Trends in Mathematics

Krishnaswami Alladi
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George E. Andrews 80 Years of Combinatory Analysis

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Trends in Mathematics

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Editors
Krishnaswami Alladi
Department of Mathematics
University of Florida
Gainesville, FL, USA

Bruce C. Berndt
Department of Mathematics
University of Illinois at Urbana-Champaign
Urbana, IL, USA

Peter Paule
Research Institute for Symbolic Computation
Johannes Kepler University Linz
Linz, Austria

James A. Sellers
Department of Mathematics and Statistics
University of Minnesota Duluth
Duluth, MN, USA

Ae Ja Yee
Department of Mathematics
Pennsylvania State University
University Park, PA, USA

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Preface

This book presents a printed testimony to the fact that George Andrews has passed the milestone age of 80.

To honor George Andrews on this occasion, the conference "Combinatory Analysis 2018" was organized at the Pennsylvania State University from June 21 to 24, 2018. As the organizers of this conference, we were planning to produce an adequate printed document of this event. When we asked Bill Chen, at that time still the Managing Editor of the *Annals of Combinatorics*, about the possibility of having this document as a special issue of his journal, he accepted immediately. We asked more than 120 participants of the conference and other colleagues of George Andrews to contribute. As the outcome of a procedure, which followed the standard rules of journal refereeing, this special issue contained 37 articles related to the mathematical interests of George Andrews.

All these articles are reprinted in this book version. In addition, this volume contains extra material more suited for a book rather than for journal publication.

First of all, the reader will find three "Personal Contributions." At the conference banquet, Amy Alznauer gave a wonderful after-dinner speech, "The Worlds of George Andrews, A Daughter's Take." Already then we felt that this would make a unique personal introductory entry to this book. To stay as close as possible to the original tone of the speech, we asked Amy to edit her text only slightly.

Another personal contribution is by Krishnaswami Alladi, "My association and collaboration with George Andrews." In his article, Krishna shares with the reader various observations of Andrews as a man and mathematician. Besides describing some aspects of joint work with Andrews, its particular focus is a discussion of joint work in connection with the Capparelli and the Göllnitz theorems.

The third personal contribution is by Bruce Berndt, "Ramanujan, his lost notebook, its importance." The "lost notebook" has generated hundreds of research and expository papers, as well as several books, in particular, the five Springer volumes produced by Andrews and Berndt. More than any other mathematician, Andrews has contributed to our understanding of the "lost notebook." The primary purpose of Bruce's contribution is to relate the history of the "lost notebook" and the events leading up to Andrews' rediscovery of it in the Wren Library of Trinity College, Cambridge.

Another aspect which makes this Andrews volume unique is a special "Photos" collection. In addition to pictures taken at "Combinatory Analysis 2018," the reader finds a variety of photos, many of them not available elsewhere; see sections "Andrews in Austria," "Andrews in China," "Andrews in Florida," "Andrews in Illinois," and "Andrews in India."

The "Articles" part contains, as already mentioned, reprints of the 37 articles of the Special Issue of the Annals of Combinatorics, Volume 23, Issue 3–4, November 2019 (in honor of Andrews' 80th birthday).

In addition, this volume contains two extra contributions: one by Bruce Berndt, Junxian Li, and Alexandru Zaharescu and one by Louis and Stephanie Kolitsch.

Last but not least, our sincere thanks go to all those who contributed to this project. In addition to authors and referees, our thanks go to William Y.C. Chen and his editorial team at the Tianjin University for their outstanding commitment and great help in the process of preparing this special issue.

Special thanks go to Tanja Gutenbrunner (RISC) who has been of essential help in the whole project since its early beginnings, and to Ralf Hemmecke (RISC) for web assistance.

Finally, we want to express kind thanks to our colleagues from Birkhäuser: to Thomas Hempfling for his enthusiasm about this project, to Sarah Annette Goob for her patience and help, and to Sabrina Hoecklin for her assistance.

Gainesville, FL — Krishnaswami Alladi
Urbana, IL — Bruce C. Berndt
Linz, Austria — Peter Paule
Duluth, MN — James A. Sellers
University Park, PA — Ae Ja Yee
July 6, 2020

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The Worlds of George Andrews, a Daughter's Take

Amy Alznauer

Abstract This conference is about the world that has grown up around my father's research, a world that he tended through his own ideas, his wide and various collaborations, and the decades of graduate students he has mentored. But there is another world he has tended with at least as much love and attention. It involves a smaller crowd, but it is a world just as beautiful and alive as the one gathered here tonight. So as his daughter, I want to tell you a bit about this other world—the world of his family.

As a child, you come into a world that is already fully formed and that you somehow mysteriously inherit. You learn of this prior world through stories—bits and pieces shared over dinner conversation or at bedtime—and your parents' lives begin to take on, or at least they did for me, the grandeur of myth. So, I want to share some of the stories that I inherited as my father's daughter.

It always amazes me, that my grandparents' lives stretch back not only into the previous century, but the one before that. My grandmother was born in 1894 into a proper and prosperous Victorian family. But by the time she was 40 years old, the Great Depression had hit, and she was letting a room in Eugene, Oregon from her unbeknownst-to-her future mother-in-law. My grandmother and her sister Mary, both strong, intellectual women, were school teachers and in letters to one another confessed that they saw themselves as lifelong spinsters.

But then, interrupting these well-laid plans, a red-headed, strapping and silent young man came to town to board at his mother's house. Eight years my grandmother's junior, he'd once been studying graduate-level economics at the University of Chicago, but now on hard times was chopping down trees in the woods of Wisconsin. He took a shine to this strong, smart woman, who always had words when he had none, and soon they were married.

A. Alznauer (✉)
Northwestern University, Chicago, IL, USA

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They moved out of town to take over her father's country home and 100-acre farm. And there, Raymond Andrews and 45-year-old Rovena Pearl brought their first and only child into the world.

George W. Eyre Andrews grew up with few playmates. Instead he had loving, if a bit non-traditional, parents. His quiet father was a deeply good man, who without rancor put aside his intellectual aspirations to labor weekly in the Oregon shipyards to supplement the meager farm income. His mother, older and more educated than almost every other mother at the time, was enamored of John Dewey, so encouraged her only child in all of his creative, solitary explorations. And in a manner that I've always thought fitting to this eccentric, old-fashioned family, George fondly called his parents, not the standard mom and dad, but Pearl and Raymond.

And so again, instead of playmates, George had his parents, and also a huge attic room in the eaves of their centuries-old farmhouse. His great bedroom windows went all the way down to the floor, so he could sit there and look out over the gardens and trees. He had endless nooks and crannies, lined with shelves, where he displayed his vast collection of arrowheads and fossils he'd found on the farm, his books, the spy rings he'd ordered from the back of cereal boxes, a powder horn out of Oregon history, and a Chemeketa Indian bowl.

And he had the farm itself with its orchards and barns and fields of corn. From the age of four he woke up early with his parents, drank black coffee, and went out to the barn for chores. They'd listen to the radio while they milked the cows, and one day, he heard this incredible music, boogie-woogie, pounding out into the barn and thought it was the saddest, most beautiful music he'd ever heard. My father says now that I've remembered that bit about sadness and beauty wrong, that I'm stealing a quote from a later blues commentator, and I'm sure he's right. But this is the intelligence of myths, which intuitively pull together different sources to make an even truer picture. At only 4-years-old, George began picking out the blues base-line, walking the same notes up and down the key-board, over and over, until his mother thought she might go crazy. She made him promise to practice only after supper.

Before he was school-age, his mother took him to see his first full-length moving picture complete with technicolor: Walt Disney's magnificent Bambi. Amazed by the giant screen, by the glorious fanfare of images, he stood up on his velvet theater seat and with arms in the air, cheered out loud. Also at that age he loved the smell of gasoline and declared one day that a gasoline pie might be a nice thing to eat.

Back then he had charge of a flock of chickens and his parents said he could sell the eggs at a tiny stand down the road. One day, proud of his full dozen, he walked across the fields to the stand, holding his carton, swinging his arms in triumph, only to finally open the lid to a mess of runny eggs. A little later with his mother he took on the WWII volunteer task of watching for enemy planes. Out there in the sunshine and rustling corn with his official guidebook and binoculars, feeling like a spy on a noble mission, he laid back in the cornstalks and watched the blue sky, mostly finding shapes in the clouds instead of planes, but loving it no less. So, at 6-years-old, when news came over the radio that the war had ended and the adults celebrated, he ran off to weep over his lost mission.

He attended a one-room school house. Once in a high school history class they read about how it used to be on the American prairie: the homesteaders with their little, white-washed schools, the desks lined up, all the grades in one room, the strict schoolmarms, the lunch pails and pot-bellied stoves to keep everyone from freezing in winter. And he wondered why a book would characterize these things as existing in some distant past, for it described his younger life perfectly.

His mother taught at that school for a few years. And even worse, the neighbors, a large group of tough farm boys also attended. George, an odd little boy full of big words and unusual interests, wasn't like them, so sometimes at lunch they'd rough him up. But during math class he'd seek revenge. His mother would read out a problem and ask one of the gang of boys to answer. A long, painful pause would follow and then a voice from across the room, a much younger voice, would pipe up. "Soooooo simple," he'd say. And the cycle of lunchtime abuse and math class ridicule continued.

As he grew older, his obsession with boogie-woogie piano never let up. He learned to play by ear and later took up the saxophone in the high-school school jazz band. At home, he'd sit in his attic, a mason-jar fitted over the arm of his turntable to make the same song—maybe Freddie Slack's Cuban Sugar Mill—repeat over and over. And he'd sit there, his giant windows open to the farm, and read Sherlock Holmes' Sign of the Four. Oh, to be like him, he'd think, to ferret out clues and solve complex, impossible problems with the power of your mind.

At 16 in recitation and debate class, where he'd win prizes for his animated renditions of James Thurber, he met this black-haired, blue-eyed girl, as smart as he. They became debate partners and soon fell in love.

But in just a few years there was a complication. In the engineering program at Oregon State University, studying to be a patent attorney, George found himself in the inspired mathematics class of Prof. Harry Goheen. Suddenly it all came together: The logic and creativity he'd first found in boogie-woogie, the steady, principle of the left-hand combined with the freedom of the right; the Holmesian search for clues and grand sweeping theories; the thrill of staring up at a blank sky and suddenly finding a plane there like a revelation, and even the joy of discovering what you had once only imagined, made real before you in technicolor. In mathematics, he found the fulfilment of all of these early passions and was hooked. If he wasn't such a dignified young man, he might have stood up in Professor Goheen's class and cheered.

So now what to do? The love of his life on one hand and his vocation on the other. He would soon graduate with a combined bachelor's and master's degree and head off to graduate school, so how to get his 19-year-old love to follow him. He'd just have to win a Fulbright that's all, he thought, for how could she resist a trip to England? So that's exactly what he did, and he took her up on a mountaintop and read her a poetic proposal, promising to shower her with diamonds. And a few months later, with a Fulbright Scholarship to Cambridge University in hand, they were married with only a pastor and their parents standing by, and soon boarded the Queen Elizabeth to sail for England.

I love to think of them there in Cambridge, that first year of their long, joyful marriage, unaware that 15 years later they would return here, and my father would come across the forgotten notebook of Ramanujan in the Wren Library. He already knew about Ramanujan, was enchanted by the biography he'd read in James Newman's *The World of Mathematics* (a book my mother had given him) and was already on his way to becoming a number theorist himself. But he didn't know that a big piece of his future was so close by, waiting for him or someone to discover it.

And the fact of that notebook—once held by the hands of Ramanujan himself and then lying there unseen but indisputable in a box—has become, for me, a metaphor for the way my father views mathematics itself. The notebook was currently undiscovered yet truly there, just as mathematics, my father thinks, is truly out there, even before people come to know its truths. Mathematics is not something he creates with his mind, my father thinks, not some intellectual game played and merely derived from axioms, but a real thing, maybe mysteriously real in its invisibility, but real nonetheless.

That brings me full circle, to the fully-formed, real world into which all of us arrive, unknowing, when we are born. In recent years my father has become enamored of the writings of Michael Polanyi, a chemist turned philosopher, who writes about this real, prior world that surrounds us. In his collection of lectures, *Tacit Knowledge*, Polanyi argues that much of our knowledge is inarticulate, that we know more than we can say, and that intuition is often our tool for tapping into the vast world that includes and precedes and encompasses us.

Belief in this real world beyond the self is one of the greatest gifts my father has given me, and I know I speak for my siblings as well. This world came to me (to all of us) first through my father's and mother's love, that existed before I was born, and has been there, undiminished throughout my entire life. And then, this world came to me through the stories that accrued into myth, of *his* life (and my mother's life), giving me one of my earliest glimpses into the reality of a person beyond the confines of my own being, and a glimpse into the history that formed me. And finally, this external world came to me through mathematics. Numbers and theorems seemed to fairly swirl about our home. I could imagine my father down in his tiny study under the stairs, listening to boogie-woogie, his blue fountain pen poised over a yellow legal pad, waiting to harness ideas from the sky. And all of this together gave me the sense of a reality, a presence even, that was large and complex, rigorous and loving, out there whether I took the time to look or not, and worthy of a lifetime of discovery.

Before I close, I want to say that just as the world represented in this conference is not his alone, for everyone here has helped in some way or another, through intellectual effort or love, to build it; the world of our family is the same. My father did not build it alone, but with the help first of his parents and then through the life-long partnership with my mother—who is one of the warmest, most intelligent, most beautiful people I have ever known.

So, in closing I want to thank you all and the organizers of this conference for giving me the chance to participate in celebrating the worlds my beloved father, George Andrews, helped to shape.

My Association and Collaboration with George Andrews

Still Going Strong at 80

Krishnaswami Alladi

Abstract This is recollection of my association with George Andrews from 1981, and a report of my joint work with him in the theory of partitions and q-series relating to the Göllnitz and Capparelli theorems starting from 1990.

1 Introduction

George Andrews is the undisputed leader on partitions and the work of Ramanujan combined. After Hardy and Ramanujan, he, more than anyone else in the modern era, is responsible for making the theory of partitions a central area of research. His book on partitions [15] published first in 1976 as Volume 2 of the Encyclopedia of Mathematics (John Wiley), is a bible in the field, and his NSF-CBMS Lectures [16] of 1984–1985 highlight the fundamental connections between partitions and Ramanujan's work with many allied fields. We definitely owe to him our present understanding of many of the deep identities in Ramanujan's Lost Notebook. I had the good fortune to collaborate with him and also interact with him very closely both at Penn State University (his home turf) where I visited often, and at the University of Florida, where he has spent the Spring term every year since 2005. I also have had the pleasure of hosting him in India several times. Thus I have come to know him really well as a mathematician, colleague, and friend. Here I will first share with you (in Sect. 2) my observations of him as a man and mathematician. I will

This is a mildly modified version of my paper "My association and collaboration with George Andrews—Torchbearer of Ramanujan and partitions" that appeared in *Lattice Paths Combinatorics and Applications* (G. E. Andrews, C. Krattenthaler, and A. Krinik, Eds.), Proc. 2015 Conf. in Pomona, *Developments in Mathematics*, **58** Springer, New York (2019). The permission to reproduce is gratefully acknowledged.

K. Alladi (✉)
Department of Mathematics, University of Florida, Gainesville, FL, USA
e-mail: alladik@ufl.edu

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then describe (in Sect. 3) some aspects of our joint work that will highlight his vast knowledge and brilliance. In Sect. 2, I will describe events chronologically rather than thematically. In Sect. 3, I will discuss my joint work with him on the Capparelli and the Göllnitz theorems.

2 Personal Recollections

First Visit to India Even though Andrews has been studying Ramanujan's work since the sixties and had been "introduced to India" through the writings of, and on, Ramanujan, his first visit to India was only in Fall 1981. That academic year, I was visiting the Institute for Advanced Study in Princeton, and he contacted me saying that he was planning a visit to India, and to Madras in particular. My father, the late Professor Alladi Ramakrishnan, was Director of MATSCIENCE, The Institute of Mathematical Sciences, that he had founded in 1962, and so I put him in touch with my father who hosted him in Madras and helped arrange a meeting for Andrews with Mrs. Janaki Ammal Ramanujan. Upon return from India, Andrews called me from Penn State, told me that it was an immensely enjoyable and fruitful visit, and that he appreciated my father's help and hospitality. To reciprocate, Andrews invited me to a Colloquium at Penn State where he was Department Chair at that time. Andrews is always a gracious host, but in his capacity as Chair, he rolled out the red carpet for me! He hosted a party for me at his house during that visit and that is how our close friendship began.

I was working at that time in analytic number theory but I wanted to learn partitions and q-series, and that aspect of the work of Ramanujan. So after I returned to India from Princeton, I wrote to Andrews and asked him for his papers. Promptly I received two large packages containing more than 100 of his reprints. So I started studying them along with his Encyclopedia, and gave a series of lectures at MATSCIENCE in Madras, the notes of which I still use today. Even after this course of lectures, I was unsure whether to venture into partitions and q-series. The infinite series formulae were beautiful, but daunting. The decision to change my field of research to the theory of partitions and q-series came during the Ramanujan Centennial in Madras in December 1987.

The Ramanujan Centennial The Ramanujan Centennial was an occasion when mathematicians from around the world gathered in India to pay homage to the Indian genius. Among the mathematical luminaries at the conference, there was a lot of attention on Andrews, Richard Askey and Bruce Berndt—jokingly referred to in the USA as the "Gang of Three" in the world of Ramanujan. I prefer to refer to them as the "Great Trinity" of the Ramanujan world, like Brahma, Vishnu, and Shiva, the three premier Hindu gods! The Great Trinity along with Nobel Laureate Astrophysicist Subrahmanyam Chandrasekhar and Fields Medalist Atle Selberg, were the stars of the Ramanujan Centennial. But Andrews occupied a special place in this elite group, because the Lost Notebook that he unearthed at the Wren Library in Cambridge University, was released in published form [22] at a grand public function in Madras on December 22, 1987, Ramanujan's 100-th birthday, by India's Prime Minister Rajiv Gandhi, who handed one copy to Janaki Ammal and another to Andrews. That definitely was a high point in the academic life of Andrews. Andrews has written a marvelous Preface to that book published by Narosa, which at that time was part of Springer, India.

December 1987 was a politically tense time in Madras because the Chief Minister of Madras, M. G. Ramachandran—MGR as he was affectionately known—a former cine hero to the millions, was terminally ill. There were several conferences in India around Ramanujan's 100-th birthday, and Andrews was a speaker in every one of them. He therefore arrived in Madras about a week before the 100-th birthday of Ramanujan and spent the first night at my house before traveling by road to a conference at Annamalai University, south of Madras. I told him that he should be very careful traveling by road in such a tense time, but he held my hand and said: "Krishna, do not worry. I am on a pilgrimage here to pay homage to Ramanujan. I will not let anything perturb me." As it turned out, one day as he, Askey, and Berndt were traveling by car on their way back to Madras, the car was suddenly encircled by a crowd of excited political activists. The car was stopped. Askey and Berndt were very nervous. But Andrews, cool as a cucumber, rolled down the window, and threw a load of cash into the air! The crowd cheered and let the car through because the foreigners had supported their cause. Andrews acted like James Bond, with tremendous presence of mind! Anyway, everyone made it safely to Madras for a one day conference I had arranged on December 21, and for the the December 22 function presided by Prime Minister Rajiv Gandhi.

The talks that Andrews gave at various conferences, including the one that I organized at Anna University on December 21, one day before the 100-th birthday of Ramanujan, were all for expert audiences. Since Andrews is a charismatic speaker, I wanted him to give a lecture to a general audience. So my father and I arranged a talk by him at our home on December 23, under the auspices of the Alladi Foundation that my father started in 1983 in memory of my grandfather Sir Alladi Krishnaswami Iyer, one of the most eminent lawyers of India. We invited the Consul General of the USA to preside over the lecture which was attended by prominent citizens of Madras in various walks of life—lawyers, judges, aristocrats, businessmen, college teachers and students. Andrews charmed them all with his inimitable description of the story of the discovery of Ramanujan's Lost Notebook. But something sensational happened that night after Andrews' lecture: Following the talk, many of us assembled at the Taj Coromandel Hotel for a dinner in honor of the conference delegates hosted by Mr. N. Ram, Editor of *The Hindu*, India's National Newspaper, based in Madras. (Ram's connection with Andrews was that in 1976, shortly after the Lost Notebook was discovered, he published a full page interview with Andrews in The Hindu.) After dinner, while we chatting over cocktails and dessert, the news came in whispers that MGR had passed away, and so the city would come to a standstill by daybreak once the general public would hear this news. So under the cover of darkness, we were asked to quietly make our way back to our hotels. And yes, as predicted, there was a complete shutdown and the Ramanujan Centenary Conference did not take place on December 24; instead all talks were squeezed into the next two days. Fortunately, Andrews had spoken at the conference on December 23. The Goddess of Namakkal had made sure that the Ramanujan Centenary celebration on December 22, and the talks the next day by the Great Trinity, would not be affected by such a tragedy!

The Frontiers of Science Lecture in Florida At the University of Florida in Gainesville, there was a public lecture series called *Frontiers of Science*. This was organized by the physics department, and students received one (hour) course credit for attending these lectures. Many world famous scientists spoke in this lecture series such as group theorist John Conway, and Johansson, the discoverer of the "Lucy" skeleton. So after my return from the Ramanujan Centennial, I suggested to the organizers to invite George Andrews. I never heard back from them and so I felt they were not interested. Quite surprisingly, three years later, in Fall 1990, they contacted me and expressed interest in Andrews delivering a Frontiers of Science Lecture. So Andrews gave such a talk in November 1990, and held the 1000 or more members of the audience in the University Auditorium in rapt attention as he described the story of the discovery of the Lost Notebook. That was his first visit to Florida, but in that visit, our collaboration began in a remarkable way. I will now relate this fascinating story that will reveal the genius of this man.

In early 1989, I got a phone call from Basil Gordon, one of my former teachers at UCLA where I did my PhD work. Gordon said that he would be on a fully paid sabbatical in 1989–1990, and that he would like to spend the Fall of 1989 in Florida. After the Ramanujan Centennial, I attempted some research on partitions and q-series, but the visit of Gordon provided me a real opportunity because Gordon was a dominant force in this domain; in the 1960s he had obtained a far-reaching generalization of the Rogers–Ramanujan identities to all odd moduli. Gordon and I first obtained a significant generalization of Schur's famous 1926 partition theorem [23] by a new technique which we called the *method of weighted words*. We then extended this method to obtain a generalization and refinement of a deep 1967 partition theorem of Göllnitz. We cast this generalization in the form of a remarkable three parameter q-hypergeometric *key identity* which we were unable to prove. When Andrews arrived in Florida for the Frontiers of Science Lecture, I went to the airport to receive him. I did not waste any time and showed him the identity right there. He said it was fascinating. During his three day stay in Gainesville, he thought of nothing else. He focused solely on the identity. In the visitors office that he occupied in our department, I saw him working on the identity, every day, and every hour. On the last day, on the way to the airport, he handed me an eight page proof of this key identity by q-hypergeometric techniques that only he could wield with such power. That is how my first paper with him (jointly also with Gordon) came about.

Sabbatical at Penn State, 1992–1993 I was having my first sabbatical in 1992–1993 and Andrews invited me to Penn State for that entire year. So I went to State College, Pennsylvania with my family. It was the most productive year of my academic life—I completed work on five papers of which two were in collaboration with Andrews. He and his wife Joy were gracious hosts. They showed us around State College and we got together as families for picnics. Most importantly, Andrews gave a year long graduate course on the theory of partitions that I attended. Although I was doing research in the theory of partitions, I never had a course on partitions and q-hypergeometric series as a student and so it was a treat for me to learn from the master. Dennis Eichhorn and Andrew Sills were also taking this course as graduate students.

The sabbatical year at Penn State gave me time to also write up work I had done previously. It was there that I finished writing my first joint paper with Andrews on the Göllnitz theorem. The story of my second joint paper with Andrews written at Penn State on the Capparelli conjecture is also equally remarkable, and demonstrates once again Andrews' power in the area of partitions and q-hypergeometric series, and so I will relate this now.

In the summer of 1992, the Rademacher Centenary Conference was held at Penn State. Andrews was a former student of Rademacher, and so he was the lead organizer of this conference. On the opening day of the conference, Jim Lepowsky gave a talk on how Lie algebras could be used to discover, and in some instances, prove, various Rogers–Ramanujan type partition identities. During the talk, he mentioned a pair of partition identities that his student Stefano Capparelli had discovered in the study of vertex operators of Lie algebras but was unable to prove. Even though Andrews was the main conference organizer, he went into hiding during the breaks to work on the Capparelli Conjecture. By the end of the conference, he had proved the conjecture; so on the last day, he changed the title of his talk and spoke about a proof of the Capparelli conjecture! This story bears similarity to the way in which he proved the three parameter identity for the Göllnitz theorem that Gordon and I had found but could not prove.

I was not present at the Rademacher Centenary Conference since I was in India at that time, just two months before reaching Penn State for my sabbatical. But Basil Gordon was at that conference and he told me this story. Actually, during Lepowsky's lecture, Gordon realized that our method of weighted words would apply to the Capparelli partition theorems and he expressed this view to me in a telephone call soon after I arrived at Penn State. So during my sabbatical, I worked out the details of this approach to obtain a two parameter refinement of the Capparelli theorems, and in that process got a combinatorial proof as well. This led to my second joint paper with Andrews, with Gordon also as a co-author.

Honorary Doctorate at UF in 2002 In view of his fundamental research and his contributions to the profession, Andrews is the recipient of numerous honors. He has received honorary doctorates from the University of Illinois and the University of Parma. In 2002, he was awarded an Honorary Doctorate by the University of Florida. I was Department Chair at that time, and it was then that we formalized the arrangement to have him as a Distinguished Visiting Professor, so that he would spend the entire Spring Term each year at the University of Florida. Some of his most important recent works have had a Florida origin, such as his work on Durfee symbols, and on the function $spt(n)$.

Visit to SASTRA University, 2003 In 2003, the recently formed SASTRA University, purchased Ramanujan's home in Kumbakonam, renovated it, and decided to maintain it as a museum. This was a major event in the preservation of Ramanujan's legacy for posterity. To mark the occasion, SASTRA decided to have an International Conference at their newly constructed Srinivasa Ramanujan Centre in Kumbakonam to coincide with Ramanujan's birthday, December 22. I was invited to organize the technical session and given funds to bring a team of mathematicians to Kumbakonam. SASTRA was a new entry in the Ramanujan world, but this conference seemed to me interesting and promising. But how to make a success of this? So I called Andrews and told him that something exciting is happening in Ramanujan's hometown, and I would like him to give the opening lecture at this conference. He readily agreed. Once he accepted, I called other mathematicians and told them that Andrews will be there. So they too accepted the invitation to the First SASTRA Conference. That shows Andrews' drawing power! That conference was inaugurated by India's President Abdul Kalam who also declared open Ramanujan's home as a museum and national treasure.

Ramanujan 125, Honorary Doctorate at SASTRA Many things developed after that 2003 SASTRA conference—the conferences at SASTRA became an annual event that I help organize, and in 2005 the SASTRA Ramanujan Prize was launched. SASTRA invited me to be Chair of the Prize Committee. I felt that Andrews' input would be crucial for the success of the prize. So I invited him to be on the Prize Committee during the first year, and he readily agreed. I then informed others about the prize and that Andrews was on the Prize Committee, and they too agreed enthusiastically. The prize as you know has become one of the most prestigious in the world, and I am grateful to Andrews for agreeing to serve on the Prize Committee during the first year.

In view of the annual conferences and the prize, SASTRA had become a major force in the world of Ramanujan by the time Ramanujan's 125-th Anniversary was celebrated in December 2012. So I suggested to the Vice-Chancellor of SASTRA, that the three greatest figures in the world of Ramanujan—namely the Trinity—should be recognized by SASTRA with honorary doctorates in Ramanujan's hometown, Kumbakonam. The Vice-Chancellor liked this suggestion, and so Andrews, Askey and Berndt were awarded honorary doctorates in a colorful ceremony with traditional Indian music being played as the recipients walked in.

Birthday Conferences Every 5 Years Andrews has remained productive defying the passage of time. In view of his enormous influence, and his charm, conferences in his honor have been organized every 5 years starting from his 60-th birthday, and I have had the privilege of participating in every one of them—in Maratea, Italy in 1998 for his 60-th, in Penn State in 2003 and 2008 for his 65-th and 70-th, in Tianjin, China in 2013 for his 75-th, and now in Penn State for his 80th.

In Indian culture, the 80th birthday is of special significance. Somewhere between the 80th and 81st birthdays, the individual would have seen 1000 crescent moons, and sighting the crescent moon is auspicious in the Hindu religion because it adorns the head of Lord Shiva. George Andrews is still going strong at 80, with no signs of slowing down. So he is full of mathematical energy and continues to be our inspiring leader.

G. H. Hardy once said that he had the unique privilege of collaborating with Ramanujan and Littlewood in something like equal terms. Although I am no Hardy, I can say proudly that I am unique in having had a close collaboration with Paul Erdős and George Andrews, two of the most influential mathematicians of our time! I next describe my joint work with Andrews on the Göllnitz and Capparelli theorems.

3 Collaboration with Andrews

Before describing my joint work with Andrews, I need to briefly provide as background, my joint work with Gordon on Schur's theorem.

One of the first results in the theory of partitions that one encounters, is a lovely theorem of Euler, namely:

Theorem E *The number of partitions $p_d(n)$ of n into distinct parts, equals the number of partitions $p_o(n)$ of n into odd parts.*

Euler's proof of this was to consider the product generating functions of these two partition functions and show they are equal by using the trick

$$1 + x = \frac{1 - x^2}{1 - x}.$$

More precisely,

$$\sum_{n=0}^{\infty} p_d(n) q^n = \prod_{m=1}^{\infty} (1 + q^m) = \prod_{m=1}^{\infty} \frac{1 - q^{2m}}{1 - q^m} = \prod_{m=1}^{\infty} \frac{1}{1 - q^{2m-1}} = \sum_{n=0}^{\infty} p_o(n) q^n. \tag{1}$$

Let us think of partitions into distinct parts as those for which the gap between the parts is ≥ 1, and partitions into odd parts as those whose parts are $\equiv \pm 1 \pmod 4$. If Euler's theorem is viewed in this fashion, then the celebrated Rogers–Ramanujan partition theorem is the "next level" result with gap ≥ 1 replaced by gap ≥ 2 between parts, and the congruence mod 4 replaced by modulus 5. More precisely, the first Rogers–Ramanujan partition theorem is:

Theorem R1 *The number of partitions of an integer n into parts that differ by ≥ 2, equals the number of partitions of n into parts $\equiv \pm 1 \pmod 5$.*

In the second Rogers–Ramanujan partition theorem (R2) we consider partitions whose parts differ by ≥ 2 but do not have 1 as a part, and equate these with partitions into parts $\equiv \pm 2 \pmod 5$. The two Rogers–Ramanujan partition identities can be cast in an analytic form, namely

$$\sum_{n=0}^{\infty} \frac{q^{n^2}}{(q)_n} = \frac{1}{(q;q^5)_\infty (q^4;q^5)_\infty}, \tag{2}$$

and

$$\sum_{n=0}^{\infty} \frac{q^{n^2+n}}{(q)_n} = \frac{1}{(q^2;q^5)_\infty (q^3;q^5)_\infty}. \tag{3}$$

In (2) and (3) and in what follows, we have used the standard notation

$$(a;q)_n = (a)_n = \prod_{j=1}^{n} (1 - aq^{j-1}),$$

and

$$(a)_\infty = \lim_{n\to\infty} (a)_n, \quad \text{for} \quad |q| < 1.$$

When the base is q, then as on the left in (2) and (3), we do not mention it, but when the base is other than q, then we always mention it, as on the right in (2) and (3).

Although the Rogers–Ramanujan identities are the next level identities beyond Euler's theorem, they are much deeper. They also have a rich history that we will not get into here. We just mention that the analytic forms of the identities (2) and (3) were first discovered by Rogers and Ramanujan independently, and it was only later that MacMahon and Schur independently provided the partition version, namely Theorems R1 and R2. Neither Rogers nor Ramanujan mentioned the partition versions of (2) and (3). So in fairness, Theorems R1 and R2 should be called the MacMahon-Schur theorems.

In the theory of partitions and q-series, a Rogers–Ramanujan (R-R) type identity is a q-hypergeometric identity in the form of an infinite (possibly multiple) series equals an infinite product. The series is the generating function of partitions whose parts satisfy certain difference conditions, whereas the product is the generating function of partitions whose parts usually satisfy certain congruence conditions. Since the 1960s, Andrews has spearheaded the study of R-R type identities (see [15], for instance). R-R type identities arise as solutions of models in statistical mechanics as first observed by Rodney Baxter in his fundamental work. After noticing the role of R-R type identities in certain physical problems, Baxter and his group approached Andrews to provide insight into the structure of such identities. Andrews then collaborated with Baxter and Peter Forrester to determine all R-R type identities that arise as solutions of the Hard-Hexagon Model in statistical mechanics. For a discussion of a theory of R-R type identities, see Andrews [15, Ch. 9]. For a discussion of connections with problems in physics, see Andrews' CBMS Lectures [16].

The partition theorem which is the combinatorial interpretation of an R-R type identity, is called a Rogers–Ramanujan type partition identity. A q-hypergeometric R-R type identity is usually discovered first and then its combinatorial interpretation as a partition theorem is given. There are important instances of Rogers–Ramanujan type partition identities being discovered first and their q-hypergeometric versions given later. Perhaps the first such significant example is the 1926 partition theorem of Schur [23].

In emphasizing the partition version of (2) and (3), Schur discovered the "next level" partition theorem, namely:

Theorem S (Schur 1926) *Let $T(n)$ denote the number of partitions of an integer n into parts* $\equiv \pm 1 \pmod 6$.

Let $S(n)$ denote the number of partitions of n into distinct parts $\equiv \pm 1 \pmod 3$.

Let $S_1(n)$ denote the number of partitions of n into parts that differ by ≥ 3*, where the inequality is strict if a part is a multiple of* 3. *Then*

$$T(n) = S(n) = S_1(n).$$

The equality $T(n) = S(n)$ is simple and follows easily by using Euler's trick on their product generating functions, namely

$$\sum_{n=0}^{\infty} T(n)q^n = \frac{1}{(q;q^6)_\infty (q^5;q^6)_\infty} = (-q;q^3)_\infty (-q^2;q^3)_\infty = \sum_{n=0}^{\infty} S(n)q^n. \tag{4}$$

Thus it is the equality $S(n) = S_1(n)$ which is the real challenge. In 1966, Andrews [11] gave a a new q-theoretic proof of $S(n) = S_1(n)$. This enabled him to discover two infinite families of identities ([12, 13]) modulo $2^k - 1$ emanating from Schur's theorem.

In 1989, in collaboration with Gordon, I obtained a generalization and two parameter refinement of the equality $S(n) = S_1(n)$ (see [10]). The main idea in [10] was to establish the *key identity*

$$\sum_{i,j} a^i b^j \sum_m \frac{q^{T_{i+j-m}+T_m}}{(q)_{i-m}(q)_{j-m}(q)_m} = (-aq)_\infty (-bq)_\infty, \tag{5}$$

and to view a two parameter refinement of the equality $S(n) = S_1(n)$ as emerging from (5) under the transformations

$$(\textit{dilation}) \quad q \mapsto q^3, \quad \text{and} \quad (\textit{translations}) \quad a \mapsto aq^{-2}, b \mapsto bq^{-1}. \tag{6}$$

In (5) and below, $T_m = m(m+1)/2$ is the m-th triangular number.

The interpretation of the product in (5) as the generating function of bi-partitions into distinct parts in two colors is clear. In [10] it was shown that the series in (5) is the generating function of partitions (= words with weights attached) into distinct parts occurring in three colors - two primary colors a and b, and one secondary color ab, and satisfying certain gap conditions. We describe this now.

We assume that the integer 1 occurs in two primary colors a and b, and that each integer $n \geq 2$ occurs in the two primary colors as well as in the secondary color ab. By a_n, b_n, and ab_n, we denote the integer n in colors a, b, and ab respectively. In order to discuss partitions, we need to impose an order on the colors, and the order that Gordon and I chose is

$$a_1 < b_1 < ab_2 < a_2 < b_2 < ab_3 < a_3 < b_3 < \cdots . \tag{7}$$

Thus for a given integer n, the order of the colors is

$$ab < a < b. \tag{8}$$

The transformations in (6) correspond to the replacements

$$a_n \mapsto 3n-2, \quad b_n \mapsto 3n-1, \quad \text{and} \quad ab_n \mapsto 3n-3, \tag{9}$$

Under (9), the ordering of the colored integers in (7) becomes

$$1 < 2 < 3 < 4 \cdots ,$$

the standard ordering among the positive integers. This is one of the reasons Gordon and I chose the ordering in (7).

Using the colored integers, Gordon and I gave the following partition interpretation for the series in (5). We defined *Type 1* partitions as those of the form $x_1 + x_2 + \cdots$, where the x_i are symbols from the sequence in (7) with the condition that the *gap* between x_i and x_{i+1}, namely the difference between the subscripts of the colored integers they represent, is ≥ 1, with strict inequality if

$$x_i \quad \text{has a lower order color compared to} \quad x_{i+1}, \tag{10a}$$

or

$$x_i, \quad x_{i+1} \quad \text{are both of secondary color.} \tag{10b}$$

In (10a), the order of the colors is as in (8).

Using (9) it can be shown that that the gap conditions of Type 1 partitions in (10a) and (10b) translate to the difference conditions of $S_1(n)$ in Schur's theorem. Two proofs of (5) were given in [10]—one combinatorial, and another using the q-Chu–Vandermonde Summation. Thus the R-R type identity for Schur's theorem came half a century later.

Gordon then suggested that we should apply the method of weighted words to generalize and refine the deep 1967 theorem of Göllnitz [19] which is:

Theorem G *Let $B(n)$ denote the number of partitions of n into parts* $\equiv 2, 5,$ *or* $11 \pmod{12}$.

Let $C(n)$ denote the number of partitions of n into distinct parts $\equiv 2, 4,$ *or* $5 \pmod 6$.

Let $D(n)$ denote the number of partitions of n into parts that differ by ≥ 6, *where the inequality is strict if a part is* $\equiv 0, 1,$ *or* $3 \pmod 6$, *and with* 1 *and* 3 *not occurring as parts. Then*

$$B(n) = C(n) = D(n).$$

The equality $B(n) = C(n)$ is easy because

$$\sum_{n=0}^{\infty} B(n)q^n = \prod_{m=1}^{\infty} \frac{1}{(1-q^{12m-10})(1-q^{12m-7})(1-q^{12m-1})}$$
$$= \prod_{m=1}^{\infty} (1+q^{6m-4})(1+q^{6m-2})(1+q^{6m-1}) = \sum_{n=0}^{\infty} C(n)q^n. \tag{11}$$

This is one reason that we focus on the deeper equality $C(n) = D(n)$, the second reason being that it is this equality which can be refined.

Göllnitz' proof of Theorem G is very intricate and difficult but he succeeded in proving Theorem G in the refined form

$$C(n;k) = D(n;k), \tag{12}$$

where $C(n;k)$ and $D(n;k)$ denote the number of partitions of the type counted by $C(n)$ and $D(n)$ respectively, with the extra condition that the number of parts is k, and with the convention that parts $\equiv 0$, 1, or 3 (mod 6) are counted twice. Andrews [14] subsequently provided a simpler proof. I think besides Göllnitz, Andrews is the only other person to have gone through the difficult details of Göllnitz' proof of Theorem G. In Chapter 10 of his famous CBMS Lectures [16], Andrews asks for a proof that will provide insights into the structure of the Göllnitz theorem.

In view of (12) and our work on Schur's theorem, Gordon suggested that we should look at Göllnitz' theorem in the context of the method of weighted words. To this end, Gordon and I first considered the product

$$(-aq)_\infty(-bq)_\infty(-cq)_\infty \tag{13}$$

and viewed the generating function of $C(n)$ as emerging out of (13) under the substitutions

$$(dilation)\quad q \mapsto q^6, \quad \text{and } (translations) \quad a \mapsto aq^{-4}, b \mapsto bq^{-2}, c \mapsto cq^{-1}. \tag{14}$$

The problem then was to find a series that would sum to this product, with the series representing the generating function of partitions into colored integers with gap conditions that would correspond to those governing $D(n)$. What Gordon and I did was to consider the integer 1 to occur in three primary colors a, b, and c, and integers $n \geq 2$ to occur in these three primary colors as well as in three secondary colors ab, ac, and bc. As before, the symbols $a_n, b_n, \cdots, bc_n$ represent n in colors $a, b, \cdots, bc$ respectively. Here too we need an ordering on the colored integers, and the one we chose is

$$a_1 < b_1 < c_1 < ab_2 < ac_2 < a_2 < bc_2 < b_2 < c_2 < ab_3 < \ldots. \tag{15}$$

The effect of the substitutions (14) is to convert the symbols to

$$\begin{cases} a_m \mapsto 6m-4,\ b_m \mapsto 6m-2,\ c_n \mapsto 6m-1, & \text{for}\, m \geq 1, \\ ab_m \mapsto 6m-6,\ ac_m \mapsto 6m-5,\ bc_n \mapsto 6m-3, & \text{for}\, m \geq 2. \end{cases} \tag{16}$$

so that the ordering (15) becomes

$$2 < 4 < 5 < 6 < 7 < 8 < 9 < 10 < 11 < 12 < \cdots, \tag{17}$$

This is one reason for the choice of the ordering of symbols in (15), because they convert to the natural ordering of the integers in (17) under the transformations (16). Notice that 1, and 3 are missing in (17), and this explains the condition that 1 and 3 do not occur as parts in the partitions enumerated by $D(n)$ in Theorem G.

To view Theorem G in this context, we think of the primary colors a, b, c as corresponding to the residue classes 2, 4 and 5 (mod 6) and so the secondary colors ab, ac, bc correspond to the residue classes $2+4 \equiv 6$, $2+5 \equiv 7$ and $4+5 \equiv 9 \pmod 6$. Note that integers of secondary color occur only when $n \geq 2$ and so ab_1, ac_1 and bc_1 are missing in (15). This is why integers $ac_1 = 1$ and $bc_1 = 3$ do not appear in (17). This explains the absence of 1 and 3 among the parts enumerated by $D(n)$ in Theorem G. Note that ab_1 corresponds to the integer 0, which is not counted as a part in ordinary partitions anyway.

In (15) for a given subscript, the ordering of the colors is

$$ab < ac < a < bc < b < c. \tag{18}$$

We use (18) to say for instance that ab is of *lower order* compared to a, or equivalently that a is of *higher order* than ab. With this concept of the order of colors, we can define *Type 1* partitions to be of the form $x_1 + x_2 + \dots$, where the x_i are symbols from (15) with the condition that the gap between x_i and x_{i+1} is ≥ 1 with strict inequality if

$$x_i \quad \text{is of lower order (color) compared to} \quad x_{i+1}, \tag{19a}$$

or

$$\text{if} \quad x_i \quad \text{and} \quad x_{i+1} \quad \text{are of the same secondary color.} \tag{19b}$$

Under the transformations given by (16), the gap conditions of Type 1 partitions become the difference conditions governing $D(n)$. Gordon and I then showed that the generating function of Type 1 partitions is

$$\sum_{i,j,k} a^i b^j c^k \sum_{\substack{s=\alpha+\beta+\gamma+\delta+\varepsilon+\phi \\ i=\alpha+\delta+\varepsilon,\ j=\beta+\delta+\phi,\ k=\gamma+\varepsilon+\phi}} \frac{q^{T_s+T_\delta+T_\varepsilon+T_{\phi-1}}(1-q^\alpha(1-q^\phi))}{(q)_\alpha(q)_\beta(q)_\gamma(q)_\delta(q)_\varepsilon(q)_\phi} \tag{20}$$

Thus our three three parameter *key identity* for the generalization and refinement of Göllnitz' theorem is

$$\begin{aligned}\sum_{i,j,k} a^i b^j c^k \sum_{\substack{s=\alpha+\beta+\gamma+\delta+\varepsilon+\phi \\ i=\alpha+\delta+\varepsilon,\ j=\beta+\delta+\phi,\ k=\gamma+\varepsilon+\phi}} &\frac{q^{T_s+T_\delta+T_\varepsilon+T_{\phi-1}}(1-q^\alpha(1-q^\phi))}{(q)_\alpha(q)_\beta(q)_\gamma(q)_\delta(q)_\varepsilon(q)_\phi} \\ &= \sum_{i,j,k} \frac{a^i b^j c^k q^{T_i+T_j+T_k}}{(q)_i(q)_j(q)_k} = (-aq)_\infty(-bq)_\infty(-cq)_\infty\,, \end{aligned} \tag{21}$$

The partition interpretation of (21) that Gordon and I had was:

Theorem 1 *Let $C(n; i, j, k)$ denote the number of vector partitions $(\pi_1; \pi_2; \pi_3)$ of n such that π_1 has i distinct parts all in color a, π_2 has j distinct parts all in color b, and π_3 has k distinct parts all in color c.*

Let $D(n; \alpha, \beta, \gamma, \delta, \varepsilon, \phi)$ denote the number of Type 1 partitions of n having α a-parts, β b-parts, ..., and ϕ bc-parts.

Then

$$C(n; i, j, k) = \sum_{\substack{i=\alpha+\delta+\varepsilon \\ j=\beta+\delta+\phi \\ k=\gamma+\varepsilon+\phi}} D(n; \alpha, \beta, \gamma, \delta, \varepsilon, \phi).$$

It is to be noted that in Theorem 1,

$$i + j + k = \alpha + \beta + \gamma + 2(\delta + \epsilon + \phi)$$

and so the parts in secondary color are counted twice. This corresponds to the condition that parts $\equiv 0, 1, 3 \pmod 6$ are counted twice in (12).

The proof in [8] that the expression in (20) is the generating function of minimal partitions is quite involved and goes by induction on $s = \alpha + \beta + \gamma + \delta + \varepsilon + \phi$, the number of parts of the Type-1 partitions, and also appeals to *minimal partitions* whose generating functions are given by multinomial coefficients (see [8] for details). Thus everything fitted perfectly, but Gordon and I had a problem: *we could not prove the key identity* (21). This is where Andrews entered into the picture. The story of how he proved (21) is described in Section 2. His ingenious proof of the remarkable key identity (21) relied on the Watson's q-analogue of Whipple's transformation and the ${}_6\psi_6$ summation of Bailey. For the proof of (21), we refer the reader to [8]. Let me just say, that there is no one in the world who can match Andrews' power in proving multi-variable q-hypergeometric identities!

One of the great advantages of the method of weighted words is that it provides a key identity for a partition theorem at the base level, and from this one can extract several partition theorems by suitable dilations and translations. I investigated in detail a variety of partition theorems that emerge from (21) (see [1, 3]), but will report here only two major developments that involved Andrews.

As noted earlier, Göllnitz' theorem pertains to the dilation $q \mapsto q^6$ in (21), and so I wanted to investigate the effect under the transformations

$$(\textit{dilation}) \quad q \mapsto q^3, \tag{22a}$$

and

$$(\textit{translations}) \quad a \mapsto aq^{-2}, \quad b \mapsto bq^{-1}, \quad c \mapsto c. \tag{22b}$$

In this case the product in (21) becomes

$$\prod_{m=1}^{\infty} (1 + aq^{3m-2})(1 + bq^{3m-1})(1 + cq^{3m}),$$

which is the three parameter generating function of partitions into distinct parts, and therefore is very interesting. The dilation $q \mapsto q^6$ converts the six colors $a, b, \cdots, bc$ into the six different residue classes mod 6, and under the dilation in (22a), one gets partitions into parts that differ by ≥ 3 but these partitions have to be counted with a weight because each positive integer ≥ 3 occurs in two colors—one primary and one secondary. Two major consequences of this weighted partition identity were (i) a new proof of Jacobi's triple product identity for theta functions, and (ii) a combinatorial proof of a variant of Göllnitz' theorem which is equivalent to it. In the course of identifying this variant, I found a new *cubic key identity* that represents it, namely

$$\sum_{i,j,k} \frac{a^i b^j c^k (-c)_i (-c)_j (-\frac{ab}{c}q)_k (-cq)_{i+j} q^{T_{i+j+k}}}{(q)_i (q)_j (q)_k (-c)_{i+j}} = (-aq)_\infty (-bq)_\infty (-cq)_\infty. \tag{23}$$

As in the case of (21), I approached Andrews for a proof of (23), and he supplied it in a matter of a few days utilizing Jackson's q-analogue of Dougall's summation. This led to our second joint paper [4]. While (23) is quite deep, it is simpler in structure compared to (21).

Next I investigated the combinatorial consequences of (21) under the

$$(\textit{dilation}) \quad q \mapsto q^4, \tag{24a}$$

but here there are four possible translations depending on which residue class modulo 4 one chooses to omit for the primary color. For example, the translations

$$a \mapsto aq^{-3}, \quad b \mapsto bq^{-1}, \quad c \mapsto cq^{-3}, \tag{24b}$$

omits the residue class 0 (mod 4) for the primary colors, and there are three other important dilations. Some very interesting weighted partition identities emerge (see [3]), but I focused on the translations in (24b) owing to the symmetry. This led me to the following *quartic key identity*:

$$\sum_{i,j,k,\ell} \frac{a^{i+\ell} b^j c^{k+\ell} q^{T_{i+j+k+\ell}+T_\ell} \left(-\frac{bc}{a}\right)_i \left(-\frac{abq}{c}\right)_k}{(q)_i (q)_j (q)_k (q)_\ell} \frac{(1+\frac{bc}{a}q^{2i-1})}{(1+\frac{bc}{a}q^{i-1})} = (-aq)_\infty (-bq)_\infty (-cq)_\infty, \tag{25}$$

Once again, I approached Andrews for a proof of (25), and he supplied it using Jackson's q-analogue of Dougall's summation. This led to my third paper with Andrews [5].

When Göllnitz proved his theorem in 1967, it was viewed as a next level result beyond Schur's theorem because the two residue classes 1, 2 (mod 3) for $S(n)$ in Schur's Theorem are replaced by three residue classes 2, 4, 5 (mod 6) for $C(n)$ in Göllnitz' theorem. Apart from this, it is not clear why Göllnitz' theorem can be considered as an extension of Schur's. But then, by our method of weighted words, one sees exactly how our generalized Göllnitz Theorem 1 is an extension of Schur's to the next level, because the key identity (5) for Schur's theorem is simply the special case $c = 0$ in the key identity (21) for Göllnitz' theorem.

So if Göllnitz' theorem is the "next level" result beyond Schur's theorem, why is it so much more difficult to prove? One reason for this is because in Göllnitz' theorem, when expanding the product in (21), we consider only the primary and secondary colors in the series and omit the ternary color abc. Actually, as early as

1968 and 1969, Andrews [12, 13], had obtained two infinite hierarchies of partition theorems to moduli $2^k - 1$ when $k \geq 2$, where he starts with k residue classes (mod $2^k - 1$) and considers the complete set of residue classes (mod $2^k - 1$) for the difference conditions. We now describe his results:

For a given integer $r \geq 2$, let $a_1, a_2, \ldots, a_r$ be r distinct positive integers such that

$$\sum_{i=1}^{k-1} a_i < a_k, \quad 1 \leq k \leq r. \tag{26}$$

Condition (26) ensures that the $2^r - 1$ sums $\sum \varepsilon_i a_i$, where $\varepsilon_i = 0$ or 1, not all $\varepsilon_i = 0$, are all distinct. Let these sums in increasing order be denoted by $\alpha_1, \alpha_2, \ldots, \alpha_{2^r-1}$.

Next let $N \geq \sum_{i=1}^{r} a_i \geq 2^r - 1$ be a modulus, and A_N denote the set of all positive integers congruent to some a_i (mod N). Similarly, letA'_N denote the set of all positive integers congruent to some α_i (mod N) Also let $\beta_N(m)$ denote the least positive residue of m (mod N). Finally, if $m = \alpha_j$ for some j, let $\phi(m)$ denote the number of terms appearing in the defining sum of m and $\psi(m)$ the smallest a_i appearing in this sum. Then the first general theorem of Andrews [12] is:

Theorem A1 *Let $C^*(A_N; n)$ denote the number of partitions of n into distinct parts taken from A_N.*

Let $D^*(A'_N; n)$ denote the number of partitions of n into parts $b_1, b_2, \ldots, b_\nu$ from A'_N such that

$$b_i - b_{i+1} \geq N\phi(\beta_N(b_{i+1})) + \psi(\beta_N(b_{i+1})) - \beta_N(b_{i+1}). \tag{27}$$

Then

$$C^*(A_N; n) = D^*(A'_N; n).$$

To describe the second general theorem of Andrews [13], let a_i, α_i and N be as above. Now let $-A_N$ denote the set of all positive integers congruent to some $-a_i$ (mod N), and $-A'_N$ the set of all positive integers congruent to some $-\alpha_i$ (mod N). The quantities $\beta_N(m)$, $\phi(m)$, $\psi(m)$ are also as above. We then have (Andrews [13]):

Theorem A2 *Let $C(-A_N; n)$ denote the number of partitions of n into distinct parts taken from $-A_N$.*

Let $D(-A'_N; n)$ denote the number of partitions of n into parts $b_1, b_2, \ldots, b_\nu$, taken from $-A'_N$ such that

$$b_i - b_{i+1} \geq N\phi(\beta_N(-b_i)) + \psi(\beta_N(-b_i)) - \beta_N(-b_i) \tag{28}$$

and also

$$b_\nu \geq N(\phi(\beta_N(-b_s) - 1)).$$

Then

$$C(-A_N; n) = D(-A'_N; n).$$

When $r = 2$, $a_1 = 1$, $a_2 = 2$, $N = 3 = 2^r - 1$, Theorems A1 and A2 both become Theorem S. Thus the two hierarchies emanate from Theorem S, and it is only when $r = 2$ that the hierarchies coincide. Thus Theorem S is its own dual. Conditions (27) and (28) can be understood better by classifying b_{i+1} (in Theorem A1) and b_i (in Theorem A2) in terms of their residue classes (mod N). In particular, with $r = 3$, $a_1 = 1$, $a_2, a_3 = 4$ and $N = 7 = 2^3 - 1$, Theorems A1 and A2 yield the following corollaries.

Corollary 1 *Let $C^*(n)$ denote the number of partitions of n into distinct parts* $\equiv 1$, 2 *or* 4 (mod 7).

Let $D^*(n)$ denote the number of partitions of n in the form $b_1 + b_2 + \cdots_\nu$ such that $b_i - b_{i+1} \geq 7, 7, 12, 7, 10, 10$ or 15 if $b_{i+1} \equiv 1, 2, 3, 4, 5, 6$ or 7 (mod 7)*. Then*

$$C^*(n) = D^*(n).$$

Corollary 2 *Let $C(n)$ denote the number of partitions of n into distinct parts* $\equiv 3$, 5 *or* 6 (mod 7).

Let $D(n)$ denote the number of partitions of n in the form $b_1 + b_2 + \cdots + b_\nu$ such that $b_i - b_{i+1} \geq 10, 10, 7, 12, 7, 7$ or 15 if $b_i \equiv 8, 9, 3, 11, 5, 6$ or 14 (mod 7) *and $b_\nu \neq 1, 2, 4$ or 7. Then*

$$C(n) = D(n).$$

Andrews' proofs of Theorems A1 and A2 are extensions of his proof [11] of Theorem S and not as difficult as the proof of Göllnitz' theorem. During the 1998 conference in Maratea, Italy, for Andrews' 60-th birthday organized by Dominique Foata, I gave a talk outlining a method of weighted words approach generalization of Theorems A1 and A2 that Gordon and I had worked out. In our approach, we obtain an "amalgamation process" that yields the weighted words generalization of the Andrews hierarchy without a q-hypergeometric key identity. In our approach we have a single partition theorem with multiple parameters, and the two Andrews hierarchies turn out to be the two extreme (special) cases under suitable choices of the parameters (for a description of the main ideas of this approach to the Andrews hierarchies, see [2], pp. 25–27). Dominique Foata then asked whether there is a hypergeometric key identity that corresponds to this generalization. Even though the proofs of Theorems A1 and A2 are simpler compared to the the proof of

Theorem G, no hypergeometric key identity has yet been found to represent the Andrews hierarchies when $2^k - 1 > 3$.

In view of the fact that with a complete set of alphabets one gets an infinite hierarchy of theorems, Andrews raised as a problem in his CBMS Lectures, whether there exists a partition theorem beyond Göllnitz' theorem in the same manner as Göllnitz' theorem goes beyond Schur. In the language of the method of weighted words, this is the same as asking whether there exists a partition theorem starting with four primary colors a, b, c, d and using only a proper subset of the complete alphabet of 15 colors, that will yield Göllnitz' theorem when we set the parameter $d = 0$. The answer to this difficult problem was found by Alladi-Andrews-Berkovich in 2000, by noticing that ALL ternary colors have to be dropped but the quaternary color $abcd$ needs to be retained. This led to a remarkable identity in four parameters a, b, c, d that went beyond (21). Our paper [7] describes the ideas behind the construction of this four parameter identity, and provides the proof as well. I just mention here a striking (mod 15) identity that emerges from this four parameter q-hypergeometric identity:

Theorem AAB *Let $P(n)$ denote the number of partitions of n into distinct parts* $\equiv -2^3, -2^2, -2^1, -2^0 \pmod{15}$.

Let $G(n)$ denote the number of partitions of n into parts $\not\equiv 2^0, 2^1, 2^2, 2^3 \pmod{15}$*, such that the difference between the parts is* ≥ 15*, with equality only if a part is* $\equiv -2^3, -2^2, -2^1, -2^0 \pmod{15}$*, parts which are* $\equiv \pm 2^0, \pm 2^1, \pm 2^2, \pm 2^3 \pmod{15}$ *are* > 15*, the difference between the multiples of* 15 *is* ≥ 60*, and the smallest multiple of* 15 *is*

$$\begin{cases} \geq 30 + 30\tau, & \text{if 7 is a part, and} \\ \geq 45 + 30\tau, & \text{otherwise,} \end{cases}$$

where τ is number of non–multiples of 15 *in the partition. Then*

$$G(n) = P(n).$$

One aspect of Göllnitz' Theorem G that escaped attention was whether it had a dual in the sense that Theorems A1 and A2 can be considered as duals. More precisely, the residue classes of Corollary 1 that constitute the primary colors are $1, 2, 4 \pmod 7$, whereas the residue classes that constitute the primary colors in Corollary 2 are $-1, -2, -4 \pmod 7$. Now one can view $2, 4, 5 \pmod 6$ as $-1, -2, -4 \pmod 6$. So the question is whether there is a dual result to Theorem G starting with $1, 2, 4 \pmod 6$. Andrews found such a theorem, namely:

Theorem A *Let $B^*(n)$ denote the number of partitions of n into parts* $\equiv 1, 7$*, or* $10 \pmod{12}$.

Let $C^*(n)$ denote the number of partitions of n into distinct parts $\equiv 1, 2$*, or* $4 \pmod 6$.

Let $D^*(n)$ *denote the number of partitions of* n *into parts that differ by at least* 6, *where the inequality is strict if the larger part is* $\equiv$ 0, 3, *or* 5 (mod 6), *with the exception that* $6+1$ *may appear in the partition. Then*

$$B^*(n) = C^*(n) = D^*(n).$$

Andrews provided a proof of Theorem A very similar to his proof of Theorem G in [14]. My role then was to construct a key identity that represented this dual, which I did. This key identity for the dual, although different from (21), is equivalent to it. This led to our most recent joint paper [6] which we published in a Special Volume of The Ramanujan Journal dedicated to the memory of Basil Gordon.

I conclude by describing my joint paper with Andrews on the Capparelli partition theorems.

In fundamental work [20, 21], Lepowsky and Wilson gave a Lie theoretic proof of the Rogers–Ramanujan identities and in that process showed how R-R type identities arise in the study of vertex operators in Lie algebras. Using vertex operator theory, Stefano Capparelli, a PhD student of Lepowsky in 1992, “discovered” two new partition results [18] which he could not prove and so he stated them as conjectures:

Conjecture C1 Let $C^*(n)$ *denote the number of partitions of* n *into parts* $\equiv \pm 2$, ± 3 *(mod* 12*).*

Let $D(n)$ *denote the number of partitions of* n *into parts* > 1 *with minimal difference* 2, *where the difference is* ≥ 4 *unless consecutive parts are both multiples of* 3 *or add up to a multiple of* 6. *Then*

$$C^*(n) = D(n).$$

He had a second partition result, Conjecture C2, which we do not state here because the conditions are more complicated; also that is not essential to what we will describe here.

As mentioned in Part I, Lepowsky stated Conjecture C1 on the opening day of the Rademacher Centenary Conference at Penn State, and by the time that conference ended, Andrews had a proof using q-recurrences (see [17]).

The first thing I did on seeing Conjecture C1 was to replace $C^*(n)$ by $C(n)$, the number of partitions of n into distinct parts $\equiv$ 2, 3, 4 or 6 (mod 6), and to note that

$$C(n) = C^*(n) \tag{29}$$

This is because by Euler’s trick

$$\begin{aligned}\sum_{n=0}^{\infty} C^*(n)q^n &= \frac{1}{(q^2;q^{12})_\infty (q^3;q^{12})_\infty (q^9;q^{12})_\infty (q^{10};q^{12})_\infty} \\ &= (-q^2;q^6)_\infty (-q^4;q^6)_\infty (-q^3;q^3)_\infty = \sum_{n=0}^{\infty} C(n)q^n. \end{aligned} \tag{30}$$

One reason for replacing $C^*(n)$ by $C(n)$ is that the equality in (29) can be refined. Another reason is that Conjecture C2 can be more elegantly stated by replacing $C(n)$ by the function $C'(n)$ which enumerates the number of partitions into distinct parts $\equiv 1, 3, 5$, or $6 \pmod 6$.

The refinement of the Capparelli Conjecture C1 that Andrews, Gordon and I [9] proved was:

Theorem 2 *Let $C(n; i, j, k)$ denote the number of partitions counted by $C(n)$ with the additional restriction that there are precisely i parts $\equiv 4 \pmod 6$, j parts $\equiv 2 \pmod 6$, and of those $\equiv 0 \pmod 3$, exactly k are $> 3(i + j)$.*

Let $D(n; i, j, k)$ denote the number of partitions counted by $D(n)$ with the additional restriction that there are precisely i parts $\equiv 1 \pmod 3$, j parts $\equiv 2 \pmod 3$, and k parts $\equiv 0 \pmod 3$. Then

$$C(n; i, j, k) = D(n; i, j, k).$$

To establish Theorem 2, we put it in the context of the method of weighted words. More precisely, let the integer 1 occur in two colors a and c and let integers ≥ 2 occur in three colors a, b and c. As before, the symbols a_j, b_j and c_j represent the integer j in colors a, b and c respectively. To discuss partitions the ordering of the symbols we used is

$$a_1 < b_2 < c_1 < a_2 < b_3 < c_2 < a_3 < b_4 < c_3 < \cdots . \tag{31}$$

The Capparelli problem corresponds to the transformations

$$a_j \mapsto 3j - 2, \qquad b_j \mapsto 3j - 4, \qquad c_j \mapsto 3j, \tag{32}$$

in which case the inequalities in (31) become

$$1 < 2 < 3 < 4 < 5 < \cdots ,$$

the natural ordering among the positive integers. With this we were able to generalize and refine Theorem 2 as follows:

Theorem 3 *Let $K(n; i, j, k)$ denote the number of vector partitions of n in the form (π_1, π_2, π_3) such that π_1 has distinct even a-parts, π_2 has distinct even b-parts, and π_3 has distinct c-parts such that $\nu(\pi_1) = i$, $\nu(\pi_2) = j$, and the number of parts of π_3 which are $> i + j$ is k.*

Let $G(n; i, j, k)$ denote the number of partitions (words) of n into symbols a_j, b_j, c_j each $> a_1$, such that the gap between consecutive symbols is given by the matrix below:

$$\begin{array}{c|ccc} & a & b & c \\ \hline a & 2 & 2 & 1 \\ b & 0 & 2 & 0 \\ c & 2 & 3 & 1 \end{array}$$

Then

$$K(n; i, j, k) = G(n; i, j, k).$$

Note The matrix above is to read row-wise. Thus if a_j is a part of the partition, and the next larger part has color b, then its weight (= subscript) must be $> j + 2$.

