PRODUCTS OF MULTIPLE-INDEX FIBONACCI NUMBERS

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Abstract. Consider the generating function for the integer sequence $(F_{mi}F_{ni} : i \in \mathbb{N}_0)$, where m and n are positive integer parameters. We may easily compute this g.f. in terms of Fibonacci/Lucas numbers using an implementation of an algorithm due to Zeilberger. However, for the case whereby the integers m and n are of the same parity, we have experimentally discovered that there is a remarkably simpler way of expressing this g.f., compared to the corresponding expression obtained via Zeilberger's procedure. We prove this equivalence via Binet's formula, and then apply our simplified g.f. evaluation to generalize a classic Fibonacci sum identity given by Freitag, and in relation to the recent work of Melham on order-2 Fibonacci-type sums. Our evaluations for finite sums over $F_{mi}F_{ni}$ are dramatically simpler compared to corresponding output obtained via Zeilberger's `Cfinite` Maple package.

1. Introduction

A C-finite sequence is a sequence that satisfies a linear recurrence equation with constant coefficients [21]. As indicated in [21], the set of all C-finite sequences is closed under multiplication, i.e., under the Hadamard product operation on sequences [21]. Given two sequences C_1 and C_2, the Hadamard product C_1C_2 refers to the sequence $(C_1(i)C_2(i) : i \in \mathbb{N}_0)$. Letting $(F_i : i \in \mathbb{N}_0)$ denote the Fibonacci sequence, if we set C_1 as the sequence $(F_{mi} : i \in \mathbb{N}_0)$ for a natural number m, and if we set C_2 as $(F_{ni} : i \in \mathbb{N}_0)$ for a parameter $n \in \mathbb{N}$, a Maple implementation of a procedure due to Zeilberger [21] allows us to easily express the g.f. for the Hadamard product

$$(F_{mi}F_{ni} : i \in \mathbb{N}_0) \tag{1.1}$$

in terms of Fibonacci/Lucas numbers. Using the Mathematica computer algebra system (CAS) and the On-line Encyclopedia of Integer Sequences (OEIS) [17], we have come to find a strikingly simpler way of evaluating the g.f. for (1.1) for the case whereby n and m are of the same parity, but state-of-the-art symbolic computation software cannot directly confirm or verify that our simplified g.f. for (1.1) is equivalent to the output obtained via Zeilberger's procedure [21] for the g.f. for (1.1). In this article, we prove this equivalence using Binet's formula; we then apply our simplified g.f. to generalize a classic result on Fibonacci sums from [4], and to build on recently introduced results on order-2 Fibonacci-type sums given in [13].

1.1. **An application of Zeilberger's procedure.** For a parameter n in $\mathbb{N}$, it is easily seen that the generating function for the sequence $(F_{ni} : i \in \mathbb{N}_0)$ is of the form

$$\frac{F_nx}{(-1)^nx^2 - L_nx + 1},$$

recalling that the sequence $(L_i : i \in \mathbb{N}_0)$ of Lucas numbers is defined, as per usual, so that $L_0 = 2$, $L_1 = 1$, and $L_n = L_{n-1}+L_{n-2}$, as in sequence `A000032` in the OEIS [17]. Implementing Zeilberger's `Cfinite` package, and then inputting

```
KefelR(Fm*x/((-1)^m*x^2 - Lm*x + 1), Fn*x/((-1)^n*x^2 - Ln*x + 1), x)
```

into Maple, letting expressions as in `Fm` denote variables or parameters in Maple, we find that the g.f. for the Hadamard product of the sequences $(F_{mi} : i \in \mathbb{N}_0)$ and $(F_{ni} : i \in \mathbb{N}_0)$ may be expressed as below:

$$\frac{-F_m\, F_n\, x\left(x^2\,(-1)^{m+n}-1\right)}{1-L_m\, L_n\, x+x^2 L_m^2\,(-1)^n+x^2 {L_n}^2\,(-1)^m-2\,x^2\,(-1)^{m+n}-L_m\, L_n\,\ (-1)^{m+n}\, x^3+x^4}.$$

