THEOREM 4.30
There exists an absolute constant c so that for all n, we have $S_n(1324) < c^n$

PROOF As there are less than 9^n classes and less than 32^n n-permutatio in each class that avoid 1324, $c = 9 \cdot 32 = 288$ will do. ∎

We point out that this is certainly not the best upper bound for $S_n(132$ With less elegant arguments, the upper bound can be decreased. Neverthele the conjecture that $S_n(1324) < 9^n$ is open. It would be interesting to dec this conjecture in either direction. As of now, the smallest constant that h a chance to play the role of c_q in the inequality $S_n(q) < c_q^n$ is $c_q = (k-1$ where k is the length of q. We know from Theorem 4.11 that no smal constant will do. A disproof of the conjecture that $S_n(1324) < 9^n$ would sh that sometimes a larger constant is needed.

Numerical evidence suggests that for any given k the value of $S_n(q)$ maximized by the pattern $1325476\cdots$. The results of this section prove t this is indeed the case for $k = 4$. If we could show that this is true for pattern lengths k, then an upper bound given for $S_n(1324576\cdots)$ would an upper bound for all patterns of length k. Exercise 32 and 31 sketch a to prove an upper bound for $S_n(1324576\cdots)$, so we would "only" need show that there is indeed no pattern of length k that is easier to avoid t $1325476\cdots$.

4.4.2 The Pattern 1342

In this subsection, we turn our attention to the pattern 1342. Interesti we will be able to provide an *exact formula* for $S_n(1342)$. This is exceptio the only other pattern longer than three for which an exact formula is kn is *1234*. The formula is given by the following theorem.

THEOREM 4.31
For all positive integers n, we have

$$S_n(1342) = (-1)^{n-1} \cdot \frac{(7n^2 - 3n - 2)}{2} + 3\sum_{i=2}^{n}(-1)^{n-i} \cdot 2^{i+1} \cdot \frac{(2i-4)!}{i!(i-2)!} \cdot \binom{n-i+2}{2},$$

This is a very surprising result. It is straightforward to prove fro formula that $S_n(1342) < 8^n$ for all n, and that $\lim_{n\to\infty} \sqrt[n]{S_n(1342)} =$

The result itself is not the only interesting aspect of the facts surro the pattern 1342. We will see that permutations avoiding 1342 are in bij with two different kinds of objects which at first look totally unrelate

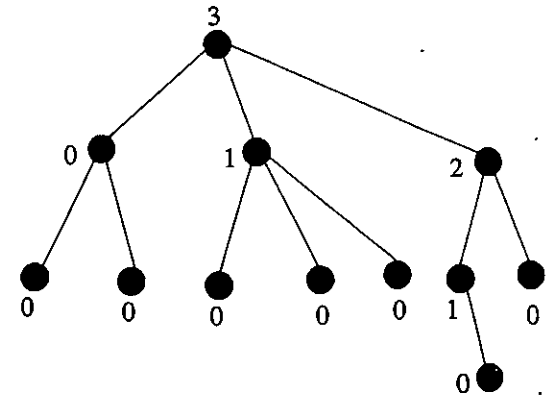

FIGURE 4.3
A $\beta(0,1)$-tree.

first, and for our purposes, more important, type of objects is a specific kind of labeled trees.

DEFINITION 4.32 [64] A rooted plane tree with non-negative integer labels $l(v)$ on each of its vertices v is called a $\beta(0,1)$-tree if it satisfies the following conditions:

- *if v is a leaf, then $l(v) = 0$, (this explains the 0 in the name of $\beta(0,1)$-trees);*
- *if v is the root and $v_1, v_2, \cdots, v_k$ are its children, then $l(v) = \sum_{i=1}^{k} l(v_k)$,*
- *if v is an internal node and $v_1, v_2, \cdots, v_k$ are its children, then $l(v) \leq 1 + \sum_{i=1}^{k} l(v_k)$ (this explains the 1 in the name of $\beta(0,1)$-trees).*

Example 4.33
Figure 4.3 shows a $\beta(0,1)$-tree on 12 vertices. □

Let us call a permutation $p = p_1p_2\cdots p_n$ *indecomposable* if there exists no $k \in [n-1]$ so that for all $i \leq k < j$, we have $p_i > p_j$. In other words, p is indecomposable if it cannot be cut into two parts so that everything before the cut is larger than everything after the cut. For instance, 3142 is indecomposable, but 43512 is not as we could choose $k = 3$, that is, we could cut between the third and fourth entries. If a permutation is not indecomposable, then we will call it *decomposable.*

The importance of $\beta(0,1)$-trees for us is explained by the following theorem.