In [9] we gave a combinatorial proof of Theorem 2 by using some ideas of Bressoud, and another proof by first showing that it is equivalent to the following *key identity*

$$\sum_{i,j,k,n} K(n; i, j, k) a^i b^j c^k q^n = \sum_{i,j} \frac{a^i b^j q^{2T_i+2T_j} (-q)_{i+j} (-cq^{i+j+1})_\infty}{(q^2; q^2)_i (q^2; q^2)_j}$$
$$= \sum_{i,j,k,n} G(n; i, j, k) a^i b^j c^k q^n = \sum_{i,j,k} \frac{a^i b^j c^k q^{2T_i+2T_j+T_k+(i+j)k}}{(q)_{i+j+k}} \times \begin{bmatrix} i+j+k \\ i+j, k \end{bmatrix}_q \begin{bmatrix} i+j \\ i, j \end{bmatrix}_{q^2}, \tag{33}$$

and then proving this identity.

The main difficulty in (33) was to show that the series on the right is the generating function of partitions with gap conditions given by the entries in the above table. This required the study of minimal partitions having a part in a specified color as the smallest part. Once the generating function of the $G(n; i, j, k)$ was shown to be the series on the right in (29), it was not difficult to establish the equality of this with the series on the left. If we take $c = 1$, then the generating function on the left in (33) becomes a product, because

$$(-q)_\infty \sum_{i,j,k} \frac{a^i b^j q^{2T_i+2T_j}}{(q^2; q^2)_i (q^2; q^2)_j} = (-q)_\infty (-aq^2; q^2)_\infty (-bq^2; q^2)_\infty. \tag{34}$$

In (30) if we replace $q \mapsto q^3$, $a \mapsto q^{-2}$, $b \mapsto q^{-4}$, we get

$$\prod_{j=0}^{\infty} (1 + q^{6j-2})(1 + q^{6j-4})(1 + q^{3j}) = \sum_{n=0}^{\infty} C(n) q^n,$$

and so Capparelli's conjecture follows.

I could say so much more about Andrews' work on partitions, q-series and Ramanujan, but here I chose to focus on an aspect of our joint work that shows that in manipulating q-hypergeometric series, he has no match in our generation. Even though he towers head and shoulders above the rest in the world of partitions, q-series and Ramanujan, he is a perfect gentleman always willing to help. It is a pleasure and a privilege for me to be his friend and collaborator.

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Ramanujan, His Lost Notebook, Its Importance

Bruce C. Berndt

To George Andrews, my close friend and collaborator, on his 80th birthday

Abstract In the spring of 1976, George Andrews discovered Ramanujan's lost notebook in the library of Trinity College, Cambridge. The present paper provides an account of Andrews' discovery of the lost notebook and its history. A general description of its contents as well as discussions of some of the most important topics therein are given. The paper is intended for a very general audience.

1 Introduction

On March 24, 1915, near the end of his first winter in Cambridge, Ramanujan wrote to his friend E. Vinayaka Row [31, pp. 116–117] in Madras, "I was not well till the beginning of this term owing to the weather and consequently I couldn't publish any thing for about 5 months." By the end of his third year in England, Ramanujan was critically ill, and, for the next 2 years, he was confined to nursing homes. In March, 1919, Ramanujan returned to India where he died on April 26, 1920 at the age of 32. It is tragic that the onset of his illness was likely caused by at least two incidences of dysentery, not tuberculosis as then diagnosed, and that his life could likely have been saved [31, pp. 116–117].

There is no other person in the history of mathematics who accomplished such a prodigious amount of beautiful, influential, and everlasting mathematics in the last years of her/his life while suffering from a debilitating illness. The music of Wagner's Götterdämmerung or Mahler's Tragic Sixth Symphony or any art form cannot begin to convey this tragedy to us. The Final Act of this drama is tragic, but it is also redemptive. According to Janaki, "he was doing his sums up until four days

B. C. Berndt (✉)
Department of Mathematics, University of Illinois at Urbana-Champaign, Urbana, IL, USA
e-mail: berndt@illinois.edu

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before he died." Ramanujan was producing his "lost notebook" for us. His beautiful identities, e.g., for theta functions, mock theta functions, Eisenstein series, other q-series, continued fractions, and partitions must have provided hours of pleasure for him while enduring insurmountable pain. We are the beneficiaries of his struggle; he has left us so much to enjoy.

The path of Ramanujan's final theorems to us was "bumpy." But George Andrews smoothed out these bumps when in the spring of 1976 he found Ramanujan's "lost notebook" while sifting through the papers of the late G. N. Watson at Trinity College, Cambridge. Perhaps not all readers are aware of many of these bumps, and so a primary purpose of this paper is to relate the history (as best as we can put it together) of the "lost notebook" and the events leading up to Andrews' discovery.

The "lost notebook" has generated hundreds (perhaps thousands) of research and expository papers, as well as several books. More than any other mathematician, Andrews has contributed to our understanding of the "lost notebook." An examination of the references in the volumes [5–9] that Andrews and the author have written on the "lost notebook" shows that (besides the self-references to our volumes) 32, 30 and 29 (not necessarily distinct) references to works of Andrews and his co-authors in, respectively, the first, second, and fifth volumes.

In the latter sections of this paper, we offer comments on a few of the topics found in the "lost notebook" and other unpublished manuscripts in [62] to which Andrews and others have contributed.

2 Discovery of the Lost Notebook

As mentioned above, in the spring of 1976, Andrews visited Trinity College Library at Cambridge University. Dr. Lucy Slater had suggested to him that there were materials deposited there from the estate of the late G. N. Watson that might be of interest to him. In one box of materials from Watson's estate, Andrews found several items written by Ramanujan, with the most interesting item being a manuscript written on 138 sides in Ramanujan's distinctive handwriting. The sheets contained over six hundred formulas without proofs. Although technically not a notebook, and although technically not "lost," as we shall see in the sequel, it was natural in view of the fame of Ramanujan's (earlier) notebooks [61] to call this manuscript *Ramanujan's lost notebook*. Almost certainly, this manuscript, or at least most of it, was written during the last year of Ramanujan's life, after his return to India from England. For an engaging personal account of his discovery of the lost notebook, see Andrews' paper [4].

The manuscript contains no introduction or covering letter. In fact, there are hardly any words in the manuscript. There are a few marks evidently made by a cataloguer, and there are also a few remarks in the handwriting of Hardy. Undoubtedly, the most famous objects examined in the lost notebook are the *mock theta functions*, about which more will be written later.

The natural, burning question now is: *How did this manuscript of Ramanujan come into Watson's possession?* We think that the manuscript's history can be traced.

3 History of the Lost Notebook

After Ramanujan died on April 26, 1920, his notebooks and unpublished papers were given by his widow, Janaki, to the University of Madras. (It should be remarked that in a conversation with the author, Janaki told him that during the funeral of her late husband, many of his papers were stolen by two persons. If Janaki's recollection is correct, these papers have evidently never been located.) After Ramanujan's death, Hardy strongly advocated bringing together all of Ramanujan's manuscripts, both published and unpublished, for publication. On August 30, 1923, Francis Dewsbury, the registrar at the University of Madras, wrote to Hardy informing him that [31, p. 266]:

> I have the honour to advise despatch to-day to your address per registered and insured parcel post of the four manuscript note-books referred to in my letter No. 6796 of the 2nd idem.
>
> I also forward a packet of miscellaneous papers which have not been copied. It is left to you to decide whether any or all of them should find a place in the proposed memorial volume. Kindly preserve them for ultimate return to this office.

(Hardy evidently never returned any of the miscellaneous papers.) Although no accurate record of this material exists, the amount sent to Hardy was doubtless substantial. It is therefore highly likely that this "packet of miscellaneous papers" contained the aforementioned "lost notebook." R. A. Rankin, in fact, opines [32, p. 124]:

> It is clear that the long MS represents work of Ramanujan subsequent to January 1920 and there can therefore be little doubt that it constitutes the whole or part of the miscellaneous papers dispatched to Hardy from Madras on 30 August 1923.

Further details can be found in Rankin's accounts of Ramanujan's unpublished manuscripts [64, 65], [32, pp. 117–142].

In 1934, Hardy passed on to Watson a considerable amount of his material on Ramanujan. However, it appears that either Watson did not possess the "lost notebook" in 1936 and 1937 when he published his papers [69, 70] on mock theta functions, or he had not thoroughly examined it. In any event, Watson [69, p. 61], [31, p. 330] writes that he believes that Ramanujan was unaware of certain third order mock theta functions. But, in his lost notebook, Ramanujan did indeed examine these third order mock theta functions. Watson's interest in Ramanujan's mathematics waned in the late 1930s, and Hardy died in 1947. In conclusion, sometime between 1934 and 1947 and probably closer to 1947, Hardy gave Watson the manuscript that we now call the "lost notebook."

Watson was Mason Professor of Pure Mathematics at the University of Birmingham for most of his career, retiring in 1951. He died in 1965 at the age of 79. Rankin, who succeeded Watson as Mason Professor but who had since become Professor of Mathematics at the University of Glasgow, was asked to write an obituary of Watson for the London Mathematical Society. Rankin wrote [64], [32, p. 120]:

> For this purpose I visited Mrs Watson on 12 July 1965 and was shown into a fair-sized room devoid of furniture and almost knee-deep in manuscripts covering the floor area. In the space of one day I had time only to make a somewhat cursory examination, but discovered a number of interesting items. Apart from Watson's projected and incomplete revision of Whittaker and Watson's *Modern Analysis* in five or more volumes, and his monograph on *Three decades of midland railway locomotives*, there was a great deal of material relating to Ramanujan, including copies of Notebooks 1 and 2, his work with B. M. Wilson on the Notebooks and much other material. ... In November 19 1965 Dr J. M. Whittaker who had been asked by the Royal Society to prepare an obituary notice [71], paid a similar visit and unearthed a second batch of Ramanujan material. A further batch was given to me in April 1969 by Mrs Watson and her son George.

Since her late husband had been a Fellow and Scholar at Trinity College and had had an abiding, lifelong affection for Trinity College, Mrs. Watson agreed with Rankin's suggestion that the library at Trinity College would be the most appropriate place to preserve her husband's papers. Since Ramanujan had also been a Fellow at Trinity College, Rankin's suggestion was even more appropriate.

During the next 3 years, Rankin sorted through Watson's papers, and dispatched Watson's and Ramanujan's papers to Trinity College in three batches on November 2, 1965; December 26, 1968; and December 30, 1969, with the Ramanujan papers being in the second shipment. Rankin did not realize the importance of Ramanujan's papers, and so when he wrote Watson's obituary [63] for the *Journal of the London Mathematical Society*, he did not mention any of Ramanujan's manuscripts. Thus, for almost 8 years, Ramanujan's "lost notebook" and some fragments of papers by Ramanujan lay in the library at Trinity College, known only to a few of the library's cataloguers, Rankin, Mrs. Watson, Whittaker, and perhaps a few others. The 138-page manuscript waited there until Andrews found it and brought it before the mathematical public in the spring of 1976. It was not until the centenary of Ramanujan's birth on December 22, 1987, that Narosa Publishing House in New Delhi published in photocopy form Ramanujan's lost notebook and his other unpublished papers [62].

4 The Origin of the Lost Notebook

Having detailed the probable history of Ramanujan's lost notebook, we return now to our earlier claim that the lost notebook was written in the last year of Ramanujan's life. On February 17, 1919, Ramanujan returned to India after almost 5 years in

England, the last two being confined to nursing homes. Despite the weakening effects of his debilitating illness, Ramanujan continued to work on mathematics. Of this intense mathematical activity, up to the discovery of the lost notebook, the mathematical community knew only of the mock theta functions. These functions were described in Ramanujan's last letter to Hardy, dated January 12, 1920 [60, pp. xxix–xxx, 354–355], [31, pp. 220–223], where he wrote:

> I am extremely sorry for not writing you a single letter up to now .... I discovered very interesting functions recently which I call "Mock" ϑ-functions. Unlike the "False" ϑ-functions (studied partially by Prof. Rogers in his interesting paper) they enter into mathematics as beautifully as the ordinary theta functions. I am sending you with this letter some examples.

In this letter, Ramanujan defines four third order mock theta functions, ten fifth order functions, and three seventh order functions. He also includes three identities satisfied by the third order functions and five identities satisfied by his first five fifth order functions. He states that the other five fifth order functions also satisfy similar identities. In addition to the definitions and formulas stated by Ramanujan in his last letter to Hardy, the lost notebook contains further discoveries of Ramanujan about mock theta functions. In particular, it contains the five identities for the second family of fifth order functions that were only mentioned but not stated in the letter.

We think that we have made the case for our assertion that the lost notebook was composed during the last year of Ramanujan's life, when, by his own words, he discovered the mock theta functions. In fact, only a fraction (perhaps less than 10%) of the notebook is devoted to the mock theta functions themselves.

5 General Content of the Lost Notebook

The next fundamental question is: *What is in Ramanujan's lost notebook besides mock theta functions?* A majority of the results fall under the purview of q-series. These include mock theta functions, theta functions, partial theta functions, false theta functions, identities connected with the Rogers–Fine identity, several results in the theory of partitions, Eisenstein series, modular equations, the Rogers–Ramanujan continued fraction, other q-continued fractions, asymptotic expansions of q-series and q-continued fractions, integrals of theta functions, integrals of q-products, and incomplete elliptic integrals. Other continued fractions, other integrals, infinite series identities, Dirichlet series, approximations, arithmetic functions, numerical calculations, Diophantine equations, and elementary mathematics are some of the further topics examined by Ramanujan in his lost notebook.

The Narosa edition [62] contains further unpublished manuscripts, portions of both published and unpublished papers, letters to Hardy written from nursing homes, and scattered sheets and fragments. The three most famous of these unpublished manuscripts are those on the partition function and Ramanujan's tau function [7, 30], forty identities for the Rogers–Ramanujan functions [7, 25], and the unpublished remainder of Ramanujan's published paper on highly composite numbers [57], [60, pp. 78–128], [7].

In the passages that follow, we select certain topics and examples to illustrate the content and importance of Ramanujan's discoveries found in his lost notebook. For an account of all of Ramanujan's discoveries in [62], consult the five volumes prepared by Andrews and the author [5–9].

6 q-Series and Theta Functions

The vast majority of entries in Ramanujan's lost notebook are on q-series. Thus, we should begin the more technical portion of this essay by giving a brief introduction to q-series. Generally, a q-series has expressions of the type

$$(a)_n := (a;q)_n := (1-a)(1-aq)\cdots(1-aq^{n-1}), \qquad n \geq 0, \tag{6.1}$$

in the summands, where we interpret $(a;q)_0 = 1$. If the *base* q is understood, we often use the notation at the far left-hand side of (6.1). Series with factors $(a;q)_n$ in their summands are also called *Eulerian series*. Our definition of a q-series is not entirely satisfactory. Some basic functions in the theory of q-series do not have expressions (6.1) in their summands. Often in the theory of q-series, we let parameters in the summands tend to 0 or to ∞, and consequently it may happen that no factors of the type $(a;q)_n$ remain in the summands. Perhaps the most important q-series without factors $(a;q)_n$ in their summands are theta functions. Following the lead of Ramanujan, we define a general theta function $f(a,b)$ by

$$f(a,b) := \sum_{n=-\infty}^{\infty} a^{n(n+1)/2} b^{n(n-1)/2}, \qquad |ab| < 1. \tag{6.2}$$

Perhaps the most useful property of theta functions is the famous Jacobi triple product identity [16, p. 35, Entry 19] given by

$$f(a,b) = (-a;ab)_\infty(-b;ab)_\infty(ab;ab)_\infty, \qquad |ab| < 1, \tag{6.3}$$

where

$$(a;q)_\infty := \lim_{n\to\infty}(a;q)_n, \qquad |q| < 1.$$

For this exposition, in Ramanujan's notation and with the use of (6.3), only one special case,

$$f(-q) := f(-q,-q^2) = \sum_{n=-\infty}^{\infty} (-1)^n q^{n(3n-1)/2} = (q;q)_\infty, \tag{6.4}$$

is relevant for us. If $q = e^{2\pi i\tau}$, where $\operatorname{Im}\tau > 0$, then $q^{1/24}f(-q) = \eta(\tau)$, the Dedekind eta function. The last equality in (6.4) renders Euler's pentagonal number theorem. For a more detailed introduction to q-series, see the author's paper [19].

7 Mock Theta Functions

As indicated above, mock theta functions are certain kinds of q-series first introduced by Ramanujan in his last letter to Hardy written on 12 January 1920. He begins his letter by examining the asymptotic behavior of two q-series as $t \to 0$, where $q = e^{-t}$. For example,

$$\sum_{n=0}^{\infty} \frac{q^{n^2}}{(q;q)_n^2} = \sqrt{\frac{t}{2\pi}} \exp\left(\frac{\pi^2}{6t} - \frac{t}{24}\right) + o(1), \tag{7.1}$$

as $t \to 0$. (We say that a function $F(x) = o(1)$ as $x \to a$ if $\lim_{x\to a} F(x) = 0$.) The series above is the reciprocal of a theta function, namely, it is the reciprocal of $f(-q)$, defined in (6.4). Ramanujan then asks if the converse is true. That is, suppose we have a q-series that exhibits an asymptotic behavior of the kind described in (7.1) as we approach any exponential singularity $e^{2\pi i m/n}$ of the function. Must the function actually be a theta function plus some easily described trivial function? Ramanujan says "not necessarily so." "When it is not so I call the function Mock ϑ-function."

Ramanujan then gives several examples of mock theta functions. For example, he defines

$$f(q) := \sum_{n=0}^{\infty} \frac{q^{n^2}}{(-q;q)_n^2} \tag{7.2}$$

and asserts that

$$f(q) + \sqrt{\frac{\pi}{t}} \exp\left(\frac{\pi^2}{24t} - \frac{t}{24}\right) \to 4, \tag{7.3}$$

as $t \to 0$, with $q = e^{-t}$. Then he remarks, "It is inconceivable that a single ϑ function could be found to cut out the singularities of $f(q)$. (The definition of $f(q)$ in (7.2) has no relation with the function $f(-q)$ defined in (6.4).) Thus, $f(q)$ is a mock theta function (of the third order). What is the *order* of a mock theta function? Ramanujan does not tell us. We emphasize that Ramanujan does not *prove* that $f(q)$ is actually a mock theta function according to his somewhat imprecise definition. Moreover, *no one* since has actually *proved* this statement, nor has anyone proved that *any* of Ramanujan's mock theta functions are really mock theta functions according to his definition. Note that the series on the left side of (7.1) is similar in appearance to the series defining $f(q)$ in (7.2); only the signs of the parameters in the summands' q-products are different. However, the two series behave quite differently, both analytically and arithmetically in regard to their coefficients. Indeed, one of the fascinating features of q-series is that making what appears to be a small modification in the series terms drastically alters the behavior of the function. A. Folsom, K. Ono, and R. C. Rhoades [47] established for the first time that the limit on the right-hand side of (7.3) is indeed equal to 4.

Ramanujan's lost notebook contains many identities involving mock theta functions. We offer two identities for fifth order mock theta functions:

$$\chi(q) := \sum_{n=0}^{\infty} \frac{q^n}{(q^{n+1})_n} = 1 + \sum_{n=0}^{\infty} \frac{q^{2n+1}}{(q^{n+1})_{n+1}} \tag{7.4}$$

and

$$f_0(q) + 2\Phi(q^2) = \frac{(q^5;q^5)_\infty (q^5;q^{10})_\infty}{(q;q^5)_\infty (q^4;q^5)_\infty}, \tag{7.5}$$

where

$$f_0(q) := \sum_{n=0}^{\infty} \frac{q^{n^2}}{(-q;q)_n}, \qquad \Phi(q) := -1 + \sum_{n=0}^{\infty} \frac{q^{5n^2}}{(q;q^5)_{n+1}(q^4;q^5)_n}.$$

All three functions $\chi(q)$, $f_0(q)$, and $\Phi(q)$ are mock theta functions. Both of these identities have interesting implications in the theory of partitions, which we address in Sect. 8.

The discovery of the lost notebook by Andrews and then the publication of the lost notebook by Narosa in 1988 [62] stimulated an enormous amount of research on mock theta functions, as researchers found proofs of the many mock theta function identities found in the lost notebook. We mention only a few of the more important contributions by: Andrews [2, 3], Andrews and F. Garvan [11], and D. Hickerson [52] on fifth order mock theta functions; Andrews and Hickerson [12] on sixth order mock theta functions; Andrews [2] and Hickerson [53] on seventh order mock theta functions; and Y.-S. Choi [39–42] on tenth order mock theta functions.

As we have seen, Ramanujan's definition of a mock theta function is somewhat vague. Can a precise, coherent theory be developed and find its place among the other great theories of our day? In 1987, at a meeting held at the University of Illinois commemorating Ramanujan on the centenary of his birth, F. J. Dyson addressed this question [44, p. 20], [32, p. 269].

> The mock theta-functions give us tantalizing hints of a grand synthesis still to be discovered. Somehow it should be possible to build them into a coherent group-theoretical structure, analogous to the structure of modular forms which Hecke built around the old theta-functions of Jacobi. This remains a challenge for the future. My dream is that I will live to see the day when our young physicists, struggling to bring the predictions of superstring theory into correspondence with the facts of nature, will be led to enlarge their analytic

machinery to include not only theta-functions but mock theta-functions. Perhaps we may one day see a preprint written by a physicist with the title "Mock Atkin-Lehner Symmetry." But before this can happen, the purely mathematical exploration of the mock-modular forms and their mock-symmetries must be carried a great deal farther.

Since we invoke the words, *modular form*, at times in the sequel, we provide here a brief definition. Let $V(\tau) = (a\tau + b)/(c\tau + d)$, where a, b, c, and d are integers such that $ad - bc = 1$, and where $\operatorname{Im} \tau > 0$. Then $f(\tau)$ is a modular form of weight k if

$$f(V(\tau)) = \epsilon(a, b, c, d)(c\tau + d)^k f(\tau), \tag{7.6}$$

where k is real (usually an integer or half of an integer) and $|\epsilon(a, b, c, d)| = 1$.

In recent years, the work of S. Zwegers [72], K. Bringmann and K. Ono [37, 38], and several others has made progress in the direction envisioned by Dyson. First it was observed, to take one example, that the infinite product on the right-hand side of (7.5) essentially coincided with the Fourier expansion of a certain weakly holomorphic modular form, where the term "weakly holomorphic" indicates that the modular form is analytic in the upper half-plane, but may have poles at what are called "cusps" on the real axis. In his doctoral dissertation [72], Zwegers related mock theta functions to real analytic vector-valued modular forms by adding to Ramanujan's mock theta functions certain non-holomorphic functions, which are called period integrals. Earlier work of Andrews [2], who used Bailey pairs to express Ramanujan's Eulerian series in terms of Hecke-type series, was also essential for Zwegers, since he applied his ideas to the Hecke-type series rather than to Ramanujan's original series. Zwegers' real analytic modular forms are examples of *harmonic Maass forms*. Briefly, a Maass form satisfies the functional equation (7.6) with $k = 0$ and is an eigenfunction of the hyperbolic Laplacian

$$\Delta := -y^2 \left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} \right), \tag{7.7}$$

where $\tau = x + iy$. A harmonic Maass form $M(\tau)$ again satisfies a functional equation of the type (7.6) when k is an integer and a slightly different functional equation if k is half of an integer, but the operator (7.7) is replaced by

$$\Delta_k := -y^2 \left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} \right) + iky \left(\frac{\partial}{\partial x} + i \frac{\partial}{\partial y} \right).$$

It is now required that $\Delta_k M = 0$.

Returning to (7.2), Bringmann and Ono [38] examined the more general function

$$R(\omega, q) := \sum_{n=0}^{\infty} \frac{q^{n^2}}{(\omega q; q)_n (\omega^{-1} q; q)_n} \tag{7.8}$$

and proved (under certain hypotheses), when ω is a root of unity, that $R(\omega, q)$ is the holomorphic part of a weight $\frac{1}{2}$ harmonic Maass form. In general, Bringmann and Ono showed that each of Ramanujan's mock theta functions is the holomorphic part of a harmonic Maass form. We cease our brief discussion of these developments and ask readers to consult the comprehensive book by Bringmann, Folsom, Ono, and Rolen [35].

In their famous paper [50], Hardy and Ramanujan developed an asymptotic series for the partition function $p(n)$, defined to be the number of ways the positive integer n can be expressed as a sum of positive integers. For example, since $4 = 3 + 1 = 2 + 2 = 2 + 1 + 1 = 1 + 1 + 1 + 1$, $p(4) = 5$. Some years later, improving on their work, H. Rademacher [56] found a convergent series representation for $p(n)$. If we write the mock theta function $f(q)$ from (7.2) as $f(q) = \sum_{n=0}^{\infty} \alpha(n)q^n$, Andrews [1] analogously found an asymptotic series for $\alpha(n)$. Bringmann and Ono [37] were able to replace the asymptotic formula by an exact formula confirming a conjecture of Andrews.

We have sketched only a few highlights among the extensive recent developments in the theory of mock theta functions. Readers are encouraged to read Ono's comprehensive description [55] of these developments. In Ono's paper, readers will also find discussions and references to the permeation of mock theta functions in physics, thus providing evidence for Dyson's prophetic vision.

8 Partitions

Recall from above the definition of the partition function $p(n)$. Inspecting a table of $p(n)$, $1 \leq n \leq 200$, calculated by P. A. MacMahon, Ramanujan was led to conjecture the congruences

$$\begin{cases} p(5n+4) & \equiv 0 \pmod{5}, \\ p(7n+5) & \equiv 0 \pmod{7}, \\ p(11n+6) & \equiv 0 \pmod{11}, \end{cases} \tag{8.1}$$

which he eventually proved [30, 59, 62]. In 1944, Dyson [43] sought to combinatorially explain (8.1) and in doing so defined the *rank of a partition* to be *the largest part minus the number of parts*. For example, the rank of 3+1 is 1. Dyson observed that the congruence classes for the rank modulo 5 and 7 appeared to divide the partitions of $p(5n+4)$ and $p(7n+5)$, respectively, into equinumerous classes. These conjectures were subsequently proved by A. O. L. Atkin and H. P. F. Swinnerton-Dyer [13]. However, for the third congruence in (8.1), the corresponding criterion failed, and so Dyson conjectured the existence of a statistic, which he called the *crank*, to combinatorially explain the congruence $p(11n + 6) \equiv 0 \pmod{11}$. The *crank of a partition* was found by Andrews and Garvan [10] and is defined to be *the largest part if the partition contains no one's, and otherwise to be the number of* *parts larger than the number of one's minus the number of one's*. The crank divides the partitions into equinumerous congruence classes modulo 5, 7, and 11 for the three congruences, respectively, in (8.1).

At roughly the same time that Andrews and Garvan found the crank, it was observed that in his lost notebook, Ramanujan had found the generating functions for both the rank and the crank. First, if $N(m,n)$ denotes the number of partitions of n with rank m, then

$$\sum_{n=0}^{\infty}\sum_{m=-\infty}^{\infty} N(m,n)z^m q^n = \sum_{n=0}^{\infty} \frac{q^{n^2}}{(zq;q)_n(z^{-1}q;q)_n}. \tag{8.2}$$

The generating function (8.2) should be compared with that in (7.8). Second, for $n > 1$, let $M(m,n)$ denote the number of partitions of n with crank m. Then,

$$\sum_{n=0}^{\infty}\sum_{m=-\infty}^{\infty} M(m,n)z^m q^n = \frac{(q;q)_\infty}{(zq;q)_\infty(z^{-1}q;q)_\infty}. \tag{8.3}$$

We do not know if Ramanujan knew the combinatorial implications of the rank and crank. However, in view of the several results on the generating functions for the rank and crank as well as calculations for cranks found in his lost notebook, it is clear that he had realized the importance of these two functions [22, 48]. There is also evidence that his very last mathematical thoughts were on cranks before he died on April 26, 1920 [23].

Many identities in the lost notebook have partition theoretic implications. As promised earlier, we now examine the partition-theoretic interpretations of (7.4) and (7.5).

We state (7.4) in an equivalent form: The number of partitions of a positive integer N where the smallest part does not repeat and the largest part is at most twice the smallest part equals the number of partitions of N where the largest part is odd and the smallest part is larger than half the largest part. As an example, take $N = 7$. Then the relevant partitions are, respectively, $7 = 4 + 3 = 2 + 2 + 2 + 1$ and $7 = 3 + 2 + 2 = 1 + 1 + 1 + 1 + 1 + 1 + 1$. A short proof can be constructed with the use of Ferrers diagrams.

To examine (7.5), we first define $\rho_0(n)$ to be the number of partitions of n with unique smallest part and all other parts $\leq$ the double of the smallest part. For example, $\rho_0(5) = 3$, with the relevant partitions being 5, $3 + 2$, and $2 + 2 + 1$. Second, let $N(a,b,n)$ denote the number of partitions of n with rank congruent to a modulo b. Then (7.5) is equivalent to *The First Mock Theta Conjecture*,

$$N(1,5,5n) = N(0,5,5n) + \rho_0(n). \tag{8.4}$$

For example, if $n = 5$, then $N(1,5,25) = 393$, $N(0,5,25) = 390$, and, as observed above, $\rho_0(5) = 3$. Although (7.5) has been proved by Hickerson, and now also by

A. Folsom [45] and by Hickerson and E. Mortenson [54], a combinatorial proof of (8.4) has never been given.

9 Further q-Series

We have discussed only two facets among Ramanujan's voluminous contributions to q-series in his lost notebook. In this short section, we briefly provide two more examples in illustration of this abundance. One of the showpieces in the theory of q-series is Heine's transformation [6, p. 6, Thm. 1.2.1]

$$\sum_{n=0}^{\infty} \frac{(a)_n(b)_n}{(c)_n(q)_n} t^n = \frac{(b)_\infty(at)_\infty}{(c)_\infty(t)_\infty} \sum_{n=0}^{\infty} \frac{(c/b)_n(t)_n}{(at)_n(q)_n} b^n, \tag{9.1}$$

where $|t|, |b| < 1$. There are many identities in the lost notebook, whose proofs naturally use Heine's transformation [6, Chapter 1]. One consequence of (9.1) is, for $|aq| < 1$ [62, p. 38],

$$\sum_{n=0}^{\infty} \frac{(-aq)^n}{(-aq^2;q^2)_n} = \sum_{n=0}^{\infty} \frac{(-1)^n a^n q^{n(n+1)/2}}{(-aq;q)_n}. \tag{9.2}$$

To see how (9.1) can be used to prove (9.2), consult [6, p. 25, Entry 1.6.4]. For partition-theoretic proofs, see [33] and [26].

Our second identity is given by [62, p. 35], [6, p. 35, Entry 1.7.9]

$$\sum_{n=0}^{\infty} \frac{(-1)^n q^{n(n+1)/2}}{(q;q)_n(1-q^{2n+1})} = f(q^3, q^5), \tag{9.3}$$

where $f(a,b)$ is defined by (6.2). We see partitions at work on the left-hand side of (9.3), but on the right-hand side, we observe that these partitions have cancelled each other out, except on a considerably thinner set of exponents. For a combinatorial proof of (9.3), see [26].

10 The Rogers–Ramanujan Continued Fraction

As the name suggests, the Rogers–Ramanujan continued fraction

$$R(q) := \frac{q^{1/5}}{1} \, \frac{}{+} \, \frac{q}{1} \, \frac{}{+} \, \frac{q^2}{1} \, \frac{}{+} \, \frac{q^3}{1} \, \frac{}{+} \, \cdots, \qquad |q| < 1, \tag{10.1}$$

was first studied by L. J. Rogers [66] in 1894 and then by Ramanujan before he departed for England, for his notebooks [61] contain many properties of this continued fraction. His study of $R(q)$ continued unabatedly in the lost notebook, with the first five chapters of [5] focusing on $R(q)$. Readers may find it strange that the first numerator of $R(q)$ in (10.1) is $q^{1/5}$. Because $R(q)$ lives in the homes of theta functions and modular forms, once we see how it relates to the other functions living in the same homes, we understand more fully why $q^{1/5}$ is a part of the continued fraction.

In his first letter to Hardy [60, p. xxvii], [31, p. 29], Ramanujan offered the elegant value

$$R(e^{-2\pi}) = \sqrt{\frac{5+\sqrt{5}}{2}} - \frac{\sqrt{5}+1}{2}$$

and also that for $R(-e^{-\pi})$, and in his second letter, he communicated the value of $R(e^{-2\pi\sqrt{5}})$ [60, p. xxviii], [31, p. 37]. In his lost notebook, Ramanujan recorded several further values; e.g., on page 46 of [62],

$$R(-e^{-\pi\sqrt{3}}) = \frac{(3+\sqrt{5}) - \sqrt{6(5+\sqrt{5})}}{4}.$$

Keys to evaluating $R(q)$ in closed form are the identity (found in Ramanujan's notebooks [16, p. 267])

$$\frac{1}{R(q)} - 1 - R(q) = \frac{f(-q^{1/5})}{q^{1/5} f(-q^5)}, \tag{10.2}$$

where $f(-q)$ is defined by (6.4), and a similar identity involving $R^5(q)$ that can be derived from (10.2). If one can evaluate the quotient of Dedekind eta functions on the right-hand side of (10.2) for a certain value of q, then one can determine $R(q)$ by solving a simple quadratic equation. The evaluation of quotients of eta functions at points $e^{-\pi\sqrt{n}}$ is usually quite difficult. To this end, evaluating appropriate *class invariants*, which are certain quotients of eta functions at $e^{-\pi\sqrt{n}}$, is often helpful. The first general theorem for determining values of $R(q)$ in this direction was developed by the author, H. H. Chan, and L.-C. Zhang [24].

The function $R(q)$ satisfies several beautiful *modular equations*. For example [5, p. 92], if $u = R(q)$ and $v = R(q^2)$, then

$$\frac{v-u^2}{v+u^2} = uv^2.$$

Because of limitations of space, we must desist from providing further properties for $R(q)$, but in closing we remark that $R(q)$ also serves as a model, in that its properties guided Ramanujan and others that followed in their quests of finding analogous theorems for other q-continued fractions. In particular, see Chapters 6–8 in [5].

11 Eisenstein Series

Continuing in his lost notebook the study of Eisenstein series made in [58], [60, pp. 136–162], Ramanujan offers many further discoveries about these series. Furthermore, published with his lost notebook are several letters that Ramanujan wrote to Hardy from nursing homes during his last 2 years in England; these letters feature Eisenstein series. An account of all of these discoveries can be found in the last six chapters in [6]. In this section, we briefly discuss some of these results.

In Ramanujan's notation, the three primary Eisenstein series are

$$P(q) := 1 - 24 \sum_{k=1}^{\infty} \frac{kq^k}{1-q^k},$$

$$Q(q) := 1 + 240 \sum_{k=1}^{\infty} \frac{k^3 q^k}{1-q^k},$$

$$R(q) := 1 - 504 \sum_{k=1}^{\infty} \frac{k^5 q^k}{1-q^k}.$$

For $q = \exp(2\pi i\tau)$, $\operatorname{Im} \tau > 0$, in more contemporary notation, $Q(q) = E_4(\tau)$ and $R(q) = E_6(\tau)$. These two functions are modular forms of weights 4 and 6, respectively, and are the "building blocks" for modular forms on the full modular group. The function $P(q)$ is not a modular form, but it is a quasi-modular form, because it satisfies the same transformation formulas as an ordinary modular form.

As emphasized in Sect. 8, Hardy and Ramanujan [50], [60, pp. 276–309] found an asymptotic series for the partition function $p(n)$, which arises from the power series coefficients of the reciprocal of the Dedekind eta-function. As they indicated near the end of their paper, their methods also apply to coefficients of further modular forms that are analytic in the upper half-plane. In their last jointly published paper [51], [60, pp. 310–321], they considered a similar problem for the coefficients of modular forms having a simple pole in a fundamental region, and, in particular, they applied their theorem to find interesting series representations for the coefficients of the reciprocal of the Eisenstein series $E_6(\tau)$. In letters from nursing homes, Ramanujan calculated formulas for the coefficients of further quotients of Eisenstein series. The formulas, which were first proved in papers by the author with P. Bialek [20] and with Bialek and A. J. Yee [21], do not fall under the purview of the general theorem from [51]. As they are too complicated to offer in a short survey, we invite readers to examine them in the aforementioned papers or to consult [6, Chapter 11]. The work of Ramanujan, the author, Bialek, and Yee has recently been considerably generalized by Bringmann and B. Kane [36].

Ramanujan recorded many beautiful identities for Eisenstein series in [62]. We close this section with one of them [6, p. 331]. If $f(-q)$ is defined by (6.4), then

$$Q(q) = \frac{f^{10}(-q)}{f^2(-q^5)} + 250qf^4(-q)f^4(-q^5) + 3125q^2\frac{f^{10}(-q^5)}{f^2(-q)}.$$

12 The Circle and Divisor Problems

Let $r_2(n)$ denote the number of representations of the positive integer n as a sum of two squares. The famous *circle problem* of Gauss is to determine the precise order of magnitude for the "error term" $P(x)$ defined by

$$\sum_{0\le n\le x}{}' r_2(n) = \pi x + P(x), \tag{12.1}$$

as $x \to \infty$, where the prime $'$ on the summation sign on the left side indicates that if x is an integer, only $\frac{1}{2}r_2(x)$ is counted. We now explain why this problem is called the *circle problem*. Each representation of n as a sum of two squares can be associated with a lattice point in the plane. For example, $5 = (-2)^2 + 1^2$ can be associated with the lattice point $(-2, 1)$. Then each lattice point can be associated with a unit square, say that unit square for which the lattice point is in the southwest corner. Thus, the sum in (12.1) is equal to the number of lattice points in the circle of radius $\sqrt{x}$ centered at the origin, or to the sum of the areas of the aforementioned squares. The area of this circle, namely πx, is a reasonable approximation to the sum of the areas of these squares. Gauss showed quite easily that the error made in this approximation is $P(x) = O(\sqrt{x})$. (We say that $F(x) = O(G(x))$, as $x \to \infty$, if there exist positive constants A and x_0, such that for all $x \ge x_0$, $|F(x)| \le A|G(x)|$.)

In 1915, Hardy [49] proved that $P(x) \ne O(x^{1/4})$, as x tends to ∞. In other words, there is a sequence of points $\{x_n\}$ tending to ∞ on which $P(x_n) \ne O(x_n^{1/4})$. (He actually proved a slightly stronger result.) In connection with his work on the *circle problem*, Hardy [49] proved that

$$\sum_{0\le n\le x}{}' r_2(n) = \pi x + \sum_{n=1}^{\infty} r_2(n)\left(\frac{x}{n}\right)^{1/2} J_1(2\pi\sqrt{nx}), \tag{12.2}$$

where $J_1(x)$ is the ordinary Bessel function of order 1. After Gauss, almost all efforts toward obtaining an upper bound for $P(x)$ have ultimately rested upon (12.2), and methods of estimating the approximating trigonometric series that is obtained from the asymptotic formula for $J_1(2\pi\sqrt{nx})$ as $n \to \infty$. In 1906, W. Sierpinski [67] proved that $P(x) = O(x^{1/3})$, as x tends to ∞, and there have been many improvements in a century of work since then, but all with the exponent greater than $3/10$. It is conjectured that $P(x) = O(x^{1/4+\epsilon})$, for every $\epsilon > 0$. In a footnote, Hardy remarks, "The form of this equation was suggested to me by Mr. S. Ramanujan, to whom I had communicated the analogous formula for $d(1) + d(2) + \cdots + d(n)$, where $d(n)$ is the number of divisors of n." In this same paper, Hardy relates a beautiful identity of Ramanujan connected with $r_2(n)$; namely, for $a, b > 0$, [49, p. 283],

$$\sum_{n=0}^{\infty} \frac{r_2(n)}{\sqrt{n+a}} e^{-2\pi\sqrt{(n+a)b}} = \sum_{n=0}^{\infty} \frac{r_2(n)}{\sqrt{n+b}} e^{-2\pi\sqrt{(n+b)a}}, \tag{12.3}$$

which is not given elsewhere in any of Ramanujan's published or unpublished work. Note that the right side of (12.3) is simply the left side with a and b interchanged. These facts indicate that Ramanujan and Hardy undoubtedly had probing conversations about the *circle problem*.

On page 335 in [62], which is *not* in Ramanujan's lost notebook, but which is in one of those fragments published *with* the lost notebook, Ramanujan offers two identities involving Bessel functions. To state Ramanujan's claims, we need to first define

$$F(x) = \begin{cases} [x], & \text{if } x \text{ is not an integer,} \\ x - \frac{1}{2}, & \text{if } x \text{ is an integer,} \end{cases} \tag{12.4}$$

where, as customary, $[x]$ is the greatest integer less than or equal to x.

Entry 12.1 (p. 335) *If* $0 < \theta < 1$, $x > 0$, *and* $F(x)$ *is defined by* (12.4), *then*

$$\sum_{n=1}^{\infty} F\left(\frac{x}{n}\right) \sin(2\pi n\theta) = \pi x \left(\frac{1}{2} - \theta\right) - \frac{1}{4}\cot(\pi\theta) \tag{12.5}$$
$$+ \frac{1}{2}\sqrt{x} \sum_{m=1}^{\infty} \sum_{n=0}^{\infty} \left\{ \frac{J_1\left(4\pi\sqrt{m(n+\theta)x}\right)}{\sqrt{m(n+\theta)}} - \frac{J_1\left(4\pi\sqrt{m(n+1-\theta)x}\right)}{\sqrt{m(n+1-\theta)}} \right\}.$$

Entry 12.1 was first proved by the author, S. Kim, and A. Zaharescu [28]. It is possible that Ramanujan did not interpret the right-hand side of (12.5) as an iterated double series, but as a series with the product mn tending to ∞. The same three authors [27] also proved (12.5) under this interpretation. In fact, Entry 12.1 was first established by the author and Zaharescu with the order of summation reversed from that prescribed by Ramanujan in (12.5) [34].

The Bessel functions in (12.5) bear a striking resemblance to those in (12.2), and so it is natural to ask if there is a connection between the two formulas. Berndt, Kim, and Zaharescu [27] proved the following corollary, as a consequence of their reinterpreted meaning of the double sum in (12.5). It had been previously established (although not rigorously) by Berndt and Zaharescu in [34] as a corollary of their theorem on twisted divisor sums arising from Entry 12.1.

Corollary 12.2 *For any $x > 0$,*

$$\sum_{0\leq n\leq x}{}' r_2(n) = \pi x \tag{12.6}$$
$$+ 2\sqrt{x}\sum_{n=0}^{\infty}\sum_{m=1}^{\infty}\left\{\frac{J_1\left(4\pi\sqrt{m(n+\frac{1}{4})x}\right)}{\sqrt{m(n+\frac{1}{4})}} - \frac{J_1\left(4\pi\sqrt{m(n+\frac{3}{4})x}\right)}{\sqrt{m(n+\frac{3}{4})}}\right\}.$$

Can Corollary 12.2 be employed in place of (12.2) to effect an improvement in the error term for the circle problem? The advantage of (12.6) is that $r_2(n)$ does not appear on the right-hand side; the disadvantage is that one needs to estimate a double sum, instead of a single sum in (12.2).

The second identity on page 335 pertains to the famous *Dirichlet divisor problem*. Let $d(n)$ denote the number of positive divisors of the integer n. Define the "error term" $\Delta(x)$, for $x > 0$, by

$$\sum_{n\leq x}{}' d(n) = x\,(\log x + (2\gamma - 1)) + \frac{1}{4} + \Delta(x), \tag{12.7}$$

where γ denotes Euler's constant, and where the prime $'$ on the summation sign on the left side indicates that if x is an integer, then only $\frac{1}{2}d(x)$ is counted. The famous *Dirichlet divisor problem* asks for the correct order of magnitude of $\Delta(x)$ as $x \to \infty$.

In [62, p. 335], Ramanujan offered an identity for

$$\sum_{n=1}^{\infty} F\left(\frac{x}{n}\right)\cos(2\pi n\theta)$$

analogous to (12.1). See the paper by the author, S. Kim, and A. Zaharescu [27], where the identity is proved with the order of summation reversed, and the paper by the author, J. Li, and Zaharescu, where the identity is proved with the order of summation as given by Ramanujan.

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Part II
Photographs

George Andrews: "Combinatory Analysis 80" Picture Book

Peter Paule

Abstract This chapter presents a collection of pictures which show George Andrews at various events related to research in combinatory analysis. The first section shows photos taken at the "Combinatory Analysis 2018" conference at the Pennsylvania State University, U.S.A. The other sections display pictures originating from research stays of Andrews in Austria, China, Florida, Illinois, and India.

1 Andrews at Combinatory Analysis 2018

These pictures were taken at "Combinatory Analysis 2018—A Conference in Honor of George Andrews' 80th Birthday", June 2018, Pennsylvania State University, U.S.A. Photos [P1+P2] by courtesy of Ae Ja Yee (Department of Mathematics, The Pennsylvania State University, U.S.A.), photos [P3+P4] by courtesy of James Sellers (University of Minnesota, Duluth).

Photo P1 Participants of "Combinatory Analysis 2018"

P. Paule (✉)
Research Institute for Symbolic Computation, Johannes Kepler University Linz, Linz, Austria
e-mail: peter.paule@risc.jku.at

K. Alladi et al. (eds.), *George E. Andrews 80 Years of Combinatory Analysis*,
Trends in Mathematics, https://doi.org/10.1007/978-3-030-57050-7_4

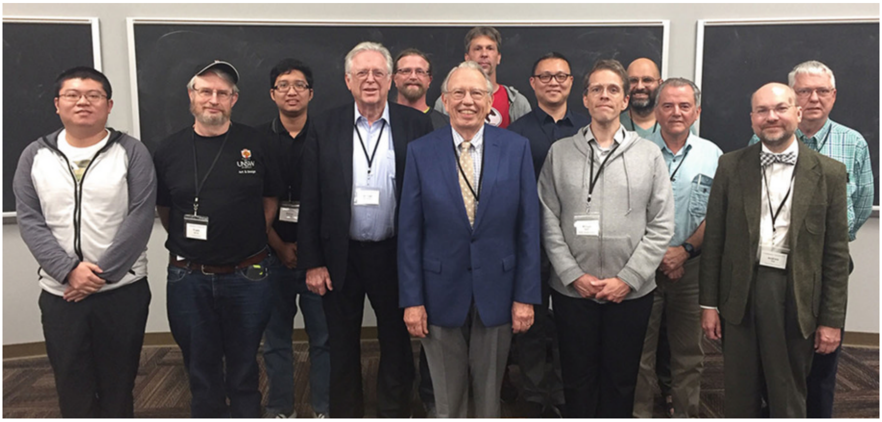

Photo P2 George Andrews with former PhD students; from left: Shane Chern, Frank Garvan, Donny Passary (co-advisor Ae Ja Yee), Mike Hirschhorn, Brandt Kronholm (co-advisor Antun Milas), Jeremy Lovejoy (co-advisor Ken Ono), Shishuo Fu (co-advisor Ae Ja Yee), William Keith, Kagan Kursungoz, Jose Plinio Santos, Andrew Sills, Louis Kolitsch

Photo P3+P4 Ae Ja Yee and James Sellers served as the local hosts among the main organizers of "Combinatory Analysis 2018"

2 Andrews in Austria

Photos [P12+P13] by courtesy of Markus Fulmek (Institut für Mathematik, University of Vienna); photos [P5–P9] by courtesy of Christoph Koutschan (Johann Radon Institute, RICAM, Linz); photos [P10+P11] by courtesy of Liangjie Ye (AVL Graz).

Photo P5 Andrews has been a regular visitor to RISC (Research Institute for Symbolic Computation). The home of RISC is the Castle of Hagenberg located northeast of Linz, Upper Austria

Photo P6 Andrews' speech on the occasion of the academic celebration "20 Years of RISC", June 2008

Photo P7 Academic celebration "20 Years of RISC" (June 2008); on the left of Andrews' row: Josef Pühringer (former Governor of Upper Austria), Bruno Buchberger (founder of RISC)

Photo P8 Andrews as key note speaker on the occasion of the 10th anniversary of the RISC collaboration with DESY (Deutsches Elektronen Synchrotron), Feb 2017

Photo P9 Key note talk at the 10th anniversary of the RISC collaboration with DESY (Feb 2017); to the left: Roger Germundsson (Mathematica), Jürgen Gerhard (Maple)

Photo P10 Andrews at the RISC Workshop “Combinatorics, Special Functions and Computer Algebra” (Hagenberg, May 2018)

Photo P11 "Combinatorics, Special Functions and Computer Algebra" conference dinner (RISC, May 2018); from left: Peter Paule, George Andrews, Johann Cigler

Photo P12 Lake Wolfgang is one of the most charming holiday regions in Austria; an ideal location for the conference center BIFEB at Strobl/St. Wolfgang. At BIFEB several meetings of the Séminaire Lotharingien de Combinatoire (SLC) were organized. Andrews attended the SLC at Strobl and at other places

Photo P13 Andrews was key note speaker at the 81st SLC, Strobl, September 2018

3 Andrews in China

Photos [P14–P20] by courtesy of William Y.C. Chen, Center for Applied Mathematics, Tianjin University, China.

Photo P14 George Andrews and Richard Askey at Tianjin Shijidayuan, China, 2004

Photo P15 Stephen Milne, Jianyi Shi, George Andrews, and James Louck at Professor Shiing-Shen Cern's home, 2004

Photo P16 Andrews at the meeting "The Renaissance of Combinatorics: Advances, Algorithms, Applications—in honor of Prof. Doron Zeilberger's 60th Birthday", Nankai University, August 2010

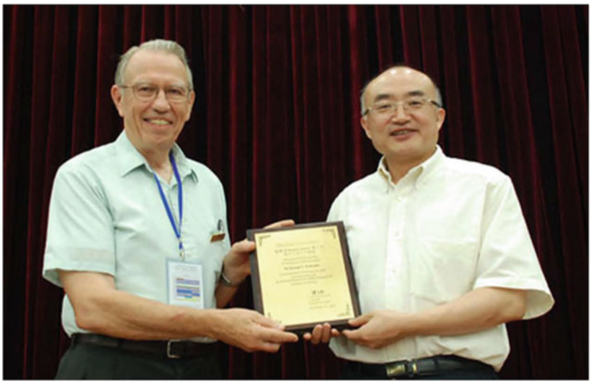

Photo P17 Andrews awarded honorary professorship at Nankai University, August 2010. To the right: William Chen

Photo P18 Andrews at the meeting "The Combinatorics of q-Series and Partitions—in honor of Prof. George Andrews' 75th Birthday", Nankai University, August 2013

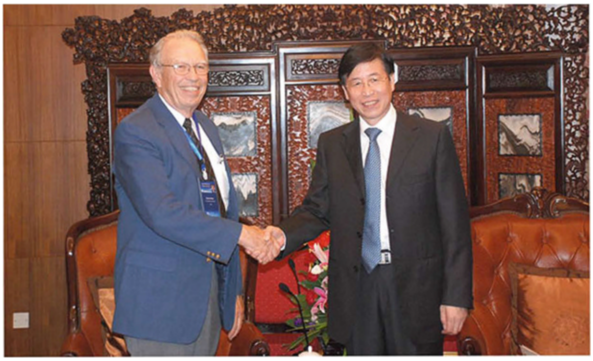

Photo P19 Andrews and the President of Tianjin University, Jiajun Li, August 2013

Photo P20 The inside-painted bottle as a gift for Prof. Andrews, Tianjin, August 2013

4 Andrews in Florida

Photos [P21–P28] by courtesy of Krishna Alladi, University of Florida, Gainesville.

Photo P21 George Andrews in front of the Science Library on the campus of the University of Florida in Nov 1990 when he came to deliver the Frontiers of Science Lecture

Photo P22 Krishna Alladi is one of Andrews' strong links to Florida. The picture shows Alladi with George Andrews in Andrews' office in the McAllister Building, The Pennsylvania State University, April 9, 1993

Photo P23 The mathematicians associated with the Capparelli partition conjecture and its resolution: seated—Jim Lepowsky (left) and Basil Gordon. Standing—Stefano Capparelli (left), George Andrews (middle), and Krishna Alladi (right)—at the Alladi House in Gainesville, Florida, Fall 2004

Photo P24 Joy and George Andrews after he received an Honorary Doctorate at the University of Florida, Dec 2002

Photo P25 Krishna Alladi, John Thompson, George Andrews, Richard Askey, and Alladi Ramakrishnan at Alladi's home in Florida, March 2005

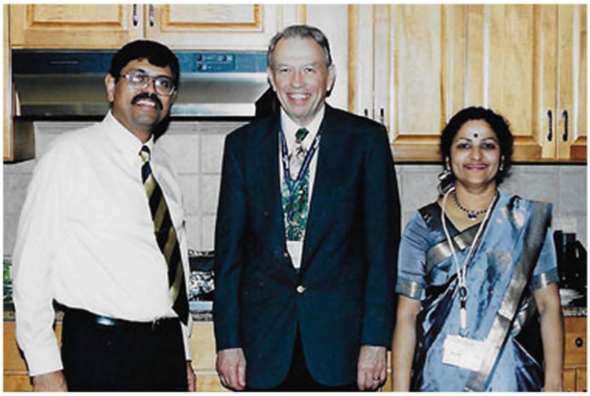

Photo P26 Krishna Alladi, George Andrews, and Mathura Alladi at the Alladi home in Gainesville, Florida, Spring 2006

Photo P27 Alladi Ramakrishnan and Krishna Alladi with Andrews who is Distinguished Visiting Professor at Florida each spring, Alladi home in Gainesville, Feb 2008

Photo P28 Frank Garvan, Krishna Alladi, and George Andrews at "Number Theory", a conference in honor of Krishna Alladi's 60th birthday, University of Florida, Gainesville, March 2016

5 Andrews in Illinois

Photos [P29+P30+P31+P33] by courtesy of Bruce Berndt (University of Illinois at Urbana-Champaign), photo [P34] by courtesy of Koustav Banerjee (RISC, Johannes Kepler University, Austria).

Photo P29 From left: Sinai Robins, Marvin Knopp, George Andrews, and Bruce Berndt; conference "Modular Forms and Related Topics" in honor of Marvin Knopp's 73rd birthday, Temple University, Jan 2006 (connection to Illinois via Bruce Berndt and Paul Bateman)

Photo P30 Picture taken 2009 in the home of Bruce and Helen Berndt; back row from left: Sun Kim, Ole Warnaar, Bruce Berndt, George Andrews, Youn-Seo Choi; front row from left: Soon-Yi Kang, Song Heng Chan, Helen Berndt, Heng Huat Chan

Photo P31 George Andrews lecturing at the Illinois Number Theory Celebration of Paul Bateman’s 90th and Bruce Berndt’s 70th birthday, University of Illinois at Urbana-Champaign, March 2009

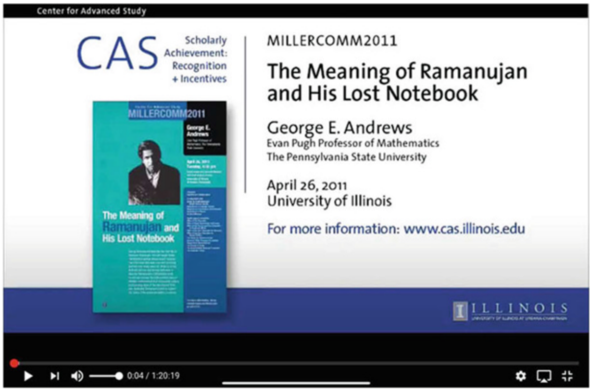

Photo P32 In April 2011 the Center for Advanced Study at the University of Illinois organized a public lecture by George Andrews; available on YouTube

Photo P33 Bruce Berndt with George Andrews, who received an honorary doctorate from the University of Illinois in 2014

Photo P34 From left: George Andrews, Peter Paule, and Krishna Alladi; banquet, "Analytic and Combinatorial Number Theory: The Legacy of Ramanujan", a conference in honor of Bruce C. Berndt's 80th birthday, University of Illinois at Urbana-Champaign, June 2019

6 Andrews in India

Photos [P35–P40] by courtesy of Krishnaswami Alladi, Department of Mathematics, University of Florida at Gainesville, U.S.A.

Photo P35 Andrews lecturing on Ramanujan's Lost Notebook at the Alladi family home in Madras, India, during the Ramanujan Centennial, Dec 23, 1987. The Portrait on the wall shows the late Sir Alladi Krishnaswami Iyer

Photo P36 Andrews answering questions after his talk (Madras, India, Dec 23, 1987). Mr. John Stempel, Consul General of the United States, who chaired the talk, is next to Andrews

Photo P37 Andrews signing the Visitors Book in the office of Alladi Ramakrishnan after his lecture (Madras, India, Dec 23, 1987)

Photo P38 Krishna Alladi and George Andrews being introduced to Dr. Abdul Kalam, President of India during the first SASTRA Ramanujan Conference in Kumbakonam, India, Dec 20, 2003

Photo P39 Krishna Alladi and George Andrews in front of Ramanujan's home in Kumbakonam, India, Dec 21, 2003

Photo P40 Andrews receiving an Honorary Doctorate Degree from SASTRA University in Kumbakonam, South India, during the Ramanujan 125 Celebrations in December 2012. In the rear is SASTRA Vice-Chancellor R. Sethuraman, and to Andrews' left is Dean Swaminathan

Part III
Articles

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Annals of Combinatorics

Dissections of Strange q-Series

Dedicated to George E. Andrews on his 80th birthday

Scott Ahlgren, Byungchan Kim and Jeremy Lovejoy

Abstract. In a study of congruences for the Fishburn numbers, Andrews and Sellers observed empirically that certain polynomials appearing in the dissections of the partial sums of the Kontsevich–Zagier series are divisible by a certain q-factorial. This was proved by the first two authors. In this paper, we extend this strong divisibility property to two generic families of q-hypergeometric series which, like the Kontsevich–Zagier series, agree asymptotically with partial theta functions.

Mathematics Subject Classification. Primary 33D15.

Keywords. Fishburn numbers, Kontsevich–Zagier strange function, q-Series, Partial theta functions, Congruences.

1. Introduction

Recall the usual q-series notation

$$(a;q)_n := (1-a)(1-aq)\cdots(1-aq^{n-1}), \tag{1.1}$$

and let $\mathcal{F}(q)$ denote the Kontsevich–Zagier "strange" function [13,14],

$$\mathcal{F}(q) := \sum_{n\geq 0} (q;q)_n.$$

This series does not converge on any open subset of $\mathbb{C}$, but it is well defined both at roots of unity and as a power series when q is replaced by $1-q$. The coefficients $\xi(n)$ of

$$\mathcal{F}(1-q) = 1 + q + 2q^2 + 5q^3 + 15q^4 + 53q^5 + \cdots$$

The first author was supported by a grant from the Simons Foundation (#426145 to Scott Ahlgren). Byungchan Kim was supported by the Basic Science Research Program through the National Research Foundation of Korea (NRF) funded by the Ministry of Education (NRF-2016R1D1A1A09917344).

are called the Fishburn numbers, and they count a number of different combinatorial objects (see [11] for references).

Andrews and Sellers [4] discovered and proved a wealth of congruences for $\xi(n)$ modulo primes p. For example, we have

$$\begin{aligned} \xi(5n+4) \equiv \xi(5n+3) \equiv 0 \pmod{5}, \\ \xi(7n+6) \equiv 0 \pmod{7}. \end{aligned} \tag{1.2}$$

In subsequent work of the first two authors [1], Garvan [6], and Straub [12], similar congruences were obtained for prime powers and for generalized Fishburn numbers.

Taking a different approach, Guerzhoy et al. [7] interpreted the coefficients in the asymptotic expansions of the functions $P^{(1)}_{a,b,\chi}(e^{-t})$ defined in (1.8) below in terms of special values of L-functions, and proved congruences for these coefficients using divisibility properties of binomial coefficients. These congruences are inherited by any function whose expansion at $q = 1$ agrees with one of these expansions; these include the function $\mathcal{F}(q)$ and, more generally, the Kontsevich–Zagier functions described in Sect. 5 below. See [7] for details.

Although the congruences (1.2) bear a passing resemblance to Ramanujan's congruences for the partition function $p(n)$, it turns out that they arise from a divisibility property of the partial sums of $\mathcal{F}(q)$. For positive integers N and s, consider the partial sums

$$\mathcal{F}(q;N) := \sum_{n=0}^{N} (q;q)_n$$

and the s-dissection

$$\mathcal{F}(q;N) = \sum_{i=0}^{s-1} q^i A_s(N,i,q^s).$$

Let $S(s) \subseteq \{0,1,\dots,s-1\}$ denote the set of reductions modulo s of the set of pentagonal numbers $m(3m+1)/2$, where $m \in \mathbb{Z}$. The key step in the proof of Andrews and Sellers is to show that if p is prime and $i \notin S(p)$ then we have

$$(1-q)^n \mid A_p(pn-1,i,q). \tag{1.3}$$

This divisibility property is also important for the proof of the congruences in [6, 12].

Andrews and Sellers [4] observed empirically that $(1-q)^n$ can be strengthened to $(q;q)_n$ in (1.3). The first two authors showed that this divisibility property holds for any s. To be precise, define

$$\lambda(N,s) = \left\lfloor \frac{N+1}{s} \right\rfloor. \tag{1.4}$$

Then we have

Theorem 1.1. [1] *Suppose that s and N are positive integers and that $i \notin S(s)$. Then*

$$(q;q)_{\lambda(N,s)} \mid A_s(N,i,q). \tag{1.5}$$

The proof of (1.5) relies on the fact that the Kontsevich–Zagier function satisfies the "strange identity"

$$\mathcal{F}(q) \text{``}=\text{''} -\frac{1}{2}\sum_{n\geq 1} n\left(\frac{12}{n}\right) q^{(n^2-1)/24}.$$

Here the symbol "=" means that the two sides agree to all orders at every root of unity (this is explained fully in Sections 2 and 5 of [13]). In this paper, we show that an analogue of Theorem 1.1 holds for a wide class of "strange" q-hypergeometric series—that is, q-series which agree asymptotically with partial theta functions.

