

A large set of permutations avoiding 1324

David Bevan[†]

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Abstract

We show that the growth rate of the class of **1324**-avoiding permutations exceeds 9.91.

Hasse graphs

To each permutation π , we associate a plane graph, which we denote H_π . To create H_π , let vertex i be the point $(i, \pi(i))$ in the plane. Now add an edge between vertices i and j , $i < j$, if and only if $\pi(i) < \pi(j)$ and there is no vertex k such that $i < k < j$ and $\pi(i) < \pi(k) < \pi(j)$. H_π corresponds to the Hasse diagram of the poset on the points $(i, \pi(i))$, so we call H_π the **Hasse graph** of π .

The subset \mathcal{W}

If π avoids the pattern **1324**, then H_π does not have the “diamond” graph H_{1324} as a minor. In particular, the subgraph of H_π induced by a left-to-right minimum of π and the points to its north-east is a tree, as is that induced by a right-to-left maximum of π and the points to its south-west.

We restrict our attention to **1324**-avoiders whose Hasse graphs are spanned by a sequence of such trees, rooted at alternate boundaries. Trees rooted at a left-to-right minimum we colour red, and trees rooted at a right-to-left maximum we colour blue. We refer to these as **red trees** and **blue trees** respectively. We denote by \mathcal{W} the set of permutations that can be partitioned in this way. See Figure 1 for an example of a permutation in \mathcal{W} . Note that \mathcal{W} does not contain every **1324**-avoider; **2143** $\notin \mathcal{W}$.

[†]Department of Mathematics and Statistics, The Open University, Milton Keynes, England.
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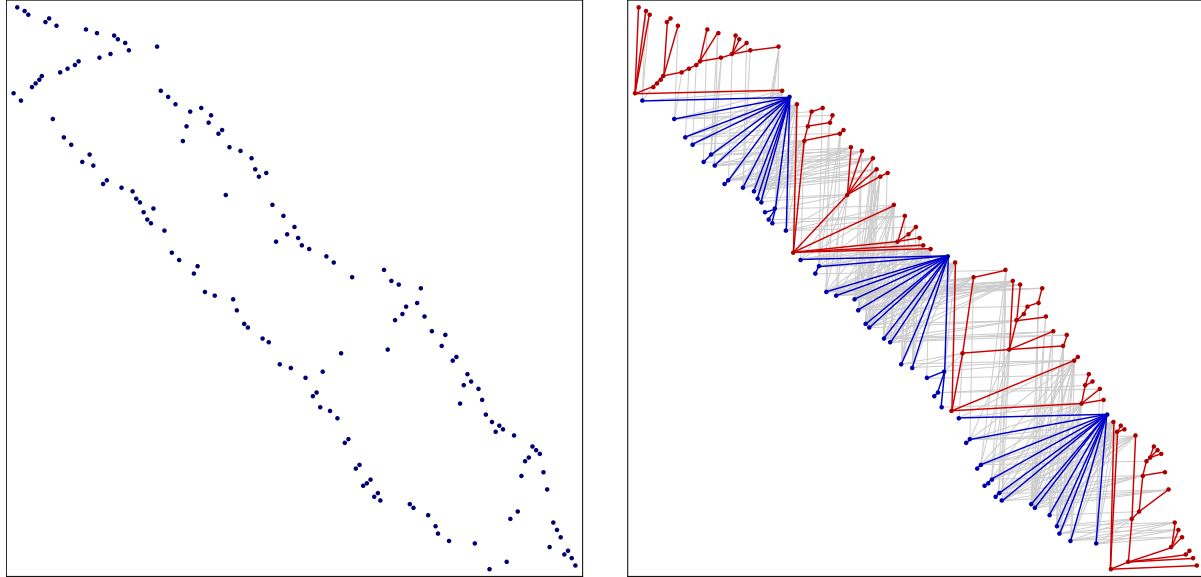


Figure 1: A permutation of length 157 in $\mathcal{W}(25, 19, 13, 3)$ and its Hasse graph

We further constrain the permutations we consider in the following ways:

- The first and last trees are red.
- Each red tree contains the same number of vertices (for which we use the variable k).
- Each blue tree contains the same number of vertices (which we denote by ℓ), and has the same root degree (denoted by d).

Note that the root of each blue tree (a right-to-left maximum) is the uppermost point below the root of the previous red tree, and the root of each non-initial red tree (a left-to-right minimum) is the leftmost point to the right of the root of the previous blue tree.