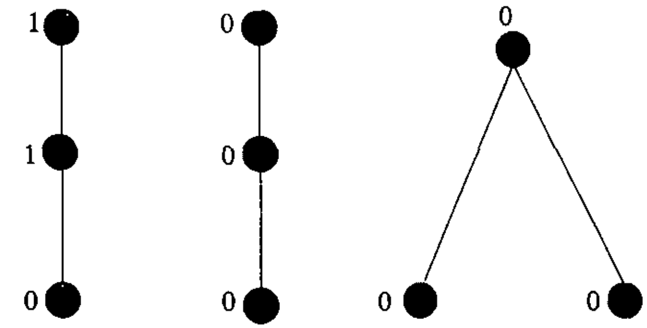

FIGURE 4.4
The three $\beta(0,1)$-trees on three vertices.

THEOREM 4.34
For all positive integers n, there is a bijection F from the set of indecomposable 1342-avoiding n-permutations to the set $D_n^{\beta(0,1)}$ of $\beta(0,1)$-trees on n vertices.

Example 4.35
Let $n = 3$. Then there are three indecomposable n-permutations, 123, 132, and 213, and they all avoid 1342. Correspondingly, there are three $\beta(0,1)$-trees on three vertices, as can be seen in Figure 4.4. □

Let $t_n = |D_n^{\beta(0,1)}|$. If we can prove Theorem 4.34, then we have made a crucial step in advance as it is known [64] that

$$t_n = 3 \cdot 2^{n-2} \cdot \frac{(2n-2)!}{(n+1)!(n-1)!}. \tag{4.6}$$

We start by treating two special types of $\beta(0,1)$-trees on n vertices. There are two things that contribute to the structure of a $\beta(0,1)$-tree, namely its (unlabeled) tree structure, and its labels. We will therefore first look at $\beta(0,1)$-trees in which one of these two ingredients is trivial, that is, $\beta(0,1)$-trees that consist of a single path only, and $\beta(0,1)$-trees in which all labels are zero.

LEMMA 4.36
There is a bijection f from the set of 1342-avoiding n-permutations starting with the entry 1 and the set of $\beta(0,1)$-trees on n vertices consisting of one single path.

Note that a simpler description of the domain of f is that it is the set of 231-avoiding permutations of the set $\{2, 3, 4, \cdots, n\}$.

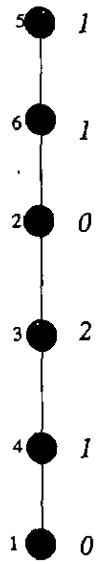

FIGURE 4.5
The $\beta(0,1)$-tree of $p = 143265$.

PROOF Let $p = p_1 p_2 \cdots p_n$ be an 1342-avoiding n-permutation so that $p_1 = 1$. Take an unlabeled tree on n nodes consisting of a single path and give the label $l(i)$ to its ith node $(1 \leq i \leq n-1)$ by the following rule:

$$l(i) = \begin{cases} |\{j \leq i \text{ so that } p_j > p_s \text{ for at least one } s > i,\}| \text{ if } i < n \\ l(n-1) \text{ if } i = n. \end{cases}$$

That is, $l(i)$ is the number of entries weakly on the left of p_i which are larger than at least one entry on the right of p_i. Note that this way we could define f on the set of *all* n-permutations starting with the entry 1, but in that case, f would not be a bijection. (For example, the images of 1342 and 1432 would be identical.)

Example 4.37
If $p = 143265$, then the labels of the nodes of $f(p)$ are, from the leaf to the root, 0,1,2,0,1,1. See Figure 4.5. For easy reference, we wrote p_i to the ith node of the path $f(p)$. To avoid confusion, in this Figure, and for the rest of this subsection, the entries of p will be written in small, Roman letters, and the labels of the nodes will be written in large italic letters. □

It is easy to see that f indeed maps into the set of $\beta(0,1)$-trees: $l(i+1) \leq l(i) + 1$ for all i because there can be at most one entry counted by $l(i+1)$ and not counted by $l(i)$, namely the entry p_{i+1}. All labels are certainly nonnegative and $l(1) = 0$.

