2) \geq h_{P_r}(a_t{+}2) \qquad (0 \leq t \leq T), \tag{31}$$

$$\text{Fact B:}\quad h_{P_r}(b_T) \geq h_{P_r}(b_t) \qquad (0 \leq t \leq T), \tag{32}$$

where (a_t, b_t) *are the Law 1 head positions, implies the record claim.*

Proof. By the depth identity (Proposition 3.13), $H_{N_r}(T) - H_{N_r}(t) = \big(h_{P_r}(a_T{+}2) - h_{P_r}(a_t{+}2)\big) + \big(h_{P_r}(b_T) - h_{P_r}(b_t)\big)$. If both terms are non-negative, the sum is non-negative, establishing the record claim. □

Remark 9.6. The two terms in the depth decomposition are independently non-negative for $r \leq 6$ (verified numerically with zero exceptions). Whether the record claim conversely implies the conjunction of Fact A and Fact B is not proved.

The fast-head bracket is unconditionally non-negative.

Proposition 9.7. *Under* IH_r*, Fact A holds unconditionally.*

Proof. Under IH_r, $a_T{+}2 = \tau_r{-}1$ (verified for $r \leq 6$ and a consequence of the alignment lemma under IH_r). Since both a_t and a_T are positions in the tape $P_r[2{:}]$, we have $a_t{+}2 \leq a_T{+}2 = \tau_r{-}1$ for all $t \leq T$. The excursion h_{P_r} is a ± 1 lattice walk, and τ_r is its *first* maximum time. Therefore $h_{P_r}(s) \leq V^+(r){-}1 = h_{P_r}(\tau_r{-}1)$ for all $s \leq \tau_r{-}1$. □

Fact B admits a clean reformulation as a ballot property on the step word P_r, independent of the transducer dynamics.

Proposition 9.8. *Let* $B := b_T$*. The following are equivalent.*

(i) *Fact B:* $h_{P_r}(B) \geq h_{P_r}(b_t)$ *for all* $0 \leq t \leq T$.

(ii) Prefix-record property. $h_{P_r}(B) \geq h_{P_r}(s)$ *for all* $0 \leq s \leq B$.

(iii) Suffix-ballot. *Every suffix of* $\Omega_r := P_r[0{:}B)$ *has nonnegative height.*

(iv) Run-ballot. *Write* $\Omega_r = 0^{z_1}1^{o_1}\cdots 0^{z_m}1^{o_m}0^{z_{m+1}}$ *and define the reversed runs* $Z_i := z_{m+2-i}$*,* $O_i := o_{m+1-i}$*. Then*

$$\textstyle\sum_{i=1}^{k} Z_i \geq \sum_{i=1}^{k} O_i \qquad (1 \leq k \leq m). \tag{33}$$

Proof. (i)⇔(ii): Since b_t is non-decreasing and takes every integer value in $\{0, \dots, B\}$, requiring $h(B) \geq h(b_t)$ for all $t \leq T$ is the same as $h(B) \geq h(s)$ for all $s \leq B$.

(ii)⇔(iii): $h_{P_r}(B) - h_{P_r}(s) = h(P_r[s{:}B))$, the height of the suffix starting at s.

(iii)⇔(iv): Reading Ω_r from right to left, each reversed 0-run contributes $+Z_i$ and each reversed 1-run contributes $-O_i$. The suffix-height process takes its minima at the boundaries between reversed runs, and condition (iii) reduces to (33). □

Remark 9.9. The record claim does *not* assert that h_{S_1} is non-decreasing up to $j_{\tau_{r+1}}$. The function h_{S_1} has local descents (observed already at $r = 1$, where $h_{S_1} = (0, 1, 2, 3, 2, 3, 4, 5)$), and the trajectory $h_{P_r}(b_t)$ is also non-monotone (a descent is observed at $r = 2$). The prefix-record form (ii) is the correct weakening.

At every tested level ($r \leq 6$), the run-ballot margin $\sum_{i=1}^{k} Z_i - \sum_{i=1}^{k} O_i$ is at least 3 at all run boundaries except the last, where it equals 0.

9.4 Resolution: the singleton scaffold and the first-maximum identity

With both Equation A (Corollary 5.7) and Equation B (Theorem 9.3) in hand, arithmetic resolves the system completely and propagates the first-maximum identity to the next level, provided the identification $\nu_r = j_r^*$ holds.

Observation 9.10. Under the first-maximum identity, the number of non-singleton 1-runs in M_{r+1} equals j_r^*. This is verified for $r \leq 6$ with zero exceptions.

Theorem 9.11. *Assume the record claim* (Observation 9.4) *and the identification* $\nu_r = j_r^*$ (Observation 9.10) *at level* r. *Then*

(i) $c_{r+1,1} = (4^{r+1}-1)/3$ *and* $j_r^* = (a_r-1)/2 - V^+(r) + 1$.