We use $\mathcal{W}(k, \ell, d, s)$ to denote the set of permutations in \mathcal{W} with s blue trees whose red trees each have k vertices and whose blue trees each have ℓ vertices and root degree d . Every permutation in $\mathcal{W}(k, \ell, d, s)$ has length $k + s(\ell + k)$.

Now, given $\lambda > 0$ and $\delta \in (0, 1)$, let $\mathcal{W}_{\lambda, \delta}(k)$ be the set of permutations $\mathcal{W}(k, \lceil \lambda k \rceil, \lceil \delta \lambda k \rceil, k)$ and $n(k, \lambda) = k + k(\lceil \lambda k \rceil + k)$ be the length of each permutation in $\mathcal{W}_{\lambda, \delta}(k)$.

Let us also use $\gamma(\lambda, \delta)$ to denote the (upper) growth rate of $\bigcup_k \mathcal{W}_{\lambda, \delta}(k)$:

$$\gamma(\lambda, \delta) = \lim_{k \rightarrow \infty} |\mathcal{W}_{\lambda, \delta}(k)|^{1/n(k, \lambda)}.$$

We will prove the following:

Theorem 0.1. *There is some $\lambda > 0$ and some $\delta \in (0, 1)$ such that $\gamma(\lambda, \delta) > 9.91$.*

Interleaving of trees

A permutation in $\mathcal{W}(k, \ell, d, s)$ can be built by starting with an arbitrary k -vertex red tree. We then take an arbitrary ℓ -vertex blue tree with root degree d and horizontally interleave its non-root vertices with the non-root vertices of the red tree in such a way as to avoid creating a **1324**. Then we take an arbitrary k -vertex red tree and vertically interleave its non-root vertices with the non-root vertices of the previous blue tree, avoiding creating a **1324**. Repeating the last two steps a total of s times creates the desired permutation.

Note that the interleaving at each step is independent of the interleaving at any previous step. The only requirement is that no **1324** is created by any of the interleavings. We will describe four ways of avoiding a **1324**, based on increasingly detailed analyses of how the vertices in red and blue trees may be interleaved.

Approach 1

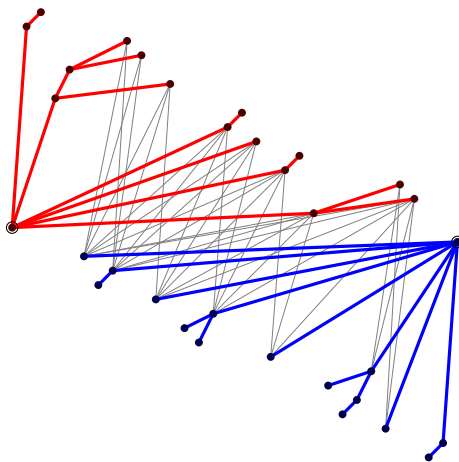


Figure 2: Interleaving red vertices with blue subtrees

In our first attempt, we arbitrarily interleave the $k - 1$ non-root red vertices with the d blue *subtrees* of the root of the blue tree, the vertices in each blue subtree being placed contiguously. This is sufficient to avoid creating a **1324**. See Figure 2 for an illustration. Let's use $\mathcal{W}^0(k, \ell, d, s)$ for elements of $\mathcal{W}(k, \ell, d, s)$ built with interleaving restricted in this way, and $\mathcal{W}_{\lambda, \delta}^0(k)$ for such elements of $\mathcal{W}_{\lambda, \delta}(k)$.

Now let $N(k, \ell, d)$ be the number of ways of building a permutation in $\mathcal{W}^0(k, \ell, d, s)$ from a permutation in $\mathcal{W}^0(k, \ell, d, s - 1)$ by adding a new ℓ -vertex blue tree with root degree d and a new k -vertex red tree. This number is independent of the starting permutation.

Indeed, since the number of ℓ -vertex trees with root degree d is $\frac{d}{\ell-1} \binom{2\ell-3-d}{\ell-2}$ (see Flajolet & Sedgewick [1] pp. 173–174),

$$N(k, \ell, d) = \binom{k-1+d}{d} \times \frac{d}{\ell-1} \binom{2\ell-3-d}{\ell-2} \times \binom{k-1+d}{d} \times \frac{1}{k} \binom{2k-2}{k-1},$$

the second and fourth terms being the number of ways of choosing the blue and red trees respectively, and the other two terms being the number of ways performing the two interleavings.