To prove that f is a bijection, it suffices to show that it has an inverse, that is, for any $\beta(0,1)$-tree T consisting of a single path, we can find the only permutation p so that $f(p) = T$. We claim that given T, we can recover the entry n of the preimage p. First note that p is 1342-avoiding and starts by 1, so any entry on the left of n must be smaller than any entry on the right of n. In particular, the node preceding n must have label 0. Moreover, as n is larger than any entry following it in p, the entry n is the leftmost entry p_i of p so that $l(j) > 0$ for all $j \geq i$ if there is such an entry at all, and $n = p_n$ if there is none. That is, n corresponds to the node which starts the uninterrupted sequence of strictly positive labels that ends in the last node as long as there is such a sequence. Otherwise, n corresponds to the last node.

Once we located where n is in p, we can simply delete the node corresponding to it from T and decrement all labels after it by 1. (If this means deleting the last node, we just change $l(n-1)$ so it is equal to $l(n-2)$ to satisfy the root-condition.) We can indeed do this because the node preceding n had label 0 and the node after n had a positive label (1 or 2), by our algorithm to locate n. Then we can proceed recursively, by finding the position of the entries $n-1, n-2, \cdots, 1$ in p. This clearly defines the inverse of f, so we have proved that f is a bijection. ∎

As we promised, we continue by explaining which indecomposable 1342-avoiding permutations correspond to $\beta(0,1)$-trees in which all labels are equal to zero.

LEMMA 4.38

There is a bijection g from the set of 132-avoiding n-permutations ending with n to the set of $\beta(0,1)$-trees on n vertices with all labels equal to zero.

Note that we could describe the domain of g as the set of *indecomposable* 132-avoiding n-permutations, or as the set of $(n-1)$-permutations that avoid 132.

PROOF In this proof, we can obviously think of our $\beta(0,1)$-trees as unlabeled rooted plane trees. A *branch* of a rooted tree is a subtree whose root is one of the root's children. Some rooted trees may have only one branch, which does not necessarily mean they consist of a single path.

We will construct g inductively. There is only one unlabeled $\beta(0,1)$-tree on 2 vertices and it is the image of the only 1-permutation; $p = 1$. Using induction, suppose we have already constructed g for all positive integers $k < n$. Let p be a 132-avoiding permutation of length n. Let $p' = p_1p_2\cdots p_{n-1}$. Then there are two possibilities.

(a) The first case is when p' is decomposable, that is, we can cut p' into several (at least two) strings $p_{<1>}, p_{<2>}, \cdots, p_{<h>}$ so that all entries of $p_{<i>}$ are larger than all entries of $p_{<j>}$ if $i < j$. In this case, $g(p)$ will have h branches, the branch b_i satisfying $g(p_{<i>}) = b_i$. We then obtain $g(p)$ by connecting all branches b_i to a common root. Given that we are in a $\beta(0,1)$-tree, the label of the root is determined by the labels of its children.

(b) The second case is when p' is indecomposable. As p avoids 132, this is equivalent to saying that p' ends with its maximal entry $n-1$. In this case, $g(p)$ will have just one branch b_1, that is, the root of $g(p)$ will have only one child. We define $b_1 = g(p')$.

Again, we prove that g is a bijection by showing that it has an inverse. Let T be an unlabeled plane tree on n vertices with root q. Let q have t children, and say that, going left-to-right, they are roots of the branches $b_1, b_2, \cdots, b_t$, which have $n_1, n_2, \cdots, n_t$ nodes. Then by induction, for each i, the branch b_i corresponds to a 132-avoiding n_i-permutation ending with n_i. Now add $\sum_{j=i+1}^{t} n_j$ to all entries of the permutation p_i associated with b_i, then concatenate all these strings and add n to the end to get the permutation p associated with T.

It is straightforward to check that this procedure always returns the original permutation, proving our claim. ∎

Example 4.39

The permutation 45631278 corresponds to the $\beta(0,1)$-tree with all labels equal to 0 shown in Figure 4.6. For easy reference, we write p_n to the root of $g(p)$, and proceeded analogously for the other entries in the recursively defined subtrees.

