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TWO NEW PROOFS OF LERCH'S FUNCTIONAL EQUATION

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ABSTRACT.

One bright Sunday morning I went to church,
And there I met a man named Lerch.
We both did sing in jubilation,
For he did show me a new equation.

Two simple derivations of the functional equation of

$$\sum_{n=0}^{\infty} \exp[2\pi inx](n+a)^{-s}$$

are given. The original proof is due to Lerch.

If x is real and $0<a\leqq 1$, define

$$\varphi(x, a, s) = \sum_{n=0}^{\infty} \exp[2\pi inx](n + a)^{-s},$$

where $\sigma = \operatorname{Re} s > 1$ if x is an integer, and $\sigma>0$ otherwise. Note that $\varphi(0, a, s) = \zeta(s, a)$, the Hurwitz zeta-function. Furthermore, if $a=1$, $\varphi(0, 1, s)=\zeta(s)$, the Riemann zeta-function.

In 1887, Lerch **[1]** derived the following functional equation for $\varphi(x, a, s)$.

THEOREM. *Let $0<x<1$. Then $\varphi(x, a, s)$ has an analytic continuation to the entire complex plane and is an entire function of s. Furthermore, for all s,*

$$\begin{aligned}\varphi(x, a, 1 - s) = (2\pi)^{-s}\Gamma(s) \\ \cdot \{\exp[\tfrac{1}{2}\pi is - 2\pi iax]\varphi(-a, x, s) \\ + \exp[-\tfrac{1}{2}\pi is + 2\pi ia(1 - x)]\varphi(a, 1 - x, s)\}.\end{aligned} \tag{1}$$

Lerch's proof **[1]** of (1) depends upon the evaluation of a certain loop integral. Our objective here is to give two simple, new proofs of (1). The first proof uses contour integration; the second employs the Euler-Maclaurin summation formula. By slight variations of our methods, one can derive the corresponding result, namely Hurwitz's formula, for $\varphi(0, a, s) = \zeta(s, a)$.

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First Proof. Assume that $s>1$ is real. With the aid of Euler's integral representation for $\Gamma(s)$, it is easy to show that [**1**, pp. 19–20]

$$\Gamma(s)\varphi(x, a, s)=\int_0^\infty \frac{\exp[(1-a)u-2\pi ix]}{\exp[u-2\pi ix]-1}\, u^{s-1}\, du. \tag{2}$$

If we put $x=b+\frac{1}{2}$, then $|b|<\frac{1}{2}$. Define

$$F(z)=\frac{\pi\exp[2\pi ibz]}{(z+a)^s\sin(\pi z)},$$

where the principal branch of $(z+a)^s$ is chosen. Choose c so that $-a<c<0$. If m is a positive integer, let C_m denote the positively oriented contour consisting of the right half of the circle with center $(c, 0)$ and radius $m+\frac{1}{2}-c$ together with the vertical diameter through $(c, 0)$. By the residue theorem,

$$\frac{1}{2\pi i}\int_{C_m} F(z)\, dz=\sum_{n=0}^{m}\exp[2\pi inx](n+a)^{-s}. \tag{3}$$

Let Γ_m denote the circular part of C_m. Since $|b|<\frac{1}{2}$, there is a constant M, independent of m, such that for z on Γ_m, $m\geqq 1$,

$$\left|\frac{\exp[2\pi ibz]}{\sin(\pi z)}\right|\leqq M.$$

Hence,

$$\left|\int_{\Gamma_m} F(z)\, dz\right|\leqq\frac{\pi^2(m+\frac{3}{2})M}{(m+\frac{1}{2})^s},$$

which tends to 0 as m tends to ∞ since $s>1$. Thus, upon letting m tend to ∞ in (3), we find that

$$\begin{aligned}\varphi(x, a, s) &= -\frac{1}{2\pi i}\int_{c-i\infty}^{c+i\infty} F(z)\, dz \\ &= \int_c^{c+i\infty}\frac{\exp[2\pi ibz-\pi iz]}{(z+a)^s(\exp[-2\pi iz]-1)}\, dz \\ &\quad + \int_c^{c-i\infty}\frac{\exp[2\pi ibz+\pi iz]}{(z+a)^s(\exp[2\pi iz]-1)}\, dz.\end{aligned} \tag{4}$$

We observe that the integrals in (4) converge uniformly in any compact set of the s-plane since $|b|<\frac{1}{2}$. Hence, (4) shows that $\varphi(x, a, s)$ can be analytically continued to an entire function of s.