To state our result, let F and G be functions of the form

$$F(q) = \sum_{n=0}^{\infty} (q;q)_n f_n(q), \tag{1.6}$$

$$G(q) = \sum_{n=0}^{\infty} (q;q^2)_n g_n(q), \tag{1.7}$$

where $f_n(q)$ and $g_n(q)$ are polynomials. (Functions of the form (1.6) are elements of the *Habiro ring*, which can be viewed as a ring of analytic functions on the set of roots of unity [8].) Note that $F(q)$ is not necessarily well defined as a power series in q, but it has a power series expansion at every root of unity ζ. In other words, $F(\zeta e^{-t})$ has a meaningful definition as a formal power series in t whose coefficients are expressed in the usual way as the "derivatives" of $F(\zeta e^{-t})$ at $t = 0$. This is explained in detail in the next section. Likewise, $G(q)$ has a power series expansion at every odd-order root of unity.

We will consider partial theta functions

$$P^{(\nu)}_{a,b,\chi}(q) := \sum_{n\geq 0} n^{\nu}\chi(n) q^{\frac{n^2-a}{b}}, \tag{1.8}$$

where $\nu \in \{0, 1\}$, $a \geq 0$ and $b > 0$ are integers, and $\chi\colon \mathbb{Z} \to \mathbb{C}$ is a function satisfying the following properties:

$$\chi(n) \neq 0 \quad \text{only if} \quad \frac{n^2-a}{b} \in \mathbb{Z}, \tag{1.9}$$

and for each root of unity ζ,

$$\text{the function } n \mapsto \zeta^{\frac{n^2-a}{b}}\chi(n) \text{ is periodic and has mean value zero.} \tag{1.10}$$

These assumptions are enough to ensure that for each root of unity ζ, the function $P^{(\nu)}_{a,b,\chi}(\zeta e^{-t})$ has an asymptotic expansion as $t \to 0^+$ (see Sect. 3 below). We note that (1.10) is satisfied by any odd periodic function. To see this, suppose that χ is odd with period T, and let ζ be a kth root of unity. Set $M = \operatorname{lcm}(T, bk)$. Then we have

$$\zeta^{\frac{(M-n)^2-a}{b}}\chi(M-n) = -\zeta^{\frac{n^2-a}{b}}\chi(n),$$

and so

$$\sum_{n=0}^{M-1} \zeta^{\frac{n^2-a}{b}} \chi(n) = 0.$$

For positive integers s and N, consider the partial sum

$$F(q;N) := \sum_{n=0}^{N} f_n(q)(q;q)_n \tag{1.11}$$

and its s-dissection

$$F(q;N) = \sum_{i=0}^{s-1} q^i A_{F,s}(N,i,q^s).$$

Define $S_{a,b,\chi}(s) \subseteq \{0,1,\ldots,s-1\}$ by

$$S_{a,b,\chi}(s) := \left\{ \frac{n^2-a}{b} \pmod{s} \colon \chi(n) \neq 0 \right\}.$$

Our first main result is the following.

Theorem 1.2. *Suppose that F is a function as in (1.6) and that $P_{a,b,\chi}^{(\nu)}$ is a function as in (1.8). Suppose that for each root of unity ζ we have the asymptotic expansion*

$$P_{a,b,\chi}^{(\nu)}(\zeta e^{-t}) \sim F(\zeta e^{-t}) \quad \text{as} \quad t \to 0^+. \tag{1.12}$$

Suppose that s and N are positive integers and that $i \notin S_{a,b,\chi}(s)$. Then we have

$$(q;q)_{\lambda(N,s)} \mid A_{F,s}(N,i,q).$$

Analogously, for positive integers s and N with s odd, consider the partial sum

$$G(q;N) := \sum_{n=0}^{N} g_n(q)(q;q^2)_n \tag{1.13}$$

and its s-dissection

$$G(q;N) = \sum_{i=0}^{s-1} q^i A_{G,s}(N,i,q^s).$$

Then the $A_{G,s}(N,i,q^s)$ also enjoy strong divisibility properties. Define

$$\mu(N,k,s) = \left\lfloor \frac{N}{s(2k-1)} + \frac{1}{2} \right\rfloor. \tag{1.14}$$

Theorem 1.3. *Suppose that G is a function as in (1.7) and that $P_{a,b,\chi}^{(\nu)}$ is a function as in (1.8). Suppose that for each root of unity ζ of odd order we have*

$$P_{a,b,\chi}^{(\nu)}(\zeta e^{-t}) \sim G(\zeta e^{-t}) \quad \text{as } t \to 0^+. \tag{1.15}$$

Suppose that s and N are positive integers with s odd and that $i \notin S_{a,b,\chi}(s)$. Then we have

$$(q;q^2)_{\mu(N,1,s)} \mid A_{G,s}(N,i,q).$$

While a generic q-series of the form (1.6) or (1.7) is not expected to be related to a partial theta function as in (1.12) or (1.15), there are a number of examples where this is the case. For example, Hikami [9] introduced a family of quantum modular forms related to torus knots, which we will discuss in Sect. 5. For now, we illustrate Theorem 1.3 with an example from Ramanujan's lost notebook. Consider the q-series

$$\mathcal{G}(q) = \sum_{n\geq 0}(q;q^2)_n q^n.$$

From [3, Entry 9.5.2], we have the identity

$$\sum_{n\geq 0}(q;q^2)_n q^n = \sum_{n\geq 0}(-1)^n q^{3n^2+2n}(1+q^{2n+1}),$$

which may be written as

$$\sum_{n\geq 0}(q;q^2)_n q^n = \sum_{n\geq 0}\chi_6(n) q^{(n^2-1)/3},$$

where

$$\chi_6(n) := \begin{cases} 1, & \text{if } n \equiv 1,2 \pmod 6, \\ -1, & \text{if } n \equiv 4,5 \pmod 6, \\ 0, & \text{otherwise.} \end{cases}$$

Therefore, for each odd-order root of unity ζ we find that

$$P^{(0)}_{1,3,\chi_6}(\zeta e^{-t}) \sim \mathcal{G}(\zeta e^{-t}) \qquad \text{as } t \to 0^+.$$

Since χ_6 is odd, it satisfies conditions (1.9) and (1.10). Thus, from Theorem 1.3, we find that for $i \notin S_{1,3,\chi_6}(s)$ we have

$$(q;q^2)_{\lfloor \frac{N}{s}+\frac{1}{2}\rfloor} \mid A_{\mathcal{G},s}(N,i,q). \tag{1.16}$$

For example, when $s = 5$ we have $S_{1,3,\chi_6}(5) = \{0,1,3\}$. For $N = 8$ we have

$$A_{\mathcal{G},5}(8,2,q) = q^2(q;q^2)_2(1+q^2-q^3+2q^4-q^5+2q^6+q^8)$$

and

$$A_{\mathcal{G},5}(8,4,q) = -q(q;q^2)_2(1-q+q^2)(1+q+q^2+q^4+q^6),$$

as predicted by (1.16), while the factorizations of $A_{\mathcal{G},5}(8,i,q)$ into irreducible factors for $i \in \{0,1,3\}$ are

$$\begin{aligned} A_{\mathcal{G},5}(8,0,q) &= (1-q)(1+q^4-2q^5+\cdots-2q^{11}+q^{12}), \\ A_{\mathcal{G},5}(8,1,q) &= 1+2q^3-q^4+\cdots+q^{13}-q^{14}, \\ A_{\mathcal{G},5}(8,3,q) &= q(-1+q^2-2q^3+2q^4-\cdots-2q^{11}+q^{12}). \end{aligned}$$

The rest of the paper is organized as follows. In the next section, we discuss power series expansions of F and G at roots of unity, and in Sect. 3, we discuss the asymptotic expansions of partial theta functions. In Sect. 4, we prove the main theorems. In Sect. 5, we give two further examples—one generalizing (1.5) and one generalizing (1.16). We close with some remarks on congruences for the coefficients of $F(1-q)$ and $G(1-q)$.

2. Power Series Expansions of F and G

Let $F(q)$ be a function as in (1.6) and $G(q)$ be a function as in (1.7). Here we collect some facts which allow us to meaningfully define $F(\zeta e^{-t})$ and $G(\zeta e^{-t})$ as formal power series.

Lemma 2.1. *Let $F(q;N)$ be as in (1.11), and let $G(q;N)$ be as in (1.13). Suppose that ζ is a kth root of unity.*

1. *The values $\left(q\frac{d}{dq}\right)^{\ell} F(q;N)\big|_{q=\zeta}$ are stable for $N \geq (\ell+1)k-1$.*
2. *If k is odd then the values $\left(q\frac{d}{dq}\right)^{\ell} G(q;N)\big|_{q=\zeta}$ are stable for $2N \geq (2\ell+1)k$.*

Proof. For each positive integer k, we have

$$(1-q^k)^{\ell+1} \mid (q;q)_N \quad \text{for} \quad N \geq (\ell+1)k,$$
$$(1-q^{2k-1})^{\ell+1} \mid (q;q^2)_N \quad \text{for} \quad 2N \geq (2\ell+1)(2k-1)+1.$$

It follows that for $0 \leq j \leq \ell$ we have

$$\left(\frac{\mathrm{d}}{\mathrm{d}q}\right)^j (q;q)_N\big|_{q=\zeta} = 0 \quad \text{for} \quad N \geq (\ell+1)k,$$
$$\left(\frac{\mathrm{d}}{\mathrm{d}q}\right)^j (q;q^2)_N\big|_{q=\zeta} = 0 \quad \text{for odd } k \text{ and} \quad 2N \geq (2\ell+1)k+1.$$

The lemma follows since for any polynomial $f(q)$, the polynomial $\left(q\frac{\mathrm{d}}{\mathrm{d}q}\right)^{\ell} f(q)$ is a linear combination (with polynomial coefficients) of $\left(\frac{q}{\mathrm{d}q}\right)^j f(q)$ with $0 \leq j \leq \ell$ (see for example [4, Lemma 2.2]). ☐

For any polynomial $f(q)$, any ζ and any $\ell \geq 0$, we have [4, Lemma 2.3]

$$\left(\frac{\mathrm{d}}{\mathrm{d}t}\right)^{\ell} f(\zeta e^{-t})\big|_{t=0} = (-1)^{\ell}\left(q\frac{\mathrm{d}}{\mathrm{d}q}\right)^{\ell} f(q)\big|_{q=\zeta}. \tag{2.1}$$

Let $F(q)$ be as in (1.6) and let ζ be a kth root of unity. The last fact together with Lemma 2.1 allows us to define

$$\left(\frac{\mathrm{d}}{\mathrm{d}t}\right)^{\ell} F(\zeta e^{-t})\big|_{t=0} := \left(\frac{\mathrm{d}}{\mathrm{d}t}\right)^{\ell} F(\zeta e^{-t};N)\big|_{t=0} \quad \text{for any } N \geq k(\ell+1)-1.$$

We, therefore, have a formal power series expansion

$$F(\zeta e^{-t}) = \sum_{\ell=0}^{\infty} \frac{\left(\frac{\mathrm{d}}{\mathrm{d}t}\right)^{\ell} F(\zeta e^{-t})\big|_{t=0}}{\ell!}\, t^{\ell}. \tag{2.2}$$

Similarly, if $G(q)$ is a function as in (1.7) and ζ is a kth root of unity with odd k, then we can define

$$\left(\frac{\mathrm{d}}{\mathrm{d}t}\right)^{\ell} G(\zeta e^{-t})\big|_{t=0} := \left(\frac{\mathrm{d}}{\mathrm{d}t}\right)^{\ell} G(\zeta e^{-t}; N)\big|_{t=0} \quad \text{for any } 2N \geq k(2\ell+1), \tag{2.3}$$

using (2.1) and Lemma 2.1. Thus, we have a formal power series expansion

$$G(\zeta e^{-t}) = \sum_{\ell=0}^{\infty} \frac{\left(\frac{\mathrm{d}}{\mathrm{d}t}\right)^{\ell} G(\zeta e^{-t})\big|_{t=0}}{\ell!}\, t^{\ell}. \tag{2.4}$$

3. The Asymptotics of $P_{a,b,\chi}^{(\nu)}$

In this section, we discuss the asymptotic expansion of the partial theta functions $P_{a,b,\chi}^{(\nu)}(q)$ defined in (1.8). Recall that

$$P_{a,b,\chi}^{(\nu)}(q) := \sum_{n\geq 0} n^{\nu}\chi(n) q^{\frac{n^2-a}{b}},$$

where $\nu \in \{0,1\}$, $a \geq 0$ and $b > 0$ are integers, and $\chi\colon \mathbb{Z} \to \mathbb{C}$ is a function satisfying properties (1.9) and (1.10).

The properties which we describe in the next proposition are more or less standard (see, for example, [10, p. 98]). For convenience and completeness we sketch a proof of the following:

Proposition 3.1. *Suppose that $P_{a,b,\chi}^{(\nu)}(q)$ is as in (1.8). Let ζ be a root of unity and let N be a period of the function $n \mapsto \zeta^{\frac{n^2-a}{b}}\chi(n)$. Then we have the asymptotic expansion*

$$P_{a,b,\chi}^{(\nu)}(\zeta e^{-t}) \sim \sum_{n=0}^{\infty} \gamma_n(\zeta) t^n, \quad t \to 0^+,$$

where

$$\gamma_n(\zeta) = \sum_{\substack{1\leq m\leq N\\ \chi(m)\neq 0}} a(m,n,N)\zeta^{\frac{m^2-a}{b}} \tag{3.1}$$

with certain complex numbers $a(m,n,N)$.

We begin with a lemma. For $n \geq 0$, let $B_n(x)$ denote the nth Bernoulli polynomial. In the rest of this section, we use s for a complex variable since there can be no confusion with the parameter s used above.

Lemma 3.2. *Let $C\colon \mathbb{Z} \to \mathbb{C}$ be a function with period N and mean value zero, and let*

$$L(s,C) := \sum_{n=1}^{\infty} \frac{C(n)}{n^s}, \quad \mathrm{Re}(s) > 0.$$

Then $L(s,C)$ has an analytic continuation to $\mathbb{C}$, and we have

$$L(-n,C)=\frac{-N^n}{n+1}\sum_{m=1}^{N}C(m)B_{n+1}\left(\frac{m}{N}\right)\qquad \textit{for } n\geq 0. \tag{3.2}$$

Proof. Let $\zeta(s,\alpha)$ denote the Hurwitz zeta function, whose properties are described, for example, in [5, Chapter 12]. We have

$$L(s,C)=N^{-s}\sum_{m=1}^{N}C(m)\zeta\left(s,\tfrac{m}{N}\right). \tag{3.3}$$

The lemma follows using the fact that each Hurwitz zeta function has only a simple pole with residue 1 at $s=1$ and the formula for the value of each function at $s=-n$ [5, Theorem 12.13]. □

Proof of Proposition 3.1. It is enough to prove the proposition for the function

$$f(t):=e^{-\frac{at}{b}}P^{(\nu)}_{a,b,\chi}(\zeta e^{-t})=\sum_{n\geq 1}n^{\nu}\chi(n)\zeta^{\frac{n^2-a}{b}}e^{-\frac{n^2t}{b}},\quad t>0.$$

Setting

$$C(n):=\zeta^{\frac{n^2-a}{b}}\chi(n), \tag{3.4}$$

we have the Mellin transform

$$\int_0^{\infty}f(t)t^{s-1}\,\mathrm{d}t=b^s\Gamma(s)L(2s-\nu,C),\quad \mathrm{Re}(s)>\frac{1}{2}.$$

Inverting, we find that

$$f(t)=\frac{1}{2\pi i}\int_{x=c}b^s\Gamma(s)L(2s-\nu,C)t^{-s}\,\mathrm{d}s,$$

for $c>\frac{1}{2}$, where we write $s=x+iy$. Using (3.3), the functional equation for the Hurwitz zeta functions, and the asymptotics of the Gamma function, we find that, for fixed x, the function $L(s,C)$ has at most polynomial growth in $|y|$ as $|y|\to\infty$. Shifting the contour to the line $x=-R-\frac{1}{2}$, we find that for each $R\geq 0$ we have

$$f(t)=\sum_{n=0}^{R}\frac{(-1)^n}{b^n n!}L(-2n-\nu,C)t^n+O\left(t^{R+\frac{1}{2}}\right),$$

from which

$$f(t)\sim\sum_{n=0}^{\infty}\frac{(-1)^n}{b^n n!}L(-2n-\nu,C)t^n.$$

The proposition follows from (3.4) and (3.2). □

4. Proofs of Theorems 1.2 and 1.3

We begin with a lemma. The first assertion is proved in [4, Lemma 2.4], and the second, which is basically equation (2.4) in [1], follows by extracting an arithmetic progression using orthogonality. (We note that there is an error in the published version of [1] which is corrected below; in that version the operators $\frac{\mathrm{d}}{\mathrm{d}q}$ and $q\frac{\mathrm{d}}{\mathrm{d}q}$ are conflated in the statements of (2.3) and (2.4). This does not affect the truth of the rest of the results.)

Let $C_{\ell,i,j}(s)$ be the array of integers defined recursively as follows:

1. $C_{0,0,0}(s) = 1$,
2. $C_{\ell,i,0}(s) = i^{\ell}$ and $C_{\ell,i,j}(s) = 0$ for $j \geq \ell + 1$ or $j < 0$,
3. $C_{\ell+1,i,j}(s) = (i + js)C_{\ell,i,j}(s) + sC_{\ell,i,j-1}(s)$ for $1 \leq j \leq \ell$.

Lemma 4.1. *Suppose that s is a positive integer and that*

$$h(q) = \sum_{i=0}^{s-1} q^i A_s(i, q^s)$$

with polynomials $A_s(i, q)$. Then the following are true:

1. *For all $\ell \geq 0$, we have*

$$\left(q\frac{d}{dq}\right)^{\ell} h(q) = \sum_{j=0}^{\ell}\sum_{i=0}^{s-1} C_{\ell,i,j}(s)q^{i+js}A_s^{(j)}(i, q^s).$$

2. *Let ζ_s be a primitive sth root of unity. Then for $\ell \geq 0$ and $i_0 \in \{0, \dots, s-1\}$, we have*

$$\sum_{j=0}^{\ell} C_{\ell,i_0,j}(s)q^{i_0+js}A_s^{(j)}(i_0, q^s) = \frac{1}{s}\sum_{k=0}^{s-1} \zeta_s^{-ki_0}\left(\left(q\frac{d}{dq}\right)^{\ell} h(q)\right)\Big|_{q\to\zeta_s^k q}. \quad (4.1)$$

Proof of Theorem 1.2. Suppose that $F(q)$ and $P_{a,b,\chi}(q)$ are as in the statement of the theorem. Suppose that s and k are positive integers, that $i \notin S_{a,b,\chi}(s)$ and that ζ_k is a primitive kth root of unity. Let $\Phi_k(q)$ be the kth cyclotomic polynomial. Recall the definition (1.4) of $\lambda(N, s)$ and note that since

$$(q; q)_n = \pm\prod_{k=1}^{n} \Phi_k(q)^{\lfloor \frac{n}{k} \rfloor} \quad (4.2)$$

and

$$\left\lfloor \frac{\lfloor \frac{x}{s} \rfloor}{k} \right\rfloor = \left\lfloor \frac{x}{ks} \right\rfloor,$$

we have

$$(q; q)_{\lambda(N,s)} = \pm\prod_{k=1}^{\lambda(N,s)} \Phi_k(q)^{\lambda(N,ks)}.$$

Therefore, Theorem 1.2 will follow once we show for each $\ell \geq 0$ that

$$A_{F,s}^{(\ell)}(N, i, \zeta_k) = 0 \quad \text{for} \quad N \geq (\ell + 1)ks - 1$$

since this implies that $\Phi_k(q)^{\lambda(N,ks)} \mid A_{F,s}(N,i,q)$ for $1 \le k \le \lambda(N,s)$.

From the definition we find that

$$A_{F,s}(N,i,q) = \sum_{j=0}^{k-1} q^j A_{F,ks}(N, i+js, q^k).$$

If $i \notin S_{a,b,\chi}(s)$, then $i + js \notin S_{a,b,\chi}(ks)$. It is, therefore, enough to show that for all s, k, and ℓ, and for $i \notin S_{a,b,\chi}(ks)$, we have

$$A^{(\ell)}_{F,ks}(N,i,1) = 0 \quad \text{for } N \ge (\ell+1)ks - 1.$$

After replacing ks by s, it is enough to show that for all s and ℓ, and for $i \notin S_{a,b,\chi}(s)$, we have

$$A^{(\ell)}_{F,s}(N,i,1) = 0 \quad \text{for } N \ge (\ell+1)s - 1. \tag{4.3}$$

We prove (4.3) by induction on ℓ. For the base case $\ell = 0$, assume that $N \ge s-1$. Using (4.1) with $q = 1$ gives

$$A_{F,s}(N,i,1) = \frac{1}{s}\sum_{j=0}^{s-1} \zeta_s^{-ji} F(\zeta_s^j; N).$$

By (1.12), (2.1), Lemma 2.1, and Proposition 3.1, we find that

$$A_{F,s}(N,i,1) = \frac{1}{s}\sum_{j=1}^{s} \zeta_s^{-ji} \gamma_0(\zeta_s^j).$$

By (3.1) and orthogonality (recalling that $i \notin S_{a,b,\chi}(s)$), we find that

$$A_{F,s}(N,i,1) = 0.$$

For the induction step, suppose that $N \ge (\ell+1)s - 1$, that $i \notin S_{a,b,\chi}(s)$, and that (4.3) holds with ℓ replaced by j for $1 \le j \le \ell - 1$. By (4.1) and the induction hypothesis we have

$$C_{\ell,i,\ell}(s) A^{(\ell)}_{F,s}(N,i,1) = \frac{1}{s}\sum_{j=1}^{s} \zeta_s^{-ji} \left(q\frac{\mathrm{d}}{\mathrm{d}q}\right)^{\ell} F(q;N)\big|_{q=\zeta_s^j}.$$

Using Proposition 3.1, (2.2), (3.1), and orthogonality, we find as above that

$$C_{\ell,i,\ell}(t) A^{(\ell)}_{F,s}(N,i,1) = 0.$$

This establishes (4.3) since $C_{\ell,i,\ell}(s) > 0$. Theorem 1.2 follows. □

Proof of Theorem 1.3. Suppose that s and k are positive integers with s odd, that $i \notin S_{a,b,\chi}(s)$ and that ζ_{2k-1} is a $(2k-1)$th root of unity. Recall the definition (1.14) of $\mu(N,k,s)$. In analogy with (4.2), we have

$$(q;q^2)_n = \pm \prod_{k=1}^{n} \Phi_{2k-1}(q)^{\lfloor \frac{(2n-1)}{2(2k-1)} + \frac{1}{2} \rfloor},$$

and as above we obtain

$$(q;q^2)_{\mu(N,1,s)} = \pm \prod_{k=1}^{\mu(N,1,s)} \Phi_{2k-1}(q)^{\mu(N,k,s)}.$$

Therefore, Theorem 1.3 follows once we show for each $\ell \geq 0$ that

$$A^{(\ell)}_{G,s}(N, i, \zeta_{2k-1}) = 0 \quad \text{for} \quad 2N \geq (2\ell+1)(2k-1)s.$$

The rest of the proof is similar to that of Theorem 1.2 (we require s to be odd because $G(q)$ has a series expansion only at odd-order roots of unity). Arguing as above, we show that for each odd s we have

$$A^{(\ell)}_{G,s}(N, i, 1) = 0 \quad \text{for} \quad 2N \geq (2\ell+1)s,$$

and the result follows. □

5. Examples

In this section, we illustrate Theorems 1.2 and 1.3 with two families of examples.

5.1. The Generalized Kontsevich–Zagier Functions

In a study of quantum modular forms related to torus knots and the Andrews–Gordon identities, Hikami [9] defined the functions

$$\begin{aligned} X_m^{(\alpha)}(q) := \sum_{k_1,k_2,\ldots,k_m \geq 0} & (q;q)_{k_m} q^{k_1^2+\cdots+k_{m-1}^2+k_{\alpha+1}+\cdots+k_{m-1}} \\ & \times \left(\prod_{\substack{i=1 \\ i\neq\alpha}}^{m-1} \begin{bmatrix} k_{i+1} \\ k_i \end{bmatrix} \right) \begin{bmatrix} k_{\alpha+1}+1 \\ k_\alpha \end{bmatrix}, \end{aligned} \tag{5.1}$$

where m is a positive integer and $\alpha \in \{0, 1, \ldots, m-1\}$. Here we have used the usual q-binomial coefficient (or Gaussian polynomial)

$$\begin{bmatrix} n \\ k \end{bmatrix} := \begin{bmatrix} n \\ k \end{bmatrix}_q := \begin{cases} \frac{(q;q)_n}{(q;q)_k (q;q)_{n-k}}, & \text{if } 0 \leq k \leq n, \\ 0, & \text{otherwise.} \end{cases}$$

The simplest example

$$X_1^{(0)}(q) = \sum_{n \geq 0} (q;q)_n$$

is the Kontsevich–Zagier function. From (5.1) we can write

$$X_m^{(\alpha)}(q) = \sum_{k_m \geq 0} (q;q)_{k_m} f^{(\alpha)}_{k_m}(q),$$

with polynomials $f^{(\alpha)}_{k_m}(q)$.

Hikami's identity [9, eq. (70)] implies that for each root of unity ζ, we have

$$P^{(1)}_{(2m-2\alpha-1)^2, 8(2m+1), \chi^{(\alpha)}_{8m+4}}(\zeta e^{-t}) \sim X_m^{(\alpha)}(\zeta e^{-t})$$

as $t \to 0^+$, where $\chi_{8m+4}^{(\alpha)}(n)$ is defined by

$$\chi_{8m+4}^{(\alpha)}(n) = \begin{cases} -1/2, & \text{if } \ n \equiv 2m-2\alpha-1 \text{ or } 6m+2\alpha+5 \pmod{8m+4}, \\ 1/2, & \text{if } \ n \equiv 2m+2\alpha+3 \text{ or } 6m-2\alpha+1 \pmod{8m+4}, \\ 0, & \text{otherwise.} \end{cases} \tag{5.2}$$

The function $\chi_{8m+4}^{(\alpha)}(n)$ satisfies condition (1.9). For (1.10) we record a short lemma.

Lemma 5.1. *Suppose that $\chi_{8m+4}^{(\alpha)}(n)$ is as defined in (5.2) and that ζ is a root of unity of order M. Define*

$$\psi(n) = \zeta^{\frac{n^2-(2m-2\alpha-1)^2}{8(2m+1)}} \chi_{8m+4}^{(\alpha)}(n).$$

Then

$$\sum_{n=1}^{M(8m+4)} \psi(n) = 0.$$

Proof. Note that ψ is supported on odd integers, so we assume in what follows that n is odd. From the definition, we have

$$\chi_{8m+4}^{(\alpha)}(n + M(4m+2)) = (-1)^M \chi_{8m+4}^{(\alpha)}(n). \tag{5.3}$$

The exponent in the ratio of the corresponding powers of ζ is $mM^2 + \frac{M^2+Mn}{2}$. So the ratio of these powers of ζ is

$$\zeta^{\frac{M^2+Mn}{2}}.$$

If M is odd then this becomes $\zeta^{M\left(\frac{M+n}{2}\right)} = 1$, while if M is even then this becomes $\zeta^{\frac{M^2}{2}}\zeta^{\frac{M}{2}n} = -1$ (since M is the order of ζ and n is odd). Therefore, the ratio in either case is $(-1)^{M+1}$. Combining this with (5.3) gives

$$\psi(n + M(4m+2)) = -\psi(n),$$

from which the lemma follows. $\square$

Therefore, $X_m^{(\alpha)}(q)$ satisfies the conditions of Theorem 1.2, and we obtain the following.

Corollary 5.2. *If s is a positive integer and $i \notin S_{(2m-2\alpha-1)^2, 8(2m+1), \chi_{8m+4}^{(\alpha)}}(s)$, then*

$$(q;q)_{\lambda(N,s)} \big| A_{X_m^{(\alpha)}, s}(N, i, q),$$

where $A_{X_m^{(\alpha)}, s}(N, i, q)$ are the coefficients in the s-dissection of the partial sums (in k_m) of $X_m^{(\alpha)}(q)$.

For example, when $s = 3$ we have $S_{9,40,\chi_{20}^{(0)}}(3) = \{0, 1\}$ and $S_{1,40,\chi_{20}^{(1)}}(3) = \{0, 2\}$. For $N = 8$ we have

$$A_{X_2^{(0)},3}(8, 2, q) = (q;q)_3(1+q)(1+q+q^2)(1-q+\cdots-q^{25}+q^{26})$$

and

$$A_{X_2^{(1)},3}(8,1,q) = (q;q)_3(1+q)(1-q+q^2) \\ \times (1+q+q^2)(1+2q+\cdots-q^{26}+q^{27}),$$

as predicted by Corollary 5.2, while

$$A_{X_2^{(0)},3}(8,0,q) = (1-q+q^2)(9+9q+\cdots+q^{33}+q^{34}),$$
$$A_{X_2^{(0)},3}(8,1,q) = -8-7q+\cdots+q^{34}-q^{35},$$
$$A_{X_2^{(1)},3}(8,0,q) = 9-7q+\cdots+2q^{36}+q^{39},$$

and

$$A_{X_2^{(1)},3}(8,2,q) = -7+3q^3-\cdots+q^{36}-q^{38}$$

are not divisible by $(q;q)_3$.

5.2. An Example with $\nu = 0$

For $k \geq 1$, let $\mathcal{G}_k(q)$ denote the q-series

$$\mathcal{G}_k(q) = \sum_{n_k \geq n_{k-1} \geq \cdots \geq n_1 \geq 0} q^{n_k+2n_{k-1}^2+2n_{k-1}+\cdots+2n_1^2+2n_1} \\ \times (q;q^2)_{n_k} \begin{bmatrix} n_k \\ n_{k-1} \end{bmatrix}_{q^2} \cdots \begin{bmatrix} n_2 \\ n_1 \end{bmatrix}_{q^2}.$$

Then we have the identity

$$\mathcal{G}_k(q) = \sum_{n \geq 0} (-1)^n q^{(2k+1)n^2+2kn}(1+q^{2n+1}), \tag{5.4}$$

which follows from Andrews' generalization [2] of the Watson–Whipple transformation

$$\sum_{m=0}^{N} \frac{(1-aq^{2m})}{(1-a)} \frac{(a,b_1,c_1,\ldots,b_k,c_k,q^{-N})_m}{(q,aq/b_1,aq/c_1,\ldots,aq/b_k,aq/c_k,aq^{N+1})_m} \left(\frac{a^k q^{k+N}}{b_1c_1\cdots b_kc_k}\right)^m$$
$$= \frac{(aq,aq/b_kc_k)_N}{(aq/b_k,aq/c_k)_N} \sum_{N \geq n_{k-1} \geq \cdots \geq n_1 \geq 0} \frac{(b_k,c_k)_{n_{k-1}} \cdots (b_2,c_2)_{n_1}}{(q;q)_{n_{k-1}-n_{k-2}} \cdots (q;q)_{n_2-n_1}(q;q)_{n_1}}$$
$$\times \frac{(aq/b_{k-1}c_{k-1})_{n_{k-1}-n_{k-2}} \cdots (aq/b_2c_2)_{n_2-n_1}(aq/b_1c_1)_{n_1}}{(aq/b_{k-1},aq/c_{k-1})_{n_{k-1}} \cdots (aq/b_1,aq/c_1)_{n_1}}$$
$$\times \frac{(q^{-N})_{n_{k-1}}(aq)^{n_{k-2}+\cdots+n_1}q^{n_{k-1}}}{(b_kc_kq^{-N}/a)_{n_{k-1}}(b_{k-1}c_{k-1})^{n_{k-2}}\cdots(b_2c_2)^{n_1}}.$$

Here we have extended the notation in (1.1) to

$$(a_1,a_2,\ldots,a_k)_n := (a_1;q)_n(a_2;q)_n\cdots(a_k;q)_n.$$

To deduce (5.4), we set $q = q^2, a = q^2, b_k = q$, and $c_k = q^2$ and then let $N \to \infty$ along with all other b_i, c_i.

The identity (5.4) may be written as

$$\mathcal{G}_k(q) = \sum_{n\geq 0} \chi_{4k+2}(n) q^{\frac{n^2-k^2}{2k+1}},$$

where

$$\chi_{4k+2}(n) := \begin{cases} 1, & \text{if } n \equiv k, k+1 \pmod{4k+2}, \\ -1, & \text{if } n \equiv -k, -k-1 \pmod{4k+2}, \\ 0, & \text{otherwise.} \end{cases}$$

This implies that for each odd-order root of unity ζ, we have

$$P^{(0)}_{k^2,2k+1,\chi_{4k+2}}(\zeta e^{-t}) \sim G_k(\zeta e^{-t}) \qquad \text{as } t \to 0^+.$$

The function $\chi_{4k+2}(n)$ satisfies conditions (1.9) and (1.10) (see the remark following (1.10)), so Theorem 1.3 gives

Corollary 5.3. *Suppose that k and N are positive integers, that s is a positive odd integer, and that $i \notin S_{k^2,2k+1,\chi_{4k+2}}(s)$. then*

$$(q;q^2)_{\lfloor \frac{N}{s}+\frac{1}{2}\rfloor} \mid A_{\mathcal{G}_k,s}(N,i,q).$$

6. Remarks on Congruences

Congruences for the coefficients of the functions $F(q)$ and $G(q)$ in Theorems 1.2 and 1.3 can be deduced from the results of [7]. In closing we mention another approach. Theorems 1.2 and 1.3 guarantee that many of the coefficients in the s-dissection are divisible by high powers of $1-q$, and the congruences follow from this fact when $s = p^r$ together with an argument as in [1, Section 3].

For example, let $\mathcal{G}_k$ be the function defined in the last section and define $\xi_{\mathcal{G}_k}(n)$ by

$$\mathcal{G}_k(1-q) = \sum_{n\geq 0} \xi_{\mathcal{G}_k}(n) q^n.$$

Consider the expansions

$$\mathcal{G}_1(1-q) = \sum_{n\geq 0} \xi_{\mathcal{G}_1}(n) q^n = 1 + q + 2q^2 + 6q^3 + 25q^4 + 135q^5 + \cdots,$$

$$\mathcal{G}_2(1-q) = \sum_{n\geq 0} \xi_{\mathcal{G}_2}(n) q^n = 1 + 2q + 6q^2 + 28q^3 + 189q^4 + 1680q^5 + \cdots.$$

Then we have such congruences as

$$\xi_{\mathcal{G}_1}(5^r n - 1) \equiv 0 \pmod{5^r},$$
$$\xi_{\mathcal{G}_1}(7^r n - 1) \equiv 0 \pmod{7^r},$$
$$\xi_{\mathcal{G}_1}(13^r n - \beta) \equiv 0 \pmod{13^r}$$

for $\beta \in \{1, 2, 3, 4\}$, and

$$\xi_{\mathcal{G}_2}(7^r n - 1) \equiv 0 \pmod{7^r},$$
$$\xi_{\mathcal{G}_2}(11^r n - 1) \equiv 0 \pmod{11^r}.$$

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Scott Ahlgren
Department of Mathematics
University of Illinois
Urbana IL61801
USA
e-mail: sahlgren@illinois.edu

Byungchan Kim
School of Liberal Arts
Seoul National University of Science and Technology
232 Gongneung-ro
Nowongu Seoul01811
Republic of Korea
e-mail: bkim4@seoultech.ac.kr

Present Address
Jeremy Lovejoy
Department of Mathematics
University of California, Berkeley
970 Evans Hall #3780
Berkeley CA94720-3840
USA
e-mail: lovejoy@math.cnrs.fr

Jeremy Lovejoy
CNRS, Université Denis Diderot-Paris 7
Case 7014
75205 Paris Cedex 13
France

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Dyson's "Favorite" Identity and Chebyshev Polynomials of the Third and Fourth Kind

George E. Andrews

Abstract. The combinatorial and analytic properties of Dyson's "favorite" identity are studied in detail. In particular, a q-series analog of the anti-telescoping method is used to provide a new proof of Dyson's results with mock theta functions popping up in intermediate steps. This leads to the appearance of Chebyshev polynomials of the third and fourth kind in Bailey pairs related to Bailey's Lemma. The natural relationship with L.J. Rogers's pioneering work is also presented.

Mathematics Subject Classification. 05A17, 05A19, 11P83.

Keywords. Partitions, Dyson's favorite identity, Bailey pairs, Bailey's lemma, Partitions, Chebyshev polynomials, Mock theta functions.

1. Introduction

Freeman Dyson, in his article, A Walk Through Ramanujan's Garden [11], describes how his study of Rogers–Ramanujan type identities helped to preserve his sanity during the dark days of World War II. Among the results he discovered was his favorite:

$$\sum_{n=0}^{\infty} \frac{q^{n^2+n} \prod_{j=1}^{n} (1+q^j+q^{2j})}{(1-q)(1-q^2)\cdots(1-q^{2n+1})} = \prod_{n=1}^{\infty} \frac{(1-q^{9n})}{(1-q^n)}. \tag{1.1}$$

Dyson's proof of (1.1) [10, pp. 8–9] and the proof subsequently provided by Slater [17, p. 161, eq. (92)] are based on what has become known as Bailey's Lemma [10, p. 3, eq. (3.1)].

We shall begin in Sect. 2 by providing a proof of (1.1) and three related identities via q-difference equations. This will necessitate a q-series analog of anti-telescoping [6] with several new intermediate q-series arising.

With an eye to understanding these new intermediate functions, we devote Sect. 3 to connections between $V_n(x)$ and $W_n(x)$ (the Chebyshev polynomials of the third and fourth kind, respectively) and surprising Bailey pairs. For example, in Sect. 4, we prove:

$$\sum_{n\geq 0} \frac{q^{n^2+n}\prod_{j=1}^{n}(1+2xq^j+q^{2j})}{(1-q)(1-q^2)\cdots(1-q^{2n+1})} = \frac{1}{\prod_{n=1}^{\infty}(1-q^n)} \sum_{n\geq 0} q^{3\binom{n+1}{2}} V_n(x),$$

where $V_n(x)$ is the Chebyshev polynomial of the third kind.

Section 4 returns to (1.1) itself to reveal how many other Rogers–Ramanujan type identities are merely specializations of the natural generalization of (1.1).

Section 5 uses the work of Sect. 3 to establish that many of the new functions are, in fact, mock theta functions or closely related.

Sections 6 and 7 are devoted to ninth-order mock theta functions and their generalizations containing Chebyshev polynomials of the third kind. For example:

$$\sum_{n\geq 1} \frac{q^{n^2}\prod_{j=1}^{n-1}(1+2xq^j+q^{2j})}{(1-q)(1-q^2)\cdots(1-q^{2n-1})}$$
$$= \prod_{n=1}^{\infty} \frac{1}{1-q^n} \sum_{m\geq 1} q^{2m^2-m}(1-q^{2m}) \sum_{j=0}^{m-1} V_j(x) q^{-j(j+1)/2}.$$

To round out a full treatment of (1.1), we provide a natural interpretation of (1.1) related to sequences in partitions [8] in Sect. 8.

Section 9 considers a companion to (1.1) arising from the quintuple product identity, and Sect. 10 considers open questions.

Although he did not use the terminology of Chebyshev polynomials, L.J. Rogers [15] tacitly used them in his combinations of Fourier series. We shall describe this relationship in Sect. 10.

I would like to thank the referee who carefully studied the original version and made numerous helpful suggestions and corrections.

2. A New Proof of Dyson's Favorite Identity

The identities to be proved are the following:

$$D_{4,4}(a;q) := \sum_{n\geq 0} \frac{a^n q^{n^2}(aq^{2n};q)_\infty}{(q;q)_n(aq^{3n};q^3)_\infty} = Q_{4,4}(a;q^3),$$

$$D_{4,3}(a;q) := \sum_{n\geq 0} \frac{a^n q^{n^2+n}(aq^{2n+2};q)_\infty}{(q;q)_n(aq^{3n+3};q^3)_\infty} = Q_{4,3}(a;q^3),$$

$$D_{4,2}(a;q) := \sum_{n\geq 0} \frac{a^n q^{n^2+2n}(aq^{2n+3};q)_\infty}{(q;q)_n(aq^{3n+3};q^3)_\infty} = Q_{4,2}(a;q^3),$$

$$D_{4,1}(a;q) := \sum_{n\geq 0} \frac{a^n q^{n^2+3n}(aq^{2n+3};q)_\infty}{(q;q)_n(aq^{3n+3};q^3)_\infty} = Q_{4,1}(a;q^3),$$

where

$$Q_{k,i}(a;q) = \sum_{n\geq 0} \frac{(-1)^n a^{kn} q^{\frac{1}{2}(k+1)n(n+1)-in}(1-a^i q^{(2n+1)i})}{(q;q)_n (aq^{n+1};q)_\infty}$$

with

$$(A;q)_N = (1-A)(1-Aq)\cdots(1-Aq^{N-1})$$

and

$$(A;q)_\infty = \prod_{n=0}^{\infty}(1-Aq^n).$$

It has been proved in [1] that the $Q_{k,i}(a;q)$ as doubly analytic functions in a and q are uniquely determined by the following initial conditions and q-difference equations:

$$Q_{k,0}(a;q) = 0, \tag{2.1}$$

$$Q_{k,i}(0;q) = Q_{k,i}(a;0) = 1 \quad \text{for } 1 \leq i \leq k, \tag{2.2}$$

and for $1 \leq i \leq k$,

$$Q_{k,i}(a;q) - Q_{k,i-1}(a;q) = (aq)^{i-1} Q_{k,k-i+1}(aq;q). \tag{2.3}$$

Theorem 2.1. *For $1 \leq i \leq 4$:*

$$D_{4,i}(a;q) = Q_{4,i}(a;q^3). \tag{2.4}$$

Proof. In light of the comments proceeding (2.1), the proof of the theorem merely requires that (2.1)–(2.3) are established (with $q \to q^3$) for $D_{4,i}(a;q)$.

First, we note that (2.1) is by definition and (2.2) follows by inspection. Indeed we see also by inspection that:

$$D_{4,1}(a;q) = D_{4,4}(aq^3;q),$$

which is (2.3) in the case $k=4$, $i=1$, $q \to q^3$.

Next

$$\begin{aligned}
&D_{4,2}(a;q) - D_{4,1}(a;q)\\
&\quad = \sum_{n\geq 0} \frac{a^n q^{n^2+2n}(aq^{2n+3};q)_\infty}{(q;q)_n (aq^{3n+3};q)_\infty} - \sum_{n\geq 0} \frac{a^n q^{n^2+3n}(aq^{2n+3};q)_\infty}{(q;q)_n (aq^{3n+3};q^3)_\infty}\\
&\quad = \sum_{n\geq 0} \frac{a^n q^{n^2+2n}(1-q^n)(aq^{2n+3};q)_\infty}{(q;q)_n (aq^{3n+3};q^3)_\infty}\\
&\quad = \sum_{n\geq 1} \frac{a^n q^{n^2+2n}(aq^{2n+3};q^3)_\infty}{(q;q)_{n-1} (aq^{3n+3};q^3)_\infty}\\
&\quad = \sum_{n\geq 0} \frac{a^{n+1} q^{n^2+4n+3}(aq^{2n+5};q)_\infty}{(q;q)_n (aq^{3n+6};q^3)_\infty}\\
&\quad = aq^3 D_{4,3}(aq^3;q),
\end{aligned} \tag{2.5}$$

which is (2.3) when $k=4$, $i=2$, and $q \to q^3$.

Before we proceed to the $k = 4$, $i = 3$ case, we shall make some observations about the simplicity of (2.5). Namely, we merely subtracted the two series term by term and the resulting new term was exactly what we wanted. As we move to $k = 4$, $i = 3$, we see that this simplicity does not exist. To produce this simplicity, we require the following intermediate functions:

$$\begin{aligned} M_{320}(a;q) &:= \sum_{n\geq 0} \frac{a^n q^{n^2+2n}(aq^{2n+2};q)_\infty}{(q;q)_n(aq^{3n+3};q^3)_\infty}, \\ M_{321}(a;q) &:= \sum_{n\geq 0} \frac{a^n q^{n^2}(aq^{2n+1};q)_\infty}{(q;q)_n(aq^{3n+3};q^3)_\infty}, \\ M_{322}(a;q) &:= \sum_{n\geq 0} \frac{a^n q^{n^2+n}(aq^{2n};q)_\infty}{(q;q)_n(aq^{3n};q^3)_\infty}. \end{aligned}$$

Hence

$$\begin{aligned} &D_{4,3}(a;q) - D_{4,2}(a;q) \\ &\quad = (D_{4,3}(a;q) - M_{320}(a;q)) + (M_{320}(a;q) - D_{4,2}(a;q)) \\ &\quad = \sum_{n\geq 0} \frac{a^n q^{n^2+n}(1-q^n)(aq^{2n+2};q)_\infty}{(q;q)_n(aq^{3n+3};q^3)_\infty} \\ &\qquad + \sum_{n\geq 0} \frac{a^n q^{n^2+2n}(aq^{2n+3};q)_\infty((1-aq^{2n+2})-1)}{(q;q)_n(aq^{3n+3};q^3)_\infty} \\ &\quad = \sum_{n\geq 0} \frac{a^{n+1} q^{n^2+3n+2}(aq^{2n+4};q)_\infty}{(q;q)_n(aq^{3n+6};q^3)_\infty} - \sum_{n\geq 0} \frac{a^{n+1} q^{n^2+4n+2}(aq^{2n+3};q)_\infty}{(q;q)_n(aq^{3n+3};q^3)_\infty} \\ &\quad = aq^2(M_{321}(aq^3;q) - M_{322}(aq^3;q)) \\ &\quad = aq^2 \sum_{n\geq 0} \frac{a^n q^{n^2+3n}(aq^{2n+4};q)_\infty}{(q;q)_n(aq^{3n+3};q^3)_\infty} \times ((1-aq^{3n+3}) - q^n(1-aq^{2n+3})) \\ &\quad = aq^2 \sum_{n\geq 0} \frac{a^n q^{n^2+3n}(1-q^n)(aq^{2n+4};q)_\infty}{(q;q)_n(aq^{3n+3};q^3)_\infty} \\ &\quad = aq^2 \sum_{n\geq 0} \frac{a^{n+1} q^{n^2+5n+4}(aq^{2n+6};q)_\infty}{(q;q)_n(aq^{3n+6};q^3)_\infty} \\ &\quad = (aq^3)^2 D_{4,2}(aq^3;q), \end{aligned}$$

which establishes (2.3) in the case $k = 4$, $i = 3$, $q \to q^3$.

Note that in all steps, the same simple term-by-term subtraction occurs. The intermediate functions provided the necessary component to allow this to take place.

To complete the final case of $k = 4$, $i = 4$, we require four new intermediate functions:

$$M_{432}(a;q) = \sum_{n\geq 0} \frac{a^n q^{n^2+n}(aq^{2n+1};q)_\infty}{(q;q)_n(aq^{3n+3};q^3)_\infty},$$

$$M_{433}(a;q) = \sum_{n\geq 0} \frac{a^n q^{n^2+2n}(aq^{2n+2};q)_\infty}{(q;q)_n(aq^{3n+3};q^3)_\infty},$$

$$M_{434}(a;q) = \sum_{n\geq 0} \frac{a^n q^{n^2+2n}(aq^{2n};q)_\infty}{(q;q)_n(aq^{3n};q^3)_\infty},$$

$$M_{435}(a;q) = \sum_{n\geq 0} \frac{a^n q^{n^2+3n}(aq^{2n+2};q)_\infty}{(q;q)_n(aq^{3n+3};q^3)_\infty}.$$

This final step is sufficiently intricate that we shall first split it into the several term-by-term subtractions that are straightforward:

$$D_{4,4}(a;q) - M_{322}(a;q) = aqM_{433}(a;q), \tag{2.6}$$

$$M_{322}(a;q) - M_{432}(a;q) = -a^2q^4M_{434}(aq^3;q), \tag{2.7}$$

$$M_{432}(a;q) - D_{4,3}(a;q) = -aqM_{435}(a;q), \tag{2.8}$$

$$M_{433}(a;q) - M_{435}(a;q) = aq^3M_{432}(aq^3;q), \tag{2.9}$$

$$M_{432}(aq^3;q) - M_{434}(aq^3;q) = aq^5D_{41}(aq^3;q). \tag{2.10}$$

Each of (2.6)–(2.10) is proved simply using term-by-term subtraction:

$$\begin{aligned}
&D_{4,4}(a;q) - M_{322}(a;q)\\
&\quad = \sum_{n\geq 0} \frac{a^n q^{n^2}(aq^{2n};q)_\infty(1-q^n)}{(q;q)_n(aq^{3n};q^3)_\infty}\\
&\quad = aq\sum_{n\geq 0} \frac{a^n q^{n^2+2n}(aq^{2n+2};q)_\infty}{(q;q)_n(aq^{3n};q^3)_\infty}\\
&\quad = aqM_{433}(a;q),\\
&M_{322}(a;q) - M_{432}(a;q)\\
&\quad = \sum_{n\geq 0} \frac{a^n q^{n^2+n}(aq^{2n+1};q)_\infty((1-aq^{2n})-(1-aq^{3n}))}{(q;q)_n(aq^{3n};q^3)_\infty}\\
&\quad = -a^2q^4\sum_{n\geq 0} \frac{a^n q^{n^2+5n}(aq^{2n+3};q)_\infty}{(q;q)_n(aq^{3n+3};q)_\infty}\\
&\quad = -a^2q^4M_{434}(aq^3;q),\\
&M_{432}(a;q) - D_{4,3}(a;q)\\
&\quad = \sum_{n\geq 0} \frac{a^n q^{n^2+n}(aq^{2n+2};q)_\infty((1-aq^{2n+1})-1)}{(q;q)_n(aq^{3n+3};q^3)_\infty}
\end{aligned}$$

$$
\begin{aligned}
&= -aqM_{435}(a;q),\\
&M_{433}(a;q) - M_{435}(a;q)\\
&= \sum_{n\geq 0} \frac{a^n q^{n^2+2n}(aq^{2n+2};q)_\infty (1-q^n)}{(q;q)_n (aq^{3n+3};q^3)_\infty}\\
&= aq^3 \sum_{n\geq 0} \frac{a^n q^{n^2+4n}(aq^{2n+4};q)_\infty}{(q;q)_n (aq^{3n+6};q^3)_\infty}\\
&= aq^3 M_{432}(aq^3;q),
\end{aligned}
$$

and finally

$$
\begin{aligned}
&M_{432}(aq^3;q) - M_{434}(aq^3;q)\\
&= \sum_{n\geq 0} \frac{a^n q^{n^2+4n}(aq^{2n+4};q)_\infty}{(q;q)_n (aq^{3n+3};q^3)_\infty}((1-aq^{3n+3}) - q^n(1-aq^{2n+3}))\\
&= \sum_{n\geq 0} \frac{a^n q^{n^2+4n}(aq^{2n+4};q)_\infty (1-q^n)}{(q;q)_n (aq^{3n+3};q^3)_\infty}\\
&= aq^5 \sum_{n\geq 0} \frac{a^n q^{n^2+6n}(aq^{2n+6};q)_\infty}{(q;q)_n (aq^{3n+6};q^3)_\infty}\\
&= aq^5 D_{4,1}(aq^3;q).
\end{aligned}
$$

Hence

$$
\begin{aligned}
&D_{4,4}(a;q) - D_{4,3}(a;q)\\
&\quad = (D_{4,4}(a;q) - M_{322}(a;q)) + (M_{322}(a;q) - M_{432}(a;q))\\
&\qquad + (M_{432}(a;q) - D_{4,3}(a;q))\\
&\quad = aq(M_{433}(a;q) - M_{435}(a;q))\\
&\qquad - a^2q^4 M_{434}(aq^3;q) \qquad \text{(by (2.6), (2.7) and (2.8))}\\
&\quad = a^2q^4(M_{432}(aq^3;q) - M_{434}(aq^3;q)) \qquad \text{(by (2.9))}\\
&\quad = a^3q^9 D_{4,1}(aq^3;q) \qquad \text{(by (2.10))}.
\end{aligned}
$$

□

Now, recall that [1, p. 408]

$$
Q_{k,i}(1;q) = \frac{(q^i;q^{2k+1})_\infty (q^{2k+1-i};q^{2k+1})_\infty (q^{2k+1};q^{2k+1})_\infty}{(q;q)_\infty}. \tag{2.11}
$$

Hence, Dyson's favorite identity follows from Theorem 2.1.

Corollary 2.2. *Identity* (1.1) *is valid.*

Proof. Take $a = 1$, $i = 3$ in Theorem 2.1 and simplify. □

In passing, we note the following instances of Theorem 2.1; $a = 1$, $i = 4$ yields [17, p. 162, eq. (93)]; $a = 1$, $i = 2$, yields [17, p. 161, eq. (91)], and $a = 1$, $i = 1$, yields [17, p. 161, eq. (90)]. See also [16, p. 109].

These observations raise the question: *Are there identities of interest for the $M_{xyz}(a;q)$ functions when $a = 1$?*

To answer this question requires that we take a short detour to study Chebyshev polynomials of the third and fourth kind.

3. Bailey Pairs and Chebyshev Polynomials

In the past, q-orthogonal polynomials have played an important role in the study of Rogers–Ramanujan type identities and mock theta functions [4,5,9].

The surprise here is that classical Chebyshev polynomials (NOT q-Chebyshev) play a central role in studying the $M_{xyz}(a;q)$ introduced in the previous section.

We recall that a sequence of pairs of rational functions $(\alpha_n, \beta_n)_{n\geq 0}$ is called a Bailey pair with respect to a provided [2, p. 278]:

$$\beta_n = \sum_{j=0}^{n} \frac{\alpha_j}{(q;q)_{n-j}(aq;q)_{n+j}}. \tag{3.1}$$

The identity (3.1) can be inverted [2, p. 278, eq. (4.1)] to the equivalent formulation:

$$\alpha_n = \frac{(1-aq^{2n})}{(1-a)} \sum_{j=0}^{n} \frac{(a;q)_{n+j}(-1)^{n-j}q^{\binom{n-j}{2}}\beta_j}{(q;q)_{n-j}}.$$

In the following, we shall also need the q-binomial coefficients:

$$\begin{bmatrix} A \\ B \end{bmatrix} = \begin{cases} 0, & \text{if } B < 0 \text{ or } B > A, \\ \frac{(q;q)_A}{(q;q)_B(q;q)_{A-B}}, & \text{otherwise.} \end{cases}$$

The Chebyshev polynomials of the third kind, $V_n(x)$ are given by [14, p. 170]:

$$V_n(x) = \begin{cases} 1, & \text{if } n = 0, \\ 2x-1, & \text{if } n = 1, \\ 2xV_{n-1}(x) - V_{n-2}(x), & \text{if } n > 1. \end{cases} \tag{3.2}$$

The Chebyshev polynomials of the fourth kind, $W_n(x)$ are given by [14, p. 170]:

$$W_n(x) = \begin{cases} 1, & \text{if } n = 0, \\ 2x+1, & \text{if } n = 1, \\ 2xW_{n-1}(x) - W_{n-2}(x), & \text{if } n > 1. \end{cases} \tag{3.3}$$

It is a simple exercise to show that

$$W_n(x) = (-1)^n V_n(-x). \tag{3.4}$$

We choose to use both $W_n(x)$ and $V_n(x)$ for simplicity of notation.

Our object in the section is to fit these Chebyshev polynomials into very natural Bailey pairs.

Theorem 3.1.

$$\prod_{j=1}^{n}(1+2xq^{j}+q^{2j}) = \sum_{j=0}^{n} q^{\binom{j+1}{2}} V_j(x) \begin{bmatrix} 2n+1 \\ n-j \end{bmatrix}. \tag{3.5}$$

Remark 3.2. Identity (3.5) is equivalent to saying that

$$\left(\frac{q^{\binom{n+1}{2}} V_n(x)}{1-q}, \frac{\prod_{j=1}^{n}(1+2xq^{j}+q^{2j})}{(q;q)_{2n+1}} \right) \tag{3.6}$$

forms a Bailey pair at $a = q$.

Proof of Theorem 3.1. Let us denote the left side of (3.5) by $L_n(x)$. Then, it is immediate that $L_n(x)$ is uniquely defined by:

$$L_n(x) = \begin{cases} 1, & \text{if } \ n = 0, \\ (1+2xq^{n}+q^{2n})L_{n-1}(x), & \text{if } \ n > 0. \end{cases}$$

Let us denote the right side of (3.5) by $R_n(x)$. Clearly, $R_0(x) = 1$. Therefore, to conclude that $L_n(x) = R_n(x)$, we need only show that for $n > 0$:

$$2xq^{n} R_{n-1}(x) = R_n(x) - (1+q^{2n}) R_{n-1}(x).$$

Now, by (3.2):

$$2xV_j(x) = V_{j+1}(x) + V_{j-1}(x).$$

Hence, we must show that

$$\begin{aligned} q^{n} \sum_{j=0}^{n} & q^{\binom{j+1}{2}} (V_{j+1}(x) + V_{j-1}(x)) \begin{bmatrix} 2n-1 \\ n-1-j \end{bmatrix} \\ &= \sum_{j \geq 0} V_j(x) q^{\binom{j+1}{2}} \left(\begin{bmatrix} 2n+1 \\ n-j \end{bmatrix} - (1+q^{2n}) \begin{bmatrix} 2n-1 \\ n-1-j \end{bmatrix} \right). \end{aligned} \tag{3.7}$$

Now, the $V_n(x)$ form a basis for the polynomials in x. Consequently, the coefficients of $V_j(x)$ on both sides of (3.7) must coincide. Thus, we need only prove:

$$\begin{aligned} q^{n+\binom{j}{2}} & \begin{bmatrix} 2n-1 \\ n-j \end{bmatrix} + q^{n+\binom{j+2}{2}} \begin{bmatrix} 2n-1 \\ n-2-j \end{bmatrix} \\ &= q^{\binom{j+1}{2}} \begin{bmatrix} 2n+1 \\ n-j \end{bmatrix} - (1+q^{2n}) \begin{bmatrix} 2n-1 \\ n-j-1 \end{bmatrix}, \end{aligned} \tag{3.8}$$

and multiplying both sides of (3.8) by $(q;q)_{n-j}(q;q)_{n-j+1}/(q;q)_{2n-1}$, we see that we finally must prove

$$\begin{aligned} q^{n+\binom{j}{2}} & (1-q^{n-j})(1-q^{n-j+1}) + q^{n+\binom{j+2}{2}} (1-q^{n-j})(1-q^{n-j+1}) \\ &= q^{\binom{j+1}{2}} (1-q^{2n+1})(1-q^{2n}) - (1+q^{2n})(1-q^{n-j})(1-q^{n+j+1}) q^{\binom{j+1}{2}}, \end{aligned} \tag{3.9}$$

and one easily expands the expressions in (3.9) to determine its validity and the truth of (3.5). □

Theorem 3.3.

$$\prod_{j=0}^{n-1}(1+2xq^j+q^{2j}) = \sum_{j=0}^{n} q^{\binom{j}{2}} W_j(x)(1-q^{2j+1})\frac{(q;q)_{2n}}{(q;q)_n(q;q)_{n+j+1}}. \tag{3.10}$$

Remark 3.4. Identity (3.10) is equivalent to saying that

$$\left(\frac{q^{\binom{n}{2}}(1-q^{2n+1})W_n(x)}{1-q}, \frac{\prod_{j=0}^{n-1}(1+2xq^j+q^{2j})}{(q;q)_{2n}}\right) \tag{3.11}$$

forms a Bailey pair at $a = q$. Also note that this result closely resembles Theorem 3.1. The change in the second entry of the Bailey pair is the shift in the index j and a shorter product in the denominator.

Proof of Theorem 3.3. As in Theorem 3.1, we denote the left side of (3.10) by $L_n(x)$. Then, it is immediate that $L_n(x)$ is uniquely defined by:

$$L_n(x) = \begin{cases} 1, & \text{if } n=0, \\ (1+2xq^{n-1}+q^{2n-2})L_{n-1}(x), & \text{if } n>0. \end{cases}$$

We now denote the right side of (3.10) by $R_n(x)$. Clearly, $R_0(x) = 1$. To conclude the proof that $L_n(x) = R_n(x)$, we need only show that for $n > 0$:

$$2xq^{n-1}R_{n-1}(x) = R_n(x) - (1+q^{2n-2})R_{n-1}(x).$$

Now, by (3.3):

$$2xW_j(x) = W_{j+1}(x) + W_{j-1}(x).$$

Hence, we must show that

$$\begin{aligned} q^{n-1}&\sum_{j=0}^{n} q^{\binom{j}{2}}(W_{j+1}(x)+W_{j-1}(x))(1-q^{2j+1})\frac{(q;q)_{2n-2}}{(q;q)_{n-j-1}(q;q)_{n+j}} \\ &= \sum_{j\geq 0} q^{\binom{j}{2}} W_j(x)(1-q^{2j+1})\left(\frac{(q;q)_{2n}}{(q;q)_{n-j}(q;q)_{n+j+1}} - \frac{(1+q^{2n-2})(q;q)_{2n-2}}{(q;q)_{n-1-j}(q;q)_{n+j}}\right). \end{aligned} \tag{3.12}$$

As before, the $W_n(x)$ form a basis for the polynomials in x. Consequently, the coefficients of $W_j(x)$ on both sides of (3.12) must coincide. Thus, we need only prove that

$$\begin{aligned} q^{n-1+\binom{j-1}{2}}&\left(\frac{(1-q^{2j-1})(q;q)_{2n-2}}{(q;q)_{n-j}(q;q)_{n+j-1}} + q^{2j-1}\frac{(1-q^{2j+3})(q;q)_{2n-2}}{(q;q)_{n-1-2}(q;q)_{n+j+1}}\right) \\ &= q^{\binom{j}{2}}(1-q^{2j+1})\left(\frac{(q;q)_{2n}}{(q;q)_{n-j}(q;q)_{n+j+1}} - \frac{(1+q^{2n-2})(q;q)_{2n-2}}{(q;q)_{n-1-j}(q;q)_{n+j}}\right), \end{aligned} \tag{3.13}$$

and multiplying both sides of (3.13) by

$$(q;q)_{n-j}(q;q)_{n+j+1}/(q)_{2n-2},$$

we see that we finally must prove

$$q^{n-1+\binom{j-1}{2}}(1-q^{n+j})(1-q^{n+j+1})(1-q^{2j-1})$$

$$+\,q^{n-1+\binom{j+1}{2}}(1-q^{n-j-1})(1-q^{n-j})(1-q^{2j+3})$$

$$=q^{\binom{j}{2}}(1-q^{2j+1})\big((1-q^{2n})(1-q^{2n-1})$$

$$-\,(1+q^{2n-2})(1-q^{n-j})(1-q^{n+j+1})\big), \tag{3.14}$$

and one easily expands the expressions in (3.14) to determine its validity and the proof of (3.10). □

4. Generalizing Dyson's Favorite Identity

This section will serve as a prototype for the types of identities that can be derived using the Bailey pairs given in Sect. 3.