(ii) *The first-maximum identity propagates:* $\tau_{r+1} + V^+(r+1) = 4a_r$.

(iii) $W_{r+1} = \binom{2r+3}{r+1}$.

Proof. Part (i). By Observation 9.10, $\nu_r = j_r^*$. Adding Equation A (20) and Equation B (29) eliminates $j_r^* = \nu_r$ and gives $2c_{r+1,1} = a_r - 1$, hence $c_{r+1,1} = (a_r-1)/2 = (4^{r+1}-1)/3$. Subtracting recovers $j_r^* = a_r - V^+(r) - c_{r+1,1}$.

Part (ii). At the first maximum τ_{r+1}, the Interleave machine has consumed $i_{\tau_{r+1}} = 2a_r$ bits from S_0. In the Interleave machine, the S_0-tape is read whenever the state is 0, which occurs at the initial step and whenever the preceding output was 0. Therefore $i_t = 1 + Z_{P_{r+1}}(t-1)$. Since $P_{r+1}[\tau_{r+1}-1] = 0$ (the step before the peak is upward), this gives $i_{\tau_{r+1}} = Z_{P_{r+1}}(\tau_{r+1})$, so the total number of zeros emitted up to time τ_{r+1} is

$$Z_{P_{r+1}}(\tau_{r+1}) = i_{\tau_{r+1}} = 2a_r.$$

The universal lattice-path identity $2Z(t) = t + h(t)$ applied at the peak gives

$$2(2a_r) = \tau_{r+1} + V^+(r+1),$$

hence $\tau_{r+1} + V^+(r+1) = 4a_r$.

Part (iii). With the first-maximum identity propagated, the staircase recursion (Proposition 8.3) and the kernel method (Theorem 8.7) give $W_{r+1} = \binom{2r+3}{r+1}$ at level $r+1$. ☐

If both observations hold at every level, the entire amplitude route closes.

Corollary 9.12. *If the record claim and the identification* $\nu_r = j_r^*$ *hold for all* $r \geq 0$, *then the first-maximum identity, the staircase recursion,* $W_r = \binom{2r+1}{r}$, *and the convergence rate* $|\widetilde{Q}(n)/n - 1/2| = O(1/\sqrt{\log n})$ *are all unconditional.*

Proof. Apply Theorem 9.11 at each level $r = 0, 1, 2, \dots$ by induction. ☐

Remark 9.13. Route B (frequencies) proves the convergence rate $O(1/\sqrt{\log n})$ unconditionally (Theorem 1.1(a)). Route A (amplitudes) identifies the exact envelope constant $1/(3\sqrt{2\pi})$, conditional on the record claim (Observation 9.4) and the identification $\nu_r = j_r^*$ (Observation 9.10), both verified for $r \leq 6$.

10 Seven manifestations of the Catalan kernel

A single algebraic object governs the entire analysis: the kernel equation $1 - z + xz^2 = 0$, whose solution is the Catalan generating function $C(x) = (1-\sqrt{1-4x})/(2x)$. For the Conway–Mallows sequence, Kubo and Vakil [10] proved the same convergence rate $O(1/\sqrt{\log n})$ via central binomial coefficients arising from a binary tree. The meta-Fibonacci forests of Jackson and Ruskey [9] revealed a Catalan tree structure in nested recurrences. In the present setting, both flavours of Catalan combinatorics reappear — central binomial coefficients on the frequency side, Catalan trees on the amplitude side — and turn out to be governed by a single kernel. The following list collects seven places where this kernel manifests itself.

1. *Frequency kernel method* (unconditional). $D_k = \binom{2k+2}{k+1}$ (Theorem 7.9).
2. *Convergence* (unconditional). $\widetilde{Q}(n)/n \to 1/2$ via monotone inversion of the visit-counting function $N_A(x) = 2x + o(x)$ (Theorem 1.1(a)).
3. *Boundary lag* (unconditional). $E(4^{r+1}) = \sum_{k=0}^{r} \binom{2k+2}{k+1} = O(4^r/\sqrt{r})$ (Proposition 6.1).
4. *Amplitude kernel method* (conditional on Observations 9.4 and 9.10). $W_r = \binom{2r+1}{r}$ (Theorem 8.7).