□

An easy way to read off the corresponding permutation once we have its entries written to the corresponding nodes is the well-known *postorder* reading: for every node, first write down the entries associated with its children from left to right, then the entry associated with the node itself, and do this recursively for all the children of the node.

Our plan is to bring Lemmas 4.36 and 4.38 together to prove Theorem 4.34. This needs some preparation. Optimally, we would take a 1342-avoiding indecomposable n-permutation p, associate its entries to the nodes of an unlabeled plane tree T, then define the labels of this tree so that it becomes a $\beta(0,1)$-tree. The question is, however, how do we know what T we should use, and if T is given, in what order we should write the entries of p to the nodes of T. In what follows, we develop the notions to decide these questions.

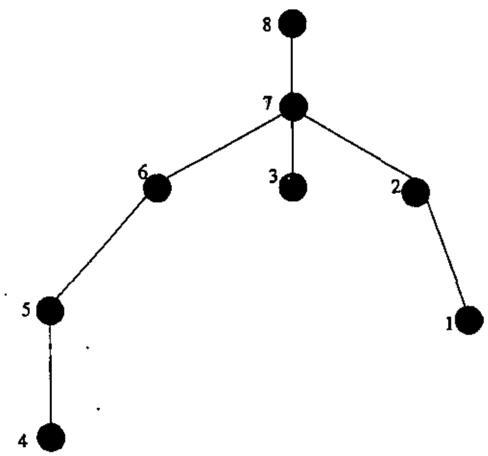

FIGURE 4.6
The $\beta(0,1)$-tree of $p = 45631278$.

DEFINITION 4.40 *Two n-permutations x and y are said to be in the same* weak class *if the left-to-right minima of x are the same as those of y, and they are in the same positions.*

Example 4.41
Permutations 456312 and 465312 are in the same weak class since their left-to-right minima are 4, 3 and 1, and they are located at the same positions. Permutations 31524 and 34152 are not in the same weak class. □

PROPOSITION 4.42
Each nonempty weak class C of n-permutations contains exactly one 132-avoiding permutation.

PROOF Take all entries which are not left-to-right minima and fill all empty positions between the left-to-right minima with them as follows: in each step place the smallest element which has not been placed yet which is larger than the previous left-to-right minimum. (This is just what we did in the proof of the Simion-Schmidt bijection in Lemma 4.3.)

On the other hand, the resulting permutation will be the only 132-avoiding permutation in this weak class because any time we deviate from this procedure, (that is, we place something else, not the smallest such entry) we create a 132-pattern. ∎

the only 132-avoiding permutation in the weak class C containing p.

Example 4.44
If $p = 356214$, then $N(p) = 345216$. □

DEFINITION 4.45 *The normalization $N(T)$ of a $\beta(0,1)$-tree T is the $\beta(0,1)$-tree which is isomorphic to T as a plane tree, with all labels equal to zero.*

It turns out that normalization preserves the indecomposable property.

PROPOSITION 4.46
A permutation p is indecomposable if and only if $N(p)$ is indecomposable.

PROOF (The author is grateful to Aaron Robertson, who found a corrected a mistake in his original argument.) We will show that whether p is decomposable or not depends only on the set and position of its left-to-right minima, which is obviously equivalent to the claim to be proved. Let C be the weak class containing p, given by the set and position of its left-to-right minima. It is clear that if $p \in C$ is decomposable, then the only way to cut it into two parts (so that everything before the cut is larger than everything after the cut) is to cut it immediately on the left of a left-to-right minimum $a < n$. Now if there is a left-to-right minimum in position $n - a + 2$, then the entries $1, 2, \cdots, a - 1$ must occupy positions $n - a + 2, n - a + 1, \cdots, n$. Therefore, we can cut immediately on the left of position $n - a + 2$, and p is decomposable.

If there is no such a, then for all left-to-right minima m, all the entries $1, 2, \cdots, m - 1$ must be to the right of m. However, in one of the positions $n - m + 2, n - m + 1, \cdots, n$, there exists an element $y > m$, implying that our permutation p is not decomposable. ∎

COROLLARY 4.47
If p is an indecomposable n-permutation, then $N(p)$ always ends in the entry n.