Let us recall the weak form of Bailey's Lemma in the case $a = q$ [3, Theorem 2]. Namely, if (α_n, β_n) form a Bailey pair at $a = q$, then

$$\sum_{n\geq 0} q^{n^2+n}\beta_n = \frac{1}{(q^2;q)_\infty}\sum_{n\geq 0} q^{n^2+n}\alpha_n. \tag{4.1}$$

Inserting the Bailey pair from (3.6) into (4.1), we obtain:

$$\sum_{n\geq 0}\frac{q^{n^2+n}\prod_{j=1}^{n}(1+2xq^j+q^{2j})}{(q;q)_{2n+1}} = \frac{1}{(q;q)_\infty}\sum_{n\geq 0} q^{3\binom{n+1}{2}}V_n(x). \tag{4.2}$$

Lemma 4.1. *For $n \geq 0$:*

$$V_n(-1) = (-1)^n(2n+1), \tag{4.3}$$

$$V_n\left(-\frac{1}{2}\right) = \begin{cases} -2, & \text{if } n \equiv 1 \pmod 3,\\ 1, & \text{otherwise,}\end{cases} \tag{4.4}$$

$$V_n(0) = \begin{cases} 1, & \text{if } n \equiv 0,3 \pmod 4,\\ -1, & \text{otherwise,}\end{cases} \tag{4.5}$$

$$V_n\left(\frac{1}{2}\right) = \begin{cases} 1, & \text{if } n \equiv 0,5 \pmod 6,\\ 0, & \text{if } n \equiv 1,4 \pmod 6,\\ -1, & \text{if } n \equiv 2,3 \pmod 6,\end{cases} \tag{4.6}$$

$$V_n(1) = 1, \tag{4.7}$$

$$V_n\left(\frac{3}{2}\right) = F_{2n+1}, \tag{4.8}$$

$$V_n\left(-\frac{3}{2}\right) = (-1)^n L_{2n+1}, \tag{4.9}$$

where F_n and L_n are the Fibonacci and Lucas numbers.

Proof. Each of (4.3)–(4.9) is a simple exercise in mathematical induction using the initial values and recurrence from (3.2). □

Theorem 4.2. *Identity (1.1), Dyson's "favorite identity" is valid.*

Proof. By (4.2) with $x = \frac{1}{2}$:

$$\sum_{n\geq 0} \frac{q^{n^2+n}\prod_{j=1}^{n}(1+q^j+q^{2j})}{(q;q)_{2n+1}}$$
$$= \frac{1}{(q;q)_\infty}\left(\sum_{n\geq 0}\left(q^{3\binom{6n+1}{2}} + q^{3\binom{6n+6}{2}} - q^{3\binom{6n+3}{2}} - q^{3\binom{6n+4}{2}}\right)\right)$$
$$= \frac{1}{(q;q)_\infty}\left(\sum_{n=-\infty}^{\infty}\left(q^{3\binom{6n+1}{2}} - q^{3\binom{6n+3}{2}}\right)\right)$$
$$= \frac{1}{(q;q)_\infty}\sum_{n\geq 0} q^{9n(3n+1)/2}(1-q^{9(2n+1)})$$
$$= \frac{1}{(q;q)_\infty}(q^9;q^9)_\infty \qquad \text{(by [2, p. 22, Corollary 2.9])}.$$

□

We isolated (1.1) and gave a detailed proof. The remaining values of $V_n(x)$ given in Lemma 4.1 yield the following identities:

Theorem 4.3.

$$\sum_{n\geq 0}\frac{q^{n^2+n}(q;q)_n}{(q^{n+1};q)_{n+1}} = \frac{(q^3;q^3)^3}{(q;q)_\infty},$$

$$\sum_{n\geq 0}\frac{q^{n^2+n}(-q^3;q^3)_n}{(-q;q)_n(q;q)_{2n+1}} = \frac{1}{(q;q)_\infty}\left(\psi(q^3) - 3q^3\psi(q^{27})\right),$$

$$\sum_{n\geq 0}\frac{q^{n^2+n}(-q^2;q^2)_n}{(q;q)_{2n+1}} = \frac{1}{(q;q)_\infty}\sum_{n=-\infty}^{\infty} q^{24n^2+6n}(1-q^{12n+3}), \tag{4.10}$$

$$\sum_{n\geq 0}\frac{q^{n^2+n}(-q;q)^2}{(q;q)_{2n+1}} = \frac{\psi(q^3)}{(q;q)_\infty},$$

$$\sum_{n\geq 0}\frac{q^{n^2+n}\prod_{j=1}^{n}(1+3q^j+q^{2j})}{(q;q)_{2n+1}} = \frac{1}{(q;q)_\infty}\sum_{n\geq 0} q^{3\binom{n+1}{2}}F_{2n+1}, \tag{4.11}$$

and

$$\sum_{n\geq 0}\frac{q^{n^2+n}\prod_{j=1}^{n}(1-3q^j+q^{2j})}{(q;q)_{2n+1}} = \frac{1}{(q;q)_\infty}\sum_{n\geq 0} q^{3\binom{n+1}{2}}(-1)^n L_{2n+1},$$

where

$$\psi(q) = \frac{(q^2;q^2)_\infty}{(q;q^2)_\infty} = \sum_{n=0}^{\infty} q^{\binom{n+1}{2}}.$$

Remark 4.4. Of these six identities, (4.10) appears in [7], and (4.11) is from [16, p. 154, eq. (22)]. The remaining four appear to be new.

Proof of Theorem 4.3. Each of these identities follows from direct substitution of the values given for $V_n(x)$ in Lemma 4.1 into (4.2). The only instance where an auxiliary result is used is (4.10) which additionally requires Jacobi's famous result [2, p. 176]:

$$\sum_{n=0}^{\infty}(-1)^n(2n+1)q^{\binom{n+1}{2}} = (q;q)_\infty^3.$$

□

5. Implications of Theorem 3.3

Just as Theorem 3.1 provided seven corollaries, so too does Theorem 3.3. We begin by inserting the Bailey pair from (3.11) into (4.1) to obtain:

$$\sum_{n\geq 0}\frac{q^{n^2+n}\prod_{j=0}^{n-1}(1+2xq^j+q^{2j})}{(q;q)_{2n}} = \frac{1}{(q;q)_\infty}\sum_{n\geq 0}q^{n(3n+1)/2}(1-q^{2n+1})W_n(x). \tag{5.1}$$

In light of (3.4), we can use Lemma 4.1 to provide the special evaluation of $W_n(x)$:

Theorem 5.1.

$$\sum_{n\geq 0}\frac{q^{n^2+n}\prod_{j=0}^{n-1}(1-3q^j+q^{2j})}{(q;q)_{2n}}$$
$$= \frac{1}{(q;q)_\infty}\sum_{n\geq 0}q^{n(3n+1)/2}(1-q^{2n+1})(-1)^nF_{2n+1},$$
$$1 = \frac{1}{(q;q)_\infty}\sum_{n\geq 0}q^{n(3n+1)/2}(1-q^{2n+1})(-1)^n,$$
$$1+\sum_{n\geq 1}\frac{q^{n^2+n}(-q^3;q^3)_{n-1}}{(-q;q)_{n-1}(q;q)_{2n}}$$
$$= \frac{-1}{(q;q)_\infty}\sum_{n\geq 0}q^{n(3n+1)/2}(1-q^{2n+1})\left(\frac{n-1}{3}\right)$$
$$= \frac{1}{(q;q)_\infty}(1-q-q^7+q^{12}+q^{15}-q^{22}-\cdots),$$
$$\sum_{n\geq 0}\frac{q^{n^2+n}(-1;q^2)_n}{(q;q)_{2n}}$$
$$= \frac{1}{(q;q)_\infty}\sum_{n=-\infty}^{\infty}q^{n(3n+1)/2}\chi_4(n)$$

(where $\chi_4(n) = +1$ if $n \equiv 0, 1 \pmod 4$ and -1 otherwise),

$$1+\sum_{n\geq 1}\frac{q^{n^2+n}(q^3;q^3)_{n-1}}{(q;q)_{n-1}(q;q)_{2n}}=\frac{1}{(q;q)_\infty}\sum_{n=-\infty}^{\infty}q^{n(3n+1)/2}\chi_{12}(n)$$

(where $\chi_{12}(n)=+1$ *if* $n\equiv 0,1,7 \pmod{12}, -1$
if $n\equiv 4,10 \pmod{12}, 0$ *otherwise),*

$$\sum_{n\geq 0}\frac{q^{n^2+n}(-1;q)_n^2}{(q;q)_{2n}}=\frac{1}{(q;q)_\infty}\sum_{n=-\infty}^{\infty}(2n+1)q^{n(3n+1)/2},$$

$$\sum_{n\geq 0}\frac{q^{n^2+n}\prod_{j=0}^{n-1}(1+3q^j+q^{2j})}{(q;q)_{2n}}$$
$$=\frac{1}{(q;q)_\infty}\sum_{n\geq 0}q^{n(3n+1)/2}(1-q^{2n+1})L_{2n+1}. \tag{5.2}$$

Proof. Each of these seven identities follows successively from the instances: $x=-\frac{3}{2},-1,-\frac{1}{2},0,\frac{1}{2},1,\frac{3}{2}$ of $W_n(x)$ in (5.1). □

6. Generalized Hecke Series

In the previous sections, we have restricted attention to results where Chebyshev polynomials have been inserted into the theta-type series, e.g., (4.2) and (5.2). In this section, we consider a similar phenomenon related to Hecke-type double series involving indefinite quadratic forms.

Throughout this section, we will require instances of the following identity:

$$\sum_{n\geq 0}\frac{q^{n^2+\alpha n}}{(q;q)_n(q;q)_{n+\beta}}=\frac{1}{(q;q)_\infty}\sum_{n\geq 0}\frac{(q^{\alpha-\beta};q)_n(-1)^n q^{\beta n+\binom{n+1}{2}}}{(q;q)_n}, \tag{6.1}$$

which follows from Heine's second transformation [12, p. 241, eq. (III.2), $a=b=\frac{1}{\tau}$, $z=q^{\alpha+1}\tau^2$, $c=q^{\beta+1}$, and $\tau\to 0$].

Theorem 6.1.

$$\sum_{n\geq 0}\frac{q^{n^2}\prod_{j=1}^{n}(1+2xq^j+q^{2j})}{(q;q)_{2n}}$$
$$=\frac{1}{(q;q)_\infty}\sum_{n\geq 0}q^{2n^2+n}(1-q^{6n+6})\sum_{j=0}^{n}V_j(x)q^{-\binom{j+1}{2}}. \tag{6.2}$$

Proof. By (3.5):

$$\sum_{n\geq 0}\frac{q^{n^2}\prod_{j=1}^{n}\left(1+2xq^j+q^{2j}\right)}{(q;q)_{2n}}=\sum_{n\geq 0}\frac{q^{n^2}\sum_{j=0}^{n}q^{\binom{j+1}{2}}V_j(x)\begin{bmatrix}2n+1\\ n-j\end{bmatrix}}{(q;q)_{2n}}. \tag{6.3}$$

Thus, to prove Theorem 6.1, we need only identify the coefficients of $V_j(x)$ in (6.3) with those in (6.2). Namely, we must prove:

$$\sum_{n\geq j}\frac{q^{n^2+\binom{j+1}{2}}\begin{bmatrix}2n+1\\n-j\end{bmatrix}}{(q;q)_{2n}}=\frac{1}{(q;q)_\infty}\sum_{n\geq j}q^{2n^2+n-\binom{j+1}{2}}\left(1-q^{6n+6}\right). \tag{6.4}$$

Now

$$\begin{aligned}\sum_{n\geq j}&\frac{q^{n^2+\binom{j+1}{2}}\begin{bmatrix}2n+1\\n-j\end{bmatrix}}{(q;q)_{2n}}\\ &=q^{j^2+\binom{j+1}{2}}\sum_{n\geq 0}\frac{q^{n^2+2nj}(1-q^{2n+2j+1})}{(q;q)_n(q;q)_{n+2j+1}}\\ &=\frac{q^{j^2+\binom{j+1}{2}}}{(q;q)_\infty}\left(\sum_{n\geq 0}\frac{(q^{-1};q)_n}{(q;q)_n}(-1)^n q^{(2j+1)n+\binom{n+1}{2}}\right.\\ &\qquad\left.-q^{2j+1}\sum_{n\geq 0}\frac{(q;q)_n}{(q;q)_n}(-1)^n q^{(2j+1)n+\binom{n+1}{2}}\right)\quad\text{(by (6.1))}\\ &=\frac{q^{j^2+\binom{j+1}{2}}}{(q;q)_\infty}\left(1+q^{2j+1}-q^{2j+1}\sum_{n\geq 0}(-1)^n q^{\binom{n+1}{2}+(2j+1)n}\right).\end{aligned} \tag{6.5}$$

Comparing the right sides of (6.4) and (6.5), we see (shifting $n\to n+j$ on the right of (6.4)) that to complete the proof, we must show that:

$$\sum_{n\geq 0}q^{2n^2+4nj+n}(1-q^{6n+6j+6})=1-q^{2j+1}\sum_{n\geq 1}(-1)^n q^{\binom{n+1}{2}+(2j+1)n}, \tag{6.6}$$

and

$$\begin{aligned}1&-q^{2j+1}\sum_{n\geq 1}(-1)^n q^{\binom{n+1}{2}+(2j+1)n}\\ &=1+\sum_{n\geq 1}q^{\binom{2n}{n}+2n(2j+1)}-\sum_{n\geq 0}q^{(2j+1)+\binom{2n+3}{2}+(2j+1)(2n+2)}\\ &=\sum_{n\geq 0}q^{2n^2+4nj+n}-\sum_{n\geq 0}q^{2n^2+4nj+7n+6j+6}\\ &=\sum_{n\geq 0}q^{2n^2+4nj+n}(1-q^{6n+6j+6}),\end{aligned}$$

and thus, (6.6) is proved and with it (6.2): □

Theorem 6.2.

$$\sum_{n\geq 0}\frac{q^{n^2+2n}\prod_{j=1}^{n}(1+2xq^j+q^{2j})}{(q;q)_{2n+1}}$$
$$=\frac{1}{(q;q)_\infty}\sum_{n\geq 0}q^{2n^2+3n}(1-q^{2n+2})\sum_{j=0}^{n}V_j(x)q^{-\binom{j+1}{2}}. \tag{6.7}$$

Proof. This is proved exactly as Theorem 6.1 is proved; so, we suppress some of the more tedious details:

$$\sum_{n\geq 0}\frac{q^{n^2+2n}\prod_{j=1}^{n}(1+2xq^j+q^{2j})}{(q;q)_{2n+1}}$$
$$=\sum_{n\geq 0}\frac{q^{n^2+2n}}{(q;q)_{2n+1}}\sum_{j=0}^{n}q^{\binom{j+1}{2}}V_j(x)\begin{bmatrix}2n+1\\ n-j\end{bmatrix}\qquad (\text{by } (3.5)). \tag{6.8}$$

Thus, to prove (6.7), we must check that the coefficients of $V_j(x)$ in (6.7) and (6.8) coincide. Hence, we must prove:

$$\sum_{n\geq j}\frac{q^{n^2+2n+\binom{j+1}{2}}}{(q;q)_{n-j}(q;q)_{n+j+1}}=\frac{1}{(q;q)_\infty}\sum_{n\geq j}q^{2n^3+3n-\binom{j+1}{2}}(1-q^{2n+2}), \tag{6.9}$$

and identity (6.9) is proved by applying (6.1) to the left side. □

7. Ninth-Order Mock Theta Functions

In this section, we shall study some of the series arising as intermediate functions in Sect. 2. We now remove the infinite products from the $M_{xyz}(a;q)$. Namely

$$m_{xyz}(a;q)=\frac{(a;q)_\infty}{(a;q^3)_\infty}M_{xyz}(a;q).$$

Theorem 7.1.

$$m_{320}(1;q):=\sum_{n\geq 0}\frac{q^{n^2+2n}(q^3;q^3)_n}{(q;q)_n(q;q)_{2n+1}}$$
$$=\frac{1}{(q;q)_\infty}\sum_{n=0}^{\infty}q^{2n^2+3n}(1-q^{2n+2})\sum_{j=-\lfloor\frac{n+1}{3}\rfloor}^{\lfloor\frac{n}{3}\rfloor}(-1)^j q^{-3j(3j+1)/2}. \tag{7.1}$$

Proof. In Theorem 6.2, set $x=\frac{1}{2}$. Then, prove

$$\sum_{j=0}^{n}V_j\left(\frac{1}{2}\right)q^{-j(j+1)/2}=\sum_{j=-\lfloor\frac{n+1}{3}\rfloor}^{\lfloor\frac{n}{3}\rfloor}(-1)^j q^{-3j(3j+1)/2} \tag{7.2}$$

by mathematical induction using (4.6). □

Theorem 7.2.

$$m_{321}(1;q) := \sum_{n\geq 0} \frac{q^{n^2}(q^3;q^3)_n}{(q;q)_n(q;q)_{2n}}$$
$$= \frac{1}{(q;q)_\infty} \sum_{n=0}^{\infty} q^{2n^2+n}(1-q^{6n+6}) \sum_{j=-\lfloor \frac{n+1}{3} \rfloor}^{\lfloor \frac{n}{3} \rfloor} (-1)^j q^{-3j(3j+1)/2}.$$

Proof. In Theorem 6.1, set $x = \frac{1}{2}$, and invoke (7.2). □

Lemma 7.3.

$$\sum_{n\geq 1} \frac{q^{n^2}(q^3;q^3)_{n-1}}{(q;q)_{n-1}(q;q)_{2n-1}}$$
$$= \frac{1}{(q;q)_\infty} \sum_{n\geq 0} q^{2n^2+3n+1}(1-q^{2n+2}) \sum_{j=-\lfloor \frac{n+1}{3} \rfloor}^{\lfloor \frac{n}{3} \rfloor} (-1)^j q^{-3j(3j+1)/2}.$$

Proof. Shift n to $n-1$ on the left side of (7.1) and multiply by q. □

Theorem 7.4.

$$m_{322}(1;q) = 1 + \sum_{n\geq 0} \frac{q^{n^2+n}(q^3;q^3)_{n-1}}{(q;q)_n(q;q)_{2n-1}}$$
$$= \frac{1}{(q;q)_\infty} \sum_{n\geq 0} q^{2n^2+n}(1-q^{2n+1}) \sum_{j=-\lfloor \frac{n}{3} \rfloor}^{\lfloor \frac{n}{3} \rfloor} (-1)^j q^{-3j(3j+1)/2}.$$

Proof. Define $d_{4,i}(a;q) = \frac{(a;q)_\infty}{(a;q)_\infty} d_{4,i}(a;q)$. Then, by [16, p. 162, eq. (93)]:

$$d_{4,4}(1;q) = 1 + \sum_{n\geq 1} \frac{q^{n^2}(q^3;q^3)_{n-1}}{(q;q)_n(q;q)_{2n-1}}$$
$$= \frac{1}{(q;q)_\infty} \sum_{n\geq 0} (-1)^n q^{3(9n^2+n)/2}(1-q^{24n+12}). \tag{7.3}$$

Therefore

$$d_{4,4}(1;q^3) - m_{322}(1;q) = \sum_{n\geq 1} \frac{q^{n^2}(q^3;q^3)_{n-1}}{(q;q)_{n-1}(q;q)_{2n-1}},$$

and thus, by (7.3) and Lemma 7.3, $m_{322}(1;q)$ is equal to:

$$\frac{1}{(q;q)_\infty}\Bigg(\sum_{n\geq 0}(-1)^n q^{(27n^2+3n)/2}(1-q^{24n+12})$$
$$- \sum_{n\geq 0} q^{2n^2+3n+1}\left(1-q^{2n+2}\right) \sum_{j=-\lfloor \frac{n+1}{3} \rfloor}^{\lfloor \frac{n}{3} \rfloor} (-1)^j q^{-3j(3j+1)/2}\Bigg)$$
$$= \frac{1}{(q;q)_\infty}\Bigg(\sum_{n\geq 0}(-1)^n q^{(27n^2+3n)/2}\left(1-q^{24n+12}\right)$$

$$
\begin{aligned}
&\quad - \sum_{n\geq 0} q^{2n^2+3n+1} \sum_{j=-\lfloor \frac{n+1}{3}\rfloor}^{\lfloor \frac{n}{3}\rfloor} (-1)^j q^{-3j(3j+1)/2} \\
&\quad + \sum_{n\geq 1} q^{2n^2+n} \sum_{j=-\lfloor \frac{n}{3}\rfloor}^{\lfloor \frac{n}{3}\rfloor} (-1)^j q^{-3j(3j+1)/2} \Bigg) \\
&= \frac{1}{(q;q)_\infty} \Bigg(\sum_{n\geq 0} (-1)^n q^{(27n^2+3n)/2} \left(1 - q^{24n+12}\right) \\
&\quad - \sum_{n\geq 0} q^{2n^2+3n+1} \sum_{j=-\lfloor \frac{n}{3}\rfloor}^{\lfloor \frac{n}{3}\rfloor} (-1)^j q^{-3k(3j+1)/2} \\
&\quad - \sum_{n\geq 0} q^{2(3n+2^2+3(3n+2)+1} (-1)^{-n-1} q^{-3(-n-1)(3(-n-1)+1)/2} \\
&\quad + \sum_{n\geq 0} q^{2n^2+n} \sum_{j=-\lfloor \frac{n}{3}\rfloor}^{\lfloor \frac{n}{3}\rfloor} (-1)^j q^{-3j(3j+1)/2} \\
&\quad - \sum_{n\geq 0} q^{2(3n)^2+3n} (-1)^n q^{-3n(3n+1)/2} \Bigg) \\
&= \frac{1}{(q;q)_\infty} \sum_{n\geq 0} q^{2n^2+n}(1-q^{2n+1}) \sum_{j=-\lfloor \frac{n}{3}\rfloor}^{\lfloor \frac{n}{3}\rfloor} (-1)^j q^{-3j(3j+1)/2}. \qquad (7.4)
\end{aligned}
$$

□

We remark that Ian Wagner has studied these and many related functions in his Ph.D. thesis (directed by Ken Ono). He observes that some are mock theta functions (Theorems 12 and 13), and some are "near misses" (Theorem 15).

8. Partition Identities

Let us recall B. Gordon's celebrated generalization of the Rogers–Ramanujan identities [13] (cf. [1]).

Gordon's Theorem. *Let $A_{k,i}(n)$ denote the number of partitions of n into parts $\not\equiv 0, \pm i \pmod{2k+1}$. Let $B_{k,i}(n)$ denote the number of partitions of n of the form $\lambda_1 + \lambda_2 + \cdots + \lambda_s$, where $\lambda_j - \lambda_{j+k-1} \geq 2$ and $\lambda_1 \geq \lambda_2 \geq \cdots \lambda_k$ and, in addition, at most $i-1$ of the λ_j are equal to 1. Then, for $1 \leq i \leq k$, $n \geq 0$:*

$$A_{k,i}(n) = B_{k,i}(n).$$

The simplest proof [1] of Gordon's theorem reveals that $Q_{k,i}(z;q)$ is the generating function for partitions of the $B_{k,i}$-type where the exponent of z counts the number of parts. Thus:

$$Q_{k,i}(1;q) = \sum_{n\geq 0} B_{k,i}(n) q^n,$$

and the result follows by invoking (2.11).

Now, one could assume that nothing more needs to be said about $Q_{4,i}(1;q^3)$. After all this is just Gordon's theorem at $k=4$ with all parts multiplied by 3.

However, the work in [8] points to a natural alternative interpretation of partitions generated by $Q_{4,i}(x;q^3)$. In the new interpretation, we begin, with R, the set of all integer partitions in which parts differ by at least 2. We also speak of *maximal sequences* of parts in such partitions. We shall say that in the partition $\lambda_1+\lambda_2+\cdots+\lambda_s$ $(\lambda_i-\lambda_{i+1}\geq 2)$:

$$\lambda_m+\lambda_{m+1}+\cdots+\lambda_{m+j}$$

is maximal if $\lambda_{m-1}-\lambda_m>2$, $\lambda_{m+j}-\lambda_{m+j+1}>2$ and $\lambda_{m+i}-\lambda_{m+i+1}=2$ for $0\leq i\leq j-1$.

Theorem 8.1. *Let $C_i(n)$ $(1\leq i\leq 4)$ denote the number of partitions of n in R with the added condition that*

$$\textit{all parts are } >4-i,\ \textit{ if } j,j+2,j+4,\ldots,j+(2r-2) \tag{8.1}$$

is a maximal sequence of parts; then

when $j\equiv 0 \pmod 3$, r must be $\equiv 0,1 \pmod 3$,
when $j\equiv 1 \pmod 3$, r must be $\equiv 0 \pmod 3$ and
when $j\equiv 2 \pmod 3$, r must be $\equiv 0,2 \pmod 3$.

Then

$$C_i(n)=A_{4,i}\left(\frac{n}{3}\right)=B_{4,i}\left(\frac{n}{3}\right) \tag{8.2}$$

(note that if $\frac{n}{3}$ is not an integer all entries in (8.2) equal 0).

Remark 8.2. As an example, in the case $n=12$ and $i=4$, $B_{4,4}(4)=4$ (the partitions considered are 4, $3+1$, $2+2$, $2+1+1$), and $C_4(12)$ also equals 4 with the relevant partitions being 12, $3+9$, $5+7$, $2+4+6$.

Proof of Theorem 8.1. We need only show that the generating function $K_i(z;q)$ for partitions with m parts among the partitions enumerated by $C_i(n)$ is $Q_{4,i}(z;q^3)$. Clearly, the initial conditions (2.1) and (2.2) hold. Thus, we need only show that (2.3) with $q\to q^3$ holds for the $K_i(z;q)$. Namely

$$K_4(z;q)-K_3(z;q)=z^3q^{1+3+5}K_1(zq^3;q), \tag{8.3}$$

$$K_3(z;q)-K_2(z;q)=z^2q^{2+4}K_2(zq^3;q), \tag{8.4}$$

$$K_2(z;q)-K_1(z;q)=zq^3K_3(zq^3;q), \tag{8.5}$$

$$K_1(z;q)=K_4(zq^3;q). \tag{8.6}$$

The proof of each of these four identities is similar so we provide full details for (8.3). We note that $K_i(z;q)-K_{i-1}(z;q)$ generates those partitions in which $5-i$ is the smallest part. Thus, when $i=4$, we can only consider partitions that have 1 as the smallest part. By (8.1), the shortest allowable sequence starting with 1 is $1+3+5$. The partitions generated by $K_1(zq^3;q)$ are K_1 partitions with 3 added to each part. Thus the smallest part is ≤ 7. Hence, either $1+3+5$ attaches to a previously maximal sequence of length now increased by 3 and thus still preserving (8.1) or else $1+3+5$ is itself a legitimate maximal sequence. Hence, (8.3) is established. Identities (8.4)–(8.6) follow in the same way.

Therefore, since (2.1)–(2.3) uniquely determine $Q_{k,i}(a;q)$, we see that for $1 \le i \le 4$:

$$K_i(z;q) = Q_{4,i}(z;q^3), \tag{8.7}$$

and Theorem 8.1 follows by setting $z = 1$ in (8.7) and comparing coefficients of q^n. $\square$

9. One More Identity

We note that Dyson's favorite identity (1.1) may be written in the form of an instance of the quintuple product identity. Namely:

$$\sum_{n=0}^{\infty} \frac{q^{n^2+n}(q^3;q^3)_n}{(q;q)_n(q;q)_{2n+1}} = \frac{(-q^3;q^9)_\infty(-q^6;q^9)_\infty(q^9;q^9)_\infty(q^3;q^{18})_\infty(q^{15};q^{18})_\infty}{(q;q)_\infty}. \tag{9.1}$$

One may naturally expect that there is an equally elegant quintuple product companion for (9.1), and this is revealed in the following:

Theorem 9.1.

$$\begin{aligned}\sum_{n\ge 0} \frac{q^{n^2}(q^3;q^3)_n}{(q;q)_n(q;q)_{2n+1}} &= \frac{1}{(q;q)_\infty} \sum_{n=-\infty}^{\infty} (-1)^n q^{3n(9n+1)/2}(1+q^{9n+1}) \\ &= (-q^8;q^9)_\infty(-q;q^9)_\infty(q^9;q^9)_\infty(q^7;q^{18})_\infty(q^{11};q^{18})_\infty/(q;q)_\infty.\end{aligned}$$

Proof. It is an exercise in mathematical induction to show that

$$\sum_{j=0}^{m} \frac{q^{j^2}(q^3;q^3)_j}{(q;q)_j(q;q)_{2j+1}} - \left(1 + \sum_{j=1}^{m} \frac{q^{j^2}(q^3;q^3)_{j-1}}{(q;q)_j(q;q)_{2j-1}}\right) - q\sum_{j=0}^{m} \frac{q^{j^2+2j}(q^3;q^3)_j}{(q;q)_j(q;q)_{2j+2}} = \frac{-q^{m^2+4m+3}(q^3;q^3)_m}{(q;q)_m(q;q)_{2m+2}}. \tag{9.2}$$

Letting $m \to \infty$ in (9.2), we see that:

$$\begin{aligned}\sum_{n\ge 0} \frac{q^{n^2}(q^3;q^3)_n}{(q;q)_n(q;q)_{2n+1}} &= 1 + \sum_{n=1}^{\infty} \frac{q^{n^2}(q^3;q^3)_{n-1}}{(q;q)_n(q;q)_{2n-1}} + q\sum_{n=0}^{\infty} \frac{q^{n^2+2n}(q^3;q^3)_n}{(q;q)_n(q;q)_{2n+2}} \\ &= \frac{1}{(q;q)_\infty}\Bigg(\sum_{n=-\infty}^{\infty} (-1)^n q^{(27n^2+3n)/2}\end{aligned}$$

$$+q\sum_{n=-\infty}^{\infty}(-1)^n q^{(27n^2+21n)/2}\Bigg)\text{(by [17, pp. 161–162, eqs. (91) and (93)])}$$

$$=\frac{1}{(q;q)_\infty}\sum_{n=-\infty}^{\infty}(-1)^n q^{3n(9n+1)/2}(1+q^{9n+1})$$

$$=\frac{1}{(q;q)_\infty}(-q^8;q^9)_\infty(-q;q^9)_\infty(q^9;q^9)_\infty(q^7;q^{18})_\infty(q^{11};q^{18})_\infty,$$

by the quintuple product identity with $q\to q^9$, $z=q$ [12, p. 134, Ex. 5.16]. □

10. Relation to L.J. Rogers's Work

We have treated all the discoveries in this paper using standard polynomial notation. This, in turn, has simplified many of our computations some of which have been extremely intricate. However, it is important to stress that the sorts of results in Sect. 3 are effectively finite versions of theorems of L.J. Rogers [15]. This is not obvious on the surface, because Rogers couched his work in terms of Fourier series.

To make this relationship clear, we reprove Theorem 3.1 in the style of L.J. Rogers.

Second Proof of Theorem 3.1

In (3.5), replace x by $\cos\theta=(\mathrm{e}^{i\theta}+\mathrm{e}^{-i\theta})/2$. Thus, (3.5) becomes:

$$\prod_{j=1}^{n}(1+\mathrm{e}^{i\theta}q^j)(1+\mathrm{e}^{-i\theta}q^j)=\sum_{j=0}^{n}q^{\binom{j+1}{2}}V_j(\cos\theta)\begin{bmatrix}2n+1\\ n-j\end{bmatrix}. \tag{10.1}$$

Now, noting

$$(-x;q)_{N+1}(-q/x;q)_N=x^{-N}q^{\binom{N+1}{2}}(-xq^{-N};q)_{2N+1}, \tag{10.2}$$

we may expand the right side of (10.2) by the q-binomial theorem [2, p. 36, eq. (3.3.6)] to obtain:

$$\begin{aligned}&(-xq;q)_N(-q/x;q)_N\\ &\quad=\frac{1}{1+x}\sum_{j=0}^{2N+1}x^j q^{\binom{j}{2}-Nj}\begin{bmatrix}2N+1\\ j\end{bmatrix}\\ &\quad=\sum_{j=0}^{N}q^{\binom{j+1}{2}}\frac{x^{j+\frac12}+x^{-j-\frac12}}{x^{\frac12}+x^{-\frac12}}\begin{bmatrix}2N+1\\ N-j\end{bmatrix}.\end{aligned} \tag{10.3}$$

Now, setting $x=\mathrm{e}^{i\theta}$ in (10.3) and noting that

$$V_j(\cos\theta)=\frac{\cos(\theta(j+\frac12))}{\cos\theta}, \tag{10.4}$$

we deduce (10.1) from (10.4). □

Now, if we let $N\to\infty$ in (10.1), we obtain the actual starting point for Rogers in his second proof of the Rogers–Ramanujan identities [15]. A similar treatment can be used for Theorem 3.3.

11. Conclusion

The entire project began in an attempt to better understand Dyson's mod 27 identities especially the favorite (1.1). The natural appearance of $M_{xyz}(a;q)$ functions naturally led to a quest for proofs of the mock theta specializations. This in turn led to the Chebyshev polynomials.

It is the latter phenomenon that is so surprising. Orthogonal polynomials have arisen several times before in the treatment of q-series (cf. [4,5,9]). However, in each instance, the orthogonal polynomials were q-analogs of classical orthogonal polynomials.

This is the first instance where classical orthogonal polynomials (namely Chebyshev polynomials of the third and fourth kinds) entered naturally into the world of q. This leaves us with at least three topics worthy of further exploration.

(11.1) Following the lead of Rogers briefly described in Sect. 10, one should be able to use the other Chebyshev polynomials in further studies of this nature.

(11.2) There are many more explicit results to be obtained for $m_{xyz}(1;q)$. The object here was to illustrate the method without obscuring the project with too many details.

(11.3) In addition to mock theta-type results for $m_{xyz}(1;q)$, there should be natural combinatorial interpretations related to the ideas in Sect. 8.

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George E. Andrews
Department of Mathematics
The Pennsylvania State University
University Park
PA 16802
USA
e-mail: gea1@psu.edu

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A q-Translation Approach to Liu's Calculus

To George Andrews, a true friend, a great mentor, and a fantastic mathematician

Hatice Aslan and Mourad E. H. Ismail

Abstract. We show that there is a concept of q-translation behind the approach used by Liu to prove summation and transformation identities for q-series. We revisit the q-translation associated with the Askey–Wilson operator introduced in Ismail (Ann Comb 5(3–4):347–362, 2001), simplify its formalism and point out new properties of this translation operator.

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Keywords. Translation operators, Askey–Wilson operators, q-Exponential functions, q^{-1}-Hermite polynomials, Addition theorems, Polynomial expansions.

1. Introduction

In this work, we introduce a concept of generalized translation. The motivation for the next step comes from the following observation. The general solution to $\frac{\partial f(x,y)}{\partial x} = \frac{\partial f(x,y)}{\partial y}$ is $f(x,y) = g(x+y)$, as can be seen from the method of characteristics [7]. Therefore, the translation by y is equivalent to solving the above-mentioned PDE.

Liu [18] gave an evaluation of a q-beta integral. In [19,20], he gave new proofs of q-series identities and derived some new ones. He studied functions satisfying $T_{q,x}F(x,y) = T_{q,y}F(x,y)$, where

$$(T_{q,x}f)(x) = \frac{f(x) - f(qx)}{x},$$

This paper started when Ismail attended a lecture by Zhi-Guo Liu where he represented some of his results. The reader can see that we tried to find a conceptual explanation of his results. Ismail also greatly acknowledges the help and hospitality of Zhi-Guo Liu and East China Normal University in Shanghai where this work was presented and completed.

and used his characterization in a very clever and creative way to evaluate q-series sums and integrals. We realized that the relationship $T_{q,x}F(x,y) = T_{q,y}F(x,y)$ is a q-analogue of $f_x = f_y$ and as such it generates a q-translation. This led us to define a concept of a generalized translation which not only incorporates the two above-mentioned cases of differential and q-difference operators but also covers divided difference operators including the Askey–Wilson operator. Our general approach also covers the q-translation introduced in [11], which was further studied by Bouzaffour [5], who defined an analogue of the Fourier transform.

In Sect. 2 we develop a general theory which explains the source of Liu's approach. This includes the definition of a general translation. Section 3 contains the definition of the corresponding exponential function. In Sect. 4, we treat a q-translation model and show that the corresponding exponential function is a function due to Euler. Section 5 is the first part of the analysis leading to an Askey–Wilson translation which is later introduced in Sect. 6.

We shall follow the standard notation for hypergeometric functions and their q-analogues as defined in [1,8,12]. The notation for the Askey–Wilson operator is as in [10,12] and is different from the original definition in [3], or in [8].

2. A Formal Construction

The Sheffer classification originated in a series of papers written by I.M. Sheffer in the 1930s, see, for example, [26]. It is treated in detail in Rainville [23] and in [12].

We say that a polynomial sequence $\{p_n(x)\}$ belongs to a linear operator T which reduces the degree of a polynomial by 1 if $Tp_n(x) = p_{n-1}(x)$. One can repeat the same arguments used in Rainville [23] and prove the following theorem.

Theorem 2.1. *Two polynomial sequences $\{p_n(x)\}$ and $\{q_n(x)\}$ belong to the same operator T if and only if there is a sequence of constants $\{a_n\}$ with $a_0 \neq 0$ such that*

$$p_n(x) = \sum_{k=0}^{n} a_k q_{n-k}(x).$$

This is equivalent to

$$\sum_{n=0}^{\infty} p_n(x)t^n = \left[\sum_{n=0}^{\infty} a_n t^n\right]\left[\sum_{k=0}^{\infty} q_k(x)t^k\right] = h(t)\sum_{k=0}^{\infty} q_k(x)t^k,$$

and $h(0) \neq 0$.

We now give a formal construction for translation operators. It must be emphasized that in any given case one has to rigorously justify the steps in the construction. We start with a linear operator T_x whose domains contain all polynomials and reduces the degree of a polynomial by 1. We also assume that we have two sequences of polynomials $\{u_n(x)\}$ and $\{v_n(x)\}$ satisfying $T_x u_n(x) = u_{n-1}(x)$, $T_x v_n(x) = v_{n-1}(x)$, that is, both $\{u_n(x)\}$ and $\{v_n(x)\}$ belong to T_x. Recall that it is tacitly assumed that both $u_n(x)$ and $v_n(x)$ are polynomials of exact degree n in x.

We now seek functions $f(x,y)$ such that

$$f(x,y) = \sum_{m,n=0}^{\infty} f_{m,n} u_m(x) v_n(y), \tag{2.1}$$

and f solves $T_x f(x,y) = T_y f(x,y)$. Substituting for f from (2.1) then we find that $f_{m+1,n} = f_{m,n+1}$. Hence, $f_{m,n} = c_{m+n}$ for some sequence $\{c_n\}$. Therefore,

$$f(x,y) = \sum_{n=0}^{\infty} c_n \sum_{k=0}^{n} u_k(x) v_{n-k}(y).$$

This means that the analogue of shifting x by y in x^n is the linear operator E^y defined by

$$E^y u_n(x) = \sum_{k=0}^{n} u_k(x) v_{n-k}(y). \tag{2.2}$$

Because we think of E^y as a translation by y we expect E^0 to be the identity operator. This happens if and only if $v_n(0) = \delta_{n,0}$. It is a fact that given T_x there is a unique sequence of polynomials $v_n(x)$ of positive leading terms, such that $T_x v_n(x) = v_{n-1}(x)$ and $v_n(0) = \delta_{n,0}$. This leads to the formal definition below.

Definition 2.2. Given an operator T_x, defined on polynomials and mapping a polynomial of degree n to a polynomial of degree $n-1$, for all n, construct the basic sequence $\{v_n(x)\}$ which belongs to T_x. Let $\{u_n(x)\}$ be any other polynomial sequence belonging to T_x. The generalized translation E^y is the linear operator defined on polynomials by

$$E^y u_n(x) = \sum_{k=0}^{n} u_k(x) v_{n-k}(y).$$

To see that E^y is well defined, assume that $\{w_n(x)\}$ is any polynomial sequence belonging to T_x. Then

$$w_n(x) = \sum_{k=0}^{n} a_{n-k} u_k(x)$$

and

$$\begin{aligned}
E^y w_n(x) &= E^y \sum_{k=0}^{n} a_{n-k} u_k(x) \\
&= \sum_{k=0}^{n} a_{n-k} E^y u_k(x) \\
&= \sum_{k=0}^{n} a_{n-k} \sum_{j=0}^{k} u_j(x) v_{k-j}(y) \\
&= \sum_{j=0}^{n} u_j(x) \sum_{k=0}^{n-j} a_{n-k-j} v_k(y) \\
&= \sum_{k=0}^{n} v_k(y) \sum_{j=0}^{n-k} a_{n-k-j} u_j(x) \\
&= \sum_{k=0}^{n} v_k(y) w_{n-k}(x),
\end{aligned}$$

which would be the generalized translation if we had started with $\{w_n(x)\}$ and $\{v_n(x)\}$.

Note that $E^z E^y = E^y E^z$ because

$$\begin{aligned}
E^z E^y u_n(x) &= E^z \sum_{k=0}^{n} v_k(y) u_{n-k}(x) \\
&= \sum_{k=0}^{n} v_k(y) \sum_{j=0}^{n-k} v_j(z) u_{n-j-k}(x) \\
&= \sum_{j,k \geq 0, j+k \leq n} v_j(z) v_k(y) u_{n-j-k}(x),
\end{aligned}$$

which is symmetric in y and z.

In the sequel, we shall use the notation

$$E^y f(x) = \sum_{n=0}^{\infty} c_n \sum_{k=0}^{n} u_k(x) v_{n-k}(y), \quad \text{if} \quad f(x) = \sum_{n=0}^{\infty} c_n u_n(x). \tag{2.3}$$

When $u_n(x) = v_n(x)$ then $E^y f(x) = E^x f(y)$, otherwise this is not true in general.

Our next goal is to describe all operators which commute with E^y. In the case $T_x = \frac{\mathrm{d}}{\mathrm{d}x}$, this corresponds to characterizing all shift invariant operators [9, 24, 25].

Let A be an operator which maps a polynomial of degree n to a polynomial of degree at most n.

Theorem 2.3. *Let A be as above. Then there is a sequence of polynomials $\{a_k(x)\}$ such that $a_k(x)$ is of degree $\leq k$ and the action of A on polynomials is given by*

$$A = \sum_{k=0}^{\infty} a_k(x) T_x^k, \tag{2.4}$$

with T_x^0 equal to the identity operator.

Proof. Let $\{u_n(x)\}$ belong to T_x, $u_0(x) = 1$, and define $a_0(x) = Au_0(x) =$ a constant. Define the a_k's inductively by

$$a_n(x) = Au_n(x) - \sum_{k=0}^{n-1} a_k(x) u_{n-k}(x).$$

This proves the theorem. □

Theorem 2.4. *Let A be an operator which maps a polynomial of degree n to a polynomial of degree at most n. Then A commutes with E^y if and only if the a_ks in (2.4) are constants.*

Proof. It is clear that if the a_ks are constants then the operator commutes with E^y. The proof of the converse is by induction. Recall that $u_0(x) = v_0(x) = 1$. We first note that

$$\begin{aligned}(AE^y - E^y A)u_1(x) &= [a_0 + a_1(x)T_x][v_1(y) + u_1(x)] - E^y[a_0 u_1(x) + a_1(x)] \\ &= a_0[v_1(y) + u_1(x)] + a_1(x) - a_0[v_1(y) + u_1(x)] - E^y a_1(x) \\ &= a_1(x) - E^y a_1(x).\end{aligned}$$

Hence, the above expression vanishes if and only if $a_1(x) = E^y a_1(x)$. Now write $a_1(x) = a_{1,0} + a_{1,1}u_1(x)$. Clearly $a_1(x)$ is invariant under E^y if and only if $a_{1,1}u_1(x) = a_{1,1}[u_1(x) + v_1(y)]$ for all y. Thus, $a_{1,1} = 0$. Now assume that $a_k(x)$ is a constant for $1 \leq k < n$. Then $\sum_{k=0}^{n-1} a_k T_x^k$ commutes with E^y. Therefore,

$$\begin{aligned}(AE^y - E^y A)u_n(x) &= a_n(x) T_x^n \sum_{j=0}^{n} u_k(x) v_{n-k}(y) - E^y(a_n(x)) \\ &= a_n(x) - E^y(a_n(x)),\end{aligned}$$

that is, $a_n(x)$ is invariant under E^y. As before, set

$$a_n(x) = \sum_{j=0}^{n} a_{n,j} u_j(x)$$

to see that $a_n(x)$ is invariant under E^y if and only if

$$\sum_{j=0}^{n} a_{n,j} u_j(x) = \sum_{k=0}^{n} a_{n,k} \sum_{j=0}^{k} u_j(x) v_{k-j}(y).$$

Clearly this holds if and only if

$$a_{n,j} = \sum_{k=j}^{n} a_{n,k} v_{k-j}(y) = \sum_{k=0}^{n-j} a_{n,k+j} v_k(y), \quad j = 0, 1, \ldots, n,$$

for all y. The case $j = 0$ implies that $a_{n,k} = 0$ for $k > 0$. □

3. Exponential Functions

We define an exponential function E by

$$E(x;t) = g(t) \sum_{n=0}^{\infty} u_n(x) t^n, \quad \text{with} \quad 1/g(t) = \sum_{n=0}^{\infty} u_n(0) t^n. \tag{3.1}$$

Of course it is assumed that the series in (3.1) converges in the x-domain and for a t-disc, $|t| < r, r > 0$. The above definition automatically implies

$$E(0;t) = 1.$$

Note that $T_x E(x;t) = tE(x;t)$. In other words, $E(x,t)$ behaves like an exponential function e^{xt}.

Theorem 3.1. *The exponential function in* (3.1) *satisfies the addition theorem*

$$E^y E(x;t) = h(t) E(x;t) E(y;t), \tag{3.2}$$

with $h(t) \left[\sum_{n=0}^{\infty} v_n(0) t^n\right] = 1$.

Proof. It is clear that

$$\begin{aligned} E^y E(x;t) &= g(t) \sum_{n=0}^{\infty} E^y u_n(x) t^n \\ &= g(t) \sum_{n=0}^{\infty} t^n \sum_{k=0}^{n} u_k(x) v_{n-k}(y) \\ &= g(t) \sum_{n=0}^{\infty} u_n(x) t^n \sum_{m=0}^{\infty} v_m(y) t^m \\ &= E(x;t) \sum_{m=0}^{\infty} v_m(y) t^m. \end{aligned}$$

Since $\{u_n(x)\}$ and $\{v_n(x)\}$ belong to the same operator, Theorem 2.1 and (3.1) then imply that there is a power series $h(t)$ with $h(0) \neq 0$ such that

$$\sum_{m=0}^{\infty} v_m(y) t^m = h(t) E(y;t).$$

Thus,

$$h(t) = \frac{1}{\sum_{m=0}^{\infty} v_m(0) t^m}$$

and the proof is complete. □

If $\{v_n(x)\}$ is chosen such that $v_n(0) = \delta_{n,0}$ then the power series $h(t) \equiv 1$ in (3.2). In the sequel, we shall always assume that

$$v_n(0) = \delta_{n,0}. \tag{3.3}$$

The concept of translation brings to mind the fact that the exponential function is multiplicative under translation, that is, $\exp(t(x+y)) = e^{tx}e^{ty}$. This motivates the following definition of $E(x,y;t)$:

$$E(x,y;t) = E^y E(x;t) = E(x;t)E(y;t).$$

The functions $E(x;t)$ and $E(x,y;t)$ depend only on the operator T and are independent of the choices of $\{u_n(x)\}$. To see this let us start with a different sequences $\{w_n(x)\}$, belonging to T_x instead of $\{u_n(x)\}$. Thus, there exists a power series $h(t)$ with

$$\sum_{n=0}^{\infty} u_n(x)t^n = h(t)\sum_{n=0}^{\infty} w_n(x)t^n.$$

Therefore,

$$h(t) = \sum_{n=0}^{\infty} u_n(0)t^n \Big/ \sum_{n=0}^{\infty} w_n(0)t^n,$$

that is,

$$\frac{\sum_{n=0}^{\infty} u_n(x)t^n}{\sum_{n=0}^{\infty} u_n(0)t^n} = \frac{\sum_{n=0}^{\infty} w_n(x)t^n}{\sum_{n=0}^{\infty} w_n(0)t^n}.$$

This shows that

$$\frac{\sum_{n=0}^{\infty} w_n(x)t^n}{\sum_{n=0}^{\infty} w_n(0)t^n} = \frac{\sum_{n=0}^{\infty} u_n(x)t^n}{\sum_{n=0}^{\infty} u_n(0)t^n}$$

holds. Hence, $E(x;t)$ is independent of the choice of the u_n's as long as they belong to T. This also defines $E(x,y;t)$ uniquely.

We now discuss infinitesimal generator. Assume that $\{v_n(x)\}$ belongs to T but $v_0(x) = 1, v_n(0) = \delta_{n,0}$. It is easy to see that this defines $\{v_n(x)\}$ uniquely. Moreover, $E(x;t) = \sum_{n=0}^{\infty} v_n(x)t^n$.

Theorem 3.2. *If f is a polynomial then*

$$(E(y;T_x))f(x) = (E^y f)(x).$$

Proof. Let $\{u_m(x)\}$ be any polynomial sequence belonging to T. Then

$$(E(y;T_x))u_m(x) = \sum_{n=0}^{\infty} v_n(y)T_x^n u_m(x) = \sum_{n=0}^{\infty} v_n(y)u_{m-n}(x) = E^y u_n(x).$$

The theorem follows from the linearity of $E(y;T_x)$. □

Theorem 3.3. *With I being the identity operator and f a polynomial, we have*

$$\lim_{y\to 0}\frac{E^y - I}{y}\, f(x) = \sum_{m=1}^{\infty} v_m'(0)T_x^m f(x).$$

Proof. Use Theorem 3.2 to see that

$$\frac{1}{y}[E^y - I]\, f(x) = \sum_{m=0}^{\infty} \frac{v_m(y) - v_m(0)}{y} T_x^m f(x) \to \sum_{m=1}^{\infty} v_m'(0) T_x^m f(x),$$

as $y \to 0$. $\square$

An immediate consequence of Theorem 3.3 is

$$\lim_{y\to 0} \frac{E^y - I}{y}\, f(x) = \left.\frac{\partial E(y;T_x)}{\partial y}\right|_{y=0} f(x). \tag{3.4}$$

We now consider the concept of polynomials of binomial type [17,24,25] in our setup.

Definition 3.4. We say that a polynomial sequence $\{p_n(x)\}$ is of binomial type relative to T_x if it has a generating function of the form

$$\sum_{n=0}^{\infty} p_n(x) t^n = E(x; H(t)), \quad H(t) = \sum_{n=1}^{\infty} h_n t^n, \quad h_1 \neq 0.$$

The dependence on T_x is implicit in the definition of the generalized exponential E.

Theorem 3.5. *Assume that $\{p_n(x)\}$ is a polynomial sequence and $p_n(0) = \delta_{n,0}$. Then $\{p_n(x)\}$ is of binomial type relative to T_x if and only if $\{p_n(x)\}$ belongs to $J = J(T_x) := \sum_{k=1}^{\infty} a_k T_x^k$, where a_k are constants, $a_1 \neq 0$ and $H(t)$ is the inverse function of $\sum_{k=1}^{\infty} a_k t^k$.*

Proof. Assume that $\{p_n(x)\}$ is of binomial type relative to T_x. Let $J(t)$ be the inverse function to $H(t)$. We assume that $\{v_n(x)\}$ belongs to T_x and satisfies (3.3). Then

$$\begin{aligned}
J(T_x) \sum_{n=0}^{\infty} p_n(x) t^n &= J(T_x) \sum_{n=0}^{\infty} v_n(x) (H(t))^n \\
&= \sum_{m=1}^{\infty} a_m T_x^m \sum_{n=0}^{\infty} v_n(x) (H(t))^n \\
&= \sum_{m=1}^{\infty} a_m \sum_{n=m}^{\infty} v_{n-m}(x) (H(t))^n \\
&= \sum_{m=1}^{\infty} a_m (H(t))^m E(x; H(t)) = tE(x; H(t)) \\
&= t \sum_{n=0}^{\infty} p_n(x) t^n.
\end{aligned}$$

This shows that $J(T_x) p_n(x) = p_{n-1}(x)$, so $\{p_n(x)\}$ belongs to $J(T_x)$. For the converse, assume that $\{v_n(x)\}$ belongs to T_x and $v_n(0) = \delta_{n,0}$. This defines the function $E(x;t)$. Define $H(t)$ to be the inverse function of $\sum_{k=1}^{\infty} a_k t^k$. Then construct $E(x;t)$ and define a polynomial sequence $\{q_n(x)\}$ of binomial type with respect to T_x by

$$\sum_{n=0}^{\infty} q_n(x)t^n = E(x;H(t)).$$

As in the first part $\{q_n(x)\}$ belongs to the same operator as $\{p_n(x)\}$. Hence, there exists a sequence of constants b_n such that

$$p_n(x) = \sum_{k=0}^{n} b_k q_{n-k}(x).$$

Now use

$$p_n(0) = q_n(0) = \delta_{n,0}$$

and find that

$$\delta_{n,0} = \sum_{k=0}^{n} b_k \delta_{n,k} = b_n.$$

Thus, $p_n(x) = q_n(x)$. □

We now come to the product of functionals. Roman and Rota [24] defined the product of functionals acting on the vector space of polynomials. Later Joni and Rota [17], and Ihrig and Ismail [9] observed that the product of functionals can be described in terms of a coalgebra map Δ. They used $\Delta x = x \otimes 1 + 1 \otimes x$. For a polynomial p, they defined the product of two functionals L and M by

$$< LM|p(x) >=< L \otimes M|\Delta p(x) >:=< L \otimes M|p(\Delta x) > .$$

Using the binomial theorem, we get

$$< LM|\frac{x^n}{n!} >= \sum_{k=0}^{n} < L|\frac{x^k}{k!} >< M|\frac{x^{n-k}}{(n-k)!} > .$$

We note that the $+$ in the definition of $\Delta x = x \otimes 1 + 1 \otimes x$ is a translation. This suggests the replacing $+$ by a generalized translation, so we define

$$< LM|u_n(x) >= \sum_{k=0}^{n} < L|u_k(x) >< M|v_{n-k}(x) > .$$

Note that the product of functional is commutative if and only if $v_n(x) = u_n(x)$.

We do not know yet how to incorporate the theory of Rota's umbral calculus within the present approach using translations.

4. First q-Translation

We now come to work of Liu who essentially utilized the set up of Section 1 to give new proofs of q-series identities and derived some new ones. He looked at solutions of $T_{q,x}F(x,y) = T_{q,y}F(x,y)$, where

$$(T_{q,x}f)(x) = \frac{f(x) - f(qx)}{x}.$$

So this is our T_x. He chose $u_n(x) = v_n(x) = x^n/(q;q)_n$. Using our approach we see that

$$E^{y_1} x^n = (q;q)_n E^{y_1} \frac{x^n}{(q;q)_n} = \sum_{k=0}^{n} \begin{bmatrix} n \\ k \end{bmatrix}_q y^k x^{n-k} := h_n(x,y).$$

At this stage we need to introduce the multivariable Rogers–Szegő polynomials

$$h_n(x_1, x_2, \ldots, x_s) := \sum_{\sum j_r = n} \frac{(q;q)_n}{\prod_{r=1}^{s} (q;q)_{j_r}} \prod_{r=1}^{s} x_r^{j_r}.$$

If $y_1, y_2, \ldots, y_s$ are distinct then it readily follows that

$$\prod_{j=1}^{r} E^{y_j} x^n = h_n(x, y_1, y_2, \ldots, y_s).$$

In the present case $u_n(0) = \delta_{n,0}$ so that $g(t) = 1$, and the corresponding exponential function becomes

$$E(x;t) = \sum_{n=0}^{\infty} \frac{x^n}{(q;q)_n} t^n = \frac{1}{(xt;q)_\infty}.$$

The large n behavior of $h_n(x,y)$ is easy to determine. It is easy to see that

$$\sum_{n=0}^{\infty} h_n(x,y) \frac{t^n}{(q;q)_n} = \frac{1}{(xt, yt;q)_\infty}.$$

If $|y/x| < 1$, then using Tannery's theorem [6, p. 316] or applying Darboux's method [22] to the above generating function

$$\lim_{n\to\infty} x^{-n} h_n(x,y) = \frac{1}{(y/x;q)_\infty}.$$

Therefore, when $|y| < |x|$, the series $\sum_{n=0}^{\infty} a_n x^n$ converges absolutely if and only if the series $\sum_{n=0}^{\infty} a_n h_n(x,y)$ converges absolutely. This means that

$$E^y \sum_{n=0}^{\infty} a_n x^n = \sum_{n=0}^{\infty} a_n h_n(x,y)$$

is a well-defined operator on the space of analytic functions on a fixed disc, say $|x| \le a$. This provides a map

$$\sum_{n=0}^{\infty} a_n x^n \to \sum_{n=0}^{\infty} a_n h_n(x,y).$$

In this way, we find summation theorems, or identities involving the Rogers–Szegő polynomials, if we have explicit representation for $\sum_{n=0}^{\infty} a_n x^n$ and can evaluate $E^y \sum_{n=0}^{\infty} a_n x^n$. Liu's work contains many instances of this process.

Hence, E^y commutes with E^z for all y, z. Moreover, this translation has the important symmetry $E^y f(x) = E^x f(y)$.

In this case, Theorem 3.3 becomes the following theorem.

Theorem 4.1. *For all polynomials f,*

$$\lim_{y\to 0}\frac{E^y-I}{y}f(x)=\frac{1}{1-q}T_{q,x}f(x).$$

5. The Askey–Wilson Operator

For completeness, we define the Askey–Wilson operator. Given a function f we set $\breve{f}\left(e^{i\theta}\right):=f(x)$, $x=\cos\theta$, that is

$$\breve{f}(z)=f((z+1/z)/2),\quad z=e^{\pm i\theta}.$$

With $e(x)=x$ the Askey–Wilson divided difference operator is defined by

$$(\mathcal{D}_q f)(x):=\frac{\breve{f}(q^{1/2}z)-\breve{f}(q^{-1/2}z)}{\breve{e}(q^{1/2}z)-\breve{e}(q^{-1/2}z)}=\frac{\breve{f}(q^{1/2}e^{i\theta})-\breve{f}(q^{-1/2}e^{i\theta})}{(q^{1/2}-q^{-1/2})(z-1/z)/2},$$

where $x=(z+1/z)/2$.

In this case, we let

$$u_n(\cos\theta):=(-i)^n\,(-iq^{(1-n)/2}e^{i\theta},-iq^{(1-n)/2}e^{-i\theta};q)_n.$$

It is straightforward to see that

$$\mathcal{D}_q u_n(x)=\frac{2(1-q^n)}{1-q}q^{(1-n)/2}u_{n-1}(x).$$

We set

$$\mathcal{T}_x=q^{-1/4}\frac{1-q}{2}\mathcal{D}_{q,x}. \tag{5.1}$$

Thus, $\{q^{n^2/4}u_n(x)/(q;q)_n\}$ belongs to $\mathcal{T}_x$. To define $E(x;t)$ we choose the u_n in (3.1) as $q^{n^2/4}u_n(x)/(q;q)_n$ and $T_x=\mathcal{T}_x$. Thus,

$$E(x;t)=g(t)\sum_{n=0}^{\infty}\frac{q^{n^2/4}u_n(x)}{(q;q)_n}t^n.$$

The requirement that $E(0;t)=1$ gives

$$\begin{aligned}1/g(t)&=\sum_{n=0}^{\infty}(-it)^n\,\frac{q^{n^2/4}}{(q;q)_n}(q^{(1-n)/2},-q^{(1-n)/2};q)_n\\&=\sum_{n=0}^{\infty}(-it)^n\,q^{n^2/4}\frac{(q^{1-n};q^2)_n}{(q;q)_n}\\&=\sum_{n=0}^{\infty}(-t^2)^n\frac{q^{n^2}(q^{1-2n};q^2)_{2n}}{(q;q)_{2n}}\\&=\sum_{n=0}^{\infty}(-t^2)^nq^{n^2}\frac{(q^{1-2n},q;q^2)_n}{(q,q^2;q^2)_n}\\&=\sum_{n=0}^{\infty}\frac{t^{2n}(q;q^2)_n}{(q^2;q^2)_n}.\end{aligned}$$

The q-binomial theorem [1,8] implies $g(t) = (t^2; q^2)_\infty/(qt^2; q^2)_\infty$. This identifies $E(x;t)$ as the q-exponential function $\mathcal{E}_q$ introduced by Ismail and Zhang in [16], namely

$$\mathcal{E}_q(x;t) = \frac{(t^2;q^2)_\infty}{(qt^2;q^2)_\infty}\sum_{n=0}^{\infty}(-ie^{i\theta}q^{(1-n)/2}, -ie^{-i\theta}q^{(1-n)/2}; q)_n \frac{(-it)^n}{(q;q)_n} q^{n^2/4}. \tag{5.2}$$

See also [12, Chapter 14].

Ismail and Stanton [15] introduced the polynomial bases

$$\phi_n(\cos\theta) = (q^{1/4}e^{i\theta}, q^{1/4}e^{-i\theta}; q^{1/2})_n, \tag{5.3}$$

$$\rho_n(\cos\theta) = (1+e^{2i\theta})(-q^{2-n}e^{2i\theta}; q^2)_{n-1}e^{-in\theta}, \tag{5.4}$$

and gave formulas for expanding entire functions in this basis. Observe that ρ_n is invariant under $q \to 1/q$. It is easy to see that

$$\mathcal{D}_q\phi_n(x) = -2q^{1/4}\frac{1-q^n}{1-q}\phi_{n-1}(x), \quad \mathcal{D}_q\rho_n(x) = 2q^{(1-n)/2}\frac{1-q^n}{1-q}\rho_{n-1}(x). \tag{5.5}$$

The q-Hermite polynomials are

$$H_n(\cos\theta|q) = \sum_{k=0}^{n}\begin{bmatrix} n \\ k\end{bmatrix}_q e^{i(n-2k)\theta},$$

and satisfy the operator formula

$$\mathcal{D}_q H_n(x|q) = 2q^{(1-n)/2}\frac{1-q^n}{1-q}H_{n-1}(x|q).$$

We note that $\rho_n(0) = \delta_{n,0}$ but $\phi_n(0) \neq 0$ for all n while $H_{2n}(0|q) \neq 0$ for any n. In the language of Sect. 2 we see that $q^{n^2/4}H_n(x|q)/(q;q)_n$ belongs to $\mathcal{T}_x$. Using the generating function

$$\sum_{n=0}^{\infty} H_n(\cos\theta|q)\frac{t^n}{(q;q)_n} = \frac{1}{(te^{i\theta}, te^{-i\theta}; q)_\infty},$$

we find that $H_{2n+1}(0|q) = 0$, and $H_{2n}(0|q) = (-1)^n(q;q)_{2n}/(q^2;q^2)_n$. This establishes the generating function

$$(qt^2;q^2)_\infty\mathcal{E}_q(x;t) = \sum_{n=0}^{\infty}\frac{t^n q^{n^2/4}}{(q;q)_n}H_n(x|q). \tag{5.6}$$

Theorem 5.1. *The polynomials $\{\rho_n(x)\}$ have the generating function*

$$\sum_{n=0}^{\infty}\frac{q^{n^2/4}t^n}{(q;q)_n}\, t^n\rho_n(x) = \mathcal{E}_q(x;t). \tag{5.7}$$

The function $\mathcal{E}_q(x;t)$ has the representation

$$\mathcal{E}_q(x;t) = \frac{(-t;q)_\infty}{(tq^{1/2};q)_\infty}\,{}_2\phi_1\left(\left.\begin{matrix} q^{1/4}e^{i\theta}, q^{1/4}e^{-i\theta} \\ -q^{1/2}\end{matrix}\right| q^{1/2}, -t\right). \tag{5.8}$$

Proof. It is clear from (5.1) and (5.5) that $w_n(x) = q^{n^2/4}\rho_n(x)/(q;q)_n$ satisfies $\mathcal{T}_x w_n(x) = w_{n-1}$, that is, $\{w_n(x)\}$ belongs to $\mathcal{T}_x$. Therefore, its generating function must be of the form $h(t)\mathcal{E}_q(x;t)$ for some power series $h(t)$. But $\rho_n(0) = \delta_{n,0}, E(0;t) = 1$. Therefore $h(t) \equiv 1$ and (5.7) follows. Similarly, (5.5) implies that $(-1)^n\phi_n(x)/(q;q)_n$ belongs to $\mathcal{T}_x$; hence,

$$\sum_{n=0}^{\infty} \frac{(-t)^n\phi_n(x)}{(q;q)_n} = h(t)\ \mathcal{E}_q(x;t),$$

and to find h we take $x = 0$. Clearly

$$\phi_n(0) = (iq^{1/4}, -iq^{1/4}; q^{1/2})_n = (-q^{1/2}; q)_n,$$

and the q-binomial theorem shows that $h(t) = (q^{1/2}t;q)_\infty/(-t;q)_\infty$. This proves (5.8). □

It must be noted that (5.7) is new but (5.8) was first proved in [14].

