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Some computations for m-dimensional partitions

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1. It was known to Euler that $p(n)$, the number of unrestricted partitions of n into non-increasing integral parts, is generated by

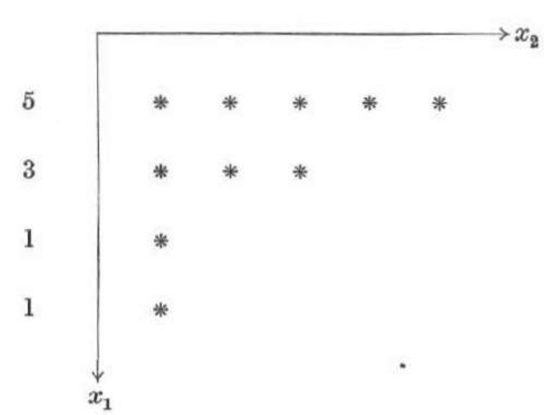
$$\sum_{n=0}^{\infty} p(n)x^n = (1-x)^{-1}(1-x^2)^{-1}(1-x^3)^{-1}\ldots \tag{1}$$

vith the usual convention that $p(0) = 1$.

We may regard a partition of n as an arrangement of nodes at integral points of the $_1, x_2)$ plane; thus

$$10 = 5+3+1+1$$

represented by



his 'Ferrers–Sylvester graph' (cf. MacMahon (1), p. 3) represents a partition of n into ntegers as a two-dimensional arrangement of nodes. We may form a natural generalization as follows.

By an 'unrestricted m-dimensional partition of n' we shall understand an arrangement of n nodes at points of Euclidean m-space with non-negative integral coordinates, with the property that if a node $(a_1, a_2, \ldots, a_m)$ occurs then so also do all the nodes $(x_1, x_2, \ldots, x_m)$ with $0 \leqslant x_i \leqslant a_i$ $(i = 1, 2, \ldots, m)$. We denote by $p_m(n)$ the number of distinct such partitions; trivially $p_1(n) = 1$ for all n. For $m \geqslant 2$ we compare $p_m(n)$ with $\pi_m(n)$ defined by

$$\sum_{n=0}^{\infty} \pi_m(n)x^n = \prod_{r=1}^{\infty} (1-x^r)^{-\binom{r+m-3}{m-2}} \tag{2}$$

where $\binom{a}{l}$ is the binomial coefficient with the usual conventions. Thus $p_2(n)$ is just the $p(n)$ of (1) above.

MacMahon (1) proved that $p_3(n) = \pi_3(n)$, i.e.

$$\sum_{n=0}^{\infty} p_3(n)\, x^n = (1-x)^{-1}(1-x^2)^{-2}(1-x^3)^{-3}\ldots, \tag{3}$$

but both his proof and that of Chaundy (2) are difficult in comparison with the straightforward proof of (1).

Presumably MacMahon was aware that (2) did not enumerate partitions correctly for four or more dimensions (or, as he regarded it, for 'solid partitions' of numbers in three or more dimensions). Nanda (3, 4) assumes that $p_4(n) = \pi_4(n)$ and writes down the form which MacMahon ((1), p. 175) states 'is shewn later not to be justified'. Thus in (4) Nanda tabulates $\pi_4(n)$ and not $p_4(n)$. Further work on the form of $p_4(n)$ is found in (5).

It is natural to enquire what $\pi_m(n)$ for $m \geqslant 4$ *does* enumerate in this context, and with this in mind we have computed a number of values of $p_m(n)$ and $\pi_m(n)$. The computation was carried out on a PDP 8 at Edinburgh University and on the Science Research Council's Atlas I at Chilton; a description of the program and an Algo' algorithm for $p_m(n)$ by Bratley and McKay will appear elsewhere (6). The tim' required to compute $p_m(n)$ from the combinatorial definition increases rapidly wit m and n, and in the absence of any clear conjecture from the first results we did ı feel justified in using any more machine time. Writing

$$E_m(n) = \pi_m(n) - p_m(n),$$

we found the values of $E_m(n)$ given in Table 2 at the end of this note.

2. If we now denote by $p_m^k(n)$ the number of unrestricted m-dimensional partition^c of n whose nodes lie in some k-dimensional hyperplane but not in any $(k-1)$-dimensional hyperplane, then we clearly have

$$p_m^k(n) = 0 \quad \text{if} \quad k > m \quad \text{or} \quad k \geqslant n, \tag{5}$$

and

$$\left.\begin{aligned} p_m(n) &= \sum_{k=1}^{n-1} p_m^k(n) = \sum_{k=1}^{n-1} \binom{m}{k} p_k^k(n), \\ p_{n-1}^{n-1}(n) &= 1. \end{aligned}\right\} \tag{6}$$