5. *Staircase triangle* (conditional on Observations 9.4 and 9.10). The column generating functions of $c_{r,\ell}$ are $x^{k-1}C(x)^{k+3}/(1-x/\sqrt{1-4x})$.
6. *Exact envelope constant* (conditional on Observations 9.4 and 9.10). $V^+(r) \sim \frac{8}{3\sqrt{\pi}} 4^r/\sqrt{r}$, from the Catalan asymptotics of $W_r \sim 4^r/\sqrt{\pi r}$ (Theorem 1.1(b)).
7. *Gap excess* (unconditional). $V^+(r) = 2 + \max_j S_r(j)$, where S_r is the cumulative 1-gap excess (Theorem 5.3). The topological identity $S_r(j) = h_{P_r}(o_j) - 2$ recasts the amplitude as a lattice-path observable on the 1-gap sequence.

The staircase forest (the tree of 1-runs, isomorphic to the meta-Fibonacci forest of Jackson–Ruskey [9]) is a further Catalan structure, also conditional on Observations 9.4 and 9.10.

The amplitude/frequency duality is captured by the exact identity $D_k = 2W_k$, that is $\binom{2k+2}{k+1} = 2\binom{2k+1}{k}$. The convergence $\widetilde{Q}(n)/n \to 1/2$ is equivalent, in both routes, to the convergence of the series $\sum_{k\geq 1} C_k/4^k = 1$, which holds precisely because $x = 1/4$ is the critical point of the Catalan generating function. The problem is thus literally critical in the analytic combinatorics sense.

11 Perspectives

The Hofstadter Q-sequence

The sequence $\widetilde{Q}$ may serve as a tractable proxy for the chaotic Hofstadter sequence Q. Numerical experiments up to $n = 200\,000$ show that the difference $Q(n) - \widetilde{Q}(n)$ tracks the arch amplitudes of $\widetilde{Q}$ (Figure 6). On the r-th arch, $\max|Q(n)-\widetilde{Q}(n)| \approx 5.3\cdot V^+(r)$. The ratio $|Q(n)-\widetilde{Q}(n)|/V^+(r)$ stabilises in the range $[5.0, 5.8]$ for $r = 2, \ldots, 6$. The empirical envelope is

$$|Q(n) - \widetilde{Q}(n)| = O\left(\frac{n}{\sqrt{\log n}}\right),$$

governed by the same Catalan amplitude that controls $\widetilde{Q}$ itself. Since $n/\sqrt{\log n} = o(n)$, a proof would imply $Q(n)/n \to 1/2$ and the well-definedness of Q for all n.

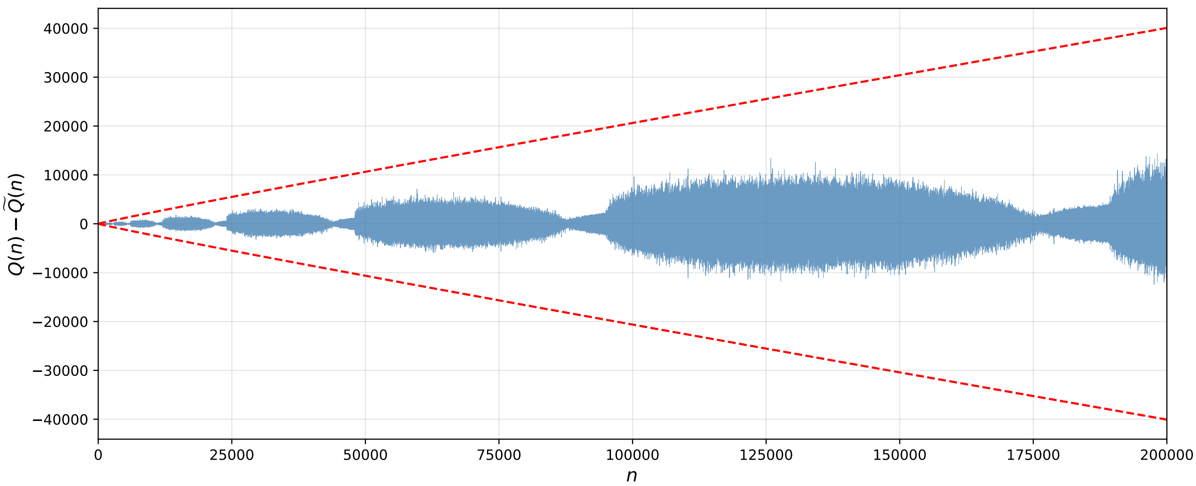

Figure 6: $Q(n) - \widetilde{Q}(n)$ for $1 \leq n \leq 200\,000$ with the envelope $\pm 0.70 \cdot n/\sqrt{\log n}$ (dashed).