Using a heavily experimental approach based on the use of both Mathematica and the OEIS [17], we came to find, conjecturally, that: If m and n are of the same parity, then

$$\frac{F_m\, F_n\, x\left(1-x^2\right)}{1-L_m\, L_n\, x+x^2 L_m^2\,(-1)^n+x^2 {L_n}^2\,(-1)^m-2\,x^2-L_m\, L_n x^3+x^4}. \tag{1.2}$$

may be written as the following remarkably simpler expression:

$$\frac{1}{5}\left(1-x^2\right)\left(\frac{1}{1-xL_{n+m}+x^2}-\frac{1}{1+(-1)^{m+1}xL_{n-m}+x^2}\right). \tag{1.3}$$

This g.f. equivalence is a main result in this article. Inputting the difference of (1.2) and (1.3) into current CAS software, and using commands as in Maple's `simplify`, we find that such software cannot confirm or "detect" the equality of (1.2) and (1.3), recalling the required parity condition. Our simplified g.f. evaluation in (1.3) is of interest in its own right, as emphasized in Proposition 1.1 below, noting the symmetry or similarity between the denominators in (1.5), in contrast to the relatively unwieldy expression in (1.4). Apart from such aesthetic considerations, in terms of concrete applications concerning our symbolic form in (1.3), since (1.3) expands as a linear combination of rational functions with quadratic denominators with closed-form coefficients, this is useful in the construction of identities for equating sums involving $F_{mi}F_{ni}$ and combinations of order-2 recurrences, with reference to our results in Section 3.2 below. As illustrated in a dramatic way in Example 3.2 below, our evaluations for sums of the form

$$\sum_{i=0}^{j} F_{mi}F_{ni}$$

are extremely simple compared to corresponding evaluations obtained via Zeilberger's `Cfinite` package [21].

Proposition 1.1. *If m and n are natural numbers of equal parity, then*

$$\frac{5\, F_m\, F_n\, x}{1-L_m\, L_n\, x+x^2 L_m^2\,(-1)^n+x^2 L_n^2\,(-1)^m-2\,x^2-L_m\, L_n x^3+x^4} \tag{1.4}$$

equals

$$\frac{1}{1-xL_{n+m}+x^2}-\frac{1}{1+(-1)^{m+1}xL_{n-m}+x^2} \tag{1.5}$$

for suitably bounded x.

2. A simplified generating function evaluation

Again, it is not clear as to why the identity shown in Proposition 1.1 holds true. We prove this Proposition in this section.

Theorem 2.1. *If $m \in \mathbb{N}_0$ and $n \in \mathbb{N}_0$ are of the same parity, then the generating function for $F_{ni}F_{mi}$ is $\frac{1}{5}\left(1-x^2\right)\left(\frac{1}{1-xL_{n+m}+x^2}-\frac{1}{1+(-1)^{m+1}xL_{n-m}+x^2}\right)$.*

Proof. Suppose that $m \in \mathbb{N}_0$ and $n \in \mathbb{N}_0$ are of the same parity. By Binet's formula, the generating function for the integer sequence $(F_{ni}F_{mi})_{i \in \mathbb{N}_0}$ is:

$$-\frac{x\left((-1)^m\phi^{2m}-1\right)\left((-1)^{n+1}\phi^{2n}+1\right)\left((-1)^{m+n+1}+x^2\right)\phi^{m+n}}{5\left((-\phi)^{m+n}-x\right)\left(x\phi^m+(-1)^{n+1}\phi^n\right)\left((-1)^{m+1}\phi^m+x\phi^n\right)\left(x\phi^{m+n}-1\right)}.$$

By assumption that m and n are of the same parity, the above expression may be rewritten as follows:

$$-\frac{x\left(x^2-1\right)\left((-1)^m\phi^{2m}-1\right)\left((-1)^{n+1}\phi^{2n}+1\right)\phi^{m+n}}{5\left(\phi^{m+n}-x\right)\left(x\phi^m+(-1)^{n+1}\phi^n\right)\left((-1)^{m+1}\phi^m+x\phi^n\right)\left(x\phi^{m+n}-1\right)}.$$

By Binet's formula for Lucas numbers, it thus remains to prove that the above expression is equal to:

$$\frac{x\left(x^2-1\right)\left(\left((-1)^m\phi^{-2m}-1\right)\phi^{m+n}-\phi^{-m-n}+(-1)^m\phi^{m-n}\right)}{5\left((-1)^{m+1}x\left(\phi^{m-n}+\phi^{n-m}\right)+x^2+1\right)\left(-x\left(\phi^{-m-n}+\phi^{m+n}\right)+x^2+1\right)}.$$