PROOF Note that the only way for a 132-avoiding n-permutation to be indecomposable is for it to end with n. If p is a 132-avoiding n-permutation and n is not the last entry, then we may cut it just after the entry n. Then the statement follows from Proposition 4.46. ∎

Now we are in a position to prove Theorem 4.34.

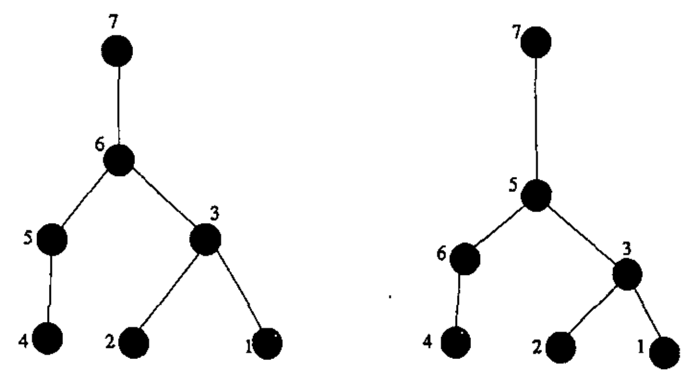

FIGURE 4.7
The unlabeled trees of $N(p) = 4521367$ and $p = 4621357$.

PROOF (of Theorem 4.34.) Let p be an indecomposable 1342-avoiding n-permutation. Take $N(p) = r$. By Corollary 4.47 its last element is n. Define $F(r)$ to be the unlabeled plane tree S associated with r by the bijection g of Lemma 4.38. So g is just the restriction of F to the set of indecomposable 132-avoiding permutations.

This unlabeled tree S is the tree we are going to work with. First, we will write the entries of p to the nodes of S. (The reader should recall that we did this in the proof of Lemma 4.36, and that the *entries* of p written to the nodes of S are not to be confused with the *labels* of the nodes.) We will do this in the order specified by p and $N(p)$. That is, $N(p)$ is a 132-avoiding permutation, so its entries are in natural bijection with the nodes of S as we saw in Lemma 4.38. We then let the permutation $p(N(p))^{-1}$ act on the entries of $N(p)$ (written to the nodes of S) to get the order in which we write the entries of p to S. Note in particular that the left-to-right minima are kept fixed.

Example 4.48
Let $p = 4621357$. Then $N(p) = 4521367$, and the unlabeled plane tree S associated with these permutations is shown in Figure 4.7, together with the order in which the entries of p are written to the nodes. Note that p and $N(p)$ only differ in the transposition (56). This is why it is these two entries whose positions have been swapped. □

Now we are going to define the label $l(v)$ of each node v for the new $\beta(0,1)$-tree $T = F(p)$ that we are constructing from S. As an unlabeled tree, T will be isomorphic to S, but its labels will be different. Let i be the ith node of T in the postorder reading, the node to which we wrote p_i. We say that p_i beats p_j if there is an element p_k so that p_k, p_i, p_j are written in this

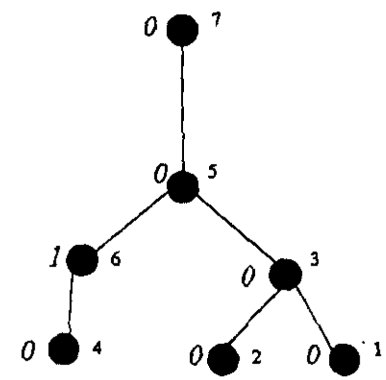

FIGURE 4.8
The image $F(p)$ of $p = 4621357$.

p_i *reaches* p_k if there is a subsequence $p_i, p_{i+a_1}, \cdots p_{i+a_t}, p_k$ of entries so that $i < i + a_1 < i + a_2 < \cdots < i + a_t < k$ and that any entry in this subsequence beats the next one. In particular, if x beats y, then x also reaches y.

Example 4.49
Let $p = 3716254$. Then 7 beats 6, 6 beats 2, therefore 7 reaches 2. □

Finally, we set

$$l(i) = |\{j \text{ is a descendant of } i \text{ (inclusive) so that there is at least one } k > i \text{ for which } p_j \text{ reaches } p_k\}|,$$

and let $F(p)$ be the $\beta(0,1)$-tree defined by these labels. A descendant of i is an element of the tree whose top element is i. Note that this rule is an extension of the labeling rule we have in Lemma 4.36.