For fixed $\lambda > 0$ and $\delta \in (0, 1)$, by applying Stirling's approximation we obtain

$$\lim_{k \rightarrow \infty} N(k, \lceil \lambda k \rceil, \lceil \delta \lambda k \rceil) = E(\lambda, \delta)^k O(k^{-3}),$$

where

$$E(\lambda, \delta) = 4 \frac{(2 - \delta)^{(2 - \delta)\lambda} (1 + \delta\lambda)^{2(1 + \delta\lambda)}}{(1 - \delta)^{(1 - \delta)\lambda} (\delta\lambda)^{2\delta\lambda}}.$$

For fixed, λ , $E(\lambda, \delta)$ achieves a maximum when δ has the value

$$\delta_\lambda = \frac{2\lambda - 1 + \sqrt{1 + 4\lambda + 8\lambda^2}}{2\lambda(2 + \lambda)}.$$

Now, from the fact that

$$|\mathcal{W}^0(k, \ell, d, s)| = \frac{1}{k} \binom{2k - 2}{k - 1} N(k, \ell, d)^s,$$

it can easily be ascertained that the growth rate is given by

$$\lim_{k \rightarrow \infty} |\mathcal{W}_{\lambda, \delta}^0(k)|^{1/n(k, \lambda)} = E(\lambda, \delta_\lambda)^{1/(1 + \lambda)}.$$

Numerically maximising by setting $\lambda \approx 0.61840$ (with $\delta_\lambda \approx 0.86238$) gives us a first lower bound for the growth rate of **Av(1324)** of **9.40399**.

Concentration of distributions

To improve on this lower bound, we depend critically on the fact that the distributions of many parameters of combinatorial classes are *concentrated* in the sense that their standard deviation is of an asymptotic order smaller than the mean. For example, the expected proportion of leaf vertices in an n -vertex plane tree is $\frac{1}{2}$ (for $n \geq 2$), with standard deviation $O(n^{-1/2})$ (see [1] p. 182).

As a result of Chebyshev's inequality, such distributions have the following concentration property (see [1] Proposition III.3):

Proposition 0.2. *If X_n is a sequence of random variables with means $\mu_n = \mathbb{E}[X_n]$ and standard deviations $\sigma_n = \sigma[X_n]$ satisfying the conditions*

$$\lim_{n \rightarrow \infty} \mu_n = \mu, \quad \lim_{n \rightarrow \infty} \sigma_n = 0,$$

then, X_n is concentrated at μ in the sense that, for any $\varepsilon > 0$, given sufficiently large n ,

$$\mathbb{P}[\mu(1 - \varepsilon) \leq X_n \leq \mu(1 + \varepsilon)] \geq 1 - \varepsilon.$$

In practice, this means that, as long as we consider only a finite number of (independent and/or mutually exclusive) concentrated parameters, we can work on the assumption that, asymptotically, the value of each parameter is entirely concentrated at its limiting mean.

Approach 2

For the second and subsequent approaches, we interpret $N(k, \ell, d)$ as giving the number of ways of choosing new red and blue trees and of choosing the *positions of the roots* of the d blue subtrees relative to the vertices of the red trees. We will refer to roots of blue subtrees simply as **blue roots**. We now consider what freedom we have in positioning the non-root vertices of the blue subtrees relative to the vertices of a red tree. Without loss of generality, we will limit our discussion to the case of horizontal interleaving (as in Figure 2).

Our first (elementary) observation is that the non-root vertices of a blue subtree must occur to the right of any blue root to its left. In an interleaving of red vertices and blue roots, let us call the number of contiguous red vertices immediately to the left of a blue root u , the **gap size** for u . The following properties enable us to establish the asymptotic behaviour.

Given $\lambda > 0$ and $\delta \in (0, 1)$, consider interleavings of $k - 1$ red vertices with the roots of $\lceil \delta \lambda k \rceil$ blue subtrees. For each $j \geq 0$, let the parameter $\varphi_k(j)$ be the expected proportion of blue roots in an interleaving which have gap size j . Also, let $\varphi_k(> j) = \sum_{i>j} \varphi_k(i)$ be the expected proportion of blue roots in an interleaving which have gap size *greater than* j .