Thus, regarding $p_m(n)$ as a function of m for fixed n, we may write

$$p_m(n) = \sum_{k=1}^{n-1} c_{kn} \binom{m}{k},$$

where the c_{kn} are integers independent of m, and $c_{n-1,n} = 1$. We also have from (2) that $\pi_m(n)$ is a polynomial in m of degree $(n-1)$ which takes integral values for $m = 1, 2, \ldots, n-1$, and so

$$\pi_m(n) = \sum_{k=1}^{n-1} \gamma_{kn} \binom{m}{k},$$

where the γ_{kn} are integers independent of m, and it is easily seen that $\gamma_{n-1,n} = 1$. Hence

$$E_m(n) = \pi_m(n) - p_m(n) = \sum_{k=1}^{n-1} e_{kn} \binom{m}{k},$$

where $e_{n-1,n} = 0$ from the above, and $e_{kn} = 0$ for $1 \leqslant k \leqslant 3$ by Euler's and MacMahon's results. Thus finally

$$E_m(n) = \sum_{k=4}^{n-2} e_{kn} \binom{m}{k}, \tag{7}$$

where the e_{kn} are integers independent of m. A more tedious calculation shows that

$$\gamma_{n-2,n} = 2^{n-3} + n - 3, \quad \text{while} \quad c_{n-2,n} = n - 2 + \binom{n-2}{2},$$

so that

$$e_{n-2,n} = 2^{n-3} - 1 - \binom{n-2}{2} = \sum_{k \geqslant 3} \binom{n-3}{k}. \tag{8}$$

Table 1. *Values of* $p_m(n)$

	$m = 2$	3	4	5	6	7	8
$n = 1$	1	1	1	1	1	1	1
2	2	3	4	5	6	7	8
3	3	6	10	15	21	28	36
4	5	13	26	45	71	105	148
5	7	24	59	120	216	357	554
6	11	48	140	326	657	1,197	2,024
7	15	86	307	835	1,907	3,857	7,134
8	22	160	684	2,145	5,507	12,300	24,796
9	30	282	1,464	5,345	15,522	38,430	84,625
10	42	500	3,122	13,220	43,352	118,874	285,784
11	56	859	6,500	32,068	119,140	362,670	953,430
12	77	1,479	13,426	76,965	323,946	1,095,430	3,151,332
13	101	2,485	27,248	181,975	869,476	3,271,751	10,314,257
14	135	4,167	54,804	425,490	2,308,071	9,673,993	
15	176	6,879	108,802	982,615	6,056,581		
16	231	11,297	214,071	2,245,444			
17	297	18,334	416,849	5,077,090			
18	385	29,601	805,124	11,371,250			
19	490	47,330	1,541,637				
20	627	75,278	2,930,329				
21	792	118,794	5,528,733				

Using now the computed values in Table 1, we find

$$E_m(n) = 0 \quad \text{if} \quad m \leqslant 3 \quad \text{or} \quad n \leqslant 5,$$

$$E_m(6) = \binom{m}{4},$$

$$E_m(7) = 3\binom{m}{4} + 5\binom{m}{5} = (m-1)\binom{m}{4},$$

$$E_m(8) = 8\binom{m}{4} + 29\binom{m}{5} + 16\binom{m}{6},$$

$$E_m(9) = 19\binom{m}{4} + 105\binom{m}{5} + 145\binom{m}{6} + 42\binom{m}{7},$$

$$E_m(10) = 40\binom{m}{4} + 321\binom{m}{5} + 755\binom{m}{6} + 545\binom{m}{7} + 99\binom{m}{8}. \tag{9}$$

The results of (9), apart from MacMahon's result for $m = 3$ and all n, are of course somewhat trivial; the difficult problem is to determine what happens for fixed m and all n. However, an immediate enquiry is whether $E_m(n) > 0$ for $m \geqslant 4$ and $n \geqslant 6$. For a fixed n, this is certainly true for large enough m by (7) and (8). A stronger form of the question is:

Are the e_{kn} in (7) always positive?

If so (and this seems to us likely), then it would appear that $\pi_m(n)$ for $m \geqslant 4$ and $n \geqslant 6$ enumerates some additional objects which do not satisfy the original partition definition. A final question is whether, at any rate, $\pi_m(n)$ gives the right order of magnitude for $p_m(n)$, i.e.

Is $E_m(n) = O\ (\pi_m(n))$ valid for fixed m and $n \to \infty$?

The numerical evidence is insufficient to justify any conjecture.

Table 2. *Values of $\pi_m(n)$ and $E_m(n)$*

	$m = 2$	3	4	5	6	7	8
$n = 1$	1	1	1	1	1	1	1
2	2	3	4	5	6	7	8
3	3	6	10	15	21	28	36
4	5	13	26	45	71	105	148
5	7	24	59	120	216	357	554
6	11	48	141	331	672	1,232	2,094
			1*	5*	15*	35*	70*
7	15	86	310	855	1,982	4,067	7,624
			3	20*	75*	210*	490*
8	22	160	692	2,214	5,817	13,301	27,428
			8	69	310*	1,001*	2,632*
9	30	282	1,483	5,545	16,582	42,357	96,231
			19	200	1,060	3,927*	11,606*
10	42	500	3,162	13,741	46,633	132,845	332,159
			40	521	3,281	13,971	46,375*

The non-zero values of $E_m(n)$ are given below the values of $\pi_m(n)$, which is easily computed. An asterisk denotes values deducible from other values using (7) and (8), which provided a check on the program.

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