From (5.3), it is easy to see that

$$\begin{aligned}\rho_{2n}(\cos\theta) &= q^{n(1-n)}(-e^{2i\theta}, -e^{-2i\theta}; q^2)_n,\\ \rho_{2n+1}(\cos\theta) &= 2q^{-n^2}\cos\theta\ (-qe^{2i\theta}, -qe^{-2i\theta}; q^2)_n,\end{aligned} \tag{5.9}$$

see page 261 in [14].

Theorem 5.2. *We have the following transformation which changes a base q to q^4:*

$$\begin{aligned}\mathcal{E}_q(\cos\theta;t) &= \frac{(-t;q)_\infty}{(tq^{1/2};q)_\infty}\,{}_2\phi_1\left(q^{1/4}e^{i\theta}, q^{1/4}e^{-i\theta}; -q^{1/2}\,\middle|\, q^{1/2}, -t\right)\\ &= {}_2\phi_1\left(-e^{2i\theta}, -e^{-2i\theta}; q\,\middle|\, q^2, qt^2\right)\\ &\quad + \frac{2tq^{1/4}\cos\theta}{1-q}\,{}_2\phi_1\left(-qe^{2i\theta}, -qe^{-2i\theta}; q^3\,\middle|\, q^2, qt^2\right).\end{aligned} \tag{5.10}$$

Proof. The proof follows from (5.7), (5.8), and (5.9). □

We now discuss possible explanations of where (5.10) comes from. Replace t by $(1-q)t$ then let $t \to 1$ in (5.10). It is straightforward to see that $(-t;q)_\infty/(tq^{1/2};q)_\infty$, being (with $t \to t(1-q)$)

$$\sum_{n=0}^{\infty}(-q^{-1/2}; q^{1/2})_n(1-q)^n t^n/(q;q)_n,$$

becomes e^{2t}. The ${}_2\phi_1$ on the right-hand side tends to

$$\sum_{n=0}^{\infty}(\frac{-t}{n!})^n(2\sin(\theta/2))^{2n},$$

which is $\exp(-4t\sin^2\theta/2)$. The right-hand side tends to

$$\begin{aligned}\sum_{n=0}^{\infty}\frac{(2t\cos\theta)^{2n}}{(2n)!} + 2t\cos\theta\sum_{n=0}^{\infty}\frac{(2t\cos\theta)^{2n}}{(2n+1)!} &= \cosh(2t\cos\theta) + \sinh(2t\cos\theta)\\ &= \exp(2t\cos\theta).\end{aligned}$$

Therefore, (5.10) seems to be a q-analogue of the trigonometric double angle formula in the form

$$\exp(2t\cos 2\theta) = \exp(2t - 4t\sin^2\theta).$$

Another explanation is to observe that the right-hand side of (5.10) is a sum of an odd and an even function of t. Since the left-hand side of (5.10) is an analogue of the exponential function $\exp(xt)$, its odd and even parts must be q-analogues of $\cosh(xt)$ and $\sinh(xt)$. This is indeed the case and these functions appeared first in [4] even before the function $\mathcal{E}_q$ was defined in [16].

The fact that $\mathcal{E}_q(x;t)$ has the three different representations (5.2), (5.6), and (5.8) leads to interesting consequences, the first of them, of course is (5.10).

Theorem 5.3. *We have the polynomial expansion*

$$\begin{aligned}\rho_n(\cos\theta) &= (1+e^{2i\theta})(-q^{2-n}e^{2i\theta};q^2)_{n-1}e^{-in\theta}\\ &= \sum_{k=0}^{\lfloor n/2\rfloor}\begin{bmatrix} n\\ 2j\end{bmatrix}_q q^{j(j+1-n)}(q;q^2)_j H_{n-2j}(\cos\theta|q),\end{aligned}$$

or equivalently

$$\begin{aligned}&\int_0^\pi (1+e^{2i\theta})(-q^{2-n}e^{2i\theta};q^2)_{n-1}e^{-in\theta}H_{n-2j}(\cos\theta|q)(e^{2i\theta},e^{-2i\theta};q)_\infty\,d\theta\\ &\quad = \frac{2\pi(q;q)_n}{(q;q)_\infty(q;q)_{2j}}\,q^{j(j+1-n)}(q;q^2)_j.\end{aligned}$$

Proof. The theorem follows from (5.6), (5.7) and the orthogonality relation [12]

$$\frac{(q;q)_\infty}{2\pi}\int_0^\pi H_m(\cos\theta|q)H_n(\cos\theta|q)(e^{2i\theta},e^{-2i\theta};q)_\infty \mathrm{d}\theta = (q;q)_n\delta_{m,n}.$$

□

We next introduce another polynomial basis. Let

$$s_n(x;a) = (aq^{-n/2}e^{i\theta}, aq^{-n/2}e^{-i\theta};q)_n.$$

Theorem 5.4. *We have the expansion*

$$\mathcal{E}_q(x;t) = g(t)\sum_{n=0}^\infty \frac{(aq^{-n/2}e^{i\theta}, aq^{-n/2}e^{-i\theta};q)_n}{(q;q)_n}\left(-\frac{t}{a}\right)^n q^{n(n+2)/4},$$

and

$$\frac{\mathcal{E}_q(x_0;t)}{g(t)} = \sum_{n=0}^\infty \frac{(aq^{-n/2}c, aq^{-n/2}/c;q)_n}{(q;q)_n}\left(-\frac{t}{a}\right)^n q^{n(n+2)/4}, \qquad (5.11)$$

where $x_0 = (c+1/c)/2$, $c \neq 0$.

Proof. It readily follows that

$$\mathcal{D}_q s_n(x;a) = \frac{-2aq^{-n/2}}{1-q}(1-q^n)s_{n-1}(x;a).$$

Therefore $\{s_n(x;a)(-a)^{-n}q^{n(n+2)/4}/(q;q)_n\}$ belongs to $\mathcal{T}_x$ which was defined in (5.1). The theorem now follows from Theorem 2.1. □

In choosing a and x_0 in Theorem 5.4 we should be able to evaluate $\mathcal{E}_q(x_0; t)$ and the series defining g. The choice $x_0 = 0$, and $a = -iq^{1/2}$ leads to (5.2) which is the original definition of $\mathcal{E}_q$ in [16]. Another choice is $x_0 = [q^{1/4} + q^{-1/4}]/2$ where the sum in (5.11) becomes

$$\begin{aligned}\sum_{n=0}^{\infty} &\frac{(aq^{(1-2n)/4}, aq^{-(1+2n)/4}; q)_n}{(q;q)_n}\left(-\frac{t}{a}\right)^n q^{n(n+2)/4} \\ &= \sum_{n=0}^{\infty}\frac{(aq^{-(1+2n)/4}; q^{1/2})_{2n}}{(q;q)_n}\left(-\frac{t}{a}\right)^n q^{n(n+2)/4}.\end{aligned}$$

Clearly (5.8) gives

$$\mathcal{E}_q(x_0; t) = (-t; q)_\infty/(q^{1/2}t; q)_\infty.$$

The above series can be summed when $a = q^{(2j+1)/4}, j = 0, 1, \dots$ because it terminates. The choices $j = 0, 1$, that is, $a = q^{1/4}$ or $a = q^{3/4}$, imply

$$g(t) = \frac{(-t;q)_\infty}{(tq^{1/2};q)_\infty}.$$

This leads to

$$\mathcal{E}_q(\cos\theta; t) = \frac{(-t;q)_\infty}{(tq^{1/2};q)_\infty}\sum_{n=0}^{\infty}\frac{(q^{(1-2n)/4}e^{i\theta}, q^{(1-2n)/4}e^{-i\theta}; q)_n}{(q;q)_n} \times (-t)^n q^{-n/4}.$$

The q^{-1}-Hermite polynomials $\{h_n(x|q)\}$ of Askey [2] and Ismail and Masson [13]. These are defined by

$$h_n(x|q) = i^{-n}H_n(ix|1/q).$$

It is a simple exercise to see that they have the generating function

$$\sum_{n=0}^{\infty} h_n(\sinh\xi|q)\frac{t^n}{(q;q)_n}\, q^{\binom{n}{2}} = (-te^{\xi}, te^{-\xi}; q)_\infty.$$

This suggests setting

$$\begin{aligned} x &:= (z - 1/z)/2, \quad f(x) = \breve{f}(z), \\ (\mathcal{D}_q f)(x) &= \frac{\breve{f}(q^{1/2}z) - \breve{f}(q^{-1/2}z)}{(q^{1/2} - q^{-1/2})(z + 1/z)/2} \end{aligned} \tag{5.12}$$

and defining another q-exponential function by

$$\tilde{\mathcal{E}}_q(x;t) := \frac{(t^2;q^2)_\infty}{(qt^2;q^2)_\infty}\sum_{n=0}^{\infty}(-zq^{(1-n)/2}, q^{(1-n)/2}/z; q)_n \frac{t^n}{(q;q)_n} q^{n^2/4}.$$

It is easy to see that

$$\begin{aligned}&\mathcal{D}_q(-zq^{(1-n)/2}, q^{(1-n)/2}/z; q)_n \\ &\qquad = 2q^{(1-n)/2}\frac{1-q^n}{1-q}\,(-zq^{(2-n)/2}, q^{(2-n)/2}/z; q)_{n-1}.\end{aligned}$$

Hence,

$$\mathcal{D}_q\tilde{\mathcal{E}}_q(x;t) = \frac{2q^{1/4}t}{1-q}\tilde{\mathcal{E}}_q(x;t).$$

On the other hand, [12, (21.6.7)]

$$\mathcal{D}_q h_n(x|q) = 2q^{(1-n)/2}\frac{1-q^n}{1-q}h_{n-1}(x|q).$$

Therefore,

$$\tilde{\mathcal{E}}_q(x;t) = g(t)\sum_{n=0}^{\infty}\frac{h_n(x|q)}{(q;q)_n}q^{n^2/4}t^n,$$

for some power series $g(t)$. It is known that

$$h_{2n+1}(0|q) = 0, \quad h_{2n}(0|q) = (-1)^n q^{-n^2}(q;q^2)_n,$$

see, for example, [12, (21.3.7)]. Therefore,

$$\frac{1}{g(t)} = \sum_{n=0}^{\infty}(-t^2)^n(q;q^2)_n(q;q)_{2n} = \sum_{n=0}^{\infty}(-t^2)^n(q^2;q^2)_n = 1/(-t^2;q^2)_\infty.$$

This establishes the expansion

$$\tilde{\mathcal{E}}_q(x;t) = (-t^2;q^2)_\infty\sum_{n=0}^{\infty}\frac{h_n(x|q)}{(q;q)_n}q^{n^2/4}t^n.$$

Next we set

$$\begin{aligned}\tilde{\rho}_n(\sinh\xi) &= e^{-n\xi}(1-e^{2\xi})(q^{2-n}e^{2\xi};q^2)_{n-1}\\ &= z^{-n}(1-z^2)(q^{2-n}z^2;q^2)_{n-1},\end{aligned}$$

and find that

$$\mathcal{D}_q\tilde{\rho}_n(x) = 2q^{(1-n)/2}\,\frac{1-q^n}{1-q}\tilde{\rho}_{n-1}(x).$$

Therefore, $\left\{\frac{h_n(x|q)}{(q;q)_n}q^{n^2/4}\right\}$ and $\left\{\frac{\tilde{\rho}_n(x)}{(q;q)_n}q^{n^2/4}\right\}$ belong to $\mathcal{D}_q$. Since $\tilde{\rho}_n(0) = \delta_{n,0}$ then

$$\tilde{\mathcal{E}}_q(x;t) = \sum_{n=0}^{\infty}\frac{\tilde{\rho}_n(x)}{(q;q)_n}q^{n^2/4}t^n.$$

From here one can easily derive formulas for $\tilde{\mathcal{E}}_q$ which are analogous to the ones we recorded for $\mathcal{E}_q$.

It must be noted that the formulas starting from (5.12) are expected as analogues of the earlier expansions when we formally replace q by $1/q$. This is so since $q \to 1/q$ in the series in $\sum_{n=0}^{\infty}x^n/(q;q)_n$ turns it into $\sum_{n=0}^{\infty}(-x)^n q^{\binom{n+1}{2}}/(q;q)_n$. In other words, $q \to 1/q$ maps $1/(x;q)_\infty$ to $(qx;q)_\infty$. This formal process, however, cannot be justified since both series have pole singularities at the nth roots of unity for all $n = 1, 2, \ldots$, so these series cannot be analytically continued from $|q| < 1$ to $|q| > 1$.

6. An Askey–Wilson Translation

In this section, we study the translation operator associated with $\mathcal{D}_q$, or $\mathcal{T}_x$ of (5.1). This translation was first considered in [11]. The treatment presented here is a major simplification of the treatment in [11].

Recall the Ismail–Zhang two variable function [16]

$$\mathcal{E}_q(\cos\theta, \cos\phi; t) := \frac{(t^2; q^2)_\infty}{(qt^2; q^2)_\infty} \sum_{n=0}^{\infty} \frac{(te^{-i\phi})^n}{(q;q)_n} q^{n^2/4} (-e^{i(\phi+\theta)} q^{(1-n)/2}, -e^{i(\phi-\theta)} q^{(1-n)/2}; q)_n. \tag{6.1}$$

Motivated by the definition (5.2) we let

$$u_n(\cos\theta, \cos\phi) := e^{-in\phi} (-e^{i(\phi+\theta)} q^{(1-n)/2}, -e^{i(\phi-\theta)} q^{(1-n)/2}; q)_n.$$

It is straightforward to see that $u_n(x, y)$ is symmetric in x and y. Moreover,

$$\mathcal{D}_{q,x} u_n(x, y) = \frac{2(1-q^n)}{1-q} q^{(1-n)/2} u_{n-1}(x, y).$$

Theorem 6.1. *The polynomials $\{u_n(x)\}$ and $\{\rho_n(x)\}$ are related via*

$$2x u_n(x) = \rho_{n+1}(x). \tag{6.2}$$

Proof. With $x = (z + 1/z)/2$ it is clear that

$$\begin{aligned} 2x u_n(x) &= (z + 1/z)(-i)^n (-izq^{(1-n)/2}; q)_n (-iz^{-1} q^{(1-n)/2}; q)_n \\ &= (z + 1/z)(-i)^n (-izq^{(1-n)/2}; q)_n (i/z)^n (1 - izq^{(n-1)/2}) \\ &\qquad \cdots (1 - izq^{(1-n)/2}) \\ &= (1 + z^2) z^{-n-1} (-izq^{(1-n)/2}; q)_n (izq^{(1-n)/2}; q)_n, \end{aligned}$$

which gives (6.2). □

It is clear that $\mathcal{E}_q(x, y; t)$ has a convergent expansion in $H_n(x|q)$, so we let

$$\mathcal{E}_q(x, y; t) = \sum_{n=0}^{\infty} a_n(y, t) \frac{q^{n^2/4}}{(q;q)_n} H_n(x|q).$$

We also know that

$$\mathcal{T}_x \mathcal{E}_q(x, y; t) = \mathcal{T}_y \mathcal{E}_q(x, y; t) = t\mathcal{E}_q(x, y; t).$$

Therefore,

$$a_{n+1}(y, t) = t a_n(y, t)$$

and we conclude that

$$\mathcal{E}_q(x, y; t) = a_0(y, t) \mathcal{E}_q(x; t).$$

But by simple manipulations on the definition (6.1) one can see that $\mathcal{E}_q(x,y;t)$ is symmetric in x and y. Therefore, $a_0(y,t) = E(y;t)g(t)$ for some power series $g(t)$. Finally,

$$\begin{aligned}
g(t) &= \mathcal{E}_q(0,y;t)/\mathcal{E}_q(0;t)\mathcal{E}_q(0;t) \\
&= \frac{(t^2;q^2)_\infty}{(qt^2;q^2)_\infty} \sum_{n=0}^{\infty} \frac{(it)^n}{(q;q)_n} q^{n^2/4}(q^{(1-n)/2}, -q^{(1-n)/2};q)_n \\
&= \frac{(t^2;q^2)_\infty}{(qt^2;q^2)_\infty} \sum_{n \text{ even}} \frac{(it)^n}{(q;q)_n} q^{n^2/4}(q^{1-n};q^2)_n.
\end{aligned}$$

The odd terms in the sum vanish; hence,

$$g(t) = \frac{(t^2;q^2)_\infty}{(qt^2;q^2)_\infty} \sum_{n=0}^{\infty} \frac{(-1)^n t^{2n}}{(q;q)_{2n}} q^{n^2}(q^{1-2n};q^2)_{2n}.$$

Now

$$(q^{1-2n};q^2)_{2n} = (q^{1-2n};q^2)_n(q;q^2)_n = (-1)^n q^{-n^2}(q,q;q^2)_n.$$

The q-binomial theorem shows that $g(t) \equiv 1$. This proves the addition theorem

$$\mathcal{E}_q(x,y;t) = \mathcal{E}_q(x;t)\mathcal{E}_q(y;t), \tag{6.3}$$

originally due to Suslov [27], see also [12]. In the later work [28], Suslov proved the more general addition theorem

$$\begin{aligned}
&\mathcal{E}_q(x;s)\mathcal{E}_q(y;t) \\
&\quad= \frac{(t^2;q^2)_\infty}{(qs^2;q^2)_\infty} \sum_{n=0}^{\infty} \frac{q^{n^2/4}t^n}{(q;q)_n} \\
&\qquad \times e^{-in\phi}(-q^{(1-n)/2}e^{i(\phi+\theta)}s/t, -q^{(1-n)/2}e^{i(\phi-\theta)}s/t;q)_n \\
&\qquad \times {}_2\phi_2\left(\left.\begin{matrix} q^{-n}, s^2/t^2 \\ -q^{(1-n)/2}e^{i(\phi+\theta)}s/t, -q^{(1-n)/2}e^{i(\phi-\theta)}s/t \end{matrix}\right| q, qe^{2i\phi}\right).
\end{aligned} \tag{6.4}$$

We expect that (6.4) can be also proved by computing the action of the Askey–Wilson operator on

$$\begin{aligned}
&\frac{q^{n^2/4}}{(q;q)_n} e^{-in\phi}(-q^{(1-n)/2}e^{i(\phi+\theta)}s/t, -q^{(1-n)/2}e^{i(\phi-\theta)}s/t;q)_n \\
&\qquad \times {}_2\phi_2\left(\left.\begin{matrix} q^{-n}, & s^2/t^2 \\ -q^{(1-n)/2}e^{i(\phi+\theta)}s/t, & -q^{(1-n)/2}e^{i(\phi-\theta)}s/t \end{matrix}\right| q, qe^{2i\phi}\right).
\end{aligned}$$

We define another divided difference operator by

$$\Delta_{q,t}f(t) = \frac{f(tq^{1/2} - f(tq^{-1/2})}{t(q^{1/2} - q^{-1/2})}.$$

We next prove the following theorem.

Theorem 6.2. *We have*

$$\Delta_{q,t}\mathcal{E}_q(x;t) = \frac{2xq^{-1/4}}{(1-q)} \frac{(qt^2;q^2)_\infty}{(t^2;q^2)_\infty} \mathcal{E}_q(x;t).$$

Proof. Using the identity (6.2), the definition of $\mathcal{E}_q$, and the generating function (5.7), we find that

$$\begin{aligned}
2xt\frac{(qt^2;q^2)_\infty}{(t^2;q^2)_\infty}\,\mathcal{E}_q(x;t) &= \sum_{n=0}^{\infty}\frac{q^{n^2/4}t^{n+1}}{(q;q)_n}\rho_{n+1}(x) \\
&= q^{1/4}\sum_{n=0}^{\infty}\frac{(t/\sqrt{q})^{n+1}(1-q^{n+1})}{(q;q)_{n+1}}q^{(n+1)^2/4}\rho_{n+1}(x|q) \\
&= q^{1/4}\mathcal{E}_q(x;tq^{-1/2}) - q^{1/4}\mathcal{E}_q(x;tq^{1/2}),
\end{aligned}$$

where we used the generating function (5.7) in the last step. □

We now apply the definition (2.2) with u_n and v_n replaced by $u_n(x)q^{n^2/4}/(q;q)_n$ and $\rho_n(x)q^{n^2/4}/(q;q)_n$. Therefore,

$$E^y\left(\frac{u_n(x)q^{n^2/4}}{(q;q)_n}\right) = \sum_{k=0}^{n}\frac{u_k(x)q^{k^2/4}}{(q;q)_k}\,\frac{\rho_{n-k}(y)q^{(n-k)^2/4}}{(q;q)_{n-k}}. \tag{6.5}$$

Indeed the above definition is nothing but

$$E^y u_n(x) = \sum_{k=0}^{n}\begin{bmatrix} n \\ k \end{bmatrix}_q q^{k(k-n)/2}u_k(x)\rho_{n-k}(y). \tag{6.6}$$

Thus, the exponential function $E(x,t)$ of Sect. 3 is $\mathcal{E}_q(x;t)$.

Theorem 6.3. *We have*

$$E^y u_n(x) = u_n(x,y). \tag{6.7}$$

Proof. Multiply (6.5) by t^n and add for $n \ge 0$. Using (5.2) and (5.7) we find that

$$\begin{aligned}
\sum_{n=0}^{\infty} t^n E^y\left(\frac{u_n(x)q^{n^2/4}}{(q;q)_n}\right) &= \sum_{k=0}^{\infty}\frac{u_k(x)q^{k^2/4}}{(q;q)_k}\,t^k\mathcal{E}_q(y;t) \\
&= \mathcal{E}_q(y;t)\frac{(qt^2;q)_\infty}{(t^2;q)_\infty}\mathcal{E}_q(x;t) \\
&= \frac{(qt^2;q)_\infty}{(t^2;q)_\infty}\mathcal{E}_q(x,y;t),
\end{aligned}$$

where we used the addition theorem (6.3). The theorem now follows from (6.1). □

In terms of products (6.7) is

$$\begin{aligned}
&E^y\left[\prod_{j=0}^{n-1}\{2x + i(q^{j+(1-n)/2} - q^{-j+(n-1)/2})\}\right] \\
&\quad = \prod_{j=0}^{n-1}\left[2x + e^{i\phi}q^{j+(1-n)/2} + e^{-i\phi}q^{-j+(n-1)/2}\right].
\end{aligned} \tag{6.8}$$

For convenience we write (6.8) in terms of x and y directly by considering even and odd n separately in the form

$$
\begin{aligned}
&E^y \prod_{j=0}^{n-1}\left[4x^2+(q^{j-n+1/2}-q^{n-j-1/2})^2\right] \\
&\quad = \prod_{j=0}^{n-1}\left[4x^2+4y^2+4xy(q^{j-n+1/2}+q^{n-j-1/2})+(q^{j-n+1/2}-q^{n-j-1/2})^2\right], \\
&E^y\, x \prod_{j=0}^{n-1}\left[4x^2+(q^{j-n}-q^{n-j})^2\right] \\
&\quad = (x+y)\prod_{j=0}^{n-1}\left[4x^2+4y^2+4xy(q^{j-n}+q^{n-j})+(q^{j-n}-q^{n-j})^2\right],
\end{aligned}
$$

respectively. Moreover,

$$E^y x^n = E^x y^n = \sum_{k=0}^{n} \begin{bmatrix} n \\ k \end{bmatrix}_q q^{k(k-n)/2} x^k y^{n-k}. \tag{6.9}$$

It is clear that (6.9) is a commutative q-binomial theorem. For non-commutative q-binomial theorem, see [21].

Theorem 6.4. *The symmetry relation $(E^y f)(x) = (E^x f)(y)$ holds.*

It must be noted that the Askey–Wilson operator is invariant under $q \to 1/q$. This is reflected in the definition of the translation in (6.6) since the Gaussian binomial coefficient $\begin{bmatrix} n \\ k \end{bmatrix}_q$ is a unimodal polynomial in q of order $k(n-k)$, whose coefficients are symmetric about the middle.

Corollary 6.5. *We have*

$$u_n(x,y) = \sum_{k=0}^{n} \begin{bmatrix} n \\ k \end{bmatrix}_q q^{k(k-n)/2} u_k(x) \rho_{n-k}(y).$$

Definition 6.6. The operator E^y is defined on all polynomials as a linear operator whose action on the basis $\{u_n(x)\}$ is given by (6.5), or (6.7). A polynomial sequence $\{p_n(x)\}$ is called of $\mathcal{D}_q$-polynomial type if

$$E^y p_n(x) = \sum_{k=0}^{n} p_k(x) p_{n-k}(y).$$

Theorem 6.7. *A polynomial sequence is of $\mathcal{D}_q$-binomial type if and only if it has a generating function of the type*

$$\sum_{n=0}^{\infty} p_n(x) t^n = \mathcal{E}_q(x; H(t)),$$

where $H(t) = \sum_{n=1}^{\infty} h_n t^n$.

Proof. For convenience we let

$$\tilde{\rho}_n(x) = q^{n^2/4}\rho_n(x)/(q;q)_n,$$

so that

$$E^y\tilde{\rho}_n(x) = \sum_{k=0}^{n}\tilde{\rho}_k(x)\tilde{\rho}_{n-k}(y).$$

We expand the generating function $\sum_{n=0}^{\infty} p_n(x)t^n$ in terms of $\{\tilde{\rho}_n(x)\}$, so we set

$$\sum_{n=0}^{\infty} p_n(x)t^n = \sum_{k=0}^{\infty}\alpha_k(t)\tilde{\rho}_n(x).$$

Thus

$$\begin{aligned}\sum_{n=0}^{\infty} t^n\sum_{k=0}^{n} p_k(x)p_{n-k}(y) &= E^y\sum_{n=0}^{\infty} p_n(x)t^n\\ &= E^y\sum_{n=0}^{\infty}\alpha_n(t)\tilde{\rho}_n(x)\\ &= \sum_{n=0}^{\infty}\alpha_n(t)\sum_{k=0}^{n}\tilde{\rho}_k(x)\tilde{\rho}_{n-k}(y)\\ &= \sum_{n,k=0}^{\infty}\alpha_{n+k}(t)\tilde{\rho}_k(x)\tilde{\rho}_n(y).\end{aligned}$$

This shows that

$$\begin{aligned}\sum_{n,k=0}^{\infty}\alpha_{n+k}(t)\tilde{\rho}_k(x)\tilde{\rho}_n(y) &= \left[\sum_{n=0}^{\infty} p_n(x)t^n\right]\left[\sum_{k=0}^{\infty} p_k(y)t^k\right]\\ &= \left[\sum_{r=0}^{\infty}\alpha_r(t)\tilde{\rho}_r(x)\right]\left[\sum_{s=0}^{\infty}\alpha_s(t)\tilde{\rho}_s(x)\right].\end{aligned}$$

Therefore, $\alpha_{n+k}(t) = \alpha_n(t)\alpha_k(t)$. This shows that $\alpha_0(t) = 1$ and $\alpha_n(t) = [\alpha_1(t)]^n$. This completes the proof. □

It readily follows from (6.5) that

$$E^y u_n(x)\frac{q^{n^2/4}}{(q;q)_n} = \sum_{k=0}^{n}\frac{q^{k^2/4}\rho_k(y)}{(q;q)_k}\mathcal{T}_x^k u_n(x)\frac{q^{n^2/4}}{(q;q)_n},$$

where $\mathcal{T}_x$ is defined in (5.1). Therefore,

$$E^y = \sum_{k=0}^{\infty}\frac{q^{k^2/4}\rho_k(y)}{(q;q)_k}\mathcal{T}_x^k = \mathcal{E}_q(y;\mathcal{T}_x).$$

We use (3.4) to see that the infinitesimal generator of E^y. We first evaluate $\rho'(0)$. Clearly

$$\rho_n'(0) = 2(-i)^{n-1}(q^{2-n};q^2)_{n-1}.$$

Therefore, $\rho'_{2n}(0) = 0$ and

$$\begin{aligned}\rho'_{2n+1}(0) &= 2(-1)^n (q^{1-2n}; q^2)_{2n} \\ &= 2(-1)^n (q^{1-2n}; q^2)_n (q; q^2)_n \\ &= 2q^{-n^2} (q; q^2)_n^2.\end{aligned}$$

Now

$$\begin{aligned}\sum_{n=0}^{\infty} \rho'_n(0) \frac{q^{n^2/4}}{(q;q)_n} \mathcal{T}_x^n &= 2q^{1/4} \sum_{n=0}^{\infty} \frac{q^n (q;q^2)_n^2}{(q;q)_{2n+1}} \\ &= 2q^{1/4} \sum_{n=0}^{\infty} \frac{q^n (q;q^2)_n}{(q^2;q^2)_n (1-q^{2n+1})} \mathcal{T}_x^{2n+1}.\end{aligned}$$

Therefore,

$$\lim_{y \to 0} \frac{1}{y} [E^y - I] = 2q^{1/4} \sum_{n=0}^{\infty} \frac{q^n (q;q^2)_n}{(q^2;q^2)_n (1-q^{2n+1})} \mathcal{T}_x^{2n+1}.$$

A more natural alternative is to let $y = (\zeta + 1/\zeta)/2$ and evaluate

$$\lim_{\zeta \to i} \frac{F(q^{1/2}\zeta) - F(q^{-1/2}\zeta)}{(q^{1/2} - q^{-1/2})(\zeta - 1/\zeta)/2},$$

where $E^y = F(\zeta)$.

Theorem 6.8.

$$\lim_{\zeta \to i} \frac{F(q^{1/2}\zeta) - F(q^{-1/2}\zeta)}{(q^{1/2} - q^{-1/2})(\zeta - 1/\zeta)/2} = \mathcal{D}_q. \tag{6.10}$$

Proof. It is clear that

$$\begin{aligned}\frac{F(q^{1/2}\zeta) - F(q^{-1/2}\zeta)}{(q^{1/2} - q^{-1/2})(\zeta - 1/\zeta)/2} &= \sum_{n=0}^{\infty} \mathcal{D}_{q,y} \frac{q^{n^2/4} \rho_n(y)}{(q;q)_n} \mathcal{T}_x^n \\ &= \frac{2q^{1/4}}{1-q} \sum_{n=0}^{\infty} \frac{q^{n^2/4} \rho_n(y)}{(q;q)_n} \mathcal{T}_x^{n+1} \to \frac{2q^{1/4}}{1-q} \mathcal{T}_x,\end{aligned}$$

as $y \to 0$. □

This means that the translation E^y is an analogue of an exponential semigroup with product rule

$$E^y E^w = \mathcal{E}_q(y; \mathcal{T}_x) \mathcal{E}_q(w; \mathcal{T}_x) = \mathcal{E}_q(y, w; \mathcal{T}_x).$$

One can then consider the left-hand side of (6.10) as the q-infinitesimal generator of E^y.

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Hatice Aslan
Department of Mathematics, Faculty of Science
Firat University
Elazig 23119
Turkey
e-mail: haslan@firat.edu.tr;
haticeasla@gmail.com

Mourad E. H. Ismail
Department of Mathematics, Faculty of Science
University of Central Florida
Orlando FL 32816
USA
e-mail: ismail@math.ucf.edu

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Combinations of Ranks and Cranks of Partitions Moduli 6, 9 and 12 and Their Comparison with the Partition Function

Zafer Selcuk Aygin and Song Heng Chan

Abstract. Let $L \in \{6, 9, 12\}$. We determine the generating functions of certain combinations of three ranks and three cranks modulo L in terms of eta quotients. Then, using the periodicity of signs of these eta quotients, we compare their values with the values of $\frac{p(n)}{L/3}$.

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1. Introduction and Notation

Dyson [3] defined the rank of a partition to be the largest part minus the number of parts. Let $p(n)$, $N(a; n)$ and $N(a, L; n)$ denote the number of partitions of n; the number of partitions of n with rank a; and the number of partitions of n with rank congruent to a modulo L, respectively. Dyson specifically conjectured that

$$N(a, 5; 5n + 4) = \frac{p(5n + 4)}{5}, \quad \text{for } 0 \le a \le 4, \tag{1.1}$$

$$N(a, 7; 7n + 5) = \frac{p(7n + 5)}{7}, \quad \text{for } 0 \le a \le 6, \tag{1.2}$$

which provide combinatorial interpretations for Ramanujan's partition congruences modulo 5 and 7. Equations (1.1) and (1.2) were proven by Atkin and Swinnerton-Dyer in [2]. The rank does not give a combinatorial interpretation

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for Ramanujan's partition congruences modulo 11. Thus, Dyson also conjectured the existence of a statistics which can and called it the 'crank'. Later Garvan defined cranks and showed that they give the desired combinatorial interpretation for congruences modulo 11 as well as 5 and 7. Let $M(a;n)$ and $M(a,L;n)$ denote the number of partitions of n with crank a and the number of partitions of n with crank congruent to a modulo L. We note the following elementary equations for future reference:

$$N(a,L;n) = N(-a,L;n), \;\; M(a,L;n) = M(-a,L;n), \tag{1.3}$$

$$\sum_{n=0}^{\infty}\sum_{a=0}^{L-1} N(a,L;n)q^n = \sum_{n=0}^{\infty} p(n)q^n, \;\; \sum_{n=0}^{\infty}\sum_{a=0}^{L-1} M(a,L;n)q^n = \sum_{n=0}^{\infty} p(n)q^n, \tag{1.4}$$

and for a shorthand notation, we use

$$\begin{aligned} N(a,b,c;L;n) &= N(a,L;n) + N(b,L;n) + N(c,L;n), \\ M(a,b,c;L;n) &= M(a,L;n) + M(b,L;n) + M(c,L;n). \end{aligned}$$

Let $q = \mathrm{e}^{2\pi i z}$ with $z \in \mathbb{H}$, thus $|q| < 1$. We define the infinite product $(q;q)_\infty = \prod_{n=1}^{\infty}(1-q^n)$. A specialized version of this product, $\eta(q) = q^{1/24}(q;q)_\infty$, is called the Dedekind eta function, whose quotients are called eta quotients.

In [5, Theorem 4.1], Kang proved the following relation between ranks and cranks:

$$\begin{aligned} &\sum_{m=-\infty}^{\infty}\sum_{n=0}^{\infty}\big(N(3m-1;n)+N(3m;n)+N(3m+1;n)\big)w^m q^n \\ &\quad = \frac{(q^3;q^3)_\infty}{(q;q)_\infty}\sum_{m=-\infty}^{\infty}\sum_{n=0}^{\infty} M(m;n)w^m q^{3n}. \end{aligned} \tag{1.5}$$

She then replaces w with -1 to obtain

$$\begin{aligned} &\sum_{n=0}^{\infty}\big(N(0,1,1;6;n) - N(2,2,3;6;n)\big)q^n \\ &\quad = \frac{(q^3;q^3)_\infty}{(q;q)_\infty}\sum_{n=0}^{\infty}\big(M(0,2;n)-M(1,2;n)\big)q^{3n} = \frac{(q^3;q^3)_\infty^4}{(q;q)_\infty (q^6;q^6)_\infty^2}. \end{aligned} \tag{1.6}$$

Inspired by (1.5), we give generating functions of two variations of three combinations of the crank function. Then letting $L \in \{6,9,12\}$, we use her formula and our results to find the generating functions of

$$N(3j-1,3j,3j+1;L;n), \quad \text{for } 0 \le j \le L/3-1, \tag{1.7}$$

$$M(3j-1,3j,3j+1;L;n), \quad \text{for } 0 \le j \le L/3-1, \tag{1.8}$$

$$M(3j-2,3j-1,3j;L;n), \quad \text{for } 0 \le j \le L/3-1, \tag{1.9}$$

in terms of the partition function and eta quotients. The signs of the coefficients of the eta quotients in the generating functions seem to be periodic (except a few cases). This lets us to compare (1.7)–(1.9) with $\frac{p(n)}{L/3}$.

In the next section, we state the main results. In Sect. 3, we define the Jacobi theta function and give some preliminary results. In Sect. 4, we give the generating functions of two variations of combinations of three cranks. The combination in (1.8) is motivated by Kang's results from [5], and the combination in (1.9) was chosen because among all other options, this one gives more elegant results. In Sect. 5, we prove Theorem 2.1. Fourier coefficients of all combinations of eta quotients that appear in Theorem 2.1 seem to be periodic in their signs. Some of these are proved in Sect. 6. The rest are left as conjectures, see Sect. 8. Section 7 is dedicated to the discussion on the connection of our results to recent results by Hickerson and Mortenson [4].

2. Main Results

Let

$$F(q) = \sum_{n=0}^{\infty} a_n q^n,$$

then we denote $\sum_{n=0}^{\infty} \mathfrak{Re}(a_n) q^n$ by $\mathfrak{Re}\big(F(q)\big)$ and $\sum_{n=0}^{\infty} \mathfrak{Im}(a_n) q^n$ by $\mathfrak{Im}\big(F(q)\big)$. Below we state the main theorems.

Theorem 2.1. *The generating functions of combinations of three ranks in* (1.7) *are given by*

$$\sum_{n=0}^{\infty} N(0,1,1;6;n) q^n = \frac{1}{2(q;q)_\infty} + \frac{(q^3;q^3)_\infty^4}{2(q;q)_\infty (q^6;q^6)_\infty^2}, \tag{2.1}$$

$$\sum_{n=0}^{\infty} N(2,2,3;6;n) q^n = \frac{1}{2(q;q)_\infty} - \frac{(q^3;q^3)_\infty^4}{2(q;q)_\infty (q^6;q^6)_\infty^2}, \tag{2.2}$$

$$\sum_{n=0}^{\infty} N(0,1,1;9;n) q^n = \frac{1}{3(q;q)_\infty} + \frac{2(q^3;q^3)_\infty^3}{3(q;q)_\infty (q^9;q^9)_\infty}, \tag{2.3}$$

$$\sum_{n=0}^{\infty} N(2,3,4;9;n) q^n = \frac{1}{3(q;q)_\infty} - \frac{(q^3;q^3)_\infty^3}{3(q;q)_\infty (q^9;q^9)_\infty}, \tag{2.4}$$

$$\sum_{n=0}^{\infty} N(2,3,4;12;n) q^n = \frac{1}{4(q;q)_\infty} - \frac{(q^3;q^3)_\infty^4}{4(q;q)_\infty (q^6;q^6)_\infty^2}, \tag{2.5}$$

$$\sum_{n=0}^{\infty} N(0,1,1;12;n) q^n = \frac{1}{4(q;q)_\infty} + \frac{(q^3;q^3)_\infty^2 (q^6;q^6)_\infty}{2(q;q)_\infty (q^{12};q^{12})_\infty} + \frac{(q^3;q^3)_\infty^4}{4(q;q)_\infty (q^6;q^6)_\infty^2}, \tag{2.6}$$

$$\sum_{n=0}^{\infty} N(5,5,6;12;n) q^n = \frac{1}{4(q;q)_\infty} - \frac{(q^3;q^3)_\infty^2 (q^6;q^6)_\infty}{2(q;q)_\infty (q^{12};q^{12})_\infty} + \frac{(q^3;q^3)_\infty^4}{4(q;q)_\infty (q^6;q^6)_\infty^2}. \tag{2.7}$$

The generating functions of combinations of three cranks in (1.8) *are given by*

$$\sum_{n=0}^{\infty} M(0,1,1;6;n)q^n = \frac{1}{2(q;q)_\infty} + \frac{(q^3;q^3)_\infty^4}{2(q;q)_\infty(q^6;q^6)_\infty^2} - 2q^2\frac{(q^3;q^3)_\infty(q^4;q^4)_\infty(q^{36};q^{36})_\infty^2}{(q^2;q^2)_\infty(q^{12};q^{12})_\infty(q^{18};q^{18})_\infty}, \tag{2.8}$$

$$\sum_{n=0}^{\infty} M(2,2,3;6;n)q^n = \frac{1}{2(q;q)_\infty} - \frac{(q^3;q^3)_\infty^4}{2(q;q)_\infty(q^6;q^6)_\infty^2} + 2q^2\frac{(q^3;q^3)_\infty(q^4;q^4)_\infty(q^{36};q^{36})_\infty^2}{(q^2;q^2)_\infty(q^{12};q^{12})_\infty(q^{18};q^{18})_\infty}, \tag{2.9}$$

$$\sum_{n=0}^{\infty} M(0,1,1;9;n)q^n = \frac{1}{3(q;q)_\infty} + \frac{2(q^9;q^9)_\infty^3}{3(q;q)_\infty(q^{27};q^{27})_\infty} - 2q^2\frac{(q^3;q^3)_\infty(q^{27};q^{27})_\infty^2}{(q;q)_\infty(q^9;q^9)_\infty}, \tag{2.10}$$

$$\sum_{n=0}^{\infty} M(2,3,4;9;n)q^n = \frac{1}{3(q;q)_\infty} - \frac{(q^9;q^9)_\infty^3}{3(q;q)_\infty(q^{27};q^{27})_\infty} + q^2\frac{(q^3;q^3)_\infty(q^{27};q^{27})_\infty^2}{(q;q)_\infty(q^9;q^9)_\infty}, \tag{2.11}$$

$$\sum_{n=0}^{\infty} M(2,3,4;12;n)q^n = \frac{1}{4(q;q)_\infty} - \frac{(q^3;q^3)_\infty^4}{4(q;q)_\infty(q^6;q^6)_\infty^2} + q^2\frac{(q^3;q^3)_\infty(q^4;q^4)_\infty(q^{36};q^{36})_\infty^2}{(q^2;q^2)_\infty(q^{12};q^{12})_\infty(q^{18};q^{18})_\infty}, \tag{2.12}$$

$$\sum_{n=0}^{\infty} M(0,1,1;12;n)q^n = \frac{1}{4(q;q)_\infty} + \frac{(q^3;q^3)_\infty^4}{4(q;q)_\infty(q^6;q^6)_\infty^2} + \frac{(q^2;q^2)_\infty^2(q^9;q^9)_\infty^2}{2(q;q)_\infty(q^4;q^4)_\infty(q^{18};q^{18})_\infty} - q^2\frac{(q^3;q^3)_\infty(q^4;q^4)_\infty(q^{36};q^{36})_\infty^2}{(q^2;q^2)_\infty(q^{12};q^{12})_\infty(q^{18};q^{18})_\infty}, \tag{2.13}$$

$$\sum_{n=0}^{\infty} M(5,5,6;12;n)q^n = \frac{1}{4(q;q)_\infty} + \frac{(q^3;q^3)_\infty^4}{4(q;q)_\infty(q^6;q^6)_\infty^2} - \frac{(q^2;q^2)_\infty^2(q^9;q^9)_\infty^2}{2(q;q)_\infty(q^4;q^4)_\infty(q^{18};q^{18})_\infty} - q^2\frac{(q^3;q^3)_\infty(q^4;q^4)_\infty(q^{36};q^{36})_\infty^2}{(q^2;q^2)_\infty(q^{12};q^{12})_\infty(q^{18};q^{18})_\infty}. \tag{2.14}$$

The generating functions of combinations of three cranks in (1.9) *are given by*

$$\sum_{n=0}^{\infty} M(0,1,2;6;n)q^n = \frac{1}{2(q;q)_\infty} + \frac{(q;q)_\infty(q^9;q^9)_\infty^2}{2(q^2;q^2)_\infty(q^{18};q^{18})_\infty}, \tag{2.15}$$

$$\sum_{n=0}^{\infty} M(3,4,5;6;n)q^n = \frac{1}{2(q;q)_\infty} - \frac{(q;q)_\infty (q^9;q^9)_\infty^2}{2(q^2;q^2)_\infty (q^{18};q^{18})_\infty}, \tag{2.16}$$

$$\sum_{n=0}^{\infty} M(0,1,2;9;n)q^n = \frac{1}{3(q;q)_\infty} + \frac{2}{3}\mathfrak{Re}\left(\frac{(q^3;q^3)_\infty (\zeta_3 q^3;\zeta_3 q^3)_\infty^2}{(q;q)_\infty (q^9;q^9)_\infty}\right.$$
$$\left. +(\zeta_3-1)q\frac{(q^3;q^3)_\infty (\zeta_3^2 q^3;\zeta_3^2 q^3)_\infty (q^{27};q^{27})_\infty}{(q;q)_\infty (q^9;q^9)_\infty}\right), \tag{2.17}$$

$$\sum_{n=0}^{\infty} M(3,4,5;9;n)q^n = \frac{1}{3(q;q)_\infty} - \frac{1}{3}\mathfrak{Re}\left(\frac{(q^3;q^3)_\infty (\zeta_3 q^3;\zeta_3 q^3)_\infty^2}{(q;q)_\infty (q^9;q^9)_\infty}\right.$$
$$\left. +(\zeta_3-1)q\frac{(q^3;q^3)_\infty (\zeta_3^2 q^3;\zeta_3^2 q^3)_\infty (q^{27};q^{27})_\infty}{(q;q)_\infty (q^9;q^9)_\infty}\right)$$
$$-\frac{1}{\sqrt{3}}\mathfrak{Im}\left(\frac{(q^3;q^3)_\infty (\zeta_3 q^3;\zeta_3 q^3)_\infty^2}{(q;q)_\infty (q^9;q^9)_\infty}\right.$$
$$\left. +(\zeta_3-1)q\frac{(q^3;q^3)_\infty (\zeta_3^2 q^3;\zeta_3^2 q^3)_\infty (q^{27};q^{27})_\infty}{(q;q)_\infty (q^9;q^9)_\infty}\right), \tag{2.18}$$

$$\sum_{n=0}^{\infty} M(6,7,8;9;n)q^n = \frac{1}{3(q;q)_\infty} - \frac{1}{3}\mathfrak{Re}\left(\frac{(q^3;q^3)_\infty (\zeta_3 q^3;\zeta_3 q^3)_\infty^2}{(q;q)_\infty (q^9;q^9)_\infty}\right.$$
$$\left. +(\zeta_3-1)q\frac{(q^3;q^3)_\infty (\zeta_3^2 q^3;\zeta_3^2 q^3)_\infty (q^{27};q^{27})_\infty}{(q;q)_\infty (q^9;q^9)_\infty}\right)$$
$$+\frac{1}{\sqrt{3}}\mathfrak{Im}\left(\frac{(q^3;q^3)_\infty (\zeta_3 q^3;\zeta_3 q^3)_\infty^2}{(q;q)_\infty (q^9;q^9)_\infty}\right.$$
$$\left. +(\zeta_3-1)q\frac{(q^3;q^3)_\infty (\zeta_3^2 q^3;\zeta_3^2 q^3)_\infty (q^{27};q^{27})_\infty}{(q;q)_\infty (q^9;q^9)_\infty}\right), \tag{2.19}$$

$$\sum_{n=0}^{\infty} M(0,1,2;12;n)q^n = \frac{1}{4(q;q)_\infty} + \frac{(q;q)_\infty (q^9;q^9)_\infty^2}{4(q^2;q^2)_\infty (q^{18};q^{18})_\infty}$$
$$+\frac{(q^4;q^4)_\infty (q^6;q^6)_\infty^4}{2(q^2;q^2)_\infty (q^3;q^3)_\infty (q^{12};q^{12})_\infty^2}, \tag{2.20}$$

$$\sum_{n=0}^{\infty} M(3,4,5;12;n)q^n = \frac{1}{4(q;q)_\infty} - \frac{(q;q)_\infty (q^9;q^9)_\infty^2}{4(q^2;q^2)_\infty (q^{18};q^{18})_\infty}$$
$$-q\frac{(q^2;q^2)_\infty^2 (q^3;q^3)_\infty (q^{18};q^{18})_\infty^2}{2(q;q)_\infty (q^4;q^4)_\infty (q^6;q^6)_\infty (q^9;q^9)_\infty}, \tag{2.21}$$

$$\sum_{n=0}^{\infty} M(6,7,8;12;n)q^n = \frac{1}{4(q;q)_\infty} + \frac{(q;q)_\infty (q^9;q^9)_\infty^2}{4(q^2;q^2)_\infty (q^{18};q^{18})_\infty}$$
$$-\frac{(q^4;q^4)_\infty (q^6;q^6)_\infty^4}{2(q^2;q^2)_\infty (q^3;q^3)_\infty (q^{12};q^{12})_\infty^2}, \tag{2.22}$$

$$\sum_{n=0}^{\infty} M(9,10,11;12;n)q^n = \frac{1}{4(q;q)_\infty} - \frac{(q;q)_\infty(q^9;q^9)_\infty^2}{4(q^2;q^2)_\infty(q^{18};q^{18})_\infty}$$
$$+ q\frac{(q^2;q^2)_\infty^2(q^3;q^3)_\infty(q^{18};q^{18})_\infty^2}{2(q;q)_\infty(q^4;q^4)_\infty(q^6;q^6)_\infty(q^9;q^9)_\infty}. \tag{2.23}$$

The following rank–crank difference formulae are direct consequences of Theorem 2.1.

Corollary 2.2. *We have*

$$\sum_{n=0}^{\infty}\big(N(0,1,1;6;n) - M(0,1,1;6;n)\big)q^n$$
$$= 2q^2\frac{(q^3;q^3)_\infty(q^4;q^4)_\infty(q^{36};q^{36})_\infty^2}{(q^2;q^2)_\infty(q^{12};q^{12})_\infty(q^{18};q^{18})_\infty},$$
$$\sum_{n=0}^{\infty}\big(N(0,1,1;9;n) - M(0,1,1;9;n)\big)q^n$$
$$= \frac{2(q^3;q^3)_\infty^3}{3(q;q)_\infty(q^9;q^9)_\infty} - \frac{2(q^9;q^9)_\infty^3}{3(q;q)_\infty(q^{27};q^{27})_\infty} + 2q^2\frac{(q^3;q^3)_\infty(q^{27};q^{27})_\infty^2}{(q;q)_\infty(q^9;q^9)_\infty},$$
$$\sum_{n=0}^{\infty}\big(N(2,3,4;12;n) - M(2,3,4;12;n)\big)q^n$$
$$= -q^2\frac{(q^3;q^3)_\infty(q^4;q^4)_\infty(q^{36};q^{36})_\infty^2}{(q^2;q^2)_\infty(q^{12};q^{12})_\infty(q^{18};q^{18})_\infty},$$
$$\sum_{n=0}^{\infty}\big(N(0,1,1;12;n) - M(0,1,1;12;n)\big)q^n$$
$$= \frac{(q^3;q^3)_\infty^2(q^6;q^6)_\infty}{2(q;q)_\infty(q^{12};q^{12})_\infty} - q\frac{(q^2;q^2)_\infty^2(q^9;q^9)_\infty^2}{2(q;q)_\infty(q^4;q^4)_\infty(q^{18};q^{18})_\infty}$$
$$+ q^2\frac{(q^3;q^3)_\infty(q^4;q^4)_\infty(q^{36};q^{36})_\infty^2}{(q^2;q^2)_\infty(q^{12};q^{12})_\infty(q^{18};q^{18})_\infty}.$$

The coefficients of the combinations of eta quotients in Theorem 2.1 seem to be alternating in sign (except a few cases), from which comparisons with $3p(n)/L$ can be deduced. In Sect. 6, we prove the following results. The rest of the comparisons are technically very challenging and thus left as conjectures, see Conjecture 8.1.

Theorem 2.3. *We have*

$$N(0,1,1;6;2n) > \frac{p(2n)}{2}, \quad \text{for all } n \in \mathbb{N}_0, \tag{2.24}$$
$$N(0,1,1;6;2n+1) < \frac{p(2n+1)}{2}, \quad \text{for all } n \in \mathbb{N}, \tag{2.25}$$
$$N(2,2,3;6;2n) < \frac{p(2n)}{2}, \quad \text{for all } n \in \mathbb{N}_0, \tag{2.26}$$

$$N(2,2,3;6;2n+1) > \frac{p(2n+1)}{2}, \quad \text{for all } n \in \mathbb{N}, \tag{2.27}$$

$$N(0,1,1;9;n) > \frac{p(n)}{3}, \quad \text{for all } n \in \mathbb{N}_0 - \{3,7\}, \tag{2.28}$$

$$N(2,3,4;9;n) < \frac{p(n)}{3}, \quad \text{for all } n \in \mathbb{N}_0 - \{3,7\}, \tag{2.29}$$

$$N(2,3,4;12;2n) < \frac{p(2n)}{4}, \quad \text{for all } n \in \mathbb{N}_0, \tag{2.30}$$

$$N(2,3,4;12;2n+1) > \frac{p(2n+1)}{4}, \quad \text{for all } n \in \mathbb{N}, \tag{2.31}$$

$$N(0,1,1;12;n) > \frac{p(n)}{4}, \quad \text{for all } n \in \mathbb{N}_0, \tag{2.32}$$

$$N(5,5,6;12;n) < \frac{p(n)}{4}, \quad \text{for all } n \in \mathbb{N}_0, \tag{2.33}$$

$$M(0,1,2;6;2n) > \frac{p(2n)}{2}, \quad \text{for all } n \in \mathbb{N}_0 - \{1\}, \tag{2.34}$$

$$M(0,1,2;6;2n+1) < \frac{p(2n+1)}{2}, \quad \text{for all } n \in \mathbb{N}_0, \tag{2.35}$$

$$M(3,4,5;6;2n) < \frac{p(2n)}{2}, \quad \text{for all } n \in \mathbb{N}_0 - \{1\}, \tag{2.36}$$

$$M(3,4,5;6;2n+1) > \frac{p(2n+1)}{2}, \quad \text{for all } n \in \mathbb{N}_0. \tag{2.37}$$

3. Further Notation and Preliminary Results

Let us define the Jacobi theta function by

$$j(w;q) = \sum_{n=-\infty}^{\infty} (-w)^n q^{\frac{n(n-1)}{2}} = \prod_{i=0}^{\infty} (1-wq^i)(1-q^{i+1}/w)(1-q^{i+1}),$$

where w is a nonzero complex number. We have

$$j(q^k;q^{3k}) = (q^k;q^k)_\infty. \tag{3.1}$$

Let ζ_N be an Nth root of unity, then from the definition we observe the following equalities which will come in handy in the remainder of the paper:

$$j(w;q)j(\zeta_3 w;q)j(\zeta_3^2 w;q) = \frac{(q;q)_\infty^3}{(q^3;q^3)_\infty} j(w^3;q^3), \tag{3.2}$$

$$j(iq^9;q^9) = (1+i)\frac{(q^9;q^9)_\infty (q^{36};q^{36})_\infty}{(q^{18};q^{18})_\infty}, \tag{3.3}$$

$$j(i;q^3) = (1-i)\frac{(q^3;q^3)_\infty (q^{12};q^{12})_\infty}{(q^6;q^6)_\infty}, \tag{3.4}$$

$$j(iq^3;q^9)j(iq^6;q^9) = \frac{(q^9;q^9)_\infty^2 (q^{12};q^{12})_\infty (q^{18};q^{18})_\infty}{(q^6;q^6)_\infty (q^{36};q^{36})_\infty}, \tag{3.5}$$

$$j(-1;q^3) = \frac{2(q^6;q^6)_\infty^2}{(q^3;q^3)_\infty}, \tag{3.6}$$

$$j(\zeta_3;q^3) = (1-\zeta_3)(q^9;q^9)_\infty, \tag{3.7}$$

$$j(\zeta_3 q^3;q^9)j(\zeta_3 q^6;q^9) = \frac{(q^9;q^9)_\infty^4}{(q^3;q^3)_\infty(q^{27};q^{27})_\infty}, \tag{3.8}$$

$$j(\zeta_3 q^9;q^9) = (1-\zeta_3^2)(q^{27};q^{27})_\infty, \tag{3.9}$$

$$j(-q^3;q^9) = j(-q^6;q^9) = \frac{(q^6;q^6)_\infty(q^9;q^9)_\infty^2}{(q^3;q^3)_\infty(q^{18};q^{18})_\infty}, \tag{3.10}$$

$$j(-q^9;q^9) = \frac{2(q^{18};q^{18})_\infty^2}{(q^9;q^9)_\infty}, \tag{3.11}$$

$$j(q^{15};q^{36})j(q^{33};q^{36}) = \frac{(q^3;q^3)_\infty(q^{18};q^{18})_\infty(q^{36};q^{36})_\infty^2}{(q^6;q^6)_\infty(q^9;q^9)_\infty}. \tag{3.12}$$

Next, we state and prove Lemmas 3.1 and 3.2 which will be used in the next section to prove Theorems 4.1 and 4.2, respectively.

Lemma 3.1. *We have*

$$\begin{aligned}&w^2 j(\zeta_3 w;q)j(\zeta_3^2 w;q) + \zeta_3^2 w^2 j(w;q)j(\zeta_3^2 w;q) + \zeta_3 w^2 j(w;q)j(\zeta_3 w;q)\\ &\quad = 3w^3 j(w^3q^3;q^9)j(w^3q^6;q^9) + 3w^6q^2 j(w^3q^9;q^9)j(w^3q^9;q^9).\end{aligned}$$

Proof. Noting that

$$\zeta_3^{m+1} + \zeta_3^{n+2m} + \zeta_3^{2n+2} = \begin{cases} 3\zeta_3, & \text{if } m \equiv 0 \pmod 3 \text{ and } n \equiv 1 \pmod 3,\\ 3\zeta_3^2, & \text{if } m \equiv 1 \pmod 3 \text{ and } n \equiv 0 \pmod 3,\\ 3, & \text{if } m \equiv 2 \pmod 3 \text{ and } n \equiv 2 \pmod 3,\\ 0, & \text{otherwise,}\end{cases}$$

we have

$$\begin{aligned}&w^2 j(\zeta_3 w;q)j(\zeta_3^2 w;q) + \zeta_3^2 w^2 j(w;q)j(\zeta_3^2 w;q) + \zeta_3 w^2 j(w;q)j(\zeta_3 w;q)\\
&\quad = w^2 \sum_{n=-\infty}^{\infty}\sum_{m=-\infty}^{\infty} (\zeta_3^{m+1} + \zeta_3^{n+2m} + \zeta_3^{2n+2})\\
&\qquad \times (-1)^{n+m} w^{n+m} q^{n(n-1)/2+m(m-1)/2}\\
&\quad = 3\zeta_3^2 w^2 \sum_{n=-\infty}^{\infty}\sum_{m=-\infty}^{\infty} (-1)^{3n+3m+1} w^{3n+3m+1} q^{3n(3n-1)/2+(3m+1)(3m)/2}\\
&\qquad + 3\zeta_3 w^2 \sum_{n=-\infty}^{\infty}\sum_{m=-\infty}^{\infty} (-1)^{3n+3m+1} w^{3n+3m+1} q^{(3m+1)3m/2+3n(3n-1)/2}\\
&\qquad + 3w^2 \sum_{n=-\infty}^{\infty}\sum_{m=-\infty}^{\infty} (-1)^{3n+3m+4} w^{3n+3m+4}\\
&\qquad \times q^{(3n+2)(3n+1)/2+(3m+2)(3m+1)/2}\\
&\quad = 3w^3 \sum_{n=-\infty}^{\infty} (-1)^n w^{3n} q^{3n} (q^9)^{(n^2-n)/2} \sum_{m=-\infty}^{\infty} (-1)^{3m} w^{3m} q^{6m} (q^9)^{(m^2-m)/2}\end{aligned}$$

$$+ 3w^6q^2 \sum_{n=-\infty}^{\infty} (-1)^n w^{3n} q^{9n} (q^9)^{(n^2-n)/2} \sum_{m=-\infty}^{\infty} (-1)^{3m} w^{3m} q^{9m} (q^9)^{(m^2-m)/2}$$
$$= 3w^3 j(w^3q^3; q^9) j(w^3q^6; q^9) + 3w^6q^2 j(w^3q^9; q^9) j(w^3q^9; q^9).$$

□

Lemma 3.2. *We have*

$$\begin{aligned} &j(\zeta_3 w; q) j(\zeta_3^2 w; q) + j(w; q) j(\zeta_3^2 w; q) + j(w; q) j(\zeta_3 w; q) \\ &\quad = 3j(w^3q^3; q^9)^2 + 3w^3 q j(w^3q^9; q^9) j(w^3q^6; q^9). \end{aligned}$$

Proof. Proceeding as in the proof of Lemma 3.1 with

$$\zeta_3^m + \zeta_3^{n+2m} + \zeta_3^{2n} = \begin{cases} 3\zeta_3, & \text{if } m \equiv 1 \pmod 3 \text{ and } n \equiv 2 \pmod 3, \\ 3\zeta_3^2, & \text{if } m \equiv 2 \pmod 3 \text{ and } n \equiv 1 \pmod 3, \\ 3, & \text{if } m \equiv 0 \pmod 3 \text{ and } n \equiv 0 \pmod 3, \\ 0, & \text{otherwise,} \end{cases}$$

we obtain

$$\begin{aligned} &j(\zeta_3 w; q) j(\zeta_3^2 w; q) + j(w; q) j(\zeta_3^2 w; q) + j(w; q) j(\zeta_3 w; q) \\ &\quad = \sum_{n=-\infty}^{\infty} \sum_{m=-\infty}^{\infty} (\zeta_3^m + \zeta_3^{n+2m} + \zeta_3^{2n}) (-1)^{n+m} w^{n+m} q^{n(n-1)/2 + m(m-1)/2} \\ &\quad = 3\zeta_3 \sum_{n=-\infty}^{\infty} \sum_{m=-\infty}^{\infty} (-1)^{3n+3m+3} w^{3n+3m+3} q^{(3n+2)(3n+1)/2 + (3m+1)3m/2} \\ &\qquad + 3\zeta_3^2 \sum_{n=-\infty}^{\infty} \sum_{m=-\infty}^{\infty} (-1)^{3n+3m+3} w^{3n+3m+3} q^{(3n+1)3n/2 + (3m+2)(3m+1)/2} \\ &\qquad + 3 \sum_{n=-\infty}^{\infty} \sum_{m=-\infty}^{\infty} (-1)^{3n+3m} w^{3n+3m} q^{3n(3n-1)/2 + 3m(3m-1)/2} \\ &\quad = 3w^3 q \sum_{n=-\infty}^{\infty} (-1)^n w^{3n} q^{9n} q^{(9n^2-9n)/2} \sum_{m=-\infty}^{\infty} (-1)^m w^{3m} q^{6m} q^{(9m^2-9m)/2} \\ &\qquad + 3 \sum_{n=-\infty}^{\infty} (-1)^n w^{3n} q^{3n} q^{(9n^2-9n)/2} \sum_{m=-\infty}^{\infty} (-1)^m w^{3m} q^{3m} q^{(9m^2-9m)/2} \\ &\quad = 3w^3 q j(w^3q^9; q^9) j(w^3q^6; q^9) + 3j(w^3q^3; q^9)^2. \end{aligned}$$

□

The following two dissections of the Jacobi theta function will come in handy in the forthcoming arguments.