First, it is easy to see that F indeed maps into the set $D_n^{\beta(0,1)}$. Indeed, let v be an internal node and let $v_1, v_2, \cdots, v_k$ be its children. Then $l(v) \leq 1 + \sum_{i=1}^{k} l(v_k)$ because there can be at most one entry counted by $l(v)$ and not counted by any of its children's labels, namely v itself. Second, all labels are certainly nonnegative and all leaves, that is, the left-to-right minima, have label 0.

Example 4.50
In Example 4.48 we have created the unlabeled tree S for $p = 4621357$. Application of the above rule shows that $F(p)$ is the $\beta(0,1)$-tree shown in Figure 4.8. Indeed, the only 132-pattern of p is 465, and that is counted only once, at the entry 6. □

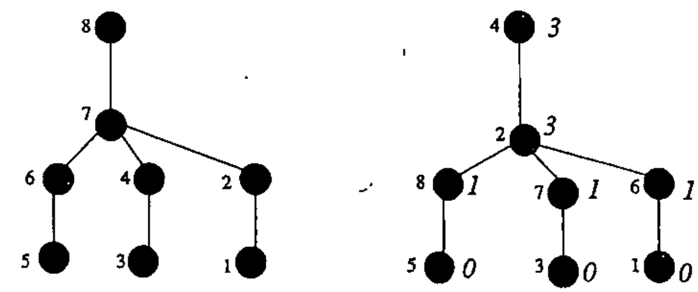

FIGURE 4.9
The tree S and $F(p)$ for $p = 58371624$.

Example 4.51
Let $p = 58371624$, then we have $N(p) = 56341278$, giving rise to the unlabeled tree shown on the left of Figure 4.9. We then compute the labels of $F(p)$ to obtain the tree shown on the right of Figure 4.9. $\Box$

To prove that F is a bijection, it suffices to show that it has an inverse. That is, it suffices to show that for any $\beta(0,1)$-tree $T \in D_n^{\beta(0,1)}$, we can find a unique permutation p so that $F(p) = T$.

We again claim that given T, we can recover the node j that has the entry n of the preimage p associated with it, and so we can recover the position of n in the preimage.

PROPOSITION 4.52
Suppose $p_n \neq n$, that is, n is not associated with the root vertex. Then each ancestor of n, including n itself, has a positive label. If $p_n = n$, then $l(n) = 0$ and thus there is no vertex with the above property.

PROOF If $p_n = n$, then there is nothing on the right of n to reach, thus $l(v)$ enumerates an empty set, yielding $p_n = 0$. Suppose now that p_n is not the root vertex.

To prove our claim it is enough to show that for any node i that is an ancestor of $p_j = n$, there is an entry p_k so that $k > i$, and $n = p_j$ reaches k. Indeed, this would imply that the entry $p_j = n$ is counted by the label $l(i)$ of i, forcing $l(i) > 0$. Now let $a_m = p_1 > a_2 > \cdots > a_1 = 1$ be the left-to-right minima of p so that n is located between a_r and a_{r+1}. (If n is located to the right of $a_1 = 1$, then $a_1 n x$ is obviously a 132-pattern for any x located to the right of n.) Then n certainly beats all elements located between a_r and a_{r+1} as a_r can play the role of 1 in the 132-pattern. Clearly, n must beat at least one entry y_1 on the right of a_{r+1} as well, otherwise p would be decomposable by cutting it right before a_{r+1}. If y_1 is on the right of i, then we are done as is on the other side of a_{r_1+1}, where y is located between a_{r_1} and a_{r_1+1} for the same reason, and so on. This way we get a subsequence $y_1, y_2, \cdots$ so that n reaches each of the y_t, and this subsequence eventually gets to the right of i, since in each step we bypass at least one left-to-right minimum. Thus the proposition is proved. ∎

The only problem is that there could be many vertices with the property that all their ancestors have a positive label. If that happens, we resort to the following Proposition to locate the vertex associated with n.