Proposition 0.3. $\varphi_k(j)$ is concentrated at $\mu_\varphi(j) = \delta \lambda (1 - \delta \lambda)^{-(j+1)}$. Similarly, $\varphi_k(> j)$ is concentrated at $\mu_\varphi(> j) = (1 + \delta \lambda)^{-(j+1)}$.

Note: Proofs that parameters satisfy the concentration conditions are currently omitted.

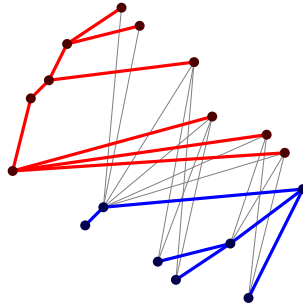


Figure 3: Interleaving red leaves with a blue subtree

Our second observation is as follows. Consider an interleaving of red vertices and blue roots. Suppose that the root u of some blue subtree τ has gap size at least j , and also that the j red vertices immediately to the left of u are all leaves. Then, any non-root vertices of τ may be arbitrarily interleaved with the j red leaves without creating a **1324**. See Figure 3 for an illustration. To determine the asymptotic behaviour, we use the following properties.

For each $j \geq 0$, let $\omega_k(j)$ be the expected proportion of positions in a red tree which have *exactly* j contiguous *leaves* immediately to their left. Also, let $\omega_k(\geq j) = \sum_{i \geq j} \omega_k(i)$ be the expected proportion of positions in a red tree which have *at least* j contiguous leaves immediately to their left.

Proposition 0.4. $\omega_k(j)$ is concentrated at $\mu_\omega(j) = 2^{-(j+1)}$. Similarly, $\omega_k(\geq j)$ is concentrated at $\mu_\omega(\geq j) = 2^{-j}$.

We need to combine $\varphi_k(j)$ and $\omega_k(j)$ as follows. Given λ and δ , consider interleavings of $k - 1$ red vertices with the roots of $\lceil \delta \lambda k \rceil$ blue subtrees. For each $j \geq 0$, let $\chi_k(j)$ be the expected proportion of blue subtrees whose non-root vertices can be interleaved arbitrarily with exactly j red leaves. If we call the j red vertices immediately to the left of a blue root, its **first** j red vertices, then this is the case if either the gap size is j and each of the first j red vertices is a leaf, or else the gap size exceeds j , the first j red vertices are all leaves but the next red vertex is an internal vertex. Hence,

$$\chi_k(j) = \varphi_k(j)\omega_k(\geq j) + \varphi_k(>j)\omega_k(j),$$

and $\chi_k(j)$ is concentrated at

$$\mu_\chi(j) = \mu_\varphi(j)\mu_\omega(\geq j) + \mu_\varphi(>j)\mu_\omega(j) = \frac{1 + 2\delta\lambda}{(2 + 2\delta\lambda)^{j+1}}.$$

We also require the distribution of blue subtree sizes. Given λ and δ , for each positive i , let $\psi_k(i)$ be the expected proportion of the subtrees in a $\lceil \lambda k \rceil$ -vertex tree with root degree $\lceil \delta \lambda k \rceil$ that have i vertices.

Proposition 0.5. $\psi_k(i)$ is concentrated at

$$\mu_\psi(i) = \frac{1}{i} \binom{2i-2}{i-1} \frac{(1-\delta)^{i-1}}{(2-\delta)^{2i-1}}.$$

Thus, for each i and j , asymptotically, i -vertex blue subtrees whose non-root vertices can be interleaved arbitrarily with exactly j red leaves contribute a factor of

$$\binom{i-1+j}{j}^{\mu_\psi(i)\mu_\chi(j)\lceil \delta \lambda k \rceil}$$

to the number of ways of interleaving a red tree with a blue tree.

So, for each n , the growth rate is at least

$$\gamma'_n(\lambda, \delta) = E(\lambda, \delta)^{1/(1+\lambda)} \times \prod_{i=2}^n \prod_{j=1}^n \binom{i-1+j}{j}^{2\mu_\psi(i)\mu_\chi(j)\delta\lambda/(1+\lambda)},$$

the factor of 2 in the exponent being due to the fact that two interleavings occur when a new blue tree and a new red tree are added.

With $n = 50$, numerical maximisation by setting $\lambda \approx 0.69161$ and $\delta \approx 0.77341$ gives us a second lower bound for the growth rate of $\text{Av}(1324)$ of **9.73384**.