Now suppose that $-1<s<0$. We wish to let c approach $-a$ in (4). In a neighborhood of $z=-a$, we have

$$\left|\frac{\exp[2\pi ibz\pm\pi iz]}{(z+a)^s(\exp[\pm 2\pi iz]-1)}\right|\leqq A\,|z+a|^{-s-1}\leqq A\,|y|^{-s-1},$$

where A is some positive number. Since $-1<s<0$,

$$\int_0^{\mp 1} |y|^{-s-1}\, dy < \infty.$$

Hence, the two integrals on the right side of (4) converge uniformly on $-a \leqq c \leqq -a+\varepsilon$, for any number $\varepsilon>0$. We may then let c tend to $-a$ in (4) to obtain

$$\begin{aligned}\varphi(x, a, s) = {} & i \exp[-\tfrac{1}{2}\pi i s - 2\pi i a b + \pi i a] \\ & \cdot \int_0^\infty \frac{\exp[-2\pi b y + \pi y]}{y^s(\exp[2\pi y + 2\pi i a] - 1)}\, dy \\ & - i \exp[\tfrac{1}{2}\pi i s - 2\pi i a b - \pi i a] \\ & \cdot \int_0^\infty \frac{\exp[2\pi b y + \pi y]}{y^s(\exp[2\pi y - 2\pi i a] - 1)}\, dy.\end{aligned}$$

If we make the substitutions $u=2\pi y$ and $b=x-\frac{1}{2}$ and replace s by $1-s$, the above becomes, for $s>1$,

$$\begin{aligned}\varphi(x, a, 1-s) = {} & \exp[\tfrac{1}{2}\pi i s - 2\pi i a(x-1)](2\pi)^{-s} \\ & \cdot \int_0^\infty \frac{\exp[-u(x-1)]u^{s-1}}{\exp[u + 2\pi i a] - 1}\, du \\ & + \exp[-\tfrac{1}{2}\pi i s - 2\pi i a x](2\pi)^{-s} \\ & \cdot \int_0^\infty \frac{\exp[ux]u^{s-1}}{\exp[u - 2\pi i a] - 1}\, du.\end{aligned}$$

If we now use (2), (1) immediately follows for $s>1$. By analytic continuation, (1) is valid for all s.

SECOND PROOF. Let f have a continuous first derivative on $[c, m]$, where m is a positive integer. Then we have the Euler-Maclaurin summation formula,

$$\begin{aligned}\sum_{c<n\leqq m} f(n) = {} & \int_c^m f(u)\, du + \tfrac{1}{2}f(m) + (c - [c] - \tfrac{1}{2})f(c) \\ & + \int_c^m (u - [u] - \tfrac{1}{2})f'(u)\, du.\end{aligned} \tag{5}$$

Put $c=1-a$ and $f(u)=(u+a)^{-s}\exp(2\pi i u x)$, where $\sigma>0$. Then, upon letting m tend to ∞ in (5), we obtain

$$\begin{aligned}\varphi(x, a, s) - a^{-s} = {} & \int_{1-a}^\infty (u+a)^{-s} \exp[2\pi i u x]\, du \\ & + (\tfrac{1}{2} - a)\exp[2\pi i x(1-a)] \\ & - s\int_{1-a}^\infty \frac{u - [u] - \frac{1}{2}}{(u+a)^{s+1}} \exp[2\pi i u x]\, du \\ & + 2\pi i x \int_{1-a}^\infty \frac{u - [u] - \frac{1}{2}}{(u+a)^s} \exp[2\pi i u x]\, du.\end{aligned} \tag{6}$$

The three integrals on the right side of (6) all converge for $\sigma>0$ by Dirichlet's test.