Lemma 3.3. *For $|A|, |B| < 1$, we have*

$$j(-A^2B; B^4) - Aj(-A^2B^3; B^4) = j(A; B).$$

Proof.

$$
\begin{aligned}
&j(-A^2B;B^4) - Aj(-A^2B^3;B^4)\\
&\quad = \sum_{n=-\infty}^{\infty} (A^2B)^n(B^4)^{n(n-1)/2} - A\sum_{n=-\infty}^{\infty} (A^2B^3)^n(B^4)^{n(n-1)/2}\\
&\quad = \sum_{n=-\infty}^{\infty} A^{2n}B^{((2n)^2-(2n))/2} - \sum_{n=-\infty}^{\infty} A^{2n+1}B^{((2n+1)^2-(2n+1))/2}\\
&\quad = j(A;B).
\end{aligned}
$$

□

4. Generating Functions

In this section, we give the generating functions for two variations of combinations of three cranks, motivated by Kang's results [5] for ranks. We use elementary manipulations to obtain our results. Noting that the generating function of cranks is given by

$$
C(w;q) = \sum_{m=-\infty}^{\infty}\sum_{n=0}^{\infty} M(m;n)w^mq^n = \frac{(1-w)(q;q)_\infty^2}{j(w;q)},
$$

we state and prove Theorems 4.1 and 4.2.

Theorem 4.1. *We have*

$$
\begin{aligned}
&\sum_{m=-\infty}^{\infty}\sum_{n=0}^{\infty} \big(M(3m-1;n) + M(3m;n) + M(3m+1;n)\big)w^mq^n\\
&\quad = \frac{(1-w)(q^3;q^3)_\infty}{(q;q)_\infty}\left(\frac{j(wq^3;q^9)j(wq^6;q^9) + wq^2j(wq^9;q^9)^2}{j(w;q^3)}\right).
\end{aligned}
$$

Proof. Recall that ζ_3 is the third root of unity. Then, we have

$$
\begin{aligned}
&\frac{C(w;q)}{w(1-w)} + \frac{C(\zeta_3 w;q)}{\zeta_3 w(1-\zeta_3 w)} + \frac{C(\zeta_3^2 w;q)}{\zeta_3^2 w(1-\zeta_3^2 w)}\\
&\quad = \frac{1}{w(1-w)}\left(\sum_{m=-\infty}^{\infty}\sum_{n=0}^{\infty} M(m;n)w^mq^n\right)\\
&\qquad + \frac{1}{\zeta_3 w(1-\zeta_3 w)}\left(\sum_{m=-\infty}^{\infty}\sum_{n=0}^{\infty} M(m;n)(\zeta_3 w)^mq^n\right)\\
&\qquad + \frac{1}{\zeta_3^2 w(1-\zeta_3^2 w)}\left(\sum_{m=-\infty}^{\infty}\sum_{n=0}^{\infty} M(m;n)(\zeta_3^2 w)^mq^n\right)\\
&\quad = \frac{3}{1-w^3}\left(\sum_{m=-\infty}^{\infty}\sum_{n=0}^{\infty}\big(M(3m-1;n)\right.\\
&\qquad \left. + M(3m;n) + M(3m+1;n)\big)w^{3m}q^n\right).
\end{aligned} \tag{4.1}
$$

On the other hand, using Lemma 3.1, we compute

$$\begin{aligned}
&\frac{C(w;q)}{w(1-w)}+\frac{C(\zeta_3 w;q)}{\zeta_3 w(1-\zeta_3 w)}+\frac{C(\zeta_3^2 w;q)}{\zeta_3^2 w(1-\zeta_3^2 w)}\\
&\quad=\frac{w^2(q^3;q^3)_\infty\left(j(\zeta_3 w;q)j(\zeta_3^2 w;q)+\zeta_3^2 j(w;q)j(\zeta_3^2 w;q)+\zeta_3 j(w;q)j(\zeta_3 w;q)\right)}{w^3(q;q)_\infty j(w^3;q^3)}\\
&\quad=\frac{(q^3;q^3)_\infty}{(q;q)_\infty j(w^3;q^3)}\left(3j(w^3q^3;q^9)j(w^3q^6;q^9)+3w^3q^2j(w^3q^9;q^9)j(w^3q^9;q^9)\right).
\end{aligned} \tag{4.2}$$

Combining (4.1) with (4.2) and replacing w^3 by w, we obtain the desired result. □

Theorem 4.2. *We have*

$$\begin{aligned}
&\sum_{m=-\infty}^{\infty}\sum_{n=0}^{\infty}(M(3m-2;n)+M(3m-1;n)+M(3m;n))w^m q^n\\
&\quad=\frac{(1-w)(q^3;q^3)_\infty}{(q;q)_\infty}\left(\frac{j(wq^3;q^9)^2+wqj(wq^9;q^9)j(wq^6;q^9)}{j(w;q^3)}\right).
\end{aligned}$$

Proof. Recalling ζ_3 is the third root of unity, then we have

$$\begin{aligned}
&\frac{C(w;q)}{(1-w)}+\frac{C(\zeta_3 w;q)}{(1-\zeta_3 w)}+\frac{C(\zeta_3^2 w;q)}{(1-\zeta_3^2 w)}\\
&\quad=\frac{1}{1-w}\left(\sum_{m=-\infty}^{\infty}\sum_{n=0}^{\infty}M(m;n)w^m q^n\right)\\
&\qquad+\frac{1}{1-\zeta_3 w}\left(\sum_{m=-\infty}^{\infty}\sum_{n=0}^{\infty}M(m;n)(\zeta_3 w)^m q^n\right)\\
&\qquad+\frac{1}{1-\zeta_3^2 w}\left(\sum_{m=-\infty}^{\infty}\sum_{n=0}^{\infty}M(m;n)(\zeta_3^2 w)^m q^n\right)\\
&\quad=\frac{3}{1-w^3}\left(\sum_{m=-\infty}^{\infty}\sum_{n=0}^{\infty}\big(M(3m-2;n)\right.\\
&\qquad\left.+M(3m-1;n)+M(3m;n)\big)w^{3m}q^n\right).
\end{aligned} \tag{4.3}$$

On the other hand, we use Lemma 3.2 and obtain

$$\begin{aligned}
&\frac{C(w;q)}{(1-w)}+\frac{C(\zeta_3 w;q)}{(1-\zeta_3 w)}+\frac{C(\zeta_3^2 w;q)}{(1-\zeta_3^2 w)}=\frac{(q;q)^2}{j(w;q)}+\frac{(q;q)^2}{j(\zeta_3 w;q)}+\frac{(q;q)^2}{j(\zeta_3^2 w;q)}\\
&\quad=\frac{(q^3;q^3)_\infty\left(j(\zeta_3 w;q)j(\zeta_3^2 w;q)+j(w;q)j(\zeta_3^2 w;q)+j(w;q)j(\zeta_3 w;q)\right)}{(q;q)_\infty j(w^3;q^3)}\\
&\quad=\frac{(q^3;q^3)_\infty}{(q;q)_\infty j(w^3;q^3)}\left(3j(w^3q^3;q^9)^2+3w^3qj(w^3q^9;q^9)j(w^3q^6;q^9)\right).
\end{aligned} \tag{4.4}$$

Then, the theorem follows from combining (4.3) with (4.4) and replacing w^3 by w. $\square$

5. Proof of Theorem 2.1

The proofs of (2.1) and (2.2) follow from (1.4) and (1.6). Next, we replace w by ζ_3 and by i in (1.5) to obtain

$$\begin{aligned}\sum_{n=0}^{\infty}&\big(N(0,1,1;9;n)-N(2,3,4;9;n)\big)q^n\\&=\frac{(q^3;q^3)_\infty}{(q;q)_\infty}\sum_{n=0}^{\infty}\big(M(0,3;n)-M(1,3;n)\big)q^{3n}\end{aligned}$$

and

$$\begin{aligned}\sum_{n=0}^{\infty}&\big(N(0,1,1;12;n)-N(5,5,6;12;n)\big)q^n\\&=\frac{(q^3;q^3)_\infty}{(q;q)_\infty}\sum_{n=0}^{\infty}\big(M(0,4;n)-M(2,4;n)\big)q^{3n},\end{aligned}$$

respectively. In [1], Andrews and Lewis proved that

$$\begin{aligned}\sum_{n=0}^{\infty}\big(M(0,3;n)-M(1,3;n)\big)q^n&=\frac{(q;q)_\infty^2}{(q^3;q^3)_\infty},\\ \sum_{n=0}^{\infty}\big(M(0,4;n)-M(2,4;n)\big)q^n&=\frac{(q;q)_\infty(q^2;q^2)_\infty}{(q^4;q^4)_\infty}.\end{aligned}$$

Thus we have

$$\sum_{n=0}^{\infty}\big(N(0,1,1;9;n)-N(2,3,4;9;n)\big)q^n=\frac{(q^3;q^3)_\infty^3}{(q;q)_\infty(q^9;q^9)_\infty},\tag{5.1}$$

$$\sum_{n=0}^{\infty}\big(N(0,1,1;12;n)-N(5,5,6;12;n)\big)q^n=\frac{(q^3;q^3)_\infty^2(q^6;q^6)_\infty}{(q;q)_\infty(q^{12};q^{12})_\infty}.\tag{5.2}$$

Hence (2.3) and (2.4) follow from (5.1) and (1.4). The relation (2.5) follows from the following observation and (2.2):

$$\sum_{n=0}^{\infty}2N(2,3,4;12;n)q^n=\sum_{n=0}^{\infty}N(2,2,3;6;n)q^n.$$

We obtain (2.6) and (2.7) from (2.1), (5.2) and the following observation:

$$\sum_{n=0}^{\infty}\big(N(0,1,1;12;n)+N(5,5,6;12;n)\big)q^n=\sum_{n=0}^{\infty}N(0,1,1;6;n)q^n.$$

Below in Lemmas 5.1–5.3, we compute the expressions in Theorems 4.1 and 4.2 when w is replaced by second, third and fourth roots of unity. We use the following three lemmas to prove the rest of the equations similarly.

Lemma 5.1. *We have*

$$\sum_{n=0}^{\infty}\big(M(-1,0,1;6;n)-M(2,3,4;6;n)\big)q^n$$
$$=\frac{(q^3;q^3)_\infty^4}{(q;q)_\infty(q^6;q^6)_\infty^2}-4q^2\frac{(q^3;q^3)_\infty(q^4;q^4)_\infty(q^{36};q^{36})_\infty^2}{(q^2;q^2)_\infty(q^{12};q^{12})_\infty(q^{18};q^{18})_\infty},$$
$$\sum_{n=0}^{\infty}\big(M(-2,-1,0;6;n)-M(1,2,3;6;n)\big)q^n=\frac{(q;q)_\infty(q^9;q^9)_\infty^2}{(q^2;q^2)_\infty(q^{18};q^{18})_\infty}.$$

Proof. We replace $w=-1$ in Theorems 4.1 and 4.2. Then, we employ (3.6), (3.10) and (3.11) to get

$$\sum_{n=0}^{\infty}\big(M(-1,0,1;6;n)-M(2,2,3;6;n)\big)q^n$$
$$=\frac{2(q^3;q^3)_\infty}{(q;q)_\infty}\left(\frac{j(-q^3;q^9)j(-q^6;q^9)-q^2j(-q^9;q^9)^2}{j(-1;q^3)}\right)$$
$$=\frac{1}{(q;q)_\infty}\left(\frac{(q^9;q^9)_\infty^4}{(q^{18};q^{18})_\infty^2}-4q^2\frac{(q^3;q^3)_\infty^2(q^{18};q^{18})_\infty^4}{(q^6;q^6)_\infty^2(q^9;q^9)_\infty^2}\right),$$
$$\sum_{n=0}^{\infty}\big(M(-2,-1,0;6;n)-M(1,2,3;6;n)\big)q^n$$
$$=\frac{2(q^3;q^3)_\infty}{(q;q)_\infty}\left(\frac{j(-q^3;q^9)^2-qj(-q^9;q^9)j(-q^6;q^9)}{j(-1;q^3)}\right)$$
$$=\frac{1}{(q;q)_\infty}\left(\frac{(q^9;q^9)_\infty^4}{(q^{18};q^{18})_\infty^2}-2q\frac{(q^3;q^3)_\infty(q^9;q^9)_\infty(q^{18};q^{18})_\infty}{(q^6;q^6)_\infty}\right).$$

The eta quotients in brackets are holomorphic. Thus, the lemma follows from the Sturm Theorem ([6, Theorem 3.13]). □

Lemma 5.2. *We have*

$$\sum_{n=0}^{\infty}\big(M(-1,0,1;9;n)-M(2,3,4;9;n)\big)q^n$$
$$=\frac{1}{(q;q)_\infty}\left(\frac{(q^9;q^9)_\infty^3}{(q^{27};q^{27})_\infty}-3q^2\frac{(q^3;q^3)_\infty(q^{27};q^{27})_\infty^2}{(q^9;q^9)_\infty}\right),$$
$$\sum_{n=0}^{\infty}\big(M(-2,-1,0;9;n)+\zeta_3M(1,2,3;9;n)+\zeta_3^2M(4,5,6;9;n)\big)q^n$$
$$=\frac{(q^3;q^3)_\infty}{(q;q)_\infty(q^9;q^9)_\infty}\left((\zeta_3q^3;\zeta_3q^3)_\infty^2+(\zeta_3-1)q(q^{27};q^{27})_\infty(\zeta_3^2q^3;\zeta_3^2q^3)_\infty\right).$$

Proof. These equations follow from employing (3.1), (3.7), (3.8) and (3.9) after replacing $w=\zeta_3$ in Theorems 4.1 and 4.2, respectively. □

Lemma 5.3. *We have*

$$\sum_{n=0}^{\infty}\big(M(-1,0,1;12;n)-M(5,6,7;12;n)\big)q^n=\frac{(q^2;q^2)_\infty^2(q^9;q^9)_\infty^2}{(q;q)_\infty(q^4;q^4)_\infty(q^{18};q^{18})_\infty},$$

$$\sum_{n=0}^{\infty}\Big(\big(M(-2,-1,0;12;n)-M(4,5,6;12;n)\big)$$
$$+i\big(M(1,2,3;12;n)-M(7,8,9;12;n)\big)\Big)q^n$$
$$=\frac{(q^4;q^4)_\infty(q^6;q^6)_\infty^4}{(q^2;q^2)_\infty(q^3;q^3)_\infty(q^{12};q^{12})_\infty^2}+iq\frac{(q^2;q^2)_\infty^2(q^3;q^3)_\infty(q^{18};q^{18})_\infty^2}{(q;q)_\infty(q^4;q^4)_\infty(q^6;q^6)_\infty(q^9;q^9)_\infty}.$$

Proof. We first substitute w by i in Theorem 4.1. Then by employing (3.3), (3.4) and (3.5) we have

$$\frac{(1-i)(q^3;q^3)_\infty}{(q;q)_\infty}\left(\frac{j(iq^3;q^9)j(iq^6;q^9)+iq^2j(iq^9;q^9)^2}{j(i;q^3)}\right)$$
$$=\frac{1}{(q;q)_\infty}\left(\frac{(q^9;q^9)_\infty^2(q^{18};q^{18})_\infty}{(q^{36};q^{36})_\infty}-2q^2\frac{(q^6;q^6)_\infty(q^9;q^9)_\infty^2(q^{36};q^{36})_\infty^2}{(q^{12};q^{12})_\infty(q^{18};q^{18})_\infty^2}\right)$$
$$=\frac{(q^2;q^2)_\infty^2(q^9;q^9)_\infty^2}{(q;q)_\infty(q^4;q^4)_\infty(q^{18};q^{18})_\infty}.$$

The last equality follows from the Sturm Theorem ([6, Theorem 3.13]).

By Lemma 3.3, we have

$$j(iq^3;q^9)=j(q^{15};q^{36})-iq^3j(q^{33};q^{36}),$$
$$j(iq^6;q^9)=j(q^{21};q^{36})-iq^6j(q^{39};q^{36}),$$

together with (3.3), (3.4) and w replaced by i in Theorem 4.2 we have

$$\frac{(1-i)(q^3;q^3)_\infty}{(q;q)_\infty}\left(\frac{j(iq^3;q^9)^2+iqj(iq^9;q^9)j(iq^6;q^9)}{j(i;q^3)}\right)$$
$$=\frac{(q^6;q^6)_\infty}{(q;q)_\infty(q^{12};q^{12})_\infty}\big(j(q^{15};q^{36})-q^3j(q^3;q^{36})\big)\big(j(q^{15};q^{36})+q^3j(q^3;q^{36})\big)$$
$$+q\frac{(q^6;q^6)_\infty(q^9;q^9)_\infty(q^{36};q^{36})_\infty}{(q;q)_\infty(q^{12};q^{12})_\infty(q^{18};q^{18})_\infty}\big(q^6j(q^{39};q^{36})-j(q^{21};q^{36})\big)$$
$$+iq\frac{(q^6;q^6)_\infty(q^9;q^9)_\infty(q^{36};q^{36})_\infty}{(q;q)_\infty(q^{12};q^{12})_\infty(q^{18};q^{18})_\infty}\big(q^6j(q^{39};q^{36})+j(q^{21};q^{36})\big)$$
$$-2iq^3\frac{(q^6;q^6)_\infty}{(q;q)_\infty(q^{12};q^{12})_\infty}\big(j(q^{15};q^{36})j(q^{33};q^{36})\big)$$
$$=\frac{(q^6;q^6)_\infty}{(q;q)_\infty(q^{12};q^{12})_\infty}j(q^3;-q^9)j(-q^3;-q^9)$$
$$+q\frac{(q^6;q^6)_\infty(q^9;q^9)_\infty(q^{36};q^{36})_\infty}{(q;q)_\infty(q^{12};q^{12})_\infty(q^{18};q^{18})_\infty}\big(-j(-q^6;-q^9)\big)$$
$$+iq\frac{(q^6;q^6)_\infty(q^9;q^9)_\infty(q^{36};q^{36})_\infty}{(q;q)_\infty(q^{12};q^{12})_\infty(q^{18};q^{18})_\infty}\big(j(-q^6;-q^9)\big)$$
$$-2iq^3\frac{(q^6;q^6)_\infty}{(q;q)_\infty(q^{12};q^{12})_\infty}\big(j(q^{15};q^{36})j(q^{33};q^{36})\big)$$
$$=\frac{(q^6;q^6)_\infty^2(q^{18};q^{18})_\infty^5}{(q;q)_\infty(q^9;q^9)_\infty^2(q^{12};q^{12})_\infty(q^{36};q^{36})_\infty^2}$$

$$- q\frac{(q^6;q^6)_\infty^4(q^9;q^9)_\infty(q^{36};q^{36})_\infty}{(q;q)_\infty(q^3;q^3)_\infty(q^{12};q^{12})_\infty^2(q^{18};q^{18})_\infty}$$
$$+ iq\left(\frac{(q^3;q^3)_\infty(q^{18};q^{18})_\infty^4}{(q;q)_\infty(q^6;q^6)_\infty(q^9;q^9)_\infty(q^{36};q^{36})_\infty}\right.$$
$$\left.-2q^2\frac{(q^3;q^3)_\infty(q^{18};q^{18})_\infty(q^{36};q^{36})_\infty^2}{(q;q)_\infty(q^9;q^9)_\infty(q^{12};q^{12})_\infty}\right).$$

In the second and third steps, we use Lemma 3.3 and (3.10)–(3.12), respectively. Then, the lemma follows from the Sturm Theorem ([6, Theorem 3.13]). □

6. Proof of Theorem 2.3

Let $f(q) = \sum_n a_n q^n$ be the Fourier series expansion of f. We use $[n]f(q)$ to denote a_n.

6.1. Proofs of (2.24)–(2.27)

The following lemma from [7, Theorem 1.1] gives the desired result.

Lemma 6.1 [7, Theorem 1.1]. *For all $n \geq 1$, we have*

$$[2n]\frac{(q^3;q^3)_\infty^4}{2(q;q)_\infty(q^6;q^6)_\infty^2} > 0,$$
$$[2n+1]\frac{(q^3;q^3)_\infty^4}{2(q;q)_\infty(q^6;q^6)_\infty^2} < 0.$$

6.2. Proofs of (2.28)–(2.29)

Lemma 6.2. *For all $n \geq 0$, except $n = 3$, or 7, we have*

$$[n]\frac{(q^3;q^3)_\infty^3}{(q;q)_\infty(q^9;q^9)_\infty} > 0.$$

Proof. We have

$$\frac{q(q^9;q^9)_\infty^3}{(q^3;q^3)_\infty} = \frac{\eta^3(q^9)}{\eta(q^3)} \in M_1(\Gamma_0(27),\chi_{-3}),$$

from which we obtain

$$\frac{q(q^9;q^9)_\infty^3}{(q^3;q^3)_\infty} = \sum_{n=0}^{\infty}\sum_{d|3n+1}\left(\frac{-3}{d}\right)q^{3n+1}.$$

On the other hand, we have $\sum_{d|3n+1}\left(\frac{-3}{d}\right) \geq 0$, since

$$\sum_{\substack{d|3n+1,\\ d\equiv 1 \pmod 3}} 1 \geq \sum_{\substack{d|3n+1,\\ d\equiv 2 \pmod 3}} 1.$$

Thus we have

$$[n]\frac{(q^3;q^3)_\infty^3}{(q;q)_\infty} \geq 0, \tag{6.1}$$

with

$$\frac{(q^3;q^3)_\infty^3}{(q;q)_\infty} = 1+q+2q^2+2q^4+q^5+2q^6+q^8+2q^9+2q^{10}+2q^{12} + 2q^{14}+3q^{16}+O(q^{17}).$$

The lemma follows from observing that

$$[9n]\frac{1}{(q^9;q^9)_\infty} > 0.$$

□

Observe that (2.28) and (2.29) is a direct consequence of Lemma 6.2.

6.3. Proofs of (2.30)–(2.33)

We obtain (2.30) and (2.31) from (2.26) and (2.27), respectively; and (2.32) and (2.33) from the following lemma combined with (6.1).

Lemma 6.3. *For all $n \geq 0$ we have*

$$2[3n]\frac{(q^6;q^6)_\infty}{(q^3;q^3)_\infty(q^{12};q^{12})_\infty} + [3n]\frac{(q^3;q^3)_\infty}{(q^6;q^6)_\infty^2} > 0,$$
$$2[3n]\frac{(q^6;q^6)_\infty}{(q^3;q^3)_\infty(q^{12};q^{12})_\infty} - [3n]\frac{(q^3;q^3)_\infty}{(q^6;q^6)_\infty^2} > 0.$$

Proof. Let

$$f_1(q) = \frac{(q^6;q^6)_\infty}{(q^3;q^3)_\infty(q^{12};q^{12})_\infty}.$$

Replacing q by $-q$, we obtain

$$f_1(-q) = \frac{(q^3;q^3)_\infty}{(q^6;q^6)_\infty^2}.$$

That is, we have

$$\begin{aligned} 2f_1(q)+f_1(-q) &= 2\frac{(q^6;q^6)_\infty}{(q^3;q^3)_\infty(q^{12};q^{12})_\infty} + \frac{(q^3;q^3)_\infty}{(q^6;q^6)_\infty^2} \\ &= \sum_{n=0}^{\infty}\left([n]f_1(q)+2[2n]f_1(q)\right)q^n, \\ 2f_1(q)-f_1(-q) &= 2\frac{(q^6;q^6)_\infty}{(q^3;q^3)_\infty(q^{12};q^{12})_\infty} - \frac{(q^3;q^3)_\infty}{(q^6;q^6)_\infty^2} \\ &= \sum_{n=0}^{\infty}\left([n]f_1(q)+2[2n+1]f_1(q)\right)q^n. \end{aligned}$$

On the other hand, we observe that

$$f_1(q) = \frac{1}{(q^3;q^6)_\infty(q^{12};q^{12})_\infty},$$

that is, we have $[3n]f_1(q) > 0$ for all $n \geq 0$, from which the lemma follows.

□

6.4. Proofs of (2.34)–(2.37)

These follow from Lemma 6.4 below.

Lemma 6.4. *For all $n > 1$ we have*

$$[2n]\frac{(q;q)_\infty (q^9;q^9)_\infty^2}{(q^2;q^2)_\infty (q^{18};q^{18})_\infty} > 0,$$
$$[2n-1]\frac{(q;q)_\infty (q^9;q^9)_\infty^2}{(q^2;q^2)_\infty (q^{18};q^{18})_\infty} < 0.$$

Proof. Let

$$f_2(q) = \frac{(q;q)_\infty (q^9;q^9)_\infty^2}{(q^2;q^2)_\infty (q^{18};q^{18})_\infty},$$

then we have

$$\begin{aligned} f_2(-q) &= \frac{(q^2;q^2)_\infty^2 (q^{18};q^{18})_\infty^5}{(q;q)_\infty (q^4;q^4)_\infty (q^9;q^9)_\infty^2 (q^{36};q^{36})_\infty^2} \\ &= \frac{(q^{18};q^{18})_\infty^5}{(q^9;q^9)_\infty^2 (q^{36};q^{36})_\infty^2} \frac{(q^2;q^2)_\infty^2 (q^3;q^3)_\infty (q^{12};q^{12})_\infty}{(q;q)_\infty (q^4;q^4)_\infty (q^6;q^6)_\infty} \\ &\quad \times \frac{(q^6;q^6)_\infty}{(q^3;q^3)_\infty (q^{12};q^{12})_\infty}. \end{aligned}$$

Let

$$\begin{aligned} f_3(q) &= \frac{(q^{18};q^{18})_\infty^5}{(q^9;q^9)_\infty^2 (q^{36};q^{36})_\infty^2} = 1 + 4q^9 + 4q^{36} + O(q^{243}), \\ f_4(q) &= \frac{(q^2;q^2)_\infty^2 (q^3;q^3)_\infty (q^{12};q^{12})_\infty}{(q;q)_\infty (q^4;q^4)_\infty (q^6;q^6)_\infty} = 1 + q + q^5 + q^8 + q^{16} + O(q^{21}), \\ f_5(q) &= \frac{(q^6;q^6)_\infty}{(q^3;q^3)_\infty (q^{12};q^{12})_\infty} = 1 + q^3 + q^6 + 2q^9 + 3q^{12} + O(q^{15}). \end{aligned}$$

It is well known that $f_3(q) = 1 + 4\sum_{n=1}^{\infty} q^{9n^2}$, thus $[n]f_3(q) \geq 0$ for all $n \in \mathbb{N}$. We observe that $f_4(-q)$ is one of the Kac identities, whose coefficients are alternating in sign, see [8, Theorem 8.2 (2)]. Thus, $[n]f_4(q) \geq 0$ for all $n \in \mathbb{N}$. We also have

$$f_5(q) = \frac{(-q^3;q^3)_\infty}{(q^{12};q^{12})_\infty},$$

from which $[3n]f_5(q) > 0$ for all $n \in \mathbb{N}_0$ follows. Thus $[n]f_4(q)f_5(q) > 0$ for all $n \geq 3$. That is, we have $[n]f_3(q)f_4(q)f_5(q) = [n]f_2(-q) > 0$ for all $n > 2$. Then we replace q by $-q$ in $f_2(-q)$ to complete the proof of the lemma. □

7. Further Discussion

In this section, we discuss the connection between Theorem 2.1 and recent results by Hickerson and Mortenson from [4]. Then, we discuss further possibilities in this direction. Let $L \in \mathbb{N}$ be divisible by 3. Let

$$D(3j-1,3j,3j+1;L) = \sum_{n=0}^{\infty} \left(N(3j-1,3j,3j+1;L;n) - \frac{3p(n)}{L} \right) q^n.$$

From [4, Theorem 4.1], one can observe that Appell–Lerch sums in $D(3j-1,3j,3j+1;L)$ cancel out, leaving the generating function to be a theta function. This explains the beauty of the formulae in (2.1)–(2.7) is not a mere coincidence. Moreover, one can give the generating functions of combinations of ranks in (1.7) in terms of Jacobi theta functions. Below we illustrate this when $j=0$ and $L = 3 \cdot 2^k$, where $k \in \mathbb{N}_0$. By [4, Theorem 4.1] we have

$$\sum_{n=0}^{\infty} N(0,1,1;L;n)q^n = \frac{3}{L(q;q)_\infty} + \frac{(q;q)_\infty (q^3;q^3)_\infty^4}{2L(q^6;q^6)_\infty^2} \sum_{j=1}^{L-1} \frac{(1-\zeta_L^j)(2+\zeta_L^j)\zeta_L^{-2j} j(-\zeta_L^{2j};q)}{j(\zeta_L^j;q) j(-\zeta_L^{3j}q;q^3) j(-\zeta_L^{3j}q^2;q^3)}. \tag{7.1}$$

Now let us consider

$$\begin{aligned}\sum_{n=0}^{\infty} N(0,1,1;2L;n)q^n &= \frac{3}{2L(q;q)_\infty} + \frac{(q;q)_\infty (q^3;q^3)_\infty^4}{4L(q^6;q^6)_\infty^2} \sum_{j=1}^{2L-1} \frac{(1-\zeta_{2L}^j)(2+\zeta_{2L}^j)\zeta_{2L}^{-2j} j(-\zeta_{2L}^{2j};q)}{j(\zeta_{2L}^j;q) j(-\zeta_{2L}^{3j}q;q^3) j(-\zeta_{2L}^{3j}q^2;q^3)} \\ &= \frac{1}{2}\sum_{n=0}^{\infty} N(0,1,1;L;n)q^n \\ &\quad + \frac{(q;q)_\infty (q^3;q^3)_\infty^4}{4L(q^6;q^6)_\infty^2} \sum_{j=0}^{L-1} \frac{(1-\zeta_{2L}^{2j+1})(2+\zeta_{2L}^{2j+1})\zeta_{2L}^{-4j-2} j(-\zeta_{2L}^{4j+2};q)}{j(\zeta_{2L}^{2j+1};q) j(-\zeta_{2L}^{6j+3}q;q^3) j(-\zeta_{2L}^{6j+3}q^2;q^3)},\end{aligned}$$

where in the last line we use (7.1). This gives a recursive relation for $N(0,1,1;L)$. In general if $L = 3 \cdot 2^k$, then we use this recursion to obtain

$$\begin{aligned}\sum_{n=0}^{\infty} N(0,1,1;L;n)q^n &= \frac{1}{2}\sum_{n=0}^{\infty} N(0,1,1;L/2;n)q^n \\ &\quad + \frac{(q;q)_\infty (q^3;q^3)_\infty^4}{2L(q^6;q^6)_\infty^2} \sum_{j=0}^{L/2-1} \frac{(1-\zeta_L^{2j+1})(2+\zeta_L^{2j+1})\zeta_L^{-4j-2} j(-\zeta_L^{4j+2};q)}{j(\zeta_L^{2j+1};q) j(-\zeta_L^{6j+3}q;q^3) j(-\zeta_L^{6j+3}q^2;q^3)}\end{aligned}$$

$$= \frac{1}{2^k}\sum_{n=0}^{\infty} N(0,1,1;3;n)q^n + \frac{(q;q)_\infty (q^3;q^3)_\infty^4}{(q^6;q^6)_\infty^2}$$
$$\times \sum_{i=0}^{k-1} \frac{2^{i-1}}{L} \sum_{j=0}^{L/2^{i+1}-1} \frac{(1-\zeta_{L/2^i}^{2j+1})(2+\zeta_{L/2^i}^{2j+1})\zeta_{L/2^i}^{-4j-2} j(-\zeta_{L/2^i}^{4j+2};q)}{j(\zeta_{L/2^i}^{2j+1};q)j(-\zeta_{L/2^i}^{6j+3}q;q^3)j(-\zeta_{L/2^i}^{6j+3}q^2;q^3)}.$$

That is by (1.4), we obtain the generating function of $N(0,1,1;L;n)$ in terms of theta functions:

$$\sum_{n=0}^{\infty} N(0,1,1;L;n)q^n$$
$$= \sum_{n=0}^{\infty} \frac{p(n)}{2^k} q^n + \frac{(q;q)_\infty (q^3;q^3)_\infty^4}{(q^6;q^6)_\infty^2}$$
$$\times \sum_{i=0}^{k-1} \frac{2^{i-1}}{L} \sum_{j=0}^{L/2^{i+1}-1} \frac{(1-\zeta_{L/2^i}^{2j+1})(2+\zeta_{L/2^i}^{2j+1})\zeta_{L/2^i}^{-4j-2} j(-\zeta_{L/2^i}^{4j+2};q)}{j(\zeta_{L/2^i}^{2j+1};q)j(-\zeta_{L/2^i}^{6j+3}q;q^3)j(-\zeta_{L/2^i}^{6j+3}q^2;q^3)}. \tag{7.2}$$

For a fixed $L \in \mathbb{N}$, it might be possible to determine the signs of the theta function in (7.2), which will yield the relations between $N(0,1,1;L;n)$ and $\frac{p(n)}{2^k}$.

Finally, similar arguments can be done for any $3|L \in \mathbb{N}$ other than $3\cdot 2^k$. So for any L divisible by 3, one can derive an equation similar to (7.2). Also, similar results can be derived for $N(3j-1,3j,3j+1;L;n)$ for all $0 \le j \le L/3-1$.

8. Remarks and Conjectures

Let

$$F(q) = \frac{(q^3;q^3)_\infty (\zeta_3 q^3;\zeta_3 q^3)_\infty^2}{(q;q)_\infty (q^9;q^9)_\infty} + (\zeta_3 - 1)q \frac{(q^3;q^3)_\infty (\zeta_3^2 q^3;\zeta_3^2 q^3)_\infty (q^{27};q^{27})_\infty}{(q;q)_\infty (q^9;q^9)_\infty}.$$

We tried linear combinations of all the eta quotients in $M_1(\Gamma_1(162))$ to represent $(q,q)_\infty \mathfrak{Re}\big(F(q)\big)$ and $(q,q)_\infty \mathfrak{Im}\big(F(q)\big)$ in terms of eta quotients to no avail. On the other hand, we have

$$j(\zeta_3 q^3;q^9) = (\zeta_3 q^3;\zeta_3 q^3)_\infty = j(q^{36};q^{81}) - \zeta_3 q^3 j(q^{63};q^{81}) + \zeta_3^2 q^{15} j(q^{90};q^{81}),$$
$$j(\zeta_3 q^6;q^9) = (\zeta_3^2 q^3;\zeta_3^2 q^3)_\infty = j(q^{45};q^{81}) - \zeta_3 q^6 j(q^{72};q^{81}) + \zeta_3^2 q^{21} j(q^{99};q^{81}),$$

from which the real and imaginary parts of $F(q)$ can be worked out in terms of Jacobi theta function. This will give alternate formulae for (2.17)–(2.19). We prefer the formulae in (2.17)–(2.19).

We note that all the theta functions in (2.1)–(2.23), except (2.17)–(2.19), are weight 1 modular forms when multiplied by $(q,q)_\infty$.

Finally, we state the conjectures for the comparisons of crank combinations with partition function.

Conjecture 8.1. *We have*

$$\begin{aligned}
&M(0,1,1;6;2n) < \frac{p(2n)}{2}, \quad \text{for all } n \in \mathbb{N},\\
&M(0,1,1;6;2n+1) > \frac{p(2n+1)}{2}, \quad \text{for all } n \in \mathbb{N}_0 - \{1\},\\
&M(2,2,3;6;2n) > \frac{p(2n)}{2}, \quad \text{for all } n \in \mathbb{N},\\
&M(2,2,3;6;2n+1) < \frac{p(2n+1)}{2}, \quad \text{for all } n \in \mathbb{N}_0 - \{1\},\\
&M(2,3,4;12;2n) > \frac{p(2n)}{4}, \quad \text{for all } n \in \mathbb{N},\\
&M(2,3,4;12;2n+1) < \frac{p(2n+1)}{4}, \quad \text{for all } n \in \mathbb{N}_0 - \{1\},\\
&M(0,1,1;12;2n) < \frac{p(2n)}{4}, \quad \text{for all } n \in \mathbb{N}_0 - \{0,3,4\},\\
&M(0,1,1;12;2n+1) > \frac{p(2n+1)}{4}, \quad \text{for all } n \in \mathbb{N}_0,\\
&M(5,5,6;12;2n) < \frac{p(2n)}{4}, \quad \text{for all } n \in \mathbb{N}_0,\\
&M(5,5,6;12;2n+1) > \frac{p(2n+1)}{4}, \quad \text{for all } n \in \mathbb{N}_0 - \{0,1,5,7\},\\
&M(0,1,2;12;2n) > \frac{p(2n)}{4}, \quad \text{for all } n \in \mathbb{N}_0,\\
&M(0,1,2;12;2n+1) < \frac{p(2n+1)}{4}, \quad \text{for all } n \in \mathbb{N}_0 - \{1,2,3,5\},\\
&M(3,4,5;12;2n) < \frac{p(2n)}{4}, \quad \text{for all } n \in \mathbb{N}_0,\\
&M(3,4,5;12;2n+1) > \frac{p(2n+1)}{4}, \quad \text{for all } n \in \mathbb{N}_0 - \{0,5\},\\
&M(6,7,8;12;2n) > \frac{p(2n)}{4}, \quad \text{for all } n \in \mathbb{N}_0 - \{0,1,2\},\\
&M(6,7,8;12;2n+1) < \frac{p(2n+1)}{4}, \quad \text{for all } n \in \mathbb{N}_0,\\
&M(9,10,11;12;2n) < \frac{p(2n)}{4}, \quad \text{for all } n \in \mathbb{N}_0 - \{1,3\},\\
&M(9,10,11;12;2n+1) > \frac{p(2n+1)}{4}, \quad \text{for all } n \in \mathbb{N}_0.
\end{aligned}$$

Additionally, the referee pointed out that the signs of theta parts of $M(0,1,1;9;n)$, $M(2,3,4;9;n)$, $M(0,1,2;9;n)$, $M(3,4,5;9;n)$ and $M(6,7,8;9;n)$ are periodic modulo 9 when $n \geq 467$. Thus, conjectures similar to Conjecture 8.1 can be stated for $M(0,1,1;9;n)$, $M(2,3,4;9;n)$, $M(0,1,2;9;n)$, $M(3,4,5;9;n)$ and $M(6,7,8;9;n)$.

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Zafer Selcuk Aygin
Department of Mathematics and Computer Science
University of Lethbridge
Lethbridge AB T1K 3M4
Canada
e-mail: selcukaygin@ntu.edu.sg;
selcukaygin@gmail.com

Song Heng Chan
Division of Mathematical Sciences
School of Physical and Mathematical Sciences
Nanyang Technological University
21 Nanyang Link
Singapore 637371
Singapore
e-mail: chansh@ntu.edu.sg

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Combinatorial Proofs of Two Euler-Type Identities Due to Andrews

To Professor George Andrews on the occasion of his 80th birthday

Cristina Ballantine and Richard Bielak

Abstract. Let $a(n)$ be the number of partitions of n, such that the set of even parts has exactly one element, $b(n)$ be the difference between the number of parts in all odd partitions of n and the number of parts in all distinct partitions of n, and $c(n)$ be the number of partitions of n in which exactly one part is repeated. Beck conjectured that $a(n) = b(n)$ and Andrews, using generating functions, proved that $a(n) = b(n) = c(n)$. We give a combinatorial proof of Andrews' result. Our proof relies on bijections between a set and a multiset, where the partitions in the multiset are decorated with bit strings. We prove combinatorially Beck's second conjecture, which was also proved by Andrews using generating functions. Let $c_1(n)$ be the number of partitions of n, such that there is exactly one part occurring three times, while all other parts occur only once and let $b_1(n)$ be the difference between the total number of parts in the partitions of n into distinct parts and the total number of different parts in the partitions of n into odd parts. Then, $c_1(n) = b_1(n)$.

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1. Introduction

In [1], following conjectures of Beck, Andrews considered what happens if one relaxes the conditions on parts in Euler's partition identity. He gave analytic proofs of related identities. In this article, we provide combinatorial proofs of his results and add another special case.

Given a non-negative integer n, a *partition* λ of n is a non-increasing sequence of positive integers $\lambda = (\lambda_1, \lambda_2, \ldots, \lambda_k)$ that add up to n, i.e., $\sum_{i=1}^{k} \lambda_i = n$. The numbers λ_i are called the *parts* of λ and n is called the *size* of λ. The number of parts of the partition is called the *length* of λ and is denoted by $\ell(\lambda)$.

When convenient, we use the exponential notation for parts in a partition. The exponent of a part is the multiplicity of the part in the partition. For example, $(7, 5^2, 4, 3^3, 1^2)$ denotes the partition $(7, 5, 5, 4, 3, 3, 3, 1, 1)$.

Let $\mathcal{O}(n)$ be the set of partitions of n with all parts odd and let $\mathcal{D}(n)$ be the set of partitions of n with distinct parts. Then, Euler's identity states that $|\mathcal{O}(n)| = |\mathcal{D}(n)|$.

For example, if $n = 7$, we have

$$\mathcal{O}(7) = \{(7), (5, 1^2), (3^2, 1), (3, 1^4), (1^7)\}$$

and

$$\mathcal{D}(7) = \{(7), (6, 1), (5, 2), (4, 3), (4, 2, 1)\}.$$

Let $\mathcal{A}(n)$ be the set of partitions of n, such that the set of even parts has exactly one element. Let $\mathcal{C}(n)$ be the set of partitions of n in which exactly one part is repeated.

Thus, for $n = 7$, we have

$$\mathcal{A}(7) = \{(6, 1), (5, 2), (4, 3), (4, 1^3), (3, 2^2), (3, 2, 1^2), (2^3, 1), (2^2, 1^3), (2, 1^5)\}$$

and

$$\mathcal{C}(7) = \{(5, 1^2), (4, 1^3), (3^2, 1), (3, 2^2), (3, 2, 1^2), (3, 1^4), (2^3, 1), (2, 1^5), (1^7)\}.$$

Let $a(n) = |\mathcal{A}(n)|$ and $c(n) = |\mathcal{C}(n)|$. Let $b(n)$ be the difference between the number of parts in all odd partitions of n and the number of parts in all distinct partitions of n. Thus, $b(n)$ is the difference between the number of parts in all partitions in $\mathcal{O}(n)$ and the number of parts in all partitions in $\mathcal{D}(n)$. In [3,4], Beck conjectured that $a(n) = b(n) = c(n)$. Andrews proved these identities in [1] using generating functions. In [2], Fu and Tang gave two generalizations of this result. For one of the generalizations, Fu and Tang gave a combinatorial proof, and as a particular case, they obtained a combinatorial proof for $a(n) = c(n)$. We give a combinatorial proof for the identities involving $b(n)$.

Theorem 1.1. *Let $n \geq 1$. Then*

(i) $a(n) = b(n)$;
(ii) $c(n) = b(n)$.

Example 1.2. When $n = 7$, from the example above, we have $a(7) = c(7) = 9$. To calculate $b(7)$, we see that the total number of parts in all partitions in $\mathcal{O}(7)$ is 19 and the total number of parts in all partitions in $\mathcal{D}(7)$ is 10. Thus, $b(7) = 19 - 10 = 9$.

The novelty of our proof of Theorem 1.1 is the use of partitions decorated with bit strings. These are defined in Sect. 2.1. This approach allows us to create bijections between a set of partitions and a multiset of partitions. We distinguish the partitions in the multiset via decorations with bit strings.

To state Beck's second conjectured identity, let $\mathcal{T}(n)$ be the subset of $\mathcal{C}(n)$ consisting of partitions of n in which one part is repeated exactly three times and all other parts occur only once. Let $c_1(n) = |\mathcal{T}(n)|$. Let $b_1(n)$ be the difference between the total number of parts in the partitions of n into distinct parts and the total number of *different* parts in the partitions of n into odd parts. Thus, $b_1(n)$ is the difference between the number of parts in all partitions in $\mathcal{D}(n)$ and the number of different parts in all partitions in $\mathcal{O}(n)$ (i.e., parts counted without multiplicity). We will prove combinatorially the following theorem.

Theorem 1.3. *Let $n \geq 1$. Then, $c_1(n) = b_1(n)$.*

Example 1.4. We continue our example for $n = 7$. We have

$$\mathcal{T}(7) = \{(4, 1^3), (2^3, 1)\}.$$

Therefore, $c_1(7) = 2$. As before, the total number of parts in all partitions in $\mathcal{D}(7)$ is 10. The total number of *different* parts in the partitions in $\mathcal{O}(7)$ is 8. Thus, $b_1(7) = 10 - 8 = 2$.

After the work for this article was finished, we found out that Yang [5] has proved these results in greater generality. However, our approach using decorations with bit strings allows us to extend Theorem 1.3 to an Euler-type identity in Theorem 3.1. Moreover, we are also able to establish the analogous result for the case when one part is repeated exactly two times.

2. Combinatorial Proofs of Theorems 1.1 and 1.3

2.1. $b(n)$ as the Cardinality of a Multiset of Partitions

First, we recall Glaisher's bijection φ used to prove Euler's identity. It is the map from the set of partitions with odd parts to the set of partitions with distinct parts which merges equal parts repeatedly.

Example 2.1. For $n = 7$, Glaisher's bijection is given by

$$\begin{array}{ccc}
(7) & \xrightarrow{\varphi} & (7) \\[2ex]
(5, \underbrace{1, 1}) & \longrightarrow & (5, 2) \\[2ex]
(\underbrace{3, 3}, 1) & \longrightarrow & (6,1) \\[2ex]
(3, \underbrace{\underbrace{1, 1}, \underbrace{1, 1}}) & \longrightarrow & (4, 3) \\[2ex]
(\underbrace{\underbrace{1, 1}, \underbrace{1, 1}}, \underbrace{1, 1}, 1) & \longrightarrow & (4, 2, 1)
\end{array}$$

Thus, each partition $\lambda \in \mathcal{O}(n)$ has at least as many parts as its image $\varphi(\lambda) \in \mathcal{D}(n)$.

When calculating $b(n)$, the difference between the number of parts in all odd partitions of n and the number of parts in all distinct partitions of n, we sum up the differences in the number of parts in each pair $(\lambda, \varphi(\lambda))$. Write each part μ_j of $\mu = \varphi(\lambda)$ as $\mu_j = 2^{k_j} \cdot m_j$ with m_j odd. Then, μ_j was obtained by merging 2^{k_j} parts in λ and thus contributes an excess of $2^{k_j} - 1$ parts to the difference. Therefore, the difference between the number of parts of λ and the number of parts of $\varphi(\lambda)$ is $\sum_{j=1}^{\ell(\varphi(\lambda))}(2^{k_j} - 1)$.

Definition 2.2. Given a partition μ with parts $\mu_j = 2^{k_j} \cdot m_j$, where m_j is odd, the *weight* of the partition is

$$wt(\mu) = \sum_{j=1}^{\ell(\mu)}(2^{k_j} - 1).$$

Thus, $wt(\mu) > 0$ if and only if μ contains at least one even part.

To illustrate Definition 2.2, let

$$\mu = (18, 9, 6, 4, 1) = (2 \cdot 9, 9, 2 \cdot 3, 2^2 \cdot 1, 1).$$

Then,

$$wt(\mu) = (2 - 1) + (2 - 1) + (2^2 - 1) = 5.$$

We denote by $\mathcal{MD}(n)$ the multiset of partitions of n with distinct parts in which every partition $\mu \in \mathcal{D}(n)$ appears exactly $wt(\mu)$ times. For example, $wt(4, 2, 1) = 4$ and $(4, 2, 1)$ appears four times in $\mathcal{MD}(7)$. Since $wt(7) = 0$, the partition (7) does not appear in $\mathcal{MD}(7)$.

The discussion above proves the following interpretation of $b(n)$.

Proposition 2.3. *Let $n \geq 1$. Then, $b(n) = |\mathcal{MD}(n)|$.*

To create bijections from $\mathcal{A}(n)$ to $\mathcal{MD}(n)$ and from $\mathcal{C}(n)$ to $\mathcal{MD}(n)$, we need to distinguish identical elements of $\mathcal{MD}(n)$ and thus view it as a set. Recall that all partitions in $\mathcal{MD}(n)$ have distinct parts and at least one even part.

Definition 2.4. A *bit string* w is a sequence of letters from the alphabet $\{0, 1\}$. The *length* of a bit string w, denoted $\ell(w)$, is the number of letters in w. We refer to *position* i in w as the ith entry from the right, where the right-most entry is counted as position 0.

Note that leading zeros are allowed and are recorded. Thus, 010 and 10 are different bit strings, even though they are the binary representation of the same number. We have $\ell(010) = 3$ and $\ell(10) = 2$. The empty bit string has length 0 and is denoted by $\emptyset$.

Definition 2.5. A *decorated partition* is a partition μ with at least one even part in which one single *even* part, called the decorated part, has a bit string w as an index. If the decorated part is $\mu_i = 2^k m$, where $k \geq 1$ and m is odd, the index w has length $0 \leq \ell(w) \leq k - 1$.

Since there are 2^t distinct bit strings of length t, there are $2^k - 1$ distinct bit strings w of length $0 \leq \ell(w) \leq k - 1$. Thus, for each even part $\mu_i = 2^k m$ of

μ, there are $2^k - 1$ possible indices, and for each partition, μ there are precisely $wt(\mu)$ possible decorated partitions with the same parts as μ.

We denote by $\mathcal{DD}(n)$ the set of decorated partitions of n with distinct parts. Note that, by definition, a decorated partition has at least one even part. Then

$$|\mathcal{MD}(n)| = |\mathcal{DD}(n)|$$

and therefore

$$b(n) = |\mathcal{DD}(n)|.$$

2.2. A Combinatorial Proof for $a(n) = b(n)$

We prove that $a(n) = b(n)$ by establishing a one-to-one correspondence between $\mathcal{A}(n)$ and $\mathcal{DD}(n)$.

From $\mathcal{DD}(n)$ to $\mathcal{A}(n)$:

Start with a decorated partition $\mu \in \mathcal{DD}(n)$. Suppose part $\mu_i = 2^k m$, with $k \geq 1$ and m odd, is decorated with bit string w of length $\ell(w)$. Then, $0 \leq \ell(w) \leq k-1$. Let d_w be the decimal value of w. We set $d_\emptyset = 0$.

1. Split part μ_i into $d_w + 1$ parts of size $2^{k-\ell(w)}m$ and parts of size m. Thus, there will be $2^k - (d_w+1)2^{k-\ell(w)}$ parts of size m. Since $d_w + 1 \leq 2^{\ell(w)}$, the resulting number of parts equal to m is non-negative. Moreover, after the split, there is at least one even part.
2. Every part of size $2^t m$, with $t > k$ (if it exists), splits completely into parts of size $2^{k-\ell(w)}m$, i.e., into $2^{t-k+\ell(w)}$ parts of size $2^{k-\ell(w)}m$.
3. Every other even part splits into odd parts of equal size, i.e., every part $2^u v$ with v odd, such that $2^u v \neq 2^s m$ for some $s \geq k$, splits into 2^u parts of size v.

The resulting partition λ is in $\mathcal{A}(n)$. Its set of even parts is $\{2^{k-\ell(w)}m\}$.

Example 2.6. Consider the decorated partition

$$\mu = (96, 35, 34, 24_{01}, 6, 2) = (2^5 \cdot 3, 35, 2 \cdot 17, (2^3 \cdot 3)_{01}, 2 \cdot 3, 2 \cdot 1) \in \mathcal{DD}(197).$$

We have $k = 3, m = 3, \ell(w) = 2, d_w = 1$.

1. Part $24 = 2^3 \cdot 3$ splits into two parts of size 6 and four parts of size 3.
2. Part $96 = 2^5 \cdot 3$ splits into 16 parts of size 6.
3. All other even parts split into odd parts. Thus, part 34 splits into two parts of size 17, part 6 splits into two parts of size 3, and part 2 splits into two parts of size 1.

The resulting partition is $\lambda = (35, 17^2, 6^{18}, 3^6, 1^2) \in \mathcal{A}(197)$.

Similarly, the transformation maps the decorated partition

$$(96, 35, 34, 24_{10}, 6, 2) \in \mathcal{DD}(197)$$

to

$$(35, 17^2, 6^{19}, 3^4, 1^2) \in \mathcal{A}(197).$$

From $\mathcal{A}(n)$ to $\mathcal{DD}(n)$:

Start with a partition $\lambda \in \mathcal{A}(n)$. Then, there is one and only one even number $2^k m$, $k \geq 1, m$ odd, among the parts of λ. Let f be the multiplicity of the even part in λ. As in Glaisher's bijection, we merge equal parts repeatedly until we obtain a partition μ with distinct parts. Since λ has an even part, μ will also have an even part.

Next, we determine the decoration of μ. Consider the parts μ_{j_i} of the form $2^{r_i} m$, with m odd (same m as in the even part of λ) and $r_i \geq k$. We have $j_1 < j_2 < \cdots$. For notational convenience, set $\mu_{j_0} = 0$. Let h be the positive integer, such that

$$\sum_{i=0}^{h-1} \mu_{j_i} < f \cdot 2^k m \leq \sum_{i=0}^{h} \mu_{j_i}. \tag{2.1}$$

Then, we will decorate part $\mu_{j_h} = 2^{r_h} m$.

To determine the decoration, let N_h be the number of parts $2^k m$ in λ that are merged to form all parts of the form $2^r m > \mu_{j_h}$. Thus

$$N_h = \frac{\sum_{i=0}^{h-1} \mu_{j_i}}{2^k m}.$$

Then, (2.1) becomes

$$2^k m N_h < f \cdot 2^k m \leq 2^k m N_h + 2^{r_h} m,$$

which in turn implies $N_h < f \leq N_h + 2^{r_h - k}$. Therefore, $0 < f - N_h \leq 2^{r_h - k}$.

Let $d = f - N_h - 1$ and $\ell = r_h - k$. We have $0 \leq \ell \leq r_h - 1$. Consider the binary representation of d and insert leading zeros to form a bit string w of length ℓ. Decorate μ_{j_h} with w. The resulting decorated partition is in $\mathcal{DD}(n)$.

Example 2.7. Consider the partition $\lambda = (35, 17^2, 6^{18}, 3^6, 1^2) \in \mathcal{A}(197)$. We have $k = 1, m = 3, f = 18$. Glaisher's bijection produces the partition $\mu = (96, 35, 34, 24, 6, 2) \in \mathcal{MD}(197)$. The parts of the form $2^{r_i} \cdot 3$ with $r_i \geq 1$ are $96, 24, 6$. Since $96 < 18 \cdot 6 \leq 96 + 24$, the decorated part will be $24 = 2^3 \cdot 3$. We have $N_h = 96/6 = 16$. To determine the decoration, let $d = 18 - 16 - 1 = 1$ and $\ell = 3 - 1 = 2$. The binary representation of d is 1. To form a bit string of length 2, we introduce one leading 0. Thus, the decoration is $w = 01$ and the resulting decorated partition is $(96, 35, 34, 24_{01}, 6, 2) \in \mathcal{DD}(197)$. Similarly, starting with $(35, 17^2, 6^{19}, 3^4, 1^2) \in \mathcal{A}(197)$, after applying Glaisher's bijection, we obtain $\mu = (96, 35, 34, 24, 6, 2) \in \mathcal{MD}(197)$. All parameters are the same as in the previous example with the exception of $f = 19$. As before, the decorated part is 24 and $\ell = 2$. We have $d = 19 - 16 - 1 = 2$, whose binary representation is 10 and already has length 2. Thus, $w = 10$ and the resulting decorated partition is $(96, 35, 34, 24_{10}, 6, 2) \in \mathcal{DD}(197)$.

2.3. A Combinatorial Proof for $c(n) = b(n)$

We could compose the bijection of Sect. 2.2 with the bijection of [2] (for $k = 2$) to obtain a combinatorial proof of part (ii) of Theorem 1.1. We give an alternative proof that $c(n) = b(n)$ by establishing a one-to-one correspondence between $\mathcal{C}(n)$ and $\mathcal{DD}(n)$ that does not involve the bijection of [2].

From $\mathcal{DD}(n)$ to $\mathcal{C}(n)$:

Start with a decorated partition $\mu \in \mathcal{DD}(n)$. Suppose part $\mu_i = 2^k m$, with $k \geq 1$ and m odd, is decorated with bit string w of length $\ell(w)$ and decimal value d_w. Then, $0 \leq \ell(w) \leq k-1$. We obtain λ from μ by performing the following steps.

1. Split μ_i into $2(d_w+1)$ parts of size $2^{k-\ell(w)-1}m$ and a part of size $2^{i+k-\ell(w)}m$ for each i such that there is a 0 in position i in w.
2. Each part $2^t m$ with $k-\ell(w)-1 \leq t < k$ splits completely into parts of size $2^{k-\ell(w)-1}m$, i.e., into $2^{t-k+\ell(w)+1}$ parts of size $2^{k-\ell(w)-1}m$.

Since $2(d_w+1) \geq 2$, the obtained partition λ is in $\mathcal{C}(n)$. The repeated part is $2^{k-\ell(w)-1}m$.

Note: If $w = \emptyset$, in Step 1, part μ_i splits into two equal parts of size $2^{k-1}m$, and Step 2 has no effect. If $w \neq \emptyset$, in Step 1, after splitting off $2(d_w+1)$ parts of size $2^{k-\ell(w)-1}m$ from μ_i, we are left with $r = 2^{k-\ell(w)}(2^{\ell(w)} - d_w - 1)m$ to split into distinct parts. We do this using Glaisher's transformation φ on r/m parts equal to m. Thus, in $\varphi((m^{r/m}))$, there is a part equal to $2^j m$ if and only if the binary representation of r/m has a 1 in position j. However, $2^{\ell(w)} - 1$ is a bit string of length $\ell(w)$ with every entry equal to 1. Then, the binary representation of $2^{\ell(w)} - d_w - 1$ (filled with leading zeros if necessary to create a bit string of length $\ell(w)$) is precisely the complement of w, i.e., the bit string obtained from w by replacing every 0 by 1 and every 1 by 0. Thus, in $\varphi((m^{r/m}))$, there will be a part of size $2^{i+k-\ell(w)}m$ if and only if there is a 0 in position i in w.

Example 2.8. Consider the decorated partition

$$\begin{aligned}\mu &= (768, 384_{0110}, 105, 96, 25, 12, 9, 6, 2)\\ &= (2^8 \cdot 3, (2^7 \cdot 3)_{0110}, 105, 2^5 \cdot 3, 25, 2^2 \cdot 3, 9, 2 \cdot 3, 2 \cdot 1) \in \mathcal{DD}(1407).\end{aligned}$$

We have $k = 7, w = 0110, \ell(w) = 4, d_w = 6$. The decorated part is μ_2.

1. Since $2(d_w+1) = 14$ and w has zeros in positions 0 and 3, μ_2 splits into 14 parts of size $2^2 \cdot 3$ and one part each of sizes $2^3 \cdot 3$ and $2^6 \cdot 3$.
2. The parts of the form $2^t \cdot 3$ with $2 \leq t < 7$ are $\mu_4 = 2^5 \cdot 3$ and $\mu_6 = 2^2 \cdot 3$. Then, μ_4 splits into 2^3 parts of size $2^2 \cdot 3$ and μ_6 "splits" into one part of size $2^2 \cdot 3$.

We obtain the partition

$$\begin{aligned}\lambda &= (2^8 \cdot 3, 2^6 \cdot 3, 105, 25, 2^3 \cdot 3, \underbrace{2^2 \cdot 3, \ldots, 2^2 \cdot 3}_{23 \text{ times}}, 9, 2 \cdot 3, 2 \cdot 1)\\ &= (768, 192, 105, 25, 24, 12^{23}, 9, 6, 2) \in \mathcal{C}(1407).\end{aligned}$$

From $\mathcal{C}(n)$ to $\mathcal{DD}(n)$:

Start with a partition $\lambda \in \mathcal{C}(n)$. Then, there is one and only one repeated part among the parts of λ. Suppose the repeated part is $2^k m$, $k \geq 0, m$ odd, and denote by $f \geq 2$ its multiplicity in λ. As in Glaisher's bijection, we merge equal parts repeatedly until we obtain a partition μ with distinct parts. Since λ has a repeated part, μ will have at least one even part.

Next, we determine the decoration of μ. In this case, we want to work with the parts of μ from right to left (i.e., from smallest to largest part). Let $\tilde{\mu}_q = \mu_{\ell(\mu)-q+1}$. Consider the parts $\tilde{\mu}_{j_i}$ of the form $2^{r_i}m$, with m odd and $r_i \geq k$. If $r_1 < r_2 < \cdots$, we have $j_1 < j_2 < \cdots$.

As before, we set $\tilde{\mu}_{j_0} = 0$. Let h be the positive integer, such that

$$\sum_{i=0}^{h-1} \tilde{\mu}_{j_i} < f \cdot 2^k m \leq \sum_{i=0}^{h} \tilde{\mu}_{j_i}. \tag{2.2}$$

Then, we will decorate part $\tilde{\mu}_{j_h} = 2^{r_h}m$. Note that $2^{r_h}m$ is the largest part of μ that is not in λ.

To determine the decoration, let N_h be the number of parts $2^k m$ in λ that merged to form all parts of the form $2^r m < \tilde{\mu}_{j_h}$. Thus

$$N_h = \frac{\sum_{i=0}^{h-1} \tilde{\mu}_{j_i}}{2^k m}. \tag{2.3}$$

Then, (2.2) becomes

$$2^k m N_h < f \cdot 2^k m \leq 2^k m N_h + 2^{r_h} m,$$

which in turn implies $N_h < f \leq N_h + 2^{r_h - k}$. Therefore, $0 < f - N_h \leq 2^{r_h-k}$.

Let

$$d = \frac{f - N_h}{2} - 1 \quad \text{and} \quad \ell = r_h - k - 1.$$

We have $0 \leq \ell \leq r_h - 1$. Consider the binary representation of d and insert leading zeros to form a bit string w of length ℓ. Decorate $\tilde{\mu}_{j_h}$ with w. The resulting decorated partition (with parts written in non-increasing order) is in $\mathcal{DD}(n)$.

Note: To see that $f - N_h$ above is always even, consider the three cases below.

(i) If $h = 1$, then $N_h = 0$. In this case, we must have $f = 2$. Thus, $f - N_h$ is even.

(ii) If f is odd, then after the merge, we have one part equal to $2^k m$ contributing to the numerator of N_h in (2.3). All other parts contributing to the numerator of N_h are divisible by $2 \cdot 2^k m$. Thus, N_h is odd and $f - N_h$ is even.

(iii) If f is even and at least 2, then after the merge we have no part equal to $2^k m$ contributing to the numerator of N_h. All parts contributing to the numerator of N_h are divisible by $2 \cdot 2^k m$. Thus, N_h is even and $f - N_h$ is even.