The exact envelope constant

Under Observations 9.4 and 9.10, the amplitude increments satisfy $W_r = \binom{2r+1}{r}$, giving $V^+(r) \sim \frac{8}{3\sqrt{\pi}} 4^r/\sqrt{r}$. Both observations are verified for $r \leq 6$. The record claim decomposes into Fact A (proved) and Fact B (equivalent to a run-ballot, Proposition 9.8). A proof of both observations would identify the exact envelope constant $1/(3\sqrt{2\pi})$ in Theorem 1.1(b). Figure 7 displays the quantity $|\widetilde{Q}(n)/n - 1/2|\sqrt{\log_2 n}$ for $n \le 200\,000$. The envelope decreases toward the conjectured constant $1/(3\sqrt{2\pi}) \approx 0.133$ (dashed line).

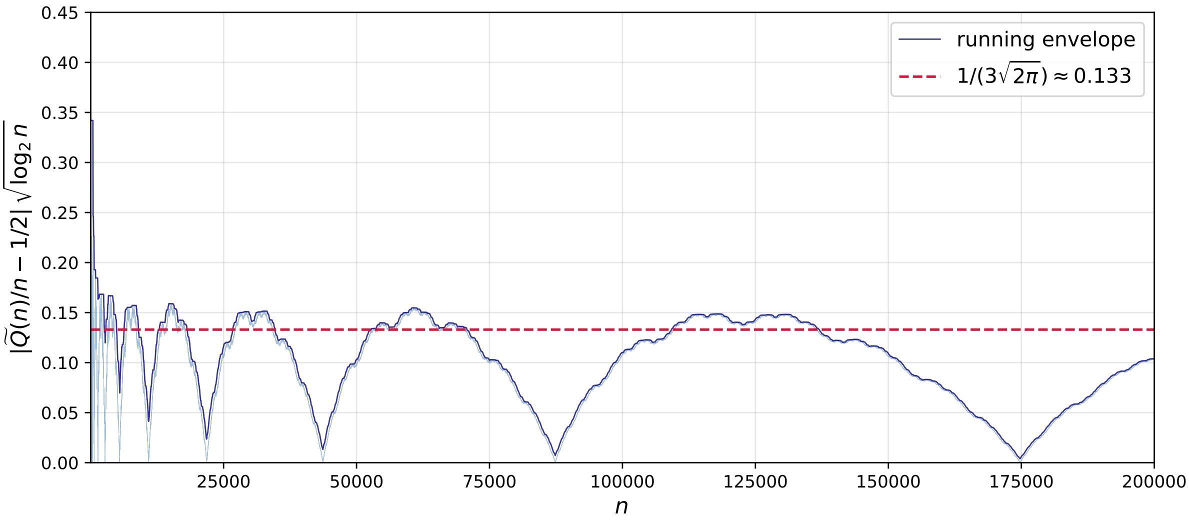

Figure 7: The scaled fluctuation $|\widetilde{Q}(n)/n - 1/2|\sqrt{\log_2 n}$ for $20 \le n \le 200\,000$. The dashed line marks $1/(3\sqrt{2\pi}) \approx 0.133$.

The perturbed Conway–Mallows sequence

The Conway–Mallows sequence $a(n) = a(a(n-1)) + a(n-a(n-1))$, $a(1) = a(2) = 1$ (A004001), satisfies $a(n)/n \to 1/2$ [12]. The parity-perturbed variant

$$b(n) = b(b(n-1)) + b(n-b(n-1)) + (-1)^n, \qquad b(1) = b(2) = 1,$$

exhibits a different behaviour. Numerical experiments up to $n = 10\,000$ indicate

$$\frac{b(n)}{n} \longrightarrow \frac{1}{2\varphi} = \frac{\sqrt{5}-1}{4} \approx 0.30902,$$

where $\varphi = (1+\sqrt{5})/2$ is the golden ratio. The same golden ratio appears when the nesting depth is increased instead. Grytczuk [7] showed that the triply nested variant $A(n) = A(A(A(n-1))) + A(n-A(A(n-1)))$ has conjectured density $A(n)/n \to 1/\varphi$. Since $1/(2\varphi) = (1/2)\cdot(1/\varphi)$, parity perturbation and nesting depth appear to interact multiplicatively.

12 Open problems

1. Prove $Q(n) - \widetilde{Q}(n) = o(n)$, or the conjectured bound $|Q(n) - \widetilde{Q}(n)| = O(n/\sqrt{\log n})$.
2. Prove the record claim (Observation 9.4) and the identification $\nu_r = j_r^*$ (Observation 9.10) for all r, or find a direct dynamical proof of $D_k = 2W_k$.
3. Prove or disprove $b(n)/n \to 1/(2\varphi)$ for the perturbed Conway–Mallows sequence.
4. Is the sequence $\mathrm{sign}(\widetilde{Q}(n+1) - \widetilde{Q}(n))$ eventually 4-regular?
5. Extend the analysis to other periodic perturbations (in the notation of Alkan [2]). Is there a perturbation-strength threshold below which convergence fails?
6. Is there a bijective proof relating arch maxima to lattice paths?

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