The rate of convergence is such that we can have high confidence that these are actually the first six decimal digits of $\lim_{n \rightarrow \infty} \max_{\lambda, \delta} \gamma'_n(\lambda, \delta)$.

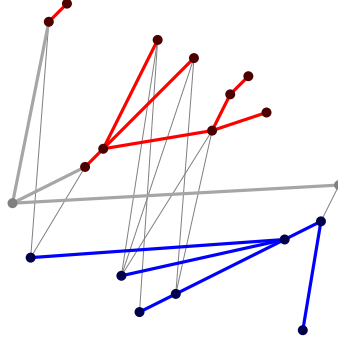


Figure 4: Interleaving a blue subtree with its red forest

Approach 3

Our third attempt is based on the following observation. Suppose v is the nearest red vertex to the *right* of the root u of some blue subtree τ . Then no vertex of τ can be positioned to the left of the *parent* of v in the red tree, since otherwise a **1324** would be created. Thus vertices of τ can only be interleaved with those red vertices positioned between u and the parent of v . We will call the graph induced by this set of red vertices the **red forest** for (the position of) u (or τ). See Figure 4 for an illustration.

If exactly the rightmost j vertices of a red forest are leaves, we will say that the forest has **right leaves**. Suppose that the red forest for an i -vertex blue subtree τ has h vertices and j right leaves. Then, assuming the gap size is at least h , the non-root vertices of τ can be interleaved with the vertices of the red forest in at least

$$Q(i, h, j) = \binom{i-1+j}{j} + (i-1)(h-j)$$

ways. The first summand comes from the fact that the non-root vertices of τ can be interleaved arbitrarily with the right leaves. The second summand counts the possible interleavings in which vertices of τ occur in precisely two positions, one of which is to the right of all the leaves of the red forest and the other of which is somewhere to the left of the right leaves.

For each positive h and each j ($0 \leq j \leq h$), let $\omega_k(h, j)$ be the expected proportion of positions in a red tree which have an h -vertex red forest with j right leaves. Also, let $\omega_k(\geq h, j) = \sum_{i \geq h} \omega_k(i, j)$ be the expected proportion of positions in a red tree which have a red forest with at least h vertices and j right leaves.

Proposition 0.6. $\omega_k(h, j)$ is concentrated at

$$\mu_\omega(h, j) = \frac{j}{2^{2s+1}(2s-j)} \binom{2s-j}{s}.$$

Similarly, $\omega_k(\geq h, j)$ is concentrated at

$$\mu_\omega(\geq h, j) = \frac{j}{(2h-j)2^{2h+1}} \binom{2h-j}{h} {}_3F_2\left(1, h - \frac{j}{2}, h + \frac{1}{2} - \frac{j}{2}; h+1, h-j+1; 1\right).$$

We combine $\varphi_k(h)$ and $\omega_k(h, j)$ as follows. Given λ and δ , consider interleavings of $k - 1$ red vertices with the roots of $\lceil \delta \lambda k \rceil$ blue subtrees. For each h and j , let $\chi_k(h, j)$ be the expected proportion of blue subtrees whose non-root vertices can be interleaved with h vertices of a red forest the rightmost j vertices of which are leaves. Then,

$$\chi_k(h, j) = \begin{cases} \varphi_k(h) \omega_k(\geq h, j) + \varphi_k(> h) \omega_k(h, j), & j < h, \\ \varphi_k(h) \omega_k(\geq h) + \varphi_k(> h) \omega_k(h, h), & j = h, \end{cases}$$

and $\chi_k(h, j)$ is concentrated at

$$\mu_\chi(h, j) = \begin{cases} \mu_\varphi(h) \mu_\omega(\geq h, j) + \mu_\varphi(> h) \mu_\omega(h, j), & j < h, \\ \mu_\varphi(h) \mu_\omega(\geq h) + \mu_\varphi(> h) \mu_\omega(h, h), & j = h. \end{cases}$$

Thus, for each n , the growth rate is at least

$$\gamma_n''(\lambda, \delta) = E(\lambda, \delta)^{1/(1+\lambda)} \times \prod_{i=2}^n \prod_{h=1}^n \prod_{j=1}^h Q(i, h, j)^{2\mu_\psi(i) \mu_\chi(h, j) \delta \lambda / (1+\lambda)}.$$

With $n = 50$, numerical maximisation by setting $\lambda \approx 0.69436$ and $\delta \approx 0.76151$ gives us a third lower bound for the growth rate of $\text{Av}(\mathbf{1324})$ of **9.80058**.