Example 2.9. Consider the partition

$$\begin{aligned}\lambda &= (2^8 \cdot 3, 2^6 \cdot 3, 105, 25, 2^3 \cdot 3, \underbrace{2^2 \cdot 3, \ldots, 2^2 \cdot 3}_{23 \text{ times}}, 9, 2 \cdot 3, 2 \cdot 1) \\ &= (768, 192, 105, 25, 24, 12^{23}, 9, 6, 2) \in \mathcal{C}(1407).\end{aligned}$$

We have $k = 2$ and $f = 23$. Glaisher's bijection transforms λ as follows:

$$(2^8 \cdot 3, 2^6 \cdot 3, 105, 25, 2^3 \cdot 3, \underbrace{2^2 \cdot 3, \ldots, 2^2 \cdot 3}_{23 \text{ times}}, 9, 2 \cdot 3, 2 \cdot 1)$$
$$\downarrow$$
$$(2^8 \cdot 3, 2^6 \cdot 3, 105, 25, \underbrace{2^3 \cdot 3, \ldots, 2^3 \cdot 3}_{12 \text{ times}}, 2^2 \cdot 3, 9, 2 \cdot 3, 2 \cdot 1)$$
$$\downarrow$$
$$(2^8 \cdot 3, 2^6 \cdot 3, 105, \underbrace{2^4 \cdot 3, \ldots, 2^4 \cdot 3}_{6 \text{ times}}, 25, 2^2 \cdot 3, 9, 2 \cdot 3, 2 \cdot 1)$$
$$\downarrow$$
$$(2^8 \cdot 3, 2^6 \cdot 3, 105, 2^5 \cdot 3, 2^5 \cdot 3, 2^5 \cdot 3, 25, 2^2 \cdot 3, 9, 2 \cdot 3, 2 \cdot 1)$$
$$\downarrow$$
$$(2^8 \cdot 3, 2^6 \cdot 3, 2^6 \cdot 3, 105, 2^5 \cdot 3, 25, 2^2 \cdot 3, 9, 2 \cdot 3, 2 \cdot 1)$$
$$\downarrow$$
$$\mu = (2^8 \cdot 3, 2^7 \cdot 3, 105, 2^5 \cdot 3, 25, 2^2 \cdot 3, 9, 2 \cdot 3, 2 \cdot 1)$$

The parts of the form $2^r \cdot 3$ with $r \geq 2$ are

$$\tilde{\mu}_4 = 2^2 \cdot 3 = 12, \tilde{\mu}_6 = 2^5 \cdot 3 = 96, \tilde{\mu}_8 = 2^7 \cdot 3 = 384,$$

and $\tilde{\mu}_9 = 2^8 \cdot 3 = 768$. Since

$$12 + 96 < 23 \cdot 2^2 \cdot 3 \leq 12 + 96 + 384,$$

the decorated part will be $2^7 \cdot 3 = 384$. We have $h = 3$ and

$$N_3 = \frac{2^2 \cdot 3 + 2^5 \cdot 3}{2^2 \cdot 3} = 1 + 2^3 = 9.$$

Thus

$$d = \frac{23 - 9}{2} - 1 = 6$$

and $\ell = 7 - 2 - 1 = 4$. Thus, $w = 0110$ and the resulting decorated partition is

$$\mu = (2^8 \cdot 3, (2^7 \cdot 3)_{0110}, 105, 2^5 \cdot 3, 25, 2^2 \cdot 3, 9, 2 \cdot 3, 2 \cdot 1)$$
$$= (768, 384_{0110}, 105, 96, 25, 12, 9, 6, 2) \in \mathcal{DD}(1407).$$

2.4. $b_1(n)$ as the Cardinality of a Set of Overpartitions

As in Sect. 2.1, we use Glaisher's bijection and calculate $b_1(n)$ by summing up the difference between the number of parts of $\varphi(\lambda)$ and the number of different parts of λ for each partition $\lambda \in \mathcal{O}(n)$. In $\varphi(\lambda)$, there is a part of size $2^i m$, with m odd if and only if there is a 1 in position i of the binary representation of the multiplicity of m in λ. After the merge, each odd part in λ creates as many parts in $\varphi(\lambda)$ as the number of ones in the binary representation of its multiplicity. Moreover, if we write the parts of $\varphi(\lambda)$ as $2^{k_i} m_i$ with m_i odd and $k_i \geq 0$, all parts $2^s m$ with the same largest odd factor m are obtained by merging parts equal to m in λ.

For each positive odd integer $2j - 1$, denote by $\text{odd}m_{\varphi(\lambda)}(2j - 1)$ the number of parts of $\varphi(\lambda)$ of the form $2^s(2j - 1)$ for some $s \geq 0$. Then, given

$\lambda \in \mathcal{O}(n)$, the difference between the number of parts of $\varphi(\lambda)$ and the number of different parts of λ equals

$$\mathrm{owt}(\varphi(\lambda)) := \sum_{\substack{j \\ \mathrm{odd}m_{\varphi(\lambda)}(2j-1)\neq 0}} (\mathrm{odd}m_{\varphi(\lambda)}(2j-1) - 1).$$

For example, if

$$\varphi(\lambda) = (18, 9, 6, 4, 1) = (2 \cdot 9, 9, 2 \cdot 3, 2^2 \cdot 1, 1),$$

we have $\mathrm{odd}m_{\varphi(\lambda)}(9) = 2$, $\mathrm{odd}m_{\varphi(\lambda)}(3) = 1$, and $\mathrm{odd}m_{\varphi(\lambda)}(1) = 2$. Thus, the contribution to $b_1(38)$ of the difference between the number of parts of $\varphi(\lambda)$ and the number of different parts of $\lambda = (9, 9, 9, 3, 3, 1, 1, 1, 1, 1)$ equals $\mathrm{owt}(\varphi(\lambda)) = (2-1) + (1-1) + (2-1) = 2$.

We denote by $\mathcal{MD}'(n)$ the multiset of partitions of n with distinct parts in which every partition $\mu \in \mathcal{D}(n)$ appears exactly $\mathrm{owt}(\mu)$ times. Then, $b_1(n) = |\mathcal{MD}'(n)|$. To distinguish equal partitions in $\mathcal{MD}'(n)$, we overline certain parts as explained below.

Let $\overline{\mathcal{D}}(n)$ be the set of overpartitions of n with distinct parts in which exactly one part is overlined. Part $2^s m$ with $s \geq 0$ and m odd may be overlined only if there is a part $2^t m$ with $t < s$. In particular, no odd part can be overlined. By an overpartition with distinct parts we mean that all parts have multiplicity one. In particular, p and $\bar{p}$ cannot both appear as parts of the overpartition. The discussion above proves the following interpretation of $b_1(n)$.

Proposition 2.10. *Let $n \geq 1$. Then, $b_1(n) = |\overline{\mathcal{D}}(n)|$.*

2.5. A Combinatorial Proof of $c_1(n) = b_1(n)$

From $\overline{\mathcal{D}}(n)$ to $\mathcal{T}(n)$:

Start with an overpartition $\mu \in \overline{\mathcal{D}}(n)$. Suppose the overlined part is $\mu_i = 2^s m$ for some $s \geq 1$ and m odd. Then, there is a part $\mu_j = 2^t m$ of μ with $t < s$. Let k be the largest positive integer, such that $2^k m$ is a part of μ and $k < s$. To obtain $\lambda \in \mathcal{T}(n)$ from μ, split μ_i into two parts equal to $2^k m$ and one part equal to $2^j m$ whenever there is a 1 in position j of the binary representation of $(2^s - 2^{k+1})$, i.e., one part equal to $2^j m$ for each $j = k+1, k+2, \ldots, s-1$.

Example 2.11. Let

$$\mu = (\overline{768}, 48, 46, 9, 6, 5, 2) = (\overline{2^8 \cdot 3}, 2^4 \cdot 3, 2 \cdot 23, 9, 2 \cdot 3, 5, 2 \cdot 1) \in \overline{\mathcal{D}}(884).$$

Then, $2^8 \cdot 3$ splits into two parts equal to $2^4 \cdot 3$ and one part each of size $2^5 \cdot 3, 2^6 \cdot 3, 2^7 \cdot 3$. Thus, we obtain the partition

$$\begin{aligned}\lambda &= (2^7 \cdot 3, 2^6 \cdot 3, 2^5 \cdot 3, 2^4 \cdot 3, 2^4 \cdot 3, 2^4 \cdot 3, 2 \cdot 23, 9, 2 \cdot 3, 5, 2 \cdot 1) \\ &= (384, 192, 96, 48^3, 46, 9, 6, 5, 2) \in \mathcal{T}(884).\end{aligned}$$

Similarly

$$(768, \overline{48}, 46, 9, 6, 5, 2) = (2^8 \cdot 3, \overline{2^4 \cdot 3}, 2 \cdot 23, 9, 2 \cdot 3, 5, 2 \cdot 1) \in \overline{\mathcal{D}}(884)$$

transforms into

$$(2^8 \cdot 3, 2 \cdot 23, 2^3 \cdot 3, 2^2 \cdot 3, 9, 2 \cdot 3, 2 \cdot 3, 2 \cdot 3, 5, 2 \cdot 1) = (768, 46, 24, 12, 9, 6^3, 5, 2) \in \mathcal{T}(884).$$

From $\mathcal{T}(n)$ to $\overline{\mathcal{D}}(n)$:

Start with a partition $\lambda \in \mathcal{T}(n)$. Merge the parts of λ repeatedly using Glaisher's bijection φ to obtain a partition μ with distinct parts. Overline the smallest part of μ that is not a part of λ. Note that if the thrice repeated part of λ is $2^k m$ for some $k \geq 0$ and m odd, then in μ, there is a part equal to $2^k m$ and the overlined part is of the form $2^t m$ for some $t > k$. Thus, we obtain an overpartition in $\overline{\mathcal{D}}(n)$.

Example 2.12. Let

$$\begin{aligned}\lambda &= (2^7 \cdot 3, 2^6 \cdot 3, 2^5 \cdot 3, 2^4 \cdot 3, 2^4 \cdot 3, 2^4 \cdot 3, 2 \cdot 23, 9, 2 \cdot 3, 5, 2 \cdot 1)\\ &= (384, 192, 96, 48^3, 46, 9, 6, 5, 2) \in \mathcal{T}(884).\end{aligned}$$

Merging equal parts as in Glaisher's bijection, we obtain the partition:

$$\begin{aligned}\mu &= (768, 48, 46, 9, 6, 5, 2)\\ &= (2^8 \cdot 3, 2^4 \cdot 3, 2 \cdot 23, 9, 2 \cdot 3, 5, 2 \cdot 1) \in \mathcal{D}(884).\end{aligned}$$

The smallest part of μ that is not a part of λ is 768. Thus, we obtain the overpartition $(\overline{768}, 48, 46, 9, 6, 5, 2) \in \overline{\mathcal{D}}(884)$.

Similarly, after applying Glaisher's bijection, the partition

$$\begin{aligned}\lambda &= (2^8 \cdot 3, 2 \cdot 23, 2^3 \cdot 3, 2^2 \cdot 3, 9, 2 \cdot 3, 2 \cdot 3, 2 \cdot 3, 5, 2 \cdot 1)\\ &= (768, 46, 24, 12, 9, 6^3, 5, 2) \in \mathcal{T}(884)\end{aligned}$$

maps to

$$\begin{aligned}\mu &= (768, 48, 46, 9, 6, 5, 2)\\ &= (2^8 \cdot 3, 2^4 \cdot 3, 2 \cdot 23, 9, 2 \cdot 3, 5, 2 \cdot 1) \in \mathcal{D}(884).\end{aligned}$$

The smallest part of μ not appearing in λ is 48. Thus, we obtain the overpartition $(768, \overline{48}, 46, 9, 6, 5, 2) \in \overline{\mathcal{D}}(884)$.

Remark 2.13. We could have obtained the transformation above from the combinatorial proof of part (ii) of Theorem 1.1. In the transformation from $\mathcal{C}(n)$ to $\mathcal{DD}(n)$, we have $f = 3$, $h = 2$, and $N_h = 1$. Thus $d = 0$ and the decorated part is the smallest part in the transformed partition μ that did not occur in the original partition λ. Then

$$r_h = 1 + k + \max\{j \mid 2^k \cdot m, 2^{k+1} \cdot m, \ldots, 2^{k+j} \cdot m \text{ are all parts of } \lambda\}.$$

Thus, in μ, the decorated part $2^{r_h} \cdot m$ is decorated with a bit string consisting of all zeros and of length $r_h - k - 1$, one less than the difference in exponents of 2 of the decorated part and the next smallest part with the same largest odd factor m. Since the decoration of a partition in $\mathcal{DD}(n)$ is completely determined by the part being decorated, we could simply just overline the part.

3. Extending Theorem 1.3 to an Euler-Type Identity

It is natural to look for the analogue of Theorem 1.1 (i) in the setting of Theorem 1.3. If in Euler's partition identity, we relax the condition in $\mathcal{D}(n)$ to allow one part to be repeated exactly three times, how do we relax the condition on $\mathcal{O}(n)$ to obtain an identity? We can search for the condition by following the proof of Theorem 1.1 part (i) but only for decorated partitions from $\mathcal{DD}(n)$, where an even part is decorated with a bit string consisting entirely of zeros as in Remark 2.13, i.e., of length one less than the difference in exponents of 2 of the decorated part and the next smallest part with the same largest odd factor m. We identify these decorated partitions with the overpartitions in $\overline{\mathcal{D}}(n)$. Following the algorithm, we see that the set $\mathcal{T}(n)$ has the same cardinality as the set of partitions with exactly one even part $2^k \cdot m$, $k \geq 1, m$ odd, which appears with odd multiplicity. Moreover, part m appears with multiplicity at least 2^{k-1} and the multiplicity of m must belong to an interval $[2^s - 2^{k-1}, 2^s - 1]$ for some $s \geq k$. Given the elegant description of the partitions in $\mathcal{T}(n)$, it would be desirable to find a nicer set of the same cardinality consisting of partitions with only one even part under some constraints.

Let $\mathcal{A}'(n)$ be the subset of $\mathcal{A}(n)$ consisting of partitions λ of n, such that the set of even parts has exactly one element and the following two conditions hold:

(1) The even part $2^k \cdot m$, $k \geq 1, m$ odd, has odd multiplicity.
(2) The largest odd factor m of the even part is a part of λ with multiplicity between 1 and $2^k - 1$.

Let $a_1(n) = |\mathcal{A}'(n)|$.

Theorem 3.1. *Let $n \geq 1$. Then, $a_1(n) = b_1(n)$.*

Proof. From $\overline{\mathcal{D}}(n)$ to $\mathcal{A}'(n)$:

Start with an overpartition $\mu \in \overline{\mathcal{D}}(n)$. Suppose the overlined part is $\mu_i = 2^s \cdot m$, $s \geq 1, m$ odd. Then, there is a part $\mu_j = 2^t m$ of μ with $0 \leq t < s$. Keep part $2^s \cdot m$ and remove its overline. Split each part of the form $2^u \cdot m$ with $u > s$ (if it exists) into 2^{u-s} parts equal to $2^s \cdot m$. Split each part of the form $2^v \cdot m$ with $0 \leq v < s$ into 2^v parts equal to m. Split every other even part into odd parts. Call the obtained partition λ. Then, the multiplicity of $2^s \cdot m$ in λ is odd. Since there is a part $\mu_j = 2^t m$ of μ with $0 \leq t < s$, there will be at least one part equal to m in λ. The largest possible multiplicity of m in λ is $2^{s-1} + 2^{s-2} + \cdots + 2 + 1 = 2^s - 1$. Thus, $\lambda \in \mathcal{A}'(n)$.

From $\mathcal{A}'(n)$ to $\overline{\mathcal{D}}(n)$:

Let $\lambda \in \mathcal{A}'(n)$. Merge equal terms repeatedly (as in Glaisher's bijection) to obtain a partition with distinct parts. Overline the part equal to the even part in λ. Call the obtained overpartition μ. Since the even part $2^k \cdot m$ in λ has odd multiplicity, there will be a part in μ equal to $2^k \cdot m$. Since m has multiplicity between 1 and $2^k - 1$ in λ, there will be a part of size $2^i \cdot m$ in μ whenever there is a 1 in position i in the binary representation of the multiplicity of m in λ. The binary representation of $2^k - 1$ is a bit string of length $k-1$ consisting entirely of ones. Thus, in μ, there is at least one part of size $2^t \cdot m$ with $0 \leq t < k$ and $\mu \in \overline{\mathcal{D}}(n)$. □

From Theorems 1.3 and 3.1, we obtain the following Euler-type identity.

Corollary 3.2. *Let $n \geq 1$. Then, $a_1(n) = c_1(n)$.*

Example 3.3. Let $n = 10$. Then

$$\mathcal{T}(20) = \{(7,1,1,1),(5,2,1,1,1),(4,3,1,1,1),(4,2,2,2),\\ (3,2,2,2,1),(3,3,3,1)\}$$

and

$$\mathcal{A}'(10) = \{(7,2,1),(3,2,2,2,1),(5,4,1),(3,4,1,1,1),(6,3,1),(8,1,1)\}.$$

4. One Part Repeated Exactly Two Times, All Other Parts Distinct

Given the specialization of Theorem 1.1 stated in Theorems 1.3 and 3.1, it is natural to ask what happens if one considers the set of partitions, such that one part is repeated exactly two times and all other parts are distinct. Let $\mathcal{S}(n)$ be the subset of $\mathcal{C}(n)$ consisting of such partitions and let $c_2(n) = |\mathcal{S}(n)|$. We would like to express $c_2(n)$ as an excess of parts between partitions in $\mathcal{D}(n)$ and $\mathcal{O}(n)$ (where parts are counted with different multiplicities) to obtain an identity similar to $c(n) = b(n)$ and $c_1(n) = b_1(n)$.

Note that $b(n)$ is the difference between the number of parts in all partitions in $\mathcal{O}(n)$ and the number of parts in all partitions in $\mathcal{D}(n)$. Thus, each part appearing in a partition in $\mathcal{O}(n)$ is counted with the multiplicity with which it appears in the partition. On the other hand, $b_1(n)$ is the difference between the number of parts in all partitions in $\mathcal{D}(n)$ and the number of *different* parts in all partitions in $\mathcal{O}(n)$. Here, each part appearing in a partition in $\mathcal{O}(n)$ is counted with multiplicity 1 for that partition.

Definition 4.1. Given a partition $\lambda \in \mathcal{O}(n)$, suppose the multiplicity of i in λ is m_i. If i appears in λ, we define the *binary order of magnitude* of the multiplicity of i in λ, denoted $bomm_\lambda(i)$, to be the number of digits in the binary representation of m_i.

Note that, if $m_i > 0$, then $\mathrm{bomm}_\lambda(i) = \lfloor \log_2(m_i) \rfloor + 1$.

Let $b_2(n)$ denote the difference between the number of parts in all partitions in $\mathcal{O}(n)$, each counted as many times as its bomm, and the number of parts in all partitions in $\mathcal{D}(n)$. Since the number of parts in all partitions in $\mathcal{D}(n)$ equals the number of ones in all binary representations of all multiplicities in all partitions of $\mathcal{O}(n)$, it follows that $b_2(n)$ equals the number of zeros in all binary representations of all multiplicities in all partitions of $\mathcal{O}(n)$. We have the following theorem.

Theorem 4.2. *Let $n \geq 1$. Then, $c_2(n) = b_2(n)$.*

Proof. Let $\mathcal{DD}'(n)$ be the subset of decorated partitions μ in $\mathcal{DD}(n)$, such that a part $2^s m$ of μ with $s \geq 1$ and m odd can be decorated only if $2^{s-1}m$ is not a part of μ. The decoration w must satisfy $d_w = 0$ and $0 \leq \ell(w) \leq s - k - 2$, where $k = \max\{j < s \mid 2^j m$ is a part of $\mu\}$ if there is a part $2^j m$ in μ with $j < s$, and $k = -1$ otherwise.

Recall that, in a bit string, the most right position of a digit is position 0 and we count positions from right to left. To see that $b_2(n) = |\mathcal{DD}'(n)|$ we argue as follows. If there is a part $2^s m$ in $\mu \in \mathcal{DD}'(n)$, with $s \geq 0$ and m odd, then there is a 1 in position s of the binary representation of $\text{mult}_\lambda(m)$ with $\lambda \in \mathcal{O}(n)$, such that $\varphi(\lambda) = \mu$. Here, $\text{mult}_\lambda(m)$ is the multiplicity of m in the partition λ. If the part $2^s m$ of μ is decorated, then $s \geq 1$ and $2^{s-1}m$ is not a part of μ. Then, digit 1 in position s of the binary representation of $\text{mult}_\lambda(m)$ is followed by $s - k - 1$ zeros, where k is defined as above. Note that in this case, $k < s - 1$, and therefore, $s - k - 1 > 0$. Since there are $s - k - 1$ possible decorations for part $2^s m$, the total number of possible decorations of parts of μ equals the number of zeros in the binary representations of all multiplicities of the corresponding $\lambda \in \mathcal{O}(n)$.

Next, we show that $c_2(n) = b_2(n)$ by creating a one-to-one correspondence between $\mathcal{S}(n)$ and $\mathcal{DD}'(n)$.

From $\mathcal{DD}'(n)$ to $\mathcal{S}(n)$:

Start with a decorated partition $\mu \in \mathcal{DD}'(n)$. Suppose part $\mu_i = 2^s m$, $s \geq 1$, and m odd is decorated with word w with $d_w = 0$ and $0 \leq \ell(w) \leq s - k - 2$, where k is defined as above. Then, we split $2^s m$ into parts $2^{s-1}m, 2^{s-2}m, \ldots, 2^{s-\ell(w)}m$ and two parts equal to $2^{s-\ell(w)-1}m$. Note that if $\ell(w) = 0$, then 2^s splits into two parts equal to $2^{s-1}m$. If there is a part $2^j m$ in μ with $j < s$, then k is defined as the largest $k < s$, such that $2^k m$ is a part of μ. Since $s - \ell(w) - 1 \geq k + 1$, part $2^{s-\ell(w)-1}m$ appears exactly twice and all other parts appear once. The obtained partition is in $\mathcal{S}(n)$.

From $\mathcal{S}(n)$ to $\mathcal{DD}'(n)$:

Let $\lambda \in \mathcal{S}(n)$. Suppose the part that appears exactly twice is $2^k m$ with $k \geq 0$ and m odd. Merge the parts of λ repeatedly using Glaisher's bijection φ to obtain a partition μ with distinct parts. There will be no part equal to $2^k m$ in μ. The decorated part will be the only part of μ that is not a part of λ. As in the proof of Theorem 1.1 (ii), $N_h = 0$, and since $f = 2$, we have

$$d = \frac{f - N_h}{2} - 1 = 0.$$

Moreover, if the decorated part is $2^s m$ with m odd, then $\ell(w) = s - k - 1$. If there is a part $2^t m$ with $t < s$ in μ, then, by construction, $t \leq k - 1$. Then, $\mu \in \mathcal{DD}'(n)$. $\square$

To establish an Euler-type identity, let $\mathcal{A}''(n)$ be the subset of $\mathcal{A}(n)$ consisting of partitions λ of n, such that the set of even parts has exactly one element and the following two conditions hold:

(1) The even part $2^k \cdot m$, $k \geq 1, m$ odd, has odd multiplicity.

(2) The largest odd factor m of the even part is a part of λ with multiplicity between 0 and $2^k - 2$.

Let $a_2(n) = |\mathcal{A}''(n)|$. Following the proof of Theorem 3.1, one can show that $a_2(n) = b_2(n)$. Then, we have the following theorem.

Theorem 4.3. *Let $n \geq 1$. Then, $a_2(n) = b_2(n) = c_2(n)$.*

Example 4.4. Let $n = 10$. Then

$$\mathcal{S}(10) = \{(8,1,1),(6,2,1,1),(5,3,1,1),(6,2,2),\\ (4,2,2,1),(4,3,3),(4,4,1),(5,5)\}$$

and

$$\mathcal{A}''(10) = \{(5,3,2),(2,2,2,2,2),(4,3,3),(5,4,1),\\ (6,3,1),(6,1,1,1,1),(8,1,1),(10)\}.$$

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Cristina Ballantine and Richard Bielak
Department of Mathematics and Computer Science
College of the Holy Cross
Worcester
MA 01610
USA
e-mail: cballant@holycross.edu

Richard Bielak
e-mail: rebiel18@g.holycross.edu

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Noncommutative Catalan Numbers

Dedicated to Professor George Andrews on the occasion of his eightieth birthday

Arkady Berenstein and Vladimir Retakh

Abstract. The goal of this paper is to introduce and study *noncommutative Catalan numbers* C_n which belong to the free Laurent polynomial algebra $\mathcal{L}_n$ in n generators. Our noncommutative numbers admit interesting (commutative and noncommutative) specializations, one of them related to Garsia–Haiman (q,t)-versions, another—to solving noncommutative quadratic equations. We also establish total positivity of the corresponding (noncommutative) Hankel matrices H_n and introduce accompanying *noncommutative binomial coefficients* $\left(\!\!\binom{n}{k}\!\!\right) \in \mathcal{L}_{n+k-1}$, $\left(\!\!\binom{n}{k}\!\!\right)' \in \mathcal{L}_n$.

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1. Introduction

Catalan numbers $c_n = \frac{1}{n+1}\binom{2n}{n}$, $n \geq 0$ are important combinatorial objects which satisfy a number of remarkable properties such as:

- The recursion

$$c_{n+1} = \sum_{k=0}^{n} c_k c_{n-k}$$

for all $n \geq 0$ (with $c_0 = c_1 = 1$).

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- The determinantal identities

$$\det\begin{pmatrix} c_m & c_{m+1} & \cdots & c_{m+n} \\ c_{m+1} & c_{m+2} & \cdots & c_{m+n+1} \\ & & \cdots & \\ c_{m+n} & c_{m+n+1} & \cdots & c_{m+2n} \end{pmatrix} = 1$$

for $n \geq 0$, $m \in \{0,1\}$.

Catalan numbers admit various q-deformations [2,9,16] and (q,t)-deformations [10,11,16]. See, for example, [1,18,19].

In this paper, we introduce and study *noncommutative Catalan numbers* C_n, $n \geq 1$, which are totally noncommutative Laurent polynomials in n variables and satisfy analogs of the recursion and the determinantal identities (Proposition 2.3; Eq. (2.9)). It turns out that specializing these variables to appropriate powers of q, we recover Garsia–Haiman $(q,1)$-Catalan numbers. Catalan numbers also satisfy a combinatorial identity [6, Eq. (4.9)] involving their truncated counterparts $c_n^k = \binom{n+k}{k} - \binom{n+k}{k-1}$ (so that $c_n = c_n^n = c_n^{n-1}$):

$$c_n = \sum_{\substack{a,b\in\mathbb{Z}_{\geq 0},\\ a+b\leq n, a-b=d}} c_{n-b}^a c_{n-a}^b \tag{1.1}$$

for each $n \in \mathbb{Z}_{\geq 0}$ and each $d \in \mathbb{Z}$ with $|d| \leq n$ (e.g., the right-hand side does not depend on d). A q-deformation of c_n^k was discussed in [7] under the name of q-ballot numbers.

We introduce noncommutative analogs of truncated Catalan numbers and establish a noncommutative version of (1.1) (Theorem 2.22). It is curious that the c_n^k satisfy three more combinatorial identities, two of which involve binomial coefficients:

$$\begin{cases} c_{n+1}^k = \sum\limits_{j=0}^{k} c_j c_{n-j}^{k-j}, \\ \sum\limits_{j=0}^{k} (-1)^j c_{n+k-j}^j \cdot \dbinom{n-j}{k-j} = 0, \\ c_{m+n}^k = \sum\limits_{\ell=0}^{n} c_{m+\ell}^{k-\ell} \cdot \dbinom{n}{\ell}, \end{cases} \tag{1.2}$$

where $0 \leq k < n$ in the first two identities and $0 \leq k \leq m+n$ in the third one.

We establish a noncommutative generalization of the first identity (1.2) (Proposition 2.20(c)), define appropriate noncommutative versions $\left(\!\!\binom{n}{k}\!\!\right)$ and $\left(\!\!\binom{n}{k}\!\!\right)'$ of binomial coefficients, and establish analogs of the last two identities (1.2) with these coefficients (Corollary 2.33; Theorem 2.34) as well as an analog of the multiplication law for both kinds of noncommutative binomial coefficients (Theorem 2.32).

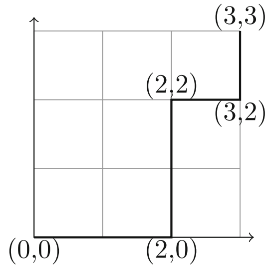

Figure 1. $M_P = x_2 x_0^{-1} x_1$ for the above path $P \in \mathcal{P}_3$

In fact, these constructions and results extend our previous work on noncommutative Laurent phenomenon [3,4], and we expect more such phenomena to emerge in combinatorics, representation theory, topology, and related fields.

The paper is organized as follows: Sect. 2 contains notation and main results and the proofs are given in Sect. 3.

2. Notation and Main Results

Let F be the free group generated by x_k, $k \in \mathbb{Z}_{\geq 0}$, and F_m be the (free) subgroup of F generated by $x_0, \ldots, x_m$.

Denote by $\tilde{\mathcal{P}}_n$ the set of all monotonic lattice paths in $[0,n] \times [0,n]$ from $(0,0)$ to (n,n). Clearly, $|\tilde{\mathcal{P}}_n| = \binom{2n}{n}$. We say that $P \in \tilde{\mathcal{P}}_n$ is *Catalan* if, for each point $p = (p_1, p_2) \in P$, one has $c(p) \geq 0$, where $c(p_1, p_2) = p_1 - p_2$ is the content of p. Denote by $\mathcal{P}_n \subset \tilde{\mathcal{P}}_n$ the set of all Catalan paths in $[0,n] \times [0,n]$. Clearly, $|\mathcal{P}_n| = \frac{1}{n+1}\binom{2n}{n}$ is the nth Catalan number, which justifies the terminology.

We say that a point $p = (p_1, p_2)$ of $P \in \tilde{\mathcal{P}}_n$ is a *southeast* (*resp. northwest*) *corner* of P if $(p_1 - 1, p_2) \in P$ and $(p_1, p_2 + 1) \in P$ (resp. $(p_1, p_2 - 1) \in P$ and $(p_1 + 1, p_2) \in P$) (Fig. 1).

To each $P \in \mathcal{P}_n$, we assign an element $M_P \in F_n$ by

$$M_P = \overrightarrow{\prod} x_{c(p)}^{\operatorname{sgn}(p)}, \tag{2.1}$$

where the product is over all corners $p \in P$ (taken in the natural order) and

$$\operatorname{sgn}(p) = \begin{cases} 1, & \text{if } p \text{ is southeast,} \\ -1, & \text{if } p \text{ is northwest.} \end{cases}$$

We define the *noncommutative Catalan number* $C_n \in \mathbb{Z}F_n$ by:

$$C_n = \sum_{P \in \mathcal{P}_n} M_P. \tag{2.2}$$

Clearly, under the counit homomorphism $\varepsilon \colon \mathbb{Z}F \to \mathbb{Z}$ $(x_k \mapsto 1)$, the image $\varepsilon(C_n)$ is $|\mathcal{P}_n|$, the ordinary Catalan number.

Noncommutative Catalan numbers exhibit some symmetries, the first of which is an anti-automorphism $\bar{\cdot}$ of $\mathbb{Z}F$, such that $\overline{x}_k = x_k$ for $k \in \mathbb{Z}_{\geq 0}$.

Proposition 2.1. *$\overline{C}_n = C_n$ for all $n \geq 0$.*

Proof. Define an involution $s_n\colon \mathbb{Z}^2 \to \mathbb{Z}^2$ by $s_n(x,y) = (n-y, n-x)$. Clearly, $s_n(\mathcal{P}_n) = \mathcal{P}_n$. It is easy to see that:

$$\overline{M}_P = M_{s_n(P)} \tag{2.3}$$

for all $P \in \mathcal{P}_n$. Therefore,

$$\overline{C}_n = \sum_{P\in\mathcal{P}_n} \overline{M}_P = \sum_{P\in\mathcal{P}_n} M_{s_n(P)} = \sum_{P\in\mathcal{P}_n} M_P = C_n$$

for all $n \geq 0$.

The proposition is proved. □

Example 2.2. $C_0 = x_0$, $C_1 = x_1$, $C_2 = x_2 + x_1x_0^{-1}x_1$,

$$\begin{aligned}
C_3 &= x_3 + x_2x_1^{-1}x_2 + x_2x_0^{-1}x_1 + x_1x_0^{-1}x_2 + x_1x_0^{-1}x_1x_0^{-1}x_1,\\
C_4 &= x_4 + x_3x_2^{-1}x_3 + x_2x_0^{-1}x_2 + x_3x_1^{-1}x_2 + x_2x_1^{-1}x_3 + x_3x_0^{-1}x_1\\
&\quad + x_1x_0^{-1}x_3 + x_2x_1^{-1}x_2x_1^{-1}x_2 + x_1x_0^{-1}x_2x_0^{-1}x_1\\
&\quad + x_2x_1^{-1}x_2x_0^{-1}x_1 + x_1x_0^{-1}x_2x_1^{-1}x_2 + x_2x_0^{-1}x_1x_0^{-1}x_1\\
&\quad + x_1x_0^{-1}x_1x_0^{-1}x_2 + x_1x_0^{-1}x_1x_0^{-1}x_1x_0^{-1}x_1.
\end{aligned}$$

It turns out that our noncommutative Catalan numbers satisfy the following generalization of the well-known classical recursion, which we prove in Sect. 3.1.

Proposition 2.3. *For $n \geq 0$, one has*

$$C_{n+1} = \sum_{k=0}^{n} C_k x_0^{-1} T(C_{n-k}),\ C_{n+1} = \sum_{k=0}^{n} T(C_k) x_0^{-1} C_{n-k} \tag{2.4}$$

for all $n \in \mathbb{Z}_{\geq 0}$, where $T\colon \mathbb{Z}F \to \mathbb{Z}F$ is an endomorphism of $\mathbb{Z}F$ given by $T(x_k) = x_{k+1}$ for all $k \in \mathbb{Z}_{\geq 0}$.

For example:

$$C_2 = T(C_1) + C_1x_0^{-1}T(C_0)$$

and

$$C_3 = T(C_2) + C_1x_0^{-1}T(C_1) + C_2x_0^{-1}T(C_0).$$

The following is an immediate corollary of Proposition 2.3.

Corollary 2.4. *The formal power series*

$$\mathbf{C}(t) = \sum_{n=0}^{\infty} C_n t^n \in (\mathbb{Z}F)[[t]]$$

satisfies:

$$\mathbf{C}(t) = x_0 + t\mathbf{C}(t)x_0^{-1}T(\mathbf{C}(t)),\ T(\mathbf{C}(t))x_0^{-1}\mathbf{C}(t) = \mathbf{C}(t)x_0^{-1}T(\mathbf{C}(t)). \tag{2.5}$$

Remark 2.5. Applying ε to (2.5), we obtain the well-known functional equation $c(t) = 1 + tc(t)^2$ for the classical generating function $c(t) = \sum_{n=0}^{\infty} \varepsilon(C_n)t^n$ of Catalan numbers.

Remark 2.6. After the first version of this paper became available, Philippe Di Francesco and Rinat Kedem pointed to us that $\mathbf{C}(t)x_0^{-1}$ is a *noncommutative Stieltjes continued fraction*, which can be computed by combining methods of [8, Section 3.3.1] and [13, Section 8] as follows:

$$\mathbf{C}(t)x_0^{-1} = \lim_{k\to\infty} \mathbf{S}(x_1x_0^{-1}, \dots, x_kx_{k-1}^{-1}, t),$$

where

$$\mathbf{S}(z_1, t) = (1 - z_1t)^{-1}, \quad \mathbf{S}(z_1, \dots, z_k, t) = \mathbf{S}(z_1, \dots, z_{k-2}, \mathbf{S}(z_k, t)z_{k-1}, t)$$

for $k \geq 2$.

Remark 2.7. In fact, there is another recursion

$$\begin{aligned} C_{n+1} &= C_nx_0^{-1}x_1 + \sum_{k=1}^{n} C_kx_1^{-1}T^2(C_{n-k}) \\ &= x_1x_0^{-1}C_n + \sum_{k=0}^{n-1} T^2(C_k)x_1^{-1}C_{n-k} \end{aligned}$$

for $n \geq 1$. For instance:

$$\begin{aligned} C_3 &= C_2x_0^{-1}x_1 + C_1x_1^{-1}T^2(C_1) + C_2x_1^{-1}T^2(C_0) \\ &= C_2x_0^{-1}x_1 + x_3 + C_2x_1^{-1}x_2. \end{aligned}$$

The recursion leads to the functional equation:

$$\mathbf{C}(t) = x_0 + t(\mathbf{C}(t)x_0^{-1}x_1 - x_0x_1^{-1}T^2(\mathbf{C}(t)) + \mathbf{C}(t)x_1^{-1}T^2(\mathbf{C}(t))),$$

which we leave as an exercise to the reader.

Remark 2.8. Equation (2.4) can be written in a matrix form:

$$Hx_0^{-1}T(H) = T(H)x_0^{-1}H = H',$$

where H (resp. H') is the lower triangular $\mathbb{Z}_{\geq 0} \times \mathbb{Z}_{\geq 0}$ Toeplitz matrix whose (i, j)th entry is C_{i-j} (resp. C_{i-j+1}) if $i \geq j$. Thus, H^{-1} is a lower triangular Toeplitz matrix whose (i, j)th entry is $-x_0^{-1}T(C_{i-j-1})x_0^{-1}$ for $i > j$.

It turns out that there is a remarkable specialization $\underline{C}_n \in \mathbb{Z}F_1$ of C_n. Indeed, let $\sigma\colon \mathbb{Z}F \to \mathbb{Z}F_1$ be a ring homomorphism given by $\sigma(x_k) = x_0^kx_1^k$, $k \in \mathbb{Z}_{\geq 0}$. Abbreviate $\underline{C}_n = \sigma(C_n)$ for $n \geq 0$.

The following result asserts, in particular, that $\underline{C}_n$ are noncommutative polynomials (rather than Laurent polynomials) and they satisfy yet another noncommutative generalization of the well-known classical recursion for Catalan numbers.

Proposition 2.9. *The elements* $\underline{C}_n \in \mathbb{Z}\langle x_0, x_1\rangle$ *are determined by the following recursion*: $\underline{C}_0 = 1$ *and*

$$\underline{C}_{n+1} = \sum_{k=0}^{n} \underline{C}_k x_0 \underline{C}_{n-k} x_1 = \sum_{k=0}^{n} x_0 \underline{C}_k x_1 \underline{C}_{n-k}, \tag{2.6}$$

for $n \geq 0$*. In particular, all* $\underline{C}_n$ *belong to the free semi-ring* $\mathbb{Z}_{\geq 0}\langle x_0, x_1\rangle \subset \mathbb{Z}_{\geq 0} F_1$*.*

Our proof of the proposition is based on the identity:

$$\sigma(T^i C_n) = x_0^i \sigma(C_n) x_1^i$$

for $i, n \geq 0$ (see Lemma 3.3).

Remark 2.10. Applying σ to the recursions from Remark 2.7 and using the same argument from the proof of Proposition 2.9, we obtain another recursion for $\underline{C}_n$:

$$\begin{aligned}\underline{C}_{n+1} &= \underline{C}_n x_0 x_1 + \sum_{k=1}^{n} \underline{C}_k x_1^{-1} x_0 \underline{C}_{n-k} x_1^2 \\ &= x_0 x_1 C_n + \sum_{k=0}^{n-1} x_0^2 \underline{C}_k x_1 x_0^{-1} \underline{C}_{n-k}.\end{aligned}$$

Remark 2.11. One can show that the "two-variable" noncommutative Catalan numbers are invariant under the anti-involution of $\mathbb{Z}F_1$ interchanging x_0 and x_1.

In fact, we can explicitly compute each $\underline{C}_n$. Indeed, assign a monomial $\underline{M}_P \in F_1$ to each $P \in \mathcal{P}_n$ by:

$$\underline{M}_P = x_0^{j_0} x_1^{j_1} x_0^{j_2} \ldots x_1^{j_{2k}},$$

where $(j_0, j_1, \ldots, j_{2k}) \in \mathbb{Z}_{>0}^{2k+1}$ is the sequence of jumps of the path P, i.e., the rth northwest corner is $(j_0 + j_2 + \cdots + j_{2r}, j_1 + j_3 + \cdots + j_{2r+1})$ and rth southeast corner of P is $(j_0 + j_2 + \cdots + j_{2r}, j_1 + j_3 + \cdots + j_{2r-1})$. One can easily see that $\sigma(M_P) = \underline{M}_P$, so we obtain the following immediate corollary.

Corollary 2.12. $\underline{C}_n = \sum_{P \in \mathcal{P}_n} \underline{M}_P$ *for all* $n \geq 1$.

Example 2.13.

$$\begin{aligned}
\underline{C}_2 &= x_0^2 x_1^2 + x_0 x_1 x_0 x_1, \\
\underline{C}_3 &= x_0^3 x_1^3 + x_0^2 x_1 x_0 x_1^2 + x_0^2 x_1^2 x_0 x_1 + x_0 x_1 x_0^2 x_1^2 + x_0 x_1 x_0 x_1 x_0 x_1, \\
\underline{C}_4 &= x_0^4 x_1^4 + x_0^3 x_1 x_0 x_1^3 + x_0^2 x_1^2 x_0^2 x_1^2 + x_0^3 x_1^2 x_0 x_1^2 + x_0^2 x_1 x_0^2 x_1^3 + x_0^3 x_1^3 x_0 x_1 \\
&\quad + x_0 x_1 x_0^3 x_1^3 + x_0^2 x_1 x_0 x_1 x_0 x_1^2 + x_0 x_1 x_0^2 x_1^2 x_0 x_1 + x_0^2 x_1 x_0 x_1^2 x_0 x_1 \\
&\quad + x_0 x_1 x_0^2 x_1 x_0 x_1^2 + x_0^2 x_1^2 x_0 x_1 x_0 x_1 + x_0 x_1 x_0 x_1 x_0^2 x_1^2 \\
&\quad + x_0 x_1 x_0 x_1 x_0 x_1 x_0 x_1.
\end{aligned}$$

The following immediate result is a "two-variable" version of Corollary 2.4.

Corollary 2.14. *The formal power series*

$$\underline{\boldsymbol{C}}(t) = \sum_{n=0}^{\infty} \underline{C}_n t^n \in \mathbb{Z}\langle x_0, x_1\rangle[[t]]$$

satisfies:

$$\underline{\boldsymbol{C}}(t) = 1 + t\underline{\boldsymbol{C}}(t)x_0\underline{\boldsymbol{C}}(t)x_1. \tag{2.7}$$

Remark 2.15. For $t = 1$, Eq. (2.7) coincides with the quadratic equation on formal series $K(x_0, x_1)$ studied in [17] where a solution of this equation was presented as a "noncommutative Rogers–Ramanujan continued fraction".

Remark 2.16. In our previous work [5] on the inversion of $\sum_{n\geq 0} x_0^n x_1^n$ in the ring of formal series $\mathbb{Z}\langle\langle x_0, x_1\rangle\rangle$ in noncommutative variables x_0, x_1, we encountered a quadratic equation $D = 1 - Dx_0x_1 + Dx_0Dx_1$ for some $D \in \mathbb{Z}\langle\langle x_0, x_1\rangle\rangle$ and noticed that it is very similar to (2.7). This was the starting point of the project.

Remark 2.17. In fact, there is another group homomorphism $\pi\colon F \to F_1$ given by $\pi(x_k) = x_0 \cdot (x_0^{-1}x_1)^k$, $k \in \mathbb{Z}_{\geq 0}$, which results in an "almost commutative" specialization of noncommutative Catalan numbers: $\pi(C_n) = \pi(x_n) \cdot \frac{1}{n+1}\binom{2n}{n}$.

For each $0 \leq k \leq n$ denote by $\mathcal{P}_n^k$, the set of all $P \in \mathcal{P}_n$, such that the rightmost southeast corner p of P satisfies $p = (n, y)$, where $y \leq k$. In particular, $\mathcal{P}_n^{n-1} = \mathcal{P}_n^n = \mathcal{P}_n$. For each $0 \leq k \leq n$, define *truncated noncommutative Catalan number* $C_n^k \in \mathbb{Z}F_n$ by

$$C_n^k = \sum_{P \in \mathcal{P}_n^k} M_P.$$

The following recursion on C_n^k is immediate.

Lemma 2.18. $C_n^k = C_n^{k-1} + C_{n-1}^k x_{n-k-1}^{-1} x_{n-k}$ *for all* $1 \leq k \leq n$ (*with the convention* $C_n^\ell = 0$ *if* $\ell > n$).

Example 2.19. $C_n^0 = x_n$, $C_n^{n-1} = C_n^n = C_n$ for all $n \geq 1$. Also

$$C_n^1 = x_n + \sum_{i=1}^{n-1} x_i x_{i-1}^{-1} x_{n-1},$$

$$C_n^2 = \sum_{1\leq i\leq j\leq n, j>1} x_i x_{i-1}^{-1} x_{j-1} x_{j-2}^{-1} x_{n-2}.$$

Sometimes, it is convenient to express C_n^k via $y_i = x_i x_{i-1}^{-1}$, $i \in \mathbb{Z}_{\geq 1}$. Indeed, denote $\tilde{C}_n^k = C_n^k x_{n-k}^{-1}$ for $k, n \in \mathbb{Z}_{\geq 0}$, $k \leq n$.

The following result generalizes a number of basic properties of truncated Catalan numbers.

Proposition 2.20. *For all* $0 \leq k \leq n$, *one has:*

(a) $\tilde{C}_n^k = \sum_{\substack{j_1\leq \cdots \leq j_k \leq n, \\ j_1 \geq 1, \ldots, j_k \geq k}} y_{j_1} y_{j_2-1} \cdots y_{j_k-k+1}$.

(b) $\tilde{C}_n^k = \tilde{C}_{n-1}^k + \tilde{C}_n^{k-1} y_{n+1-k}$ (*with the convention* $\tilde{C}_n^\ell = 0$ *if* $\ell > n$).

(c) $\tilde{C}_{n+1}^k = \sum_{i=0}^k \tilde{C}_i^i T(\tilde{C}_{n-i}^{k-i})$.

A proof follows from Lemmas 3.1 and 3.2.

Example 2.21. $\tilde{C}_n^0 = 1$, $\tilde{C}_n^1 = y_1 + \cdots + y_n$, and $\tilde{C}_n^n = \tilde{C}_n^{n-1} y_1$ for all $n \geq 1$.

$$\tilde{C}_n^2 = \sum_{1 \leq i \leq j \leq n, j>1} y_i y_{j-1}, \ \tilde{C}_n^3 = \sum_{\substack{1 \leq i \leq j \leq k \leq n, \\ j>1, \ k>2}} y_i y_{j-1} y_{k-2}.$$

However, the following recursion is rather non-trivial (and we could not find its classical analog in the literature).

Theorem 2.22.

$$C_n = \sum_{\substack{a,b \in \mathbb{Z}_{\geq 0}, \\ a+b \leq n, a-b=d}} C_{n-b}^a x_{n-a-b}^{-1} \overline{C_{n-a}^b}, \tag{2.8}$$

for each $n \in \mathbb{Z}_{\geq 0}$ and each $d \in \mathbb{Z}$ with $|d| \leq n$ (e.g., the right-hand side does not depend on d).

A proof is given by Lemmas 3.4–3.6 in Sect. 3.1.

Remark 2.23. In particular, Theorem 2.22 provides another confirmation $\overline{\cdot}$-invariance of noncommutative Catalan numbers (established in Proposition 2.1).

It turns out that the above "two-variable specialization" σ is also of interest for truncated noncommutative Catalan numbers. Indeed, in the notation as above, denote $\underline{C}_n^k = \sigma(C_n^k)$ and $\underline{\underline{C}}_n^k = \underline{C}_n^k x_1^{k-n}$.

The following is immediate.

Corollary 2.24. *In the notation of Proposition 2.9, one has*

(a) $\underline{C}_n^k = \sum_{P \in \mathcal{P}_n^k} \underline{M}_P$ *for all $k, n \in \mathbb{Z}_{\geq 0}$, $k \leq n$.*

(b) $\underline{\underline{C}}_n^k = \underline{\underline{C}}_n^{k-1} x_1 + \underline{\underline{C}}_{n-1}^k x_0$ *for all $1 \leq k \leq n$ (with the convention $\underline{\underline{C}}_n^\ell = 0$ if $\ell > n$). In particular, each $\underline{\underline{C}}_n^k$ is a noncommutative polynomial in x_0, x_1 of degree $n+k$.*

Example 2.25. $\underline{\underline{C}}_n^0 = x_0^n$, $\underline{\underline{C}}_n^1 = x_0^n x_1 + \sum_{i=1}^{n-1} x_0^i x_1 x_0^{n-i}$,

$$\underline{\underline{C}}_n^2 = \underline{\underline{C}}_n^1 x_1 + \sum_{1 \leq i \leq j \leq n-1, j>1} x_0^i x_1 x_0^{j-i} x_1 x_0^{n-j}.$$

It turns out that our (truncated) noncommutative Catalan numbers $\tilde{C}_n^k$ admit another specialization into certain polynomials in $\mathbb{Z}_{\geq 0}[q]$ defined by Garsia and Haiman in [10]. Namely, let $\chi_q \colon \mathbb{Z}F \to \mathbb{Z}[q, q^{-1}]$ be a ring homomorphism defined by $\chi_q(x_k) = q^{\frac{k(k-1)}{2}}$ for $k \geq 0$, i.e., $\chi_q(y_k) = q^{k-1}$ for $k \in \mathbb{Z}_{\geq 1}$.

Define polynomials $c_n^k(q,t) \in \mathbb{Z}_{\geq 0}[q,t]$, $0 \leq k \leq n$ recursively by $c_n^0(q,t) = 1$ and

$$c_n^k(q,t) = \sum_{r=1}^k \begin{bmatrix} r+n-k \\ r \end{bmatrix}_q t^{k-r} q^{\frac{r(r-1)}{2}} c_{k-1}^{k-r}(q,t),$$

where $\begin{bmatrix} n \\ k \end{bmatrix}_q$ denotes the q-binomial coefficient $\frac{[n]_q!}{[k]_q![n-k]_q!}$,

$$[n]_q! = [1]_q \dots [n]_q, \quad [k]_q = \frac{1-q^n}{1-q} = 1 + q + \cdots q^{k-1}.$$

These polynomials are closely related to polynomials $H_{n,k}(q,t)$ introduced by Garsia and Haglund ([11, Equation I.24]), namely,

$$c_n^k(q,t) = t^{-k} q^{-\frac{(n+1-k)(n-k)}{2}} H_{n+1,n+1-k}(q,t).$$

In particular, $c_n^n(q,t) = c_n(q,t)$ is the celebrated (q,t)-Catalan number introduced in [10].

The following result shows that our (truncated) noncommutative Catalan numbers are noncommutative deformations of $(q,1)$-Catalan numbers.

Theorem 2.26. *$\chi_q(\tilde{C}_n^k) = c_n^k(q,1)$ for all $k \leq n$. In particular, $\chi_q(C_n) = c_n(q,1)$ for $n \geq 0$.*

We prove Theorem 2.26 in Sect. 3.1.

Example 2.27. $\chi_q(\tilde{C}_n^1) = [n+1]_q$ and

$$\chi_q(\tilde{C}_n^k) = \chi_q(\tilde{C}_n^{k-1}) q^{n-k} + \chi_q(\tilde{C}_{n-1}^k),$$

for $1 \leq k \leq n$.

Remark 2.28. It is curious that for another class of q-Catalan numbers:

$$q^{\frac{n(n-1)}{2}} c_n(q, q^{-1}) = \frac{1}{[n+1]_q} \begin{bmatrix} 2n \\ n \end{bmatrix}_q,$$

there is no analog of Theorem 2.26. Also, it would be interesting to find an appropriate noncommutative deformations of (q,t)-Catalan numbers.

The following result is a generalization of the well-known property of Hankel determinants of q-Catalan numbers.

Theorem 2.29. *For $n \geq 1$, $m \in \{0,1\}$ the determinant of the $(n+1) \times (n+1)$ matrix $(c_{i+j+m}(q,1))$, $i,j = 0, \dots, n$, is $q^{\frac{n(n+1)(4n-1+6m)}{6}}$.*

We prove Theorem 2.29 in Sect. 3.4.

Define the *noncommutative binomial coefficients* $\left\| \begin{matrix} n \\ k \end{matrix} \right\| \in \mathbb{Z}F_{n+k-1}$, $\left\| \begin{matrix} n \\ k \end{matrix} \right\|' \in \mathbb{Z}F_n$ by:

$$\left\| \begin{matrix} n \\ k \end{matrix} \right\| = \sum y_J, \quad \left\| \begin{matrix} n \\ k \end{matrix} \right\|' = \sum y'_J,$$

where each summation is over all subsets $J = \{j_1 < j_2 < \cdots < j_k\}$ of $[1,n]$ and we abbreviated

$$y_J = y_{j_k+k-1} \dots y_{j_2+1} y_{j_1}, \quad y'_J = y_{j_1+k-1} y_{j_2+k-3} \dots y_{j_k+1-k}$$

for $j \in \mathbb{Z}_{\geq 1}$.

Remark 2.30. The q-binomial coefficients can be expressed as:

$$\begin{bmatrix} n \\ k \end{bmatrix}_q = \sum q^{j_1+\cdots+j_k - \frac{k(k+1)}{2}},$$

where the summation is over all subsets $J = \{j_1 < j_2 < \cdots < j_k\}$ of $[1, n]$. Therefore, under the above specialization $\chi_q \colon \mathbb{Z}F \to \mathbb{Z}[q, q^{-1}]$, we have:

$$\chi_q\left(\left(\!\!\binom{n}{k}\!\!\right)\right) = q^{k(k-1)} \begin{bmatrix} n \\ k \end{bmatrix}_q, \quad \chi_q\left(\left(\!\!\binom{n}{k}\!\!\right)'\right) = q^{\frac{k(k-1)}{2}} \begin{bmatrix} n \\ k \end{bmatrix}_q$$

for all $k, n \in \mathbb{Z}_{\geq 0}$.

Example 2.31.

$$\left(\!\!\binom{n}{0}\!\!\right) = \left(\!\!\binom{n}{0}\!\!\right)' = 1, \quad \left(\!\!\binom{n}{1}\!\!\right) = \left(\!\!\binom{n}{1}\!\!\right)' = \sum_{i=1}^{n} y_i,$$

$$\left(\!\!\binom{n}{2}\!\!\right) = \sum_{1 \leq i < j \leq n} y_{j+1} y_i, \quad \left(\!\!\binom{n}{n}\!\!\right) = y_{2n-1} \cdots y_3 y_1 = y_{[1,n]},$$

$$\left(\!\!\binom{n}{n-1}\!\!\right) = \sum_{i=1}^{n} y_{[1,n]\setminus\{i\}}, \quad \left(\!\!\binom{n}{n-2}\!\!\right) = \sum_{1 \leq i < j \leq n} y_{[1,n]\setminus\{i,j\}},$$

$$\left(\!\!\binom{n}{2}\!\!\right)' = \sum_{1 \leq i < j \leq n} y_{i+1} y_{j-1}, \quad \left(\!\!\binom{n}{n}\!\!\right)' = y_n y_{n-1} \cdots y_1 = y'_{[1,n]},$$

$$\left(\!\!\binom{n}{n-1}\!\!\right)' = \sum_{i=1}^{n} y'_{[1,n]\setminus\{i\}}, \quad \left(\!\!\binom{n}{n-2}\!\!\right)' = \sum_{1 \leq i < j \leq n} y'_{[1,n]\setminus\{i,j\}}.$$

Clearly, $\varepsilon\left(\left(\!\!\binom{n}{k}\!\!\right)\right) = \varepsilon\left(\left(\!\!\binom{n}{k}\!\!\right)'\right) = \binom{n}{k}$ and $\left(\!\!\binom{n}{k}\!\!\right) = \left(\!\!\binom{n}{k}\!\!\right)' = 0$ if $k \notin [0, n]$.

Similarly to the classical case, we have an analog of the Pascal triangle and the multiplication law for noncommutative binomial coefficients.

Theorem 2.32.

$$\left(\!\!\binom{m+n}{k}\!\!\right) = \sum_{\substack{a,b \in \mathbb{Z}_{\geq 0},\\ a+b=k}} T^{n+b}\left(\left(\!\!\binom{m}{a}\!\!\right)\right) \left(\!\!\binom{n}{b}\!\!\right),$$

$$\left(\!\!\binom{m+n}{k}\!\!\right)' = \sum_{\substack{a,b \in \mathbb{Z}_{\geq 0},\\ a+b=k}} T^{b}\left(\left(\!\!\binom{m}{a}\!\!\right)'\right) T^{m-a}\left(\left(\!\!\binom{n}{b}\!\!\right)'\right),$$

for $m, n, k \in \mathbb{Z}_{\geq 0}$. In particular:

$$\left(\!\!\binom{n+1}{k}\!\!\right) = \left(\!\!\binom{n}{k}\!\!\right) + y_{n+k}\left(\!\!\binom{n}{k-1}\!\!\right), \quad \left(\!\!\binom{n+1}{k}\!\!\right)' = T\left(\left(\!\!\binom{n}{k}\!\!\right)'\right) + y_k\left(\left(\!\!\binom{n}{k-1}\!\!\right)'\right),$$

for all $n, k \in \mathbb{Z}_{\geq 0}$.

Actually, Theorem 2.32 which is proved in Sect. 3.2 together with the recursion from Proposition 2.20(b) implies the following analog of the multiplication law for the truncated noncommutative Catalan numbers, which justified the introduction of noncommutative binomial coefficients of the "second kind".

Corollary 2.33.

$$\tilde{C}_{m+n}^{k}=\sum_{\ell=0}^{n}\tilde{C}_{m+\ell}^{k-\ell}\cdot T^{m-k+\ell}\left(\left\|\begin{matrix}n\\ \ell\end{matrix}\right\|'\right),$$

for all $m,n,k\in\mathbb{Z}_{\geq 0}$.

The following relation between truncated noncommutative Catalan numbers and the binomial coefficients of the "first kind" is rather surprising.

Theorem 2.34.

$$\sum_{j=0}^{k}(-1)^{j}\tilde{C}_{n+k-j}^{j}\cdot\left\|\begin{matrix}n-j\\ k-j\end{matrix}\right\|=0,$$

for any $0<k\leq n$.

We prove Theorem 2.34 in Sect. 3.2 (Lemmas 3.7, 3.8).

Remark 2.35. In fact, there is an accompanying identity:

$$\sum_{j=0}^{k}(-1)^{j}\left\|\begin{matrix}n+k-j\\ j\end{matrix}\right\|\cdot\tilde{C}_{n-j}^{k-j}=0$$

for any $0<k\leq n$, which follows from Theorem 2.43 below. We leave this as an exercise to the readers.

This turns out to be equivalent to the following "determinantal" identities between noncommutative truncated Catalan numbers and binomial coefficients (whose classical analogs also seem to be new).

Theorem 2.36. *For all* $k,n\in\mathbb{Z}_{\geq 0}$, $k\leq n$, *one has:*

$$\tilde{C}_{n}^{k}=\sum_{J}(-1)^{k+1-|J|}M_{n,J},\qquad\left\|\begin{matrix}n\\ k\end{matrix}\right\|=\sum_{J}(-1)^{k+1-|J|}\tilde{M}_{n,J},$$

where each summation is over all subsets $J=\{0=j_0<\cdots<j_\ell=k\}$ *of* $[0,k]$ *and*

$$M_{n,J}=\left\|\begin{matrix}n+j_{\ell-1}+j_\ell-k\\ j_\ell-j_{\ell-1}\end{matrix}\right\|\cdots\left\|\begin{matrix}n+j_1+j_2-k\\ j_2-j_1\end{matrix}\right\|\left\|\begin{matrix}n+j_0+j_1-k\\ j_1-j_0\end{matrix}\right\|,$$

$$\tilde{M}_{n,J}=\tilde{C}_{n+j_0+j_1-k}^{j_1-j_0}\cdot\tilde{C}_{n+j_1+j_2-k}^{j_2-j_1}\cdots\tilde{C}_{n+j_{\ell-1}+j_\ell-k}^{j_\ell-j_{\ell-1}}.$$

We prove Theorem 2.36 in Sect. 3.4.

Actually, Theorems 2.26, 2.34, and 2.36 hint to some remarkable properties of Hankel matrices with noncommutative Catalan numbers as entries.

For $m\in\mathbb{Z}_{\geq 0}$, define the $\mathbb{Z}_{\geq 0}\times\mathbb{Z}_{\geq 0}$ matrix H_m over $\mathbb{Z}F$ whose (i,j)th entry is C_{m+i+j}, $i,j\in\mathbb{Z}_{\geq 0}$, and for each $n\geq 0$, denote by $H_{m,n}$ the principal $[0,n]\times[0,n]$ submatrix of H_m.

Example 2.37.

$$H_{0,1} = \begin{pmatrix} C_0 & C_1 \\ C_1 & C_2 \end{pmatrix}, \qquad H_{1,1} = \begin{pmatrix} C_1 & C_2 \\ C_2 & C_3 \end{pmatrix},$$

$$H_{0,2} = \begin{pmatrix} C_0 & C_1 & C_2 \\ C_1 & C_2 & C_3 \\ C_2 & C_3 & C_4 \end{pmatrix}, \quad H_{1,2} = \begin{pmatrix} C_1 & C_2 & C_3 \\ C_2 & C_3 & C_4 \\ C_3 & C_4 & C_5 \end{pmatrix}.$$

We refer to all H_m and H_m^n as *noncommutative Hankel–Catalan matrices* by analogy with its classical counterpart $\varepsilon(H_{m,n}) \in Mat_{n+1,n+1}(\mathbb{Z})$.

We will finish the section by showing that each $H_{m,n}$, $m \in \{0,1\}$, $n \geq 0$ admits a Gauss factorization over $\mathbb{Z}F$ involving truncated noncommutative Catalan numbers and its inverse (which is also a matrix over $\mathbb{Z}F$) is given by an interesting combinatorial formula involving our noncommutative binomial coefficients.

For $m \in \{0,1\}$, let L_m be the lower unitriangular $\mathbb{Z}_{\geq 0} \times \mathbb{Z}_{\geq 0}$ matrix whose (j,i)th entry, $0 \leq i \leq j$, is $\tilde{C}_{i+j+m}^{j-i}$, and let U_m be the upper triangular $\mathbb{Z}_{\geq 0} \times \mathbb{Z}_{\geq 0}$ matrix whose (i,j)th entry, $0 \leq i \leq j$, is $\overline{C_{i+j+m}^{j-i}}$.

Theorem 2.38. *$H_m = L_m \cdot U_m$ for each $m \in \{0,1\}$.*

We prove Theorem 2.38 in Sect. 3.3.

Remark 2.39. A classical version of this result, $\varepsilon(H_m) = \varepsilon(L_m) \cdot \varepsilon(U_m)$, was established in [1].

Theorem 2.38 and [12, Theorem 4.9.7] imply the following immediate corollary.