PROPOSITION 4.53

Suppose $p_n \neq n$. Then n is the leftmost *entry of p which has the property that each of its ancestors has a positive label.*

PROOF Suppose p_k and n both have this property and that p_k is on the left of n. (If there are several candidates for the role of p_k, choose the rightmost one). If p_k beats an element y on the right of n by participating in the 132-pattern xp_ky, then xp_kny is a 1342-pattern, which is a contradiction. So p_k does not beat such an element y. In other words, all elements after n are smaller than all elements before p_k. Still, p_k must reach elements on the right of n, thus it beats some element v between p_k and n. This element v in turn beats some element w on the right of n by participating in some 132-pattern tvw. However, this would imply that $tvnw$ is a 1342-pattern, a contradiction, which proves our claim. ∎

Therefore, we can recover the entry n of p from T. Then we can proceed as in the proof of Lemma 4.36, that is, just delete n, subtract 1 from the labels of its ancestors and iterate this procedure to get p. If at any time during this procedure we find that the current root is associated with the maximal entry that has not been associated with other vertices yet, then there are two possibilities.

(a) If the tree has only one branch at this moment, then simply remove its root (and the maximal entry with it), and adjust the label of the new root so that it is the sum of the labels of its children.

(b) If the tree has more than one branch at this moment, then deleting the root vertex will split the tree into smaller trees. Then we continue the same procedure on each of them. The set of the entries associated to each of these smaller trees is uniquely determined. Indeed, the fact that our current tree T' has more than one branch is equivalent to the fact that the current partial permutation p' becomes decomposable when the maximal element (the one associated to the root of T') is removed. We have seen this for unlabeled trees in the proof of Lemma 4.38, and we know from Proposition 4.46 that p is indecomposable if and only if $N(p)$ is indecomposable. So the entries are assigned to the subtrees so that each subtree consists of larger entries than the subtrees on its right.

Therefore, we can always recover p in this way from T. This proves that F is a bijection, completing the proof of Theorem 4.34. ▮

COROLLARY 4.54
The number of indecomposable 1342-avoiding n-permutations is

$$|D_n^{\beta(0,1)}| = t_n = 3 \cdot 2^{n-2} \cdot \frac{(2n-2)!}{(n+1)!(n-1)!}. \tag{4.7}$$

PROOF Follows from (4.6) and Theorem 4.34. ▮

Computing the numbers $S_n(1342)$ is now a breeze, (well, if you like generating functions).

LEMMA 4.55
Let $s_n = S_n(1342)$ and let $H(x) = \sum_{n=0}^{\infty} s_n x^n$. Furthermore, let $F(x) = \sum_{n=1}^{\infty} t_n x^n$, that is, let $F(x)$ be the generating function of the numbers of indecomposable 1342-avoiding permutations. Then

$$H(x) = \sum_{i \geq 0} F^i(x) = \frac{1}{1-F(x)} = \frac{32x}{-8x^2+20x+1-(1-8x)^{3/2}}. \tag{4.8}$$

PROOF Tutte [191] has computed the ordinary generating function of the numbers t_n and obtained

$$F(x) = \sum_{n=1}^{\infty} t_n x^n = \sum_{n=1}^{\infty} 3 \cdot 2^{n-1} \cdot \frac{(2n-2)!}{(n+1)!(n-1)!} x^n \tag{4.9}$$

$$= \frac{8x^2+12x-1+(1-8x)^{3/2}}{32x}. \tag{4.10}$$

The coefficients of $F(x)$ are the numbers of indecomposable 1342-avoiding n-permutations. Any 1342-avoiding permutation has a unique decomposition into indecomposable permutations. This can consist of $1, 2, \cdots, n$ blocks, implying that $s_n = \sum_{i=1}^{n} t_i s_{n-i}$. Therefore, $H(x) = 1/(1-F(x))$ as claimed. ▮

It is time that we mentioned the other kind of objects that are in bijection with these permutations. These are *rooted bicubic maps*, that is, planar maps root), and the underlying graph is bipartite. Tutte was enumerating these maps (according to the number $2(n+1)$ of vertices) when he obtained formula (4.9), and Cori, Jacquard, and Schaeffer then used the $\beta(0,1)$-trees to find a more combinatorial proof of Tutte's result.

Now that we have the generating function of the numbers $S_n(1342)$, we are in a position to obtain an explicit formula for their number. That formula will prove Theorem 4.31.