The rate of convergence is such that we can have high confidence that these are actually the first six decimal digits of $\lim_{n \rightarrow \infty} \max_{\lambda, \delta} \gamma_n''(\lambda, \delta)$.

Approach 4

So far, we have taken no account of the structure of the blue subtrees, and very little account of the structure of the red forests (only considering the number of right leaves). For our final attempt, we consider each distinct subtree and forest individually.

For each forest π , let $\omega_k(\pi)$ be the expected proportion of positions in a red tree whose red forest is π . $\omega_k(\pi)$ depends only on the size of π . Also, let $\omega_k(\geq \pi)$ be the expected proportion of positions in a red tree whose red forest has at least $|\pi|$ vertices, and for which the graph induced by the rightmost $|\pi|$ vertices of the forest is π . $\omega_k(\geq \pi)$ depends on both the size of π and also on the number of components in π .

Proposition 0.7. *Let $h = |\pi|$. $\omega_k(\pi)$ is concentrated at*

$$\mu_\omega(\pi) = \frac{h!(h+1)!}{2^{h+1}(2h)!}.$$

Similarly, $\omega_k(\geq \pi)$ is concentrated at

$$\mu_\omega(\geq \pi) = \frac{1}{2^{2h-c}},$$

where c is the number of components of (i.e. the number of trees in) the forest π .

We combine $\varphi_k(h)$ and $\omega_k(\pi)$ as follows. Given λ and δ , consider interleavings of $k - 1$ red vertices with the roots of $\lceil \delta \lambda k \rceil$ blue subtrees. For each forest π , let $\chi_k(\pi)$ be the expected proportion of blue subtrees for whom the red vertices with which the blue non-root vertices can be interleaved induce the forest π . Then,

$$\chi_k(\pi) = \varphi_k(|\pi|)\omega_k(\geq \pi) + \varphi_k(>|\pi|)\omega_k(\pi),$$

and $\chi_k(\pi)$ is concentrated at

$$\mu_\chi(\pi) = \mu_\varphi(|\pi|)\mu_\omega(\geq \pi) + \mu_\varphi(>|\pi|)\mu_\omega(\pi).$$

Given λ and δ , for each tree τ , let $\psi_k(\tau)$ be the expected proportion of the subtrees in a $\lceil \lambda k \rceil$ -vertex tree with root degree $\lceil \delta \lambda k \rceil$ that are isomorphic to τ . $\psi_k(\tau)$ depends only on the size of τ .

Proposition 0.8. *Let $i = |\tau|$. $\psi_k(\tau)$ is concentrated at*

$$\mu_\psi(\tau) = \frac{(1 - \delta)^{i-1}}{(2 - \delta)^{2i-1}}.$$

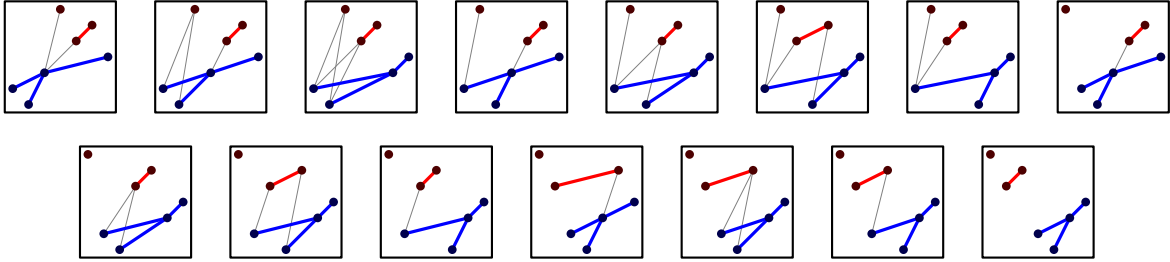


Figure 5: The fifteen ways of interleaving blue subtree 2134 with red forest 312 without creating a 1324

Finally, let $Q(\tau, \pi)$ be the number of ways of interleaving (the non-root vertices of) blue subtree τ with (the vertices in) red forest π without creating a 1324.