First, assume that $0<\sigma<1$. Since

$$[u]-u+\tfrac{1}{2}=\sum_{n=1}^{\infty}\frac{\sin(2\pi nu)}{\pi n} \tag{7}$$

if u is not an integer, we have formally

$$\begin{aligned}
&\int_{-a}^{\infty}\frac{u-[u]-\frac{1}{2}}{(u+a)^s}\exp[2\pi iux]\,du \\
&\quad=-\frac{1}{\pi}\int_0^{\infty}u^{-s}\exp[2\pi i(u-a)x]\sum_{n=1}^{\infty}\frac{1}{n}\sin(2\pi n\{u-a\})\,du \\
&\quad=\frac{\exp[-2\pi iax]}{2\pi i}\sum_{n=1}^{\infty}\frac{1}{n}\Big\{\exp[2\pi ina]\int_0^{\infty}u^{-s}\exp[2\pi iu(x-n)]\,du \\
&\qquad\qquad-\exp[-2\pi ina]\int_0^{\infty}u^{-s}\exp[2\pi iu(x+n)]\,du\Big\}.
\end{aligned} \tag{8}$$

We must justify the inversion in order of summation and integration. Since the Fourier series in (7) is boundedly convergent, the inversion is justified if we integrate over $(0, b)$, where b is any finite number [**2**, p. 41]. We need then only show that, for $0<\sigma<1$,

$$\begin{aligned}
&\lim_{b\to\infty}\sum_{n=1}^{\infty}\frac{1}{n}\Big\{\exp[2\pi ina]\int_b^{\infty}u^{-s}\exp[2\pi iu(x-n)]\,du \\
&\qquad-\exp[-2\pi ina]\int_b^{\infty}u^{-s}\exp[2\pi iu(x+n)]\,du\Big\}=0.
\end{aligned} \tag{9}$$

Upon an integration by parts,

$$\begin{aligned}
\int_b^{\infty}u^{-s}\exp[2\pi iu(x-n)]\,du&=O(b^{-\sigma}/n) \\
&\quad+\frac{s}{2\pi i(x-n)}\int_b^{\infty}u^{-s-1}\exp[2\pi iu(x-n)]\,du=O(b^{-\sigma}/n),
\end{aligned}$$

as b tends to ∞. By the same argument we obtain the same O-estimate for the integrals involving $\exp\{2\pi iu(x+n)\}$. Hence, (9) now easily follows.

Now, if $0<\sigma<1$ and $d\neq0$ is real, we have [**2**, pp. 107–108]

$$\int_0^{\infty}u^{-s}\exp[idu]\,du=\Gamma(1-s)\,|d|^{s-1}\exp[\tfrac{1}{2}\pi i(1-s)\operatorname{sgn} d]. \tag{10}$$

Thus, (8) becomes

$$\begin{aligned}\int_{-a}^{\infty} \frac{u-[u]-\frac{1}{2}}{(u+a)^s} \exp[2\pi iux]\, du \\ = -(2\pi)^{s-2}\Gamma(1-s)\exp[\tfrac{1}{2}\pi is - 2\pi iax] \\ \cdot \sum_{n=1}^{\infty}\left\{\frac{\exp[2\pi ina]}{n(n-x)^{1-s}} + \frac{\exp[-\pi is - 2\pi ina]}{n(n+x)^{1-s}}\right\}.\end{aligned} \tag{11}$$

Using (10) again, we have, for $0<\sigma<1$,

$$\begin{aligned}\int_{-a}^{\infty} (u+a)^{-s} \exp[2\pi iux]\, du \\ = \exp[-2\pi iax]\int_{0}^{\infty} u^{-s}\exp[2\pi iux]\, du \\ = \Gamma(1-s)(2\pi x)^{s-1}\exp[\tfrac{1}{2}\pi i(1-s) - 2\pi iax].\end{aligned} \tag{12}$$