Equivalently, it remains to prove that

$$-\frac{\left((-1)^m\phi^{2m}-1\right)\left((-1)^{n+1}\phi^{2n}+1\right)\phi^{m+n}}{\left(\phi^{m+n}-x\right)\left(x\phi^m+(-1)^{n+1}\phi^n\right)\left((-1)^{m+1}\phi^m+x\phi^n\right)\left(x\phi^{m+n}-1\right)}$$

is equal to:

$$\frac{\left(\left((-1)^m\phi^{-2m}-1\right)\phi^{m+n}-\phi^{-m-n}+(-1)^m\phi^{m-n}\right)}{\left((-1)^{m+1}x\left(\phi^{m-n}+\phi^{n-m}\right)+x^2+1\right)\left(-x\left(\phi^{-m-n}+\phi^{m+n}\right)+x^2+1\right)}.$$

Now consider the ratio of the former numerator to the latter numerator:

$$\frac{\left((-1)^m\phi^{2m}-1\right)\left((-1)^{n+1}\phi^{2n}+1\right)\phi^{m+n}}{\left(\left((-1)^m\phi^{-2m}-1\right)\phi^{m+n}-\phi^{-m-n}+(-1)^m\phi^{m-n}\right)}$$

It is easily seen that the above quotient is equal to ϕ^{2m+2n}, as may be verified by expanding the above numerator and expanding the expression $\phi^{2m+2n}((((-1)^m\phi^{-2m}-1)\phi^{m+n}-\phi^{-m-n}+(-1)^m\phi^{m-n}))$. So, it remains to prove that

$$\frac{1}{-\left(\phi^{m+n}-x\right)\left(x\phi^m+(-1)^{n+1}\phi^n\right)\left((-1)^{m+1}\phi^m+x\phi^n\right)\left(x\phi^{m+n}-1\right)}$$

is equal to:

$$\frac{1}{\phi^{2m+2n}\left((-1)^{m+1}x\left(\phi^{m-n}+\phi^{n-m}\right)+x^2+1\right)\left(-x\left(\phi^{-m-n}+\phi^{m+n}\right)+x^2+1\right)}.$$

Expand the former denominator as follows:

$$x^4\phi^{2m+2n}-x^3\phi^{m+n}+(-1)^{m+1}x^3\phi^{3m+n}+(-1)^{n+1}x^3\phi^{m+3n}-x^3\phi^{3m+3n}+x^2(-1)^{m+n+2}\phi^{2m+2n}+x^2\phi^{2m+2n}+(-1)^{m+2}x^2\phi^{4m+2n}+(-1)^{n+2}x^2\phi^{2m+4n}+x(-1)^{m+n+3}\phi^{m+n}+(-1)^{m+3}x\phi^{3m+n}+(-1)^{n+3}x\phi^{m+3n}+x(-1)^{m+n+3}\phi^{3m+3n}+(-1)^{m+n+4}\phi^{2m+2n}+(-1)^{m+2}x^2\phi^{2m}+(-1)^{n+2}x^2\phi^{2n}$$

Expand the latter denominator as follows:

$$x^4\phi^{2m+2n}-x^3\phi^{m+n}+(-1)^{m+1}x^3\phi^{3m+n}+(-1)^{m+1}x^3\phi^{m+3n}-x^3\phi^{3m+3n}+(-1)^{m+2}x^2\phi^{2n}+2x^2\phi^{2m+2n}+(-1)^{m+2}x^2\phi^{4m+2n}+(-1)^{m+2}x^2\phi^{2m+4n}-x\phi^{m+n}+(-1)^{m+1}x\phi^{3m+n}+(-1)^{m+1}x\phi^{m+3n}-x\phi^{3m+3n}+\phi^{2m+2n}+(-1)^{m+2}x^2\phi^{2m}.$$

Simplifying the former denominator using the fact that m and n are of the same parity, it is easily seen that the above two expressions are equal. □

Zeilberger's implementation of an algorithm for computing Hadamard products of g.f.'s [21] gives us a computer proof that the g.f. for (1.1) is as given by the `KefelR` function output shown in Section 1.1. However, we have shown, as above, that this same g.f. is as in Theorem 2.1.