With these definitions complete, we can now present the following inequality for $\gamma(\lambda, \delta)$:

$$\gamma(\lambda, \delta) \geq E(\lambda, \delta)^{1/(1+\lambda)} \times \prod_{\tau, \pi} Q(\tau, \pi)^{2\mu_\psi(\tau)\mu_\chi(\pi)\delta\lambda/(1+\lambda)},$$

where the product is over all trees τ and forests π .

Determining $Q(\tau, \pi)$, $\mu_\psi(\tau)$ and $\mu_\chi(\pi)$ for all except the smallest forests and trees is clearly infeasible. However, for each n and m (with $n \leq m$), the growth rate is at least

$$\begin{aligned} \gamma_{n,m}(\lambda, \delta) &= E(\lambda, \delta)^{1/(1+\lambda)} \times \prod_{|\tau|+|\pi| \leq n} Q(\tau, \pi)^{2\mu_\psi(\tau)\mu_\chi(\pi)\delta\lambda/(1+\lambda)} \\ &\quad \times \prod_{\substack{i+h > n \\ i, h \leq m}} \prod_{j \leq h} Q(i, h, j)^{2\mu_\psi(i)\mu_\chi(h, j)\delta\lambda/(1+\lambda)}. \end{aligned}$$

Here, the contribution from pairs consisting of a subtree and a forest the sum of whose lengths does not exceed n are handled exactly. For larger pairs, we use the approximate results from our third attempt above.

With $n = 14$ and $m = 32$, numerical maximisation by setting $\lambda \approx 0.70869$ and $\delta \approx 0.73482$ gives us a final lower bound for the growth rate of $\text{Av}(\mathbf{1324})$ of **9.91466**, thus proving Theorem 0.1.

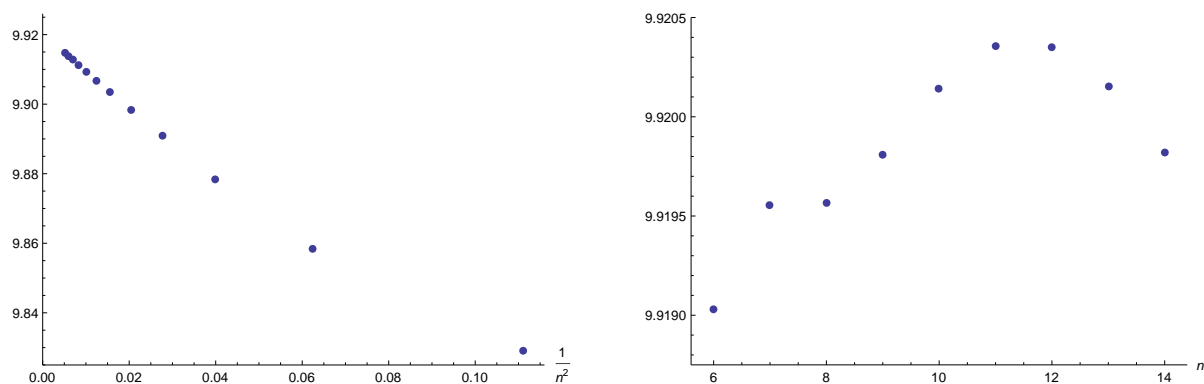


Figure 6: A plot of $\max_{\lambda, \delta} \gamma_{n,32}(\lambda, \delta)$ against $1/n^2$, and a plot of the estimation for the limiting growth rate resulting from linearly extrapolating from the pair of points for n and $n - 1$ in the first plot

Determining this value requires handling over 1.6 million pairs consisting of a subtree and a forest. Unfortunately, the rate of convergence at $n = 14$ is still quite slow, but the data suggests that $\lim_{n, m \rightarrow \infty} \max_{\lambda, \delta} \gamma_{n, m}(\lambda, \delta)$ is close to (and probably less than) 9.92. See Figure 6.

A final question

If we let $\mathcal{W}^* = \bigcup_{k, \ell, d, s} \mathcal{W}(k, \ell, d, s)$, then we have the following sequence of inequalities:

$$\text{gr}(\text{Av}(\mathbf{1324})) \geq \text{gr}(\mathcal{W}) \geq \text{gr}(\mathcal{W}^*) \geq \max_{\lambda, \delta} \gamma(\lambda, \delta) > 9.91.$$

Which (if any) of these inequalities are actually identities?

Acknowledgements

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References

- [1] Philippe Flajolet and Robert Sedgewick. *Analytic combinatorics*. Cambridge University Press, Cambridge, 2009.