PROOF (of Theorem 4.31). Multiply both the numerator and the denominator of $H(x)$ by $(-8x^2+20x+1)+(1-8x)^{3/2}$. After simplifying we get

$$H(x) = \frac{(1-8x)^{3/2} - 8x^2 + 20x + 1}{2(x+1)^3}. \qquad (4.11)$$

As $(1-8x)^{3/2} = 1 - 12x + \sum_{n\geq 2} 3 \cdot 2^{n+2} x^n \frac{(2n-4)!}{n!(n-2)!}$, formula (4.11) implies our claim. ∎

So the first few values of $S_n(1342)$ are 1, 2, 6, 23, 103, 512, 2740, 15485, 91245, 555662.

In particular, one sees easily that the formula for $S_n(1342)$ given by Theorem (4.31) is dominated by the last summand; in fact, the alternation in sign assures that this last summand is larger than the whole right hand side if $n \geq 8$. As $\frac{(2n-4)!}{n!(n-2)!} < \frac{8^{n-2}}{n^{2.5}}$ by Stirling's formula, we have proved the following exponential upper bound for $S_n(1342)$.

COROLLARY *4.56*
For all n, we have $S_n(1342) < 8^n$.

On the other hand, it is routine to verify that the numbers t_n satisfy the recurrence $t_n = (8n-12)t_{n-1}/(n+1)$. As we explained immediately after Conjecture 4.9, the fact that $S_n(1342) < 8^n$ implies that the limit $\sqrt[n]{S_n(1342)}$ exists. Therefore, by the Squeeze Principle, we obtain the following Corollary.

COROLLARY *4.57*
We have

$$\lim_{n\to\infty} \sqrt[n]{S_n(1342)} = 8.$$

This result is again striking for two different reasons. On one hand, this is the third time that we can compute $\lim_{n\to\infty} \sqrt[n]{S_n(q)}$ for some pattern q. In fact, we have seen that

$$\lim_{n\to\infty} \sqrt[n]{S_n(q)} = \begin{cases} 4 \text{ if } q \text{ is of length 3,} \\ (k-1)^2 \text{ if } q = 123\cdots k, \\ 8 \text{ if } q = 1342, \end{cases}$$

In other words, in every case when we saw an exact answer, the exact answer was an integer. In general, however, that does not hold. Present author [41] has recently proved that $\lim_{n\to\infty} \sqrt[n]{S_n(12453)} = 9+4\sqrt{2}$. So these limits are not even always *rational*.

The other surprise provided by Corollary 4.57 is that

$$\lim_{n\to\infty} \sqrt[n]{S_n(1342)} = 8 \neq \lim_{n\to\infty} \sqrt[n]{S_n(1234)} = 9.$$

That is, even in the logarithmic sense, the sequences $S_n(1342)$ and $S_n(1234)$ are different. This phenomenon is not well understood. If we could understand why the pattern 1342 is *really* so easy to avoid, then maybe we could use that information to find other, longer patterns that are easy to avoid.

4.4.3 The Pattern 1234

The pattern 1234 is a monotone pattern, therefore Theorem 4.11, that provides an asymptotic formula and a very good upper bound for the numbers $S_n(123\cdots k)$, applies to it. We would like to point out, however, that using certain techniques beyond the scope of this book, Ira Gessel [105] proved the following *exact formula* for these numbers

$$S_n(1234) = 2 \cdot \sum_{k=0}^{n} \binom{2k}{k}\binom{n}{k}^2 \frac{3k^2+2k+1-n-2nk}{(k+1)^2(k+2)(n-k+1)}. \tag{4.12}$$

The alert reader has probably noticed that the summands on the right-hand side are not always non-negative, which decreases the hopes for a combinatorial proof. However, a few years later Gessel found the following alternative form for his formula [104]

$$S_n(1234) = \frac{1}{(n+1)^2(n+2)} \sum_{k=0}^{n} \binom{2k}{k}\binom{n+1}{k+1}\binom{n+2}{k+1}. \tag{4.13}$$

In this new form, all terms are nonnegative, but there is still a division, suggesting that a direct combinatorial proof is probably difficult to find.

We will return to the surprising complexity of Gessel's formulae in the next