Hence, substituting (11) and (12) into (6), we obtain, for $0<\sigma<1$,

$$\begin{aligned}\varphi(x, a, x) - a^{-s} \\ = \Gamma(1-s)(2\pi x)^{s-1}\exp[\tfrac{1}{2}\pi i(1-s) - 2\pi iax] \\ - \int_{-a}^{1-a} (u+a)^{-s}\exp[2\pi iux]\{1 + 2\pi ix(u-[u]-\tfrac{1}{2})\}\, du \\ + (\tfrac{1}{2}-a)\exp[2\pi ix(1-a)] - s\int_{1-a}^{\infty}\frac{u-[u]-\frac{1}{2}}{(u+a)^{s+1}}\exp[2\pi iux]\, du \\ + x(2\pi)^{s-1}\Gamma(1-s)\exp[\tfrac{1}{2}\pi i(s-1) - 2\pi iax] \\ \cdot \sum_{n=1}^{\infty}\left\{\frac{\exp[2\pi ina]}{n(n-x)^{1-s}} + \frac{\exp[-\pi is - 2\pi ina]}{n(n+x)^{1-s}}\right\}.\end{aligned} \tag{13}$$

We next observe that the infinite series on the right side of (13) converges absolutely and uniformly on any compact subset of the strip $-1<\sigma<1$. By Dirichlet's test, the integrals on the right side of (13) converge uniformly on any compact subset of the strip $-1<\sigma<1$. Hence, by analytic continuation, (13) is valid for $-1<\sigma<1$. Assume now that $-1<\sigma<0$. Replacing s by $s+1$ in (11), we have

$$\begin{aligned}\int_{-a}^{\infty} \frac{u-[u]-\frac{1}{2}}{(u+a)^{s+1}} \exp[2\pi iux]\, du \\ = (2\pi)^{s-1}\Gamma(-s)\exp[\tfrac{1}{2}\pi i(s-1) - 2\pi iax] \\ \cdot \sum_{n=1}^{\infty}\left\{\frac{\exp[2\pi ina]}{n(n-x)^{-s}} - \frac{\exp[-i\pi s - 2\pi ina]}{n(n+x)^{-s}}\right\}.\end{aligned} \tag{14}$$

Substitute (14) into (13), use the functional equation of $\Gamma(s)$, and observe that $x/\{n(n-x)\}+1/n=1/(n-x)$ and $x/\{n(n+x)\}-1/n=-1/(n+x)$. For $-1<\sigma<0$ we arrive at

$$\begin{aligned}
&\varphi(x, a, s) - a^{-s}\\
&\qquad = \Gamma(1-s)(2\pi x)^{s-1}\exp[\tfrac{1}{2}\pi i(1-s) - 2\pi iax]\\
&\qquad\quad - \int_{-a}^{1-a} (u+a)^{-s}\exp[2\pi iux]\\
&\qquad\quad \cdot \{1 + 2\pi ix(u - [u] - \tfrac{1}{2}) - s(u+a)^{-1}(u - [u] - \tfrac{1}{2})\}\, du \qquad (15)\\
&\qquad\quad + (\tfrac{1}{2} - a)\exp[2\pi ix(1-a)]\\
&\qquad\quad + (2\pi)^{s-1}\Gamma(1-s)\exp[\tfrac{1}{2}\pi i(s-1) - 2\pi iax]\\
&\qquad\quad \cdot \left\{\sum_{n=1}^{\infty}\frac{\exp[2\pi ina]}{(n-x)^{1-s}} - \sum_{n=1}^{\infty}\frac{\exp[-\pi is - 2\pi ina]}{(n+x)^{1-s}}\right\}.
\end{aligned}$$

Observe that the first expression on the right side of (15) corresponds to the term $n=0$ for the second series on the right side of (15). In the first series replace n by $n+1$. By an elementary calculation the second expression on the right side of (15) is seen to be

$$-a^{-s} - (\tfrac{1}{2} - a)\exp[2\pi ix(1-a)].$$

Upon these simplifications, (15) now becomes, for $-1<\sigma<0$,

$$\begin{aligned}
&\varphi(x, a, x)\\
&\quad = (2\pi)^{s-1}\Gamma(1-s)\exp[\tfrac{1}{2}\pi i(s-1) - 2\pi iax] \qquad (16)\\
&\quad\quad \cdot \{\exp[2\pi ia]\varphi(a, 1-x, 1-s) - \exp[-\pi is]\varphi(-a, x, 1-s)\}.
\end{aligned}$$

By analytic continuation (16) is valid for all s. Now replace s by $1-s$ in (16) to obtain (1).

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