3. Order-2 Fibonacci sums

By multiplying our simplified g.f. in Theorem 2.1 by expressions such as $\frac{1}{1-x}$ and then using the Cauchy product for g.f.'s, this gives us interesting results, as in Theorems 3.1–3.4 below. To begin with, we find it worthwhile to describe how these Theorems build upon and otherwise relate to relevant background material.

3.1. Background.

In 1973, Freitag [4] proved the summation identity

$$\sum_{i=1}^{j} F_{ni} = \frac{(-1)^n F_{jn} + F_n - F_{(j+1)n}}{(-1)^n + 1 - L_n}.$$

As described in [4], it is a natural mathematical problem to consider finite sums involving entries in the Fibonacci sequence that "skip" by a fixed period. With reference to the terms *double-index harmonic number* [19] and *Euler sum with multiple argument* [18], we consider, in this article, sums over products of *multiple-index* Fibonacci numbers. It seems that our results on sums involving $F_{ni}F_{mi}$ have not appeared previously, as in references concerning Freitag's work in [4], as in [3, 15]. Letting u and v denote Fibonacci-type or Lucas-type sequences, the evaluation of sums of the form

$$\sum_{i=1}^{j} u_{a+bi} v_{c+di}$$

is a topic that has been explored in many past references, as stated in [10], with reference to publications as in [8, 9, 14].

For integer values a_ℓ and b_ℓ, an algorithm was given by Greene and Wilf [5] (cf. [21]) that may be used to express

$$\sum_{i=0}^{j} F_{a_1 j + b_1 i + c_1} F_{a_2 j + b_2 i + c_2} \cdots F_{a_k j + b_k i + c_k} \tag{3.1}$$

in closed form with Fibonacci numbers. With regard to Zeilberger's Maple implementation of an algorithm for evaluating finite sums as in (3.1), it is unclear as to how this can be used to obtain our parity-dependent results on sums involving $F_{mi}F_{ni}$ in Section 3, with specific reference to the `FindCHvTN` program from Zeilberger's `Cfinite` Maple package [21]. In particular, it seems that `FindCHvTN` does not apply to summands with additional parameters apart from the upper limit of a given finite sum under consideration. For example, by applying Zeilberger's `Cfinite` package and inputting

```
FindCHvTN(Fn(), 1, 5, [2*j, 3*j], n, j, 1)
```

this gives an evaluation for $\sum_{j=0}^{n-1} F_{2j} F_{3j}$, but inputting an expression such as

```
FindCHvTN(Fn(), 1, 5, [m*j, n*j], n, j, 1)
```

results in an error message. It is unclear as to how the algorithms in [5] may be applied to obtain our results as in Theorem 2.1, relative to the corresponding `Cfinite` output under consideration in Section 1.1.

Finite sums of the form indicated in (3.1) for the case whereby $k = 2$ are often referred to as order-2 Fibonacci sums [11, 13]. Our proofs for the results given in Section 3.2 significantly build upon past results on order-2 Fibonacci sums, as in the identity

$$\sum_{i=0}^{j} F_{2i}^2 = \frac{F_{4j+2} - 2j - 1}{5}$$

given in [20, p. 70] (cf. [11]).

In 2017 [13], Melham introduced a method for evaluating certain order-2 Fibonacci-type sums. More specifically, writing

$$W_n(a, b, p) = W_n = pW_{n-1} + W_{n-2}, W_0 = a, W_1 = b$$

as in [13], a method for evaluating

$$\sum_{i=1}^{j} W_{ai+b_1} W_{ai+b_2}$$

is given, noting that the coefficient of i in the above summand is the same for both of the indices of the above W-expressions. So, it is unclear as to how the material from [13] could be reformulated so as to be applicable to sums as in Theorem 3.1 below.

3.2. **Finite sums of products of multiple-index Fibonacci numbers.** A remarkable aspect about our identity highlighted as Theorem 3.1 below is due to how the right-hand side of this identity is very simple compared to corresponding output from the above referenced `FindCHvTN` program [21], as in Example 3.2 below.

Theorem 3.1. *If $m \in \mathbb{N}_0$ and $n \in \mathbb{N}_0$ are distinct and of the same parity, then*

$$\sum_{i=0}^{j} F_{mi} F_{ni} = \frac{1}{5} \left(\frac{(-1)^{jm+1} \left((-1)^m F_{j(n-m)} + F_{(j+1)(n-m)} \right)}{F_{n-m}} + \frac{F_{j(m+n)} + F_{(j+1)(m+n)}}{F_{m+n}} \right).$$

Proof. This can be shown to follow in a direct way by applying the partial sum operator $\frac{1}{1-x}$ to the g.f. in Theorem 2.1, under the given assumptions on m and n □

Example 3.2. *We input*

```
FindCHvTN(Fn(), 1, 6, [2*j, 4*j], n, j, 1)
```

into Maple, using Zeilberger's `Cfinite` *package* [21]*. This gives us that the sum*

$$\sum_{i=0}^{j} F_{2i} F_{4i} \tag{3.2}$$

equals the following for all $j \in \mathbb{N}_0$:

$$\frac{275\, F_{j+1}^6}{4} - \frac{315\, F_j F_{j+1}^5}{2} + \frac{41\, F_j F_{j+1}}{2} - 30\, F_j^2 F_{j+1}^4 + 170\, F_j^3 F_{j+1}^3 - \frac{275\, F_{j+1}^2}{4} - 3\, F_j^2 + F_{2\,j} F_{4\,j}$$

In stark contrast, simply by setting $m = 2$ and $n = 4$ in Theorem 3.1, we obtain that the sum in (3.2) *also equals*

$$\frac{F_{6j} + F_{6j+6}}{40} - \frac{F_{2j} + F_{2j+2}}{5} \tag{3.3}$$

for all $j \in \mathbb{N}_0$. It does not seem to be possible to evaluate (3.2) *with less terms compared to the above* `FindCHvTN` *output. Without knowing that the second-to-last displayed expression and the dramatically simpler expression in* (3.3) *are related via the sum in* (3.2)*, it would be far* *from clear as to why these two expressions should be equal for all $j \in \mathbb{N}_0$. This is indicative of the computational usefulness of our identities as in Theorem 3.1.*

Theorem 3.3. *If $m \in \mathbb{N}_0$ is even, then the identity*

$$\sum_{i=0}^{j} F_{mi}^2 = \frac{1}{5}\left(\frac{F_{2jm}}{F_{2m}} + \frac{F_{2(j+1)m}}{F_{2m}} - (2j+1)\right)$$

holds true (cf. **Theorem 2.3** *in* [13]*).*

Proof. Again, this follows in a direct way by applying the partial sum operator to the g.f. in Theorem 2.1. □

Theorem 3.4. *If $m \in \mathbb{N}_0$ is odd, then the identity*

$$\sum_{i=0}^{j} F_{mi}^2 = \frac{1}{5}\left(\frac{F_{2jm}}{F_{2m}} + \frac{F_{2(j+1)m}}{F_{2m}} - (-1)^j\right)$$

holds (cf. **Theorem 2.3** *in* [13]*).*

Proof. This follows from Theorem 2.1 as in the above proofs. □

We may obtain many similarly elegant results by applying operators such as $\frac{1}{1+x}$ and $\frac{1}{1-x^2}$ to our g.f. evaluation in Theorem 2.1. For the sake of brevity, we leave a full exploration of this kind of topic to a separate research endeavour. Also consider the repeated application of the partial sum operator to our g.f., to build on John Ivie's work on *multiple Fibonacci sums*, as in [7], along with Chu's very recent work in [2]. The material in [12] motivates extending our results to k-Fibonacci number sequences.

4. Conclusion

Generating function-based methods and results continue to be frequently involved in research on Fibonacci/Fibonacci-type sequences; in this regard, see publications as in [1, 16, 23]. We encourage the use of the concepts and methods involved in our article in relation to past research efforts on g.f.-based results on Fibonacci/Fibonacci-type sequences.

Combinatorial proofs for evaluating g.f.'s for products of second-order recurrences are the subject of the Ph.D. thesis [22]. We encourage the development of combinatorial methods for proving and generalizing Proposition 1.1.

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