Corollary 2.40. *C_{m+i+j}^{j-i} equals the quasideterminant:*

$$\begin{vmatrix} C_m & C_{m+1} & \cdots & C_{m+i} \\ C_{m+1} & C_{m+2} & \cdots & C_{m+i+1} \\ & & \cdots & \\ C_{m+i-1} & C_{m+i} & \cdots & C_{m+2i-1} \\ C_{m+j} & C_{m+j+1} & \cdots & \boxed{C_{m+i+j}} \end{vmatrix}$$

for $0 \leq i \leq j$, $m \in \{0,1\}$ (see [14,15] *for notation*). *In particular:*

$$\begin{vmatrix} C_m & C_{m+1} & \cdots & C_{m+n} \\ C_{m+1} & C_{m+2} & \cdots & C_{m+n+1} \\ & & \cdots & \\ C_{m+n} & C_{m+n+1} & \cdots & \boxed{C_{m+2n}} \end{vmatrix} = x_{m+2n} \tag{2.9}$$

for all $n \in \mathbb{Z}_{\geq 0}$, $m \in \{0,1\}$.

Remark 2.41. In fact, (2.9) is a noncommutative generalization of the well-known fact that $\det(\varepsilon(H_{0,n})) = \det(\varepsilon(H_{1,n})) = 1$ for $n \geq 0$. Moreover, similarly to the classical case, noncommutative Catalan numbers are uniquely determined by (2.9) for $n \in \mathbb{Z}_{\geq 0}$, $m \in \{0,1\}$.

Remark 2.42. Noncommutative Hankel quasideterminants were introduced in [13] in the context of inversion of noncommutative power series. In fact, [13, Corollary 8.3] asserts that such an inverse can be expressed via continued fractions involving such quasideterminants of the coefficients of the series in question. This correlates with Remark 2.6 above.

For $m \in \{0,1\}$, let L_m^- be the lower unitriangular $\mathbb{Z}_{\geq 0}\times\mathbb{Z}_{\geq 0}$ matrix whose (j,i)th entry, $0 \leq i \leq j$, is $(-1)^{i+j}\left(\!\!\binom{i+j+m}{j-i}\!\!\right)$, and let U_m^- be the upper triangular $\mathbb{Z}_{\geq 0}\times\mathbb{Z}_{\geq 0}$ matrix whose (i,j)th entry, $0 \leq i \leq j$, is $(-1)^{i+j}\overline{\left(\!\!\binom{i+j+m}{j-i}\!\!\right)}x_{2j+m}^{-1}$.

For any $\mathbb{Z}_{\geq 0}\times\mathbb{Z}_{\geq 0}$ matrix M, denote by $M|_n$ the principal $(n+1)\times(n+1)$-submatrix of M (e.g., $H_{m,n} = H_m|_n$).

Theorem 2.43. $(U_m)^{-1} = U_m^-$ *and* $(L_m)^{-1} = L_m^-$, *and hence,* $(H_{m,n})^{-1} = U_m^-|_n \cdot L_m^-|_n$ *for* $m \in \{0,1\}$, $n \geq 1$.

Remark 2.44. Similar to Remark 2.39, the classical version of this result, $\varepsilon(H_{m,n})^{-1} = \varepsilon(L_m^-|_n)\cdot\varepsilon(U_m^-)|_n$, seems to be new.

Computation of H_m^{-1} for $m \geq 2$ is a more challenging task, which we will perform elsewhere.

3. Proofs of Main Results

3.1. Proofs of Propositions 2.3, 2.9, 2.20 and Theorems 2.22, 2.26

We start with a proof of Proposition 2.20. Then, specializations will lead to Propositions 2.3 and 2.9.

Proof of Proposition 2.20. Prove (a) first. Denote by $\mathbf{J}_n^k$ the set of all sequences $\mathbf{j} = (j_1,\ldots,j_k) \in \mathbb{Z}^k$, such that $j_1 \leq \cdots \leq j_k \leq n$ and $j_1 \geq 1,\ldots,j_k \geq k$.

For each $P \in \mathcal{P}_n^k$ and $s \in [1,k]$, denote by $j_s(P)$ the minimum of x-coordinates of all points in P whose y-coordinate is s. For each $\mathbf{j} = (j_1,\ldots,j_k) \in \mathbb{Z}^k$ with $j_s \geq s$, $s \in [1,k]$, we abbreviate $y_{\mathbf{j}} = y_{j_1}y_{j_2-1}\cdots y_{j_k-k+1}$.

The following is immediate.

Lemma 3.1. *For all* $k,n \in \mathbb{Z}_{\geq 0}$, $k \leq n$, *one has:*

(a) *The assignment* $P \mapsto \mathbf{j}(P) = (j_1(P),\ldots,j_k(P))$ *defines a bijection* $\mathcal{P}_n^k \widetilde{\to} \mathbf{J}_n^k$.
(b) *For each* $P \in \mathcal{P}_n^k$, *we have:* $M_P x_{n-k}^{-1} = y_{\mathbf{j}(P)}$.

Using Lemma 3.1(b), we obtain $\tilde{C}_n^k = \sum\limits_{\mathbf{j}\in\mathbf{J}_n^k} y_{\mathbf{j}}$ and, thus, finish the proof of (a).

Prove (b). It is easy to see that $\mathbf{J}_n^k = \mathbf{J}_{n-1}^k \sqcup (\mathbf{J}_n^{k-1}, n)$. Therefore,

$$\tilde{C}_n^k = \sum_{\mathbf{j}\in\mathbf{J}_n^k} y_{\mathbf{j}} = \sum_{\mathbf{j}\in\mathbf{J}_n^k} y_{\mathbf{j}} + \sum_{\mathbf{j}\in(\mathbf{J}_n^{k-1},n)} y_{\mathbf{j}} = \tilde{C}_{n-1}^k + \tilde{C}_n^{k-1}y_{n+1-k}.$$

This proves (b).

To prove (c), we need the following result.

Lemma 3.2. *$\mathbf{J}_{n+1}^k = \bigsqcup_{i=0}^k \mathbf{J}_i^i \times T^{i+1}(\mathbf{J}_{n-i}^{k-i})$ for all $k, n \in \mathbb{Z}_{\geq 0}$, $0 \leq k \leq n$, where $T = T_r : \mathbb{Z}^r \to \mathbb{Z}^r$, and $r \geq 1$ is the translation given by $x \mapsto x + \underbrace{(1, \ldots, 1)}_{r}$.*

Proof. For each $\mathbf{j} = (j_1, \ldots, j_k) \in \mathbf{J}_{n+1}^k$, denote by $i_{\mathbf{j}}$ the largest $i \in [1, k]$, such that $j_i = i$ and set $i(\mathbf{j}) = 0$ if such an i does not exist. This implies that:

$$\{\mathbf{j} \in \mathbf{J}_{n+1}^k : i_{\mathbf{j}} = i\} = \mathbf{J}_i^i \times T^{i+1}(\mathbf{J}_{n-i-1}^{k-i})$$

for all $i \in [0, k]$ (the first factor is empty for $i = 0$).

The lemma is proved. $\square$

Taking into account that for $\mathbf{j} = (\mathbf{j}', T^{i+1}(\mathbf{j}'')) \in \mathbf{J}_i^i \times T^{i+1}(\mathbf{J}_{n-i}^{k-i})$, we have $y_{\mathbf{j}} = y_{\mathbf{j}'} T(y_{\mathbf{j}''})$, we obtain:

$$\tilde{C}_{n+1}^k = \sum_{\mathbf{j} \in \mathbf{J}_{n+1}^k} y_{\mathbf{j}} = \sum_{i \in [0,k], \mathbf{j}' \in \mathbf{J}_i^i, \mathbf{j}'' \in \mathbf{J}_{n-i}^{k-i}} y_{\mathbf{j}'} T(y_{\mathbf{j}''}) = \sum_{i=0}^k \tilde{C}_i^i T(\tilde{C}_{n-i}^{k-i}).$$

This proves (c).

Proposition 2.20 is proved. $\square$

Proof of Theorem 2.26. Applying χ_q to $\tilde{C}_{n+1}^k$ given by Proposition 2.20(c) and using the fact that $\chi_q(T(y)) = q^d \chi_q(y)$ for any homogeneous noncommutative polynomial of degree d in $y_1, y_2, \ldots$, we obtain:

$$\chi_q(\tilde{C}_{n+1}^k) = \sum_{i=0}^k q^{k-i} \chi_q(\tilde{C}_i^i) \chi_q(\tilde{C}_{n-i}^{k-i})$$

for all $0 \leq k \leq n$. In view of [16, Eq. (3.41)] and that $F_{n,k}(q,t) = H_{n,k}(q,t) = t^{n-k} q^{\frac{k(k-1)}{2}} c_{n-1}^{n-k}(q,t)$ for all $0 \leq k < n$, we obtain the same recursion $c_{n+1}^k = \sum_{i=0}^k c_i^i(q,1) q^{k-i} c_{n-i}^{k-i}(q,1)$ for all $0 \leq k \leq n$. Using this and taking into account that $\chi_q(\tilde{C}_{n+1}^n) = \chi_q(\tilde{C}_{n+1}^{n+1})$, we conclude that $\chi_q(\tilde{C}_n^k) = c_n^k(q,1)$ for all $0 \leq k \leq n$.

The theorem is proved. $\square$

Proof of Proposition 2.3. Indeed, taking into account that $C_r = \tilde{C}_r^r \cdot x_0 = \tilde{C}_r^{r-1} y_1 x_0$ for all $r \geq 1$, we see that the first identity (2.4) is equivalent to $\tilde{C}_{n+1}^n = \sum_{k=0}^n \tilde{C}_k^k T(\tilde{C}_{n-k}^{n-k})$ which coincides with the assertion of Proposition 2.20(c) with $k = n$.

The second identity (2.4) follows from the first one and Proposition 2.1 by applying the anti-involution $\bar{\cdot}$.

Proposition 2.3 is proved. $\square$

Proof of Proposition 2.9. We say that $x \in F$ is alternating if it is of the form $x_{i_1} x_{i_2}^{-1} x_{i_3} \ldots x_{i_{s-1}}^{-1} x_{i_s}$ for some $i_1, \ldots, i_s \in \mathbb{Z}_{\geq 0}$ and denote by F^{alt} the set of all alternating elements in F. We also denote by $\mathbb{Z}F^{alt}$ the $\mathbb{Z}$-linear span of F^{alt} in $\mathbb{Z}F$. We need the following fact.

Lemma 3.3. *$\sigma(T(x)) = x_0 \sigma(x) x_1$ for all $x \in \mathbb{Z}F^{alt}$.*

Proof. We first prove the assertion for all $x \in F^{alt}$. Indeed, let

$$x = x_{i_1} x_{i_2}^{-1} x_{i_3} \cdots x_{i_{s-1}}^{-1} x_{i_s}$$

for some $i_1, i_2, \ldots, i_s \geq 0$. We have:

$$\begin{aligned}
\sigma(T(x)) &= \sigma(x_{i_1+1} x_{i_2+1}^{-1} x_{i_3+1} \cdots x_{i_{s-1}+1}^{-1} x_{i_s+1}) \\
&= (x_0^{i_1+1} x_1^{i_1+1})(x_0^{i_2+1} x_1^{i_2+1})^{-1}(x_0^{i_3+1} x_1^{i_3+1}) \\
&\quad \cdots (x_0^{i_{s-1}+1} x_1^{i_{s-1}+1})^{-1}(x_0^{i_s+1} x_1^{i_s+1}) \\
&= x_0 \cdot (x_0^{i_1} x_1^{i_1})(x_0^{i_2} x_1^{i_2})^{-1}(x_0^{i_3} x_1^{i_3}) \cdots (x_0^{i_{s-1}} x_1^{i_{s-1}})^{-1}(x_0^{i_s} x_1^{i_s}) \cdot x_1 \\
&= x_0 \sigma(x) x_1.
\end{aligned}$$

By linearity of σ, we obtain the assertion for all $x \in \mathbb{Z}F^{alt}$.

The lemma is proved. □

Since each C_k belongs to $\mathbb{Z}F^{alt}$, Lemma 3.3 implies that $\sigma(T(C_k)) = x_0\sigma(C_k)x_1 = x_0\underline{C}_k x_1$ for all $k \geq 0$. Using this and applying σ to the first identity (2.4), we obtain (2.6).

Proposition 2.9 is proved. □

Proof of Theorem 2.22. In the notation of the proof of Proposition 2.20, for all $0 \leq k \leq n$, denote by $\overline{\mathbf{J}_n^k}$ the set of all $\mathbf{j} = (j_1, \ldots, j_n) \in \mathbf{J}_n^n$, such that $j_1 \geq n - k$.

Lemma 3.4. $\overline{C}_n^k \cdot x_0^{-1} = \sum_{\mathbf{j} \in \overline{\mathbf{J}_n^k}} y_{\mathbf{j}}$ *for all* $0 \leq k \leq n$.

Proof. Indeed, in view of (2.3), we obtain using Lemma 3.1(b):

$$\overline{C_n^k} x_0^{-1} = \sum_{P \in \mathcal{P}_n^k} \overline{M}_P \cdot x_0^{-1} = \sum_{P \in \mathcal{P}_n^k} M_{s_n(P)} \cdot x_0^{-1} = \sum_{P \in \mathcal{P}_n^k} y_{\mathbf{j}(s_n(P))} = \sum_{\mathbf{j} \in \overline{\mathbf{J}}_n^k} y_{\mathbf{j}},$$

because $\overline{\mathbf{J}_n^k} = \mathbf{j}(s_n(\mathcal{P}_n^k))$.

The lemma is proved. □

Furthermore, after multiplying x_0^{-1} on the right hand of (2.8), the assertion of Theorem 2.22 is equivalent to:

$$\tilde{C}_n^n = \sum_{\substack{a,b \in \mathbb{Z}_{\geq 0}, \\ a+b \leq n, a-b=d}} \tilde{C}_{n-b}^a \cdot (\overline{C_{n-a}^b} x_0^{-1}), \tag{3.1}$$

for each $n \in \mathbb{Z}_{\geq 0}$ and each $d \in \mathbb{Z}$ with $|d| \leq n$.

Lemma 3.5. *Let* $d \in [1-n, n-1]$. *For each* $\mathbf{j} = (j_1, \ldots, j_n) \in \mathbf{J}_n^n$, *there exists a unique* $a = a(\mathbf{j}, d) \in [\max(0, d), n]$ *such that* $j_a \leq n + d - a \leq j_{a+1}$ *(with the convention* $j_0 = 0$, $j_{n+1} = \infty$*).*

Proof. Consider the graph of the linear function $y = n+d-x$ on the coordinate plain. Set $a = k$ if there exists $1 \leq k \leq n$, such the point with coordinates (k, j_k) is closest to the graph from the left. Otherwise, set $a = 0$. □

For $a \in [\max(0,d),n]$, denote by $\mathbf{J}_n^n(a,d)$ the set of all $\mathbf{j} \in \mathbf{J}_n^n$ such that $a(\mathbf{j},d) = a$.

We need the following fact (in the notation of Lemmas 3.2 and 3.4).

Lemma 3.6. $\mathbf{J}_n^n(a,d) = \mathbf{J}_{n+d-a}^a \times T^a(\overline{\mathbf{J}_{n-a}^{a-d}})$.

Proof. Clearly, for any sequence $\mathbf{j} \in \mathbf{J}_n^n(a,d)$, its subsequence $\mathbf{j}' = (j_1,\ldots,j_a)$ belongs to $\mathbf{J}_{n+d-a}^a$ and the subsequence $\mathbf{j}'' = (j_{a+1},\ldots,j_n)$ belongs to $T^a(\overline{\mathbf{J}_{n-a}^{a-d}})$.

Conversely, it is also clear that for any sequences $\mathbf{j}' \in \mathbf{J}_{n+d-a}^a$ and $\mathbf{j}'' \in T^a(\overline{\mathbf{J}_{n-a}^{a-d}})$, their concatenation $\mathbf{j} = (\mathbf{j}',\mathbf{j}'')$ belongs to $\mathbf{J}_n^n(a,d)$.

The lemma is proved. $\square$

For any two sequences of integers $\mathbf{j}' = (j_1',\ldots,j_k')$ and $\mathbf{j}'' = (j_1'',\ldots,j_\ell'')$, define the shifted concatenation by $\mathbf{j}' \bullet \mathbf{j}'' = (\mathbf{j}', T^k(\mathbf{j}''))$. We use now an obvious fact that if $\mathbf{j}' = (j_1',\ldots,j_k')$, $\mathbf{j}'' = (j_1'',\ldots,j_\ell'')$, then

$$y_{\mathbf{j}'\bullet\mathbf{j}''} = y_{\mathbf{j}'}y_{\mathbf{j}''}.$$

Then, applying this formula to $\mathbf{j} = (\mathbf{j}', T^a(\mathbf{j}'')) \in \mathbf{J}_{n+d-a}^a \times T^a(\overline{\mathbf{J}_{n-a}^{a-d}})$, we obtain:

$$\begin{aligned}\tilde{C}_{n+1}^n = \sum_{\mathbf{j}\in\mathbf{J}_n^n} y_{\mathbf{j}} &= \sum_{\substack{a\in[\max(0,d),n],\\ \mathbf{j}'\in\mathbf{J}_{n+d-a}^a,\mathbf{j}''\in\overline{\mathbf{J}_{n-a}^{a-d}}}} y_{\mathbf{j}'}y_{\mathbf{j}''}\\ &= \sum_{a\in[\max(0,d),n]} \tilde{C}_{n+d-a}^a \cdot (\overline{C_{n-a}^{a-d}}x_0^{-1}).\end{aligned}$$

This proves (3.1).

Theorem 2.22 is proved. $\square$

3.2. Proofs of Theorems 2.32 and 2.34

Proof of Theorem 2.32. For any set X and $k \geq 0$, denote by $\begin{Bmatrix} X \\ k \end{Bmatrix}$ the set of all subsets $J \subset X$ of cardinality $|J| = k$. Clearly,

$$\begin{Bmatrix} [1,m+n] \\ k \end{Bmatrix} = \bigsqcup_{\substack{a,b\in\mathbb{Z}_{\geq 0},\\ a+b=k}} \begin{Bmatrix} [1,m] \\ a \end{Bmatrix} \times T^m\left(\begin{Bmatrix} [1,n] \\ b \end{Bmatrix}\right),$$

for all $m,n,k \in \mathbb{Z}_{\geq 0}$ in the notation of Lemma 3.2, where we view each $J \in \begin{Bmatrix} [1,n] \\ k \end{Bmatrix}$ naturally as an element of $\mathbb{Z}^b$.

Taking into account that for

$$J = (J', T^m(J'')) \in \begin{Bmatrix} [1,m] \\ a \end{Bmatrix} \times T^m\left(\begin{Bmatrix} [1,n] \\ b \end{Bmatrix}\right),$$

$a+b=k$, we have $y_J = T^{m+a}(y_{J''})y_{J'}$ and $y'_J = T^b(y'_{J'})T^{m-a}(y'_{J''})$, we obtain for $m, n, k \in \mathbb{Z}_{\geq 0}$:

$$\left(\!\!\left(\begin{matrix} m+n \\ k \end{matrix}\right)\!\!\right) = \sum_{J \in \left\{ \begin{matrix} [1, m+n] \\ k \end{matrix} \right\}} y_J = \sum_{\substack{a,b \in \mathbb{Z}_{\geq 0}, \\ a+b=k, \\ J' \in \left\{ \begin{smallmatrix} [1,m] \\ k \end{smallmatrix} \right\}, J'' \in \left\{ \begin{smallmatrix} [1,n] \\ k \end{smallmatrix} \right\}}} T^{m-a}(y_{J''})y_{J'} = \sum_{\substack{a,b \in \mathbb{Z}_{\geq 0}, \\ a+b=k}} T^{m+a}\left(\left(\!\!\left(\begin{matrix} n \\ b \end{matrix}\right)\!\!\right)\right)\left(\!\!\left(\begin{matrix} m \\ a \end{matrix}\right)\!\!\right),$$

$$\left(\!\!\left(\begin{matrix} m+n \\ k \end{matrix}\right)\!\!\right)' = \sum_{J \in \left\{ \begin{matrix} [1, m+n] \\ k \end{matrix} \right\}} y'_J = \sum_{\substack{a,b \in \mathbb{Z}_{\geq 0}, \\ a+b=k, \\ J' \in \left\{ \begin{smallmatrix} [1,m] \\ k \end{smallmatrix} \right\}, J'' \in \left\{ \begin{smallmatrix} [1,n] \\ k \end{smallmatrix} \right\}}} T^b(y'_{J'})T^{m-a}(y'_{J''}) = \sum_{\substack{a,b \in \mathbb{Z}_{\geq 0}, \\ a+b=k}} T^b\left(\left(\!\!\left(\begin{matrix} m \\ a \end{matrix}\right)\!\!\right)'\right) T^{m-a}\left(\left(\!\!\left(\begin{matrix} n \\ b \end{matrix}\right)\!\!\right)'\right).$$

Theorem 2.32 is proved. □

Proof of Theorem 2.34. For each $0 \leq j \leq k \leq n$, denote by $\mathbf{I}_{j,k;n}$ the set of all $\mathbf{i} = (i_1, \ldots, i_k) \in \mathbb{Z}_{\geq 1}^k$, such that $i_j \leq n+k+1-2j$, $i_{j+1} \leq n+k-1-2j$, $i_s \leq i_{s+1}+1$ for all $s \in [1, j]$, and $i_s > i_{s+1}+1$ for all $s \in [j+1, k]$. (with the convention that if $j \in \{0, k\}$, then meaningless inequalities are omitted and $\mathbf{I}_{-1,k;n} = \mathbf{I}_{k+1,k;n} = \emptyset$).

The following statement is straightforward.

Lemma 3.7. $\tilde{C}^j_{n+k-j} \cdot \left(\!\!\left(\begin{matrix} n-j \\ k-j \end{matrix}\right)\!\!\right) = \sum_{\mathbf{i} \in \mathbf{I}_{j,k;n}} Y_{\mathbf{i}}$ *for all* $0 \leq j \leq k$, *where we abbreviate* $Y_{\mathbf{i}} = y_{i_1} \ldots y_{i_k}$.

For $j \in [0, k+1]$, denote $\mathbf{I}^-_{j,k;n} = \mathbf{I}_{j-1,k;n} \cap \mathbf{I}_{j,k;n}$. By definition, $\mathbf{I}^-_{0,k} = \mathbf{I}^-_{k+1,k} = \emptyset$ and the following is immediate.

Lemma 3.8. $\mathbf{I}^-_{j,k;n}$ *is the set of all* $\mathbf{i} = (i_1, \ldots, i_m) \in \mathbf{I}_{j,k;n}$, *such that* $i_j \leq i_{j+1}+1$ *for all* $j \in [0, k]$. *In particular,* $\mathbf{I}_{j,k;n} = \mathbf{I}^-_{j,k;n} \sqcup \mathbf{I}^-_{j+1,k;n}$ *for* $j \in [0, k]$.

Using Lemmas 3.7 and 3.8, we obtain for all $0 < k \le n$:

$$\begin{aligned}\sum_{j=0}^{k}(-1)^j\tilde{C}^j_{n+k-j}\cdot\left(\!\!\binom{n-j}{k-j}\!\!\right) &= \sum_{j\in[0,k],\mathbf{i}\in\mathbf{I}_{j,k;n}}(-1)^jY_{\mathbf{i}}\\ &= \sum_{j\in[0,k],\mathbf{i}\in\mathbf{I}^-_{j,k;n}}(-1)^jY_{\mathbf{i}}+\sum_{j\in[0,k],\mathbf{i}\in\mathbf{I}^-_{j+1,k;n}}(-1)^jY_{\mathbf{i}}\\ &= 0.\end{aligned}$$

Theorem 2.34 is proved. □

3.3. Proofs of Theorems 2.38 and 2.43

Proof of Theorem 2.38. We prove Theorem 2.38 first. Indeed, the assertion is equivalent to:

$$(H_m)_{ij} = \sum_{k=0}^{\min(i,j)}(L_m)_{ik}(U_m)_{kj},$$

i.e., to

$$C_{m+i+j} = \sum_{k=0}^{\min(i,j)} C^{i-k}_{i+k+m}\cdot x^{-1}_{2k+m}\overline{C^{j-k}_{k+j+m}}$$

for all $i, j \in \mathbb{Z}_{\ge 0}$, $m \in \{0,1\}$. This identity coincides with that from Theorem 2.22 taken with $n = m+i+j$, $a = i-k$, $b = j-k$, and $d = i-j$.

Theorem 2.38 is proved. □

Proof of Theorem 2.43. It suffices to do so only for L^-_m (the argument for U^-_m is identical). Indeed, the assertion is equivalent to:

$$\sum_{k'=i'}^{j'}(L_m)_{j'k'}(L^-_m)_{k'i'} = \delta_{i'j'},$$

i.e., to

$$\sum_{k'=i'}^{j'}\tilde{C}^{j'-k'}_{j'+k'+m}\cdot(-1)^{i'+k'}\left(\!\!\binom{i'+k'+m}{k'-i'}\!\!\right) = 0$$

for all $0 \le i' < j'$. It is easy to show that this identity coincides with that from Theorem 2.34 taken with $n = i'+j'+m$, $j = j'-k'$, and $k = j'-i'$.

Theorem 2.43 is proved. □

3.4. Proofs of Theorems 2.29 and 2.36

Proof of Theorem 2.36. We start with a proof of Theorem 2.36. The following is well known.

Lemma 3.9. *Any lower unitriangular $\mathbb{Z}_{\ge 0}\times\mathbb{Z}_{\ge 0}$ matrix $A = (a_{ij})$ over an associative unital ring $\mathcal{A}$ is invertible and*

$$(A^{-1})_{ji} = \sum_{j=i_1>i_2>\cdots>i_k=i,k\ge 1}(-1)^{k-1}a_{i_1,i_2}\dots a_{i_{k-1},i_k}$$

for all $1 \le i \le j \le n$.

Applying Lemma 3.9 with $A = L_m^-$, i.e., $a_{ji} = \tilde{C}_{i+j+m}^{i-j}$ and using Theorem 2.43 in the form $(A^{-1})_{ji} = (-1)^{i+j}\left(\!\!\binom{i+j+m}{j-i}\!\!\right)$, we obtain the first identity. Swapping A and A^{-1}, we obtain the second one.

Theorem 2.36 is proved. □

Proof of Theorem 2.29. Recall from [15] that for any matrix over a commutative ring, its determinant equals the product of its principal quasiminors. Let $\underline{H}_m^n = \chi_q(H_m^n) = (c_{i+j+m}(q,1))$, $i,j = 0,\dots,n$, where $\chi_q\colon \mathbb{Z}F \to \mathbb{Z}[q,q^{-1}]$ is defined in Sect. 2. Since all principal submatrices of $\underline{H}_m^n$ are $\underline{H}_m^k$, $k = 0,1,\dots,n$, these and Corollary 2.40 guarantee that

$$\det(\underline{H}_m^n) = \prod_{k=0}^{n} \chi_q(x_{m+2k}) = q^{\sum_{k=0}^{m} \frac{(m+2k)(m+2k-1)}{2}} = q^{\frac{n(n+1)(4n-1+6m)}{6}}.$$

Theorem 2.29 is proved. □

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Arkady Berenstein
Department of Mathematics
University of Oregon
Eugene
OR 97403
USA
e-mail: arkadiy@math.uoregon.edu

Vladimir Retakh
Department of Mathematics
Rutgers University
Piscataway
NJ 08854
USA
e-mail: vretakh@math.rutgers.edu

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Elementary Polynomial Identities Involving q-Trinomial Coefficients

To George E. Andrews, on the occasion of his 80th birthday

Alexander Berkovich and Ali Kemal Uncu

Abstract. We use the q-binomial theorem to prove three new polynomial identities involving q-trinomial coefficients. We then use summation formulas for the q-trinomial coefficients to convert our identities into another set of three polynomial identities, which imply Capparelli's partition theorems when the degree of the polynomial tends to infinity. This way we also obtain an interesting new result for the sum of the Capparelli's products. We finish this paper by proposing an infinite hierarchy of polynomial identities.

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Keywords. Happy Birthday, Capparelli's partition theorems, q-Trinomial coefficients, q-Series, Polynomial identities.

1. Introduction

George E. Andrews is known for his many accomplishments and impeccable leadership in both his mathematical contributions and in service to the community of mathematics. His influence on research keeps opening new horizons, and at the same time, new doors to young researchers. There are many areas of study he introduced that are now saturated with world-class mathematicians, yet there are many more that the community is only catching up studying. Here, we look at one of these lesser-studied objects: q-trinomial coefficients. Introduced by Andrews in collaboration with Baxter, the q-trinomial coefficients are defined by

$$\binom{L,\ b}{a};q\Big)_2 := \sum_{n\geq 0} q^{n(n+b)} \frac{(q;q)_L}{(q;q)_n (q;q)_{n+a} (q;q)_{L-2n-a}}, \tag{1.1}$$

where, for any non-negative integer n, $(a;q)_n$ is the standard q-Pochhammer symbol [3]:

$$(a;q)_n := (1-a)(1-aq)(1-aq^2)\dots(1-aq^{n-1}).$$

Here and throughout $|q|<1$.

It is easy to verify that

$$\sum_{a=-L}^{L} \binom{L,\ b}{a};1\Big)_2 t^a = \left(t+1+\frac{1}{t}\right)^L,$$

which implies the generalized Pascal Triangle for (1.1) with $q=1$:

$$\begin{array}{ccccccccccc}
 & & & & & 1 & & & & & \\
 & & & & 1 & 1 & 1 & & & & \\
 & & & 1 & 2 & 3 & 2 & 1 & & & \\
 & & 1 & 3 & 6 & 7 & 6 & 3 & 1 & & \\
 & 1 & 4 & 10 & 16 & 19 & 16 & 10 & 4 & 1 & \\
\cdot^{\cdot^{\cdot}} & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \ddots
\end{array}$$

The q-trinomial coefficients were studied in [1,2,4–10,14–17]. Nevertheless, it appears that the following identities are new.

Theorem 1.1.

$$\begin{aligned}
&\sum_{n\geq 0}(-1)^n q^{(3n^2+n)/2}\frac{(q^3;q^3)_L}{(q;q)_{L-2n}(q^3;q^3)_n} \\
&\qquad + q^{2L+1}\sum_{n\geq 0}(-1)^n q^{(3n^2-n)/2}\frac{(q^3;q^3)_L}{(q;q)_{L-2n}(q^3;q^3)_n} \\
&\quad = \sum_{j=-L}^{L}\left\{q^{L+j+1}\left(\begin{matrix}L,\ j+1\\ j\end{matrix};q^3\right)_2 + q^{L+4j}\left(\begin{matrix}L,\ j\\ j-1\end{matrix};q^3\right)_2\right\},
\end{aligned} \tag{1.2}$$

$$\sum_{n\geq 0}(-1)^n q^{(3n^2-n)/2}\frac{(q^3;q^3)_L}{(q;q)_{L-2n}(q^3;q^3)_n} = \sum_{j=-L}^{L} q^{2L-j}\left(\begin{matrix}L,\ j-1\\ j\end{matrix};q^3\right)_2, \tag{1.3}$$

$$\sum_{n\geq 0}(-1)^n q^{(3n^2+n)/2}\frac{(q^3;q^3)_L}{(q;q)_{L-2n}(q^3;q^3)_n} = \sum_{j=-L}^{L} q^{L-j}\left(\begin{matrix}L,\ j\\ j\end{matrix};q^3\right)_2. \tag{1.4}$$

From the left-hand sides of (1.2)–(1.4), we can easily discover the limiting formulas:

$$\frac{1}{(q;q^3)_\infty},\ \frac{1}{(q^2;q^3)_\infty},\ \text{and}\ \frac{1}{(q;q^3)_\infty},$$

respectively, as $L\to\infty$ with the aid of the q-binomial theorem. However, this is not as easy to see that from the right-hand sides of these identities.

These identities are related to combinatorics and partition theory. As an example, we will show that (1.2) and (1.4) in Theorem 1.1 imply Capparelli's partition theorems. Moreover, (1.3) implies the following new interesting result.

Theorem 1.2.

$$\sum_{m,n\geq 0}\frac{q^{2m^2+6mn+6n^2-2m-3n}}{(q;q)_m(q^3;q^3)_n}=(-q^2,-q^4;q^6)_\infty(-q^3;q^3)_\infty$$
$$+(-q,-q^5;q^6)_\infty(-q^3;q^3)_\infty. \tag{1.5}$$

In Sect. 2, we give a comprehensive list of definitions and identities that will be used in this paper. Section 3 has the proof of Theorem 1.1. This section also includes the dual of these identities and a necessary version of Bailey's lemma for the q-trinomial coefficients. We find new polynomial identities that yield Capparelli's partition theorem in Sect. 4. Theorem 1.2, which includes the sum of the two Capparelli's theorem's products, is also proven in Sect. 4. This section also contains a comparison of the mentioned polynomial identities and the previously found polynomial identities [10] that also imply Capparelli's partition theorems. The outlook section, Sect. 5, includes two new results the authors are planning on presenting soon: a doubly bounded identity involving Warnaar's refinement of the q-trinomial coefficients, and also an infinite hierarchy of q-series identities.

2. Necessary Definitions and Identities

We use the standard notation as in [3]. For formal variables a_i and q, and a non-negative integer n:

$$(a;q)_\infty := \lim_{n\to\infty}(a;q)_n, \tag{2.1}$$
$$(a_1,a_2,\ldots,a_k;q)_n := (a_1;q)_n(a_2;q)_n\cdots(a_k;q)_n. \tag{2.2}$$

We can extend the definition of the q-Pochhamer symbol to negative n as follows:

$$(a;q)_n=\frac{(a;q)_\infty}{(aq^n;q)_\infty}. \tag{2.3}$$

Observe that (2.1) implies

$$\frac{1}{(q;q)_n}=0 \quad \text{if} \quad n<0. \tag{2.4}$$

In addition, observe that for non-negative n, we have

$$(q^{-1};q^{-1})_n=(-1)^n q^{-\binom{n+1}{2}}(q;q)_n. \tag{2.5}$$

We define the q-binomial coefficients in the classical manner as

$$\begin{bmatrix} m+n \\ m \end{bmatrix}_q := \begin{cases} \frac{(q)_{m+n}}{(q)_m(q)_n}, & \text{for } m,n\geq 0,\\ 0, & \text{otherwise.}\end{cases} \tag{2.6}$$

It is well known that for $m\in\mathbb{Z}_{\geq 0}$

$$\lim_{N\to\infty}\begin{bmatrix} N \\ m \end{bmatrix}_q=\frac{1}{(q;q)_m}, \tag{2.7}$$

for any $j \in \mathbb{Z}_{\geq 0}$ and $\nu = 0$ or 1

$$\lim_{M \to \infty} \begin{bmatrix} 2M+\nu \\ M-j \end{bmatrix}_q = \frac{1}{(q;q)_\infty}. \tag{2.8}$$

We define another q-trinomial coefficient for any integer n:

$$T_n\left(\begin{matrix} L \\ a \end{matrix}; q\right) := q^{(L(L-n)-a(a-n))/2} \left(\begin{matrix} L,\, a-n \\ a \end{matrix}; \frac{1}{q}\right)_2. \tag{2.9}$$

Theorem 2.1 (q-Binomial theorem). *For variables a, q, and z*

$$\sum_{n\geq 0} \frac{(a;q)_n}{(q;q)_n} z^n = \frac{(az;q)_\infty}{(z;q)_\infty}. \tag{2.10}$$

In addition, note that the q-exponential sum

$$\sum_{n\geq 0} \frac{q^{n(n-1)/2}}{(q;q)_n} z^n = (-z;q)_\infty \tag{2.11}$$

is a limiting case ($a \to \infty$ after the variable change $z \mapsto -z/a$) of (2.10).

Another ingredient we will use here is the Jacobi Triple Product Identity [3].

Theorem 2.2 (Jacobi triple product identity).

$$\sum_{j=-\infty}^{\infty} z^j q^{j^2} = \left(q^2, -zq, -\frac{q}{z}; q^2\right)_\infty. \tag{2.12}$$

3. Proof of Theorem 1.1 and Some q-Trinomial Summation Formulas

We start with the following lemma.

Lemma 3.1. *For any integer n, we have*

$$\sum_{L\geq 0} \sum_{j=-\infty}^{\infty} \frac{x^j t^L}{(q;q)_L} \left(\begin{matrix} L,\, j-n \\ j \end{matrix}; q\right)_2 = \frac{(t^2 q^{-n};q)_\infty}{(t, \frac{t}{x} q^{-n}, tx; q)_\infty}. \tag{3.1}$$

The $n = 0$ case of (3.1) first appeared in the work of Andrews [2, p. 153, (6.6)].

Proof of Lemma 3.1. We start by writing definition (1.1) on the left-hand side of formula (3.1). After a simple cancellation, one sees that the triple sum can be untangled by the change of the summation variables $\nu = k + j$, and $\mu = L - 2k - j$. This change of summation variables, keeping (2.3) in mind, shows that the left-hand side sum of (3.1) can be written as:

$$\sum_{k\geq 0} \frac{x^{-k} t^k q^{-nk}}{(q;q)_k} \sum_{\nu\geq 0} \frac{x^\nu t^\nu q^{k\nu}}{(q;q)_\nu} \sum_{\mu\geq 0} \frac{t^\mu}{(q;q)_\mu}. \tag{3.2}$$

One can apply the q-Binomial Theorem starting from the innermost sum of (3.2). After applying the q-binomial Theorem 2.1 with $(a,z) = (0,t)$, and $(a,z) = (0, xtq^k)$, we get

$$\frac{1}{(t;q)_\infty} \sum_{k\geq 0} \frac{x^{-k} t^k q^{-nk}}{(q;q)_k (xtq^k;q)_\infty}. \tag{3.3}$$

We rewrite $(xtq^k;q)_\infty$ using (2.1), take the k-free portion out of the summation, and use (2.10) once again with $(a,z) = (xt, x^{-1}tq^n)$ to finish the proof. □

We can prove Theorem 1.1 using Lemma 3.1.

Proof of Theorem 1.1. Instead of proving these identities directly, we will prove the equality of their generating functions. It is clear that one can prove the equality of the two sides of polynomial identities of the form

$$A_L(q) = B_L(q)$$

by a multi-variable generating function equivalence

$$\sum_{L\geq 0} \frac{t^L}{(q^3;q^3)_L} A_L(q) = \sum_{L\geq 0} \frac{t^L}{(q^3;q^3)_L} B_L(q). \tag{3.4}$$

On the right-hand side of (3.4) with the choice of $B_L(q)$ being the right-hand sides of (1.2)–(1.4), we get

$$\frac{(t^2q^2;q^3)_\infty}{(t;q)_\infty} \frac{(1+q)}{(1+tq)}, \frac{(t^2q;q^3)_\infty}{(t;q)_\infty}, \text{ and } \frac{(t^2q^2;q^3)_\infty}{(t;q)_\infty}, \tag{3.5}$$

respectively, by Lemma 3.1. Hence, all we need to do is to show that the left-hand side of (3.4) with the choice of $A_L(q)$ being the left-hand sides of (1.2)–(1.4) yields the same products.

The left-hand side of (1.2) has two sums. The first sum of the left-hand side of (1.2) after being multiplied by $t^L/(q^3;q^3)_L$, summing over L as suggested in (3.4), and after simple cancellations turns into

$$\sum_{L,n\geq 0} (-1)^n \frac{q^{\frac{3n^2+n}{2}} t^L}{(q;q)_{L-2n} (q^3;q^3)n}. \tag{3.6}$$

We introduce the new summation variable $\nu = L - 2n$. This factors the double sum fully. Keeping (2.3) in mind, we rewrite (3.6) as

$$\sum_{\nu\geq 0} \frac{t^\nu}{(q;q)_\nu} \sum_{n\geq 0} \frac{q^{3n(n-1)/2}}{(q^3;q^3)_n} (-t^2q)^n.$$

Then, using (2.10) and (2.11) on the two sums, respectively, we see that (3.6) is equal to

$$\frac{(t^2q;q^3)_\infty}{(t;q)_\infty}.$$

The same exact calculation can be done for the second sum on the left-hand side of (1.2), and the left-hand side sums of (1.3) and (1.4). After the simplifications, we see that the products we get from the left-hand side sums after (3.4) is applied to them, are the same as the products (3.5). $\square$

In the identities of Theorem 1.1, we replace $q \mapsto 1/q$, multiply both sides of the equations by $q^{3L^2/2}$, use (2.4) and (2.9), and do elementary simplifications to get the following theorem.

Theorem 3.2.

$$\sum_{n\geq 0} q^{\binom{L-2n}{2}} \frac{(q^3;q^3)_L}{(q;q)_{L-2n}(q^3;q^3)_n} + q^{L+1}\sum_{n\geq 0} q^{\binom{L-2n+1}{2}+n} \frac{(q^3;q^3)_L}{(q;q)_{L-2n}(q^3;q^3)_n}$$
$$= \sum_{j=-L}^{L} q^{\frac{3j^2+j}{2}} \left\{ T_{-1}\left(\begin{matrix} L \\ j \end{matrix}; q^3\right) + T_{-1}\left(\begin{matrix} L \\ j+1 \end{matrix}; q^3\right)\right\}, \tag{3.7}$$

$$\sum_{n\geq 0} q^{\binom{L-2n}{2}} \frac{(q^3;q^3)_L}{(q;q)_{L-2n}(q^3;q^3)_n} = \sum_{j=-L}^{L} q^{\frac{3j^2-j}{2}} T_1\left(\begin{matrix} L \\ j \end{matrix}; q^3\right), \tag{3.8}$$

$$\sum_{n\geq 0} q^{\frac{(L-2n)^2}{2}} \frac{(q^3;q^3)_L}{(q;q)_{L-2n}(q^3;q^3)_n} = \sum_{j=-L}^{L} q^{\frac{3j^2+2j}{2}} T_0\left(\begin{matrix} L \\ j \end{matrix}; q^3\right). \tag{3.9}$$

Building on the development in [7,9,14], Warnaar [15, eqs. (10), (14)] proved the following summation formulas.

Theorem 3.3 (Warnaar).

$$\sum_{i\geq 0} q^{\frac{i^2}{2}} \begin{bmatrix} L \\ i \end{bmatrix}_q T_0\left(\begin{matrix} i \\ a \end{matrix}; q\right) = q^{\frac{a^2}{2}} \begin{bmatrix} 2L \\ L-a \end{bmatrix}_q, \tag{3.10}$$

$$\sum_{i\geq 0} q^{\binom{i}{2}}(1+q^L) \begin{bmatrix} L \\ i \end{bmatrix}_q T_1\left(\begin{matrix} i \\ a \end{matrix}; q\right) = (1+q^a)q^{\binom{a}{2}} \begin{bmatrix} 2L \\ L-a \end{bmatrix}_q. \tag{3.11}$$

We found a similar new summation formula:

Theorem 3.4.

$$\sum_{i\geq 0} q^{\binom{i+1}{2}} \begin{bmatrix} L \\ i \end{bmatrix}_q \left\{ T_{-1}\left(\begin{matrix} i \\ a \end{matrix}; q\right) + T_{-1}\left(\begin{matrix} i \\ a+1 \end{matrix}; q\right)\right\} = q^{\binom{a+1}{2}} \begin{bmatrix} 2L+1 \\ L-a \end{bmatrix}_q. \tag{3.12}$$

Proof. To prove (3.12), we need the following identity of Berkovich–McCoy–Orrick [8, p. 815, (4.8)]:

$$T_{-1}\left(\begin{matrix} L \\ a \end{matrix}; q\right) + T_{-1}\left(\begin{matrix} L \\ a+1 \end{matrix}; q\right)$$
$$= \frac{1}{1-q^{L+1}} \left\{ T_1\left(\begin{matrix} L+1 \\ a \end{matrix}; q\right) - q^{(L+1-a)/2} T_0\left(\begin{matrix} L+1 \\ a \end{matrix}; q\right)\right\}. \tag{3.13}$$

After the use of (3.13) on the left-hand side of (3.12), we employ

$$\frac{1}{1-q^{i+1}}\begin{bmatrix}L\\ i\end{bmatrix}_q = \frac{1}{1-q^{L+1}}\begin{bmatrix}L+1\\ i+1\end{bmatrix}_q, \tag{3.14}$$

and summations (3.10) and (3.11). This yields the right-hand side of (3.12) after some elementary simplifications. ☐

Theorem 3.5. *Let $F_j(L)$ and $\alpha_j(a)$ be sequences, depending on L and a, respectively, for $j=-1$, 0 or 1. If*

$$F_0(L) = \sum_{a=-\infty}^{\infty} \alpha_0(a) T_0\left(\begin{matrix}L\\ a\end{matrix};q\right), \tag{3.15}$$

$$F_1(L) = \sum_{a=-\infty}^{\infty} \alpha_1(a) T_1\left(\begin{matrix}L\\ a\end{matrix};q\right), \tag{3.16}$$

$$F_{-1}(L) = \sum_{a=-\infty}^{\infty} \alpha_{-1}(a) \left\{T_{-1}\left(\begin{matrix}L\\ a\end{matrix};q\right) + T_{-1}\left(\begin{matrix}L\\ a+1\end{matrix};q\right)\right\} \tag{3.17}$$

hold, then

$$\sum_{i\geq 0} q^{\frac{i^2}{2}}\begin{bmatrix}L\\ i\end{bmatrix}_q F_0(i) = \sum_{a=-\infty}^{\infty} \alpha_0(a) q^{\frac{a^2}{2}}\begin{bmatrix}2L\\ L-a\end{bmatrix}_q, \tag{3.18}$$

$$(1+q^L)\sum_{i\geq 0} q^{\binom{i}{2}}\begin{bmatrix}L\\ i\end{bmatrix}_q F_1(i) = \sum_{a=-\infty}^{\infty} \alpha_1(a)(1+q^a) q^{\binom{a}{2}}\begin{bmatrix}2L\\ L-a\end{bmatrix}_q, \tag{3.19}$$

$$\sum_{i\geq 0} q^{\binom{i+1}{2}}\begin{bmatrix}L\\ i\end{bmatrix}_q F_{-1}(i) = \sum_{a=-\infty}^{\infty} \alpha_{-1}(a) q^{\binom{a+1}{2}}\begin{bmatrix}2L+1\\ L-a\end{bmatrix}_q \tag{3.20}$$

are true.

Proof. We apply (3.10)–(3.12) to (3.15)–(3.17) and get (3.18)–(3.20), respectively. ☐

4. New Polynomial Identities Implying Capparelli's Partition Theorems

We apply (3.18) to (3.9) to get

$$\sum_{L,n\geq 0} q^{\frac{(L-2n)^2+3L^2}{2}} \frac{(q^3;q^3)_M}{(q;q)_{L-2n}(q^3;q^3)_n(q^3;q^3)_{M-L}} = \sum_{j=-M}^{M} q^{3j^2+j}\begin{bmatrix}2M\\ M+j\end{bmatrix}_{q^3}. \tag{4.1}$$

We introduce the new variable $m = L-2n$, and let

$$Q(m,n) := 2m^2+6mn+6n^2,$$

and observe that

$$Q(m,n) = \frac{(L-2n)^2+3L^2}{2},$$

after the change of variable. Hence, (4.1) can be written as follows.

Theorem 4.1.

$$\sum_{m,n\geq 0} \frac{q^{Q(m,n)}(q^3;q^3)_M}{(q;q)_m(q^3;q^3)_n(q^3;q^3)_{M-2n-m}} = \sum_{j=-M}^{M} q^{3j^2+j} \begin{bmatrix} 2M \\ M+j \end{bmatrix}_{q^3}. \tag{4.2}$$

Recall that (4.2) is Theorem 7.1 in [10]. Letting $M \to \infty$ in (4.2), using (2.8), and the Jacobi Triple Product Identity (2.12) on the right-hand side, we obtain the following.

Theorem 4.2.

$$\sum_{m,n\geq 0} \frac{q^{Q(m,n)}}{(q;q)_m(q^3;q^3)_n} = (-q^2,-q^4;q^6)_\infty(-q^3;q^3)_\infty. \tag{4.3}$$

The authors recently discovered another polynomial identity [10, Theorem 1.3, (1.12)] that implies the same q-series identity (4.3) as $N \to \infty$:

Theorem 4.3. *For any non-negative integer N, we have*

$$\sum_{m,n\geq 0} q^{Q(m,n)} \begin{bmatrix} 3(N-2n-m) \\ m \end{bmatrix}_q \begin{bmatrix} 2(N-2n-m)+n \\ n \end{bmatrix}_{q^3}$$
$$= \sum_{l=0}^{N} q^{3\binom{N-2l}{2}} \begin{bmatrix} N \\ 2l \end{bmatrix}_{q^3} (-q^2,-q^4;q^6)_l.$$

The identity (4.3) was independently proposed by Kanade–Russell [12] and Kurşungöz [13]. They showed that (4.3) is equivalent to the following partition theorem.

Theorem 4.4 (Capparelli's first partition theorem [11]). *For any integer n, the number of partitions of n into distinct parts where no part is congruent to ± 1 modulo 6 is equal to the number of partitions of n into parts, not equal to 1, where the minimal difference between consecutive parts is 2. In fact, the difference between consecutive parts is greater than or equal to 4 unless consecutive parts are $3k$ and $3k+3$ (yielding a difference of 3), or $3k-1$ and $3k+1$ (yielding a difference of 2) for some $k \in \mathbb{N}$.*

Theorem 4.4 was first proven by Andrews in [2].

Analogously, we apply (3.19) to (3.8). This way we are led to the theorem:

Theorem 4.5.

$$\sum_{m,n\geq 0} \frac{q^{Q(m,n)-2m-3n}(q^3;q^3)_M}{(q;q)_m(q^3;q^3)_n(q^3;q^3)_{M-2n-m}}(1+q^{3M})$$
$$= \sum_{j=-M}^{M} q^{3j^2-2j}(1+q^{3j}) \begin{bmatrix} 2M \\ M+j \end{bmatrix}_{q^3}. \tag{4.4}$$

Letting M tend to infinity, and using (2.8) and (2.12) on the right-hand side proves Theorem 1.2.

Similar to the above calculations, we apply (3.20) to (3.7) and get the theorem:

Theorem 4.6.

$$\sum_{m,n\geq 0}\frac{q^{Q(m,n)+m+3n}(q^3;q^3)_M}{(q;q)_m(q^3;q^3)_n(q^3;q^3)_{M-2n-m}}+\sum_{m,n\geq 0}\frac{q^{Q(m,n)+3m+6n+1}(q^3;q^3)_M}{(q;q)_m(q^3;q^3)_n(q^3;q^3)_{M-2n-m}}$$
$$=\sum_{j=-M-1}^{M}q^{3j^2+2j}\begin{bmatrix}2M+1\\ M-j\end{bmatrix}_{q^3}. \tag{4.5}$$

Letting $M\to\infty$ and using (2.8) and (2.12) on the right-hand side, we get the result:

Theorem 4.7.

$$\sum_{m,n\geq 0}\frac{q^{Q(m,n)+m+3n}}{(q;q)_m(q^3;q^3)_n}+\sum_{m,n\geq 0}\frac{q^{Q(m,n)+3m+6n+1}}{(q;q)_m(q^3;q^3)_n}=(-q,-q^5;q^6)_\infty(-q^3;q^3)_\infty. \tag{4.6}$$

It is instructive to compare (4.5) with the following polynomial identity [10, Theorem 1.3, (1.12)], which also implies (4.6) as $N\to\infty$:

Theorem 4.8. *For any non-negative integer N, we have*

$$\sum_{m,n\geq 0}q^{Q(m,n)+m+3n}\begin{bmatrix}3(N-2n-m)+2\\ m\end{bmatrix}_q\begin{bmatrix}2(N-2n-m)+n+1\\ n\end{bmatrix}_{q^3}$$
$$+\sum_{m,n\geq 0}q^{Q(m,n)+3m+6n+1}\begin{bmatrix}3(N-2n-m)\\ m\end{bmatrix}_q\begin{bmatrix}2(N-2n-m)+n\\ n\end{bmatrix}_{q^3}$$
$$=\sum_{l=0}^{N}q^{3\binom{N-2l}{2}}\begin{bmatrix}N+1\\ 2l+1\end{bmatrix}_{q^3}(-q;q^6)_{l+1}(-q^5;q^6)_l.$$

We note that (4.6) first appeared in Kurşungöz [13]. In fact, this is equivalent to the Cappareli's second partition theorem.

Theorem 4.9 (Capparelli's second partition theorem [11]). *For any integer n, the number of partitions of n into distinct parts, where no part is congruent to ± 2 modulo 6 is equal to the number of partitions of n into parts, not equal to 2, where the minimal difference between consecutive parts is 2. In fact, the difference between consecutive parts is greater than or equal to 4 unless consecutive parts are $3k$ and $3k+3$ (yielding a difference of 3), or $3k-1$ and $3k+1$ (yielding a difference of 2) for some $k\in\mathbb{N}$.*

We note that Kurşungöz [13] showed the equivalence of (4.6) to Theorem 4.9. On the other hand, Kanade–Russell [12] showed the equivalence of a slightly different (yet equivalent) double sum identity to the Capparelli's Second Partition Theorem.

Comparing Theorems 4.1 and 4.3, and Theorems 4.6 and 4.8, we see that the identities proven here look somewhat simpler. On the other hand, the objects that appear on both sides of the identities from [10] clearly come with combinatorial interpretations and are made up of objects with manifestly positive coefficients. It is not necessarily clear that the left-hand sides of (4.2), (4.4), and (4.5) have positive coefficients at first sight. A combinatorial study of these objects as generating functions, which would also show the non-negativity of the coefficients of these polynomials, is a task for the future.

5. Outlook

Identity (3.9) is the special case $M \to \infty$ of the following doubly bounded identity.

Theorem 5.1.

$$\sum_{\substack{m\geq 0,\\ L\equiv m(\mathrm{mod} 2)}} q^{\frac{m^2}{2}} \begin{bmatrix} 3M \\ m \end{bmatrix}_q \begin{bmatrix} 2M + \frac{L-m}{2} \\ 2M \end{bmatrix}_{q^3} = \sum_{j=-\infty}^{\infty} q^{\frac{3j^2+2j}{2}} \mathcal{T}\left(\begin{matrix} L,\ M \\ j,\ j \end{matrix}; q^3\right), \quad (5.1)$$

where

$$\mathcal{T}\left(\begin{matrix} L,\ M \\ a,\ b \end{matrix}; q\right) := \sum_{\substack{n\geq 0,\\ L-a\equiv n(\mathrm{mod} 2)}} q^{\frac{n^2}{2}} \begin{bmatrix} M \\ n \end{bmatrix}_q \begin{bmatrix} M+b+\frac{L-a-n}{2} \\ M+b \end{bmatrix}_q \begin{bmatrix} M-b+\frac{L+a-n}{2} \\ M-b \end{bmatrix}_q . \quad (5.2)$$

The refinement (5.2) of the q-trinomial coefficients was first introduced by Warnaar [16,17].

In the forthcoming paper, we will show that Theorem 5.1 implies the following infinite hierarchy of identities.

Theorem 5.2. *Let ν be a positive integer, and let $N_k = n_k + n_{k+1} + \cdots + n_\nu$, for $k = 1, 2, \ldots, \nu$. Then*

$$\sum_{\substack{i,m,n_1,n_2,\ldots,n_\nu\geq 0,\\ i+m\equiv N_1+N_2+\cdots+N_\nu (\mathit{mod}\ 2)}} q^{\frac{m^2+3(i^2+N_1^2+N_2^2+\cdots+N_\nu^2)}{2}} \begin{bmatrix} L-N_1 \\ i \end{bmatrix}_{q^3} \begin{bmatrix} 3n_\nu \\ m \end{bmatrix}_q$$
$$\times \begin{bmatrix} 2n_\nu + (i-N_1-N_2-\cdots-N_\nu-m)/2 \\ 2n_\nu \end{bmatrix}_{q^3} \prod_{j=1}^{\nu-1} \begin{bmatrix} i-\sum_{k=1}^{j} N_k + n_j \\ n_j \end{bmatrix}_{q^3}$$
$$= \sum_{j=\infty}^{\infty} q^{3\binom{\nu+2}{2}j^2+j} \left(\begin{matrix} L,\ (\nu+2)j \\ (\nu+2)j \end{matrix}; q^3\right)_2 . \quad (5.3)$$

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Alexander Berkovich
Department of Mathematics
University of Florida
358 Little Hall
Gainesville
FL 32611
USA
e-mail: alexb@ufl.edu

Ali Kemal Uncu
Johann Radon Institute for Computational and Applied Mathematics (RICAM)
Austrian Academy of Sciences (OEAW)
Altenbergerstrasse 69
4040 Linz
Austria
e-mail: akuncu@risc.jku.at;
akuncu@ricam.oeaw.ac.at

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A Partial Theta Function Borwein Conjecture

Dedicated to George Andrews on the occasion of his 80th birthday

Gaurav Bhatnagar and Michael J. Schlosser

Abstract. We present an infinite family of Borwein type $+--$ conjectures. The expressions in the conjecture are related to multiple basic hypergeometric series with Macdonald polynomial argument.

Mathematics Subject Classification. Primary 11B65; Secondary 05A20, 11B83, 33D52.

Keywords. q-Series, Borwein conjecture, Non-negativity, Multiple basic hypergeometric series with Macdonald polynomial argument.

1. Introduction

The so-called Borwein conjectures, due to Peter Borwein (circa 1990), were popularized by Andrews [1]. The first of these concerns the expansion of finite products of the form

$$(1-q)(1-q^2)(1-q^4)(1-q^5)(1-q^7)(1-q^8)\ldots$$

into a power series in q and the sign pattern displayed by the coefficients. In June 2018, in a conference at Penn State celebrating Andrews' 80th birthday, Chen Wang, a young Ph.D. student studying at the University of Vienna, announced that he has vanquished the first of the Borwein conjectures. In this paper, we propose another set of Borwein-type conjectures. The conjectures here are consistent with the first two Borwein conjectures, and one given by Ismail et al. [5,11]. At the same time, they do not appear to be very far from these conjectures in form and content. However, they are on different lines from other extensions of Borwein conjectures considered in [2,3,5,10,11,13,14].

Borwein's first conjecture may be stated as follows: the polynomials $A_n(q)$, $B_n(q)$, and $C_n(q)$ defined by

$$\prod_{i=0}^{n-1}(1-q^{3i+1})(1-q^{3i+2}) = A_n(q^3) - qB_n(q^3) - q^2C_n(q^3), \tag{1.1}$$

each have non-negative coefficients. This is the one now settled by Wang [12]. We say that the polynomial on the left-hand side satisfies the Borwein $+--$ condition.

Our first conjecture considers products of the form

$$\prod_{i=0}^{n-1}(1-q^{3i+1})(1-q^{3i+2})\prod_{j=1}^{m}\prod_{i=-n}^{n-1}(1-p^jq^{3i+1})(1-p^jq^{3i+2}).$$

Computational evidence suggests that for fixed k, the coefficient of p^k (a Laurent polynomial in q) satisfies the Borwein $+--$ condition for n large enough. For $m=0$, this reduces to the left-hand side of (1.1).

This paper is organized as follows. In Sect. 2 we present a precise statement of this conjecture and outline the computational evidence for this conjecture. We also make another—even more general—conjecture, which is motivated by the first two Borwein conjectures, and Andrews' refinement of these conjectures. Our third and most general conjecture is motivated by Ismail, Kim and Stanton [5, Conjecture 1] (see also Stanton [11, Conjecture 3]). In Sect. 3, we make some remarks concerning the connection to multiple basic hypergeometric series with Macdonald polynomial argument.

2. The Conjectures

Let a, p and q be formal variables. We shall work in the ring of Laurent polynomials in q. For n being a non-negative integer or infinity, the q-shifted factorial is defined as follows:

$$(a;q)_n = \prod_{j=0}^{n-1}(1-aq^j).$$

For convenience, we write

$$(a_1,\ldots,a_m;q)_n = \prod_{k=1}^{m}(a_k;q)_n$$

for products of q-shifted factorials. With this notation, our first conjecture can be stated as follows.

Conjecture 2.1. *Let m and k be non-negative integers. Let the Laurent polynomials $A_{m,n,k}(q)$, $B_{m,n,k}(q)$, and $C_{m,n,k}(q)$ be defined by*

$$\begin{aligned}(q,q^2;q^3)_n &\prod_{j=1}^{m}(p^jq,p^jq^2;q^3)_n(p^jq^{-1},p^jq^{-2};q^{-3})_n\\ &= \sum_{k\geq 0}p^k\left[A_{m,n,k}(q^3)-qB_{m,n,k}(q^3)-q^2C_{m,n,k}(q^3)\right].\end{aligned} \tag{2.1}$$

Then for each $m, k \geq 0$, there is a non-negative integer $N_{m,k}$ such that if $n \geq N_{m,k}$ then the Laurent polynomials $A_{m,n,k}(q)$, $B_{m,n,k}(q)$, and $C_{m,n,k}(q)$ have non-negative coefficients.

Further, for $m = 1$ we have $N_{1,k} = 0$ for $k \leq 4$, and $N_{1,k} = \lceil \frac{k}{4} \rceil$ for $k \geq 5$, while for $m > 1$, $N_{m,k} \equiv N_k$ is independent of m.

Notes

1. The case $m = 0$ or $k = 0$ of Conjecture 2.1 is consistent with the first Borwein conjecture, see [1, Equation (1.1)].
2. For given m and n, the summation index k is bounded by
$$k \leq 4n\binom{m+1}{2} = 2m(m+1)n.$$
3. For $m = 1$, we must have $n \geq k/4$. Indeed, $n = \lceil \frac{k}{4} \rceil$ are the values of $N_{m,k}$ in Table 1 for $m = 1$ for $k \geq 5$. For $k < 5$, $\lceil \frac{k}{4} \rceil = 1$, so we have $N_{m,k} = 0$, since for $n = 0$ the statement of the conjecture holds trivially.
4. We examined the products for $m = 1, 2, \ldots, 10$; $k = 0, 1, 2, \ldots, 15$; and $n = 0, 1, 2, \ldots, 25$. For fixed m and k, the value of $N_{m,k}$ such that the coefficient of p^k in the products satisfies the Borwein $+--$ condition for $N_{m,k} \leq n \leq 25$ (for $m \leq 5$) is recorded in Table 1. The values for $m = 6, 7, \ldots, 10$ were the same as for $m = 5$. Thus for $m > 1$, the values of $N_{m,k}$ appear to be independent of m.
5. The coefficients of $A_{m,n,k}(q)$ were non-negative for all the values of m, n, and k that we computed.
6. The coefficients of powers of q in $q^2 C_{m,n,k}(q^3)$ are the same as those of $qB_{m,n,k}(q^3)$, but in reverse order, that is, we have,
$$q^{n^2-1} B_{m,n,k}(q^{-1}) = C_{m,n,k}(q).$$
This can be seen by replacing q by q^{-1} in (2.1) and comparing the two sides.
7. One can ask, as did Stanton for [11, Conjecture 3], whether Conjecture 2.1 holds for $n = \infty$. However, this question is not applicable here, since the product on the left-hand side of (2.1) is not defined at $n = \infty$.

We now make a few remarks about the form of Conjecture 2.1. The modified theta function is defined as
$$\theta(a; p) = (a; p)_\infty (p/a; p)_\infty.$$
Here we take $n = \infty$ and replace q by p in the definition of the q-shifted factorial. This product is convergent if $|p| < 1$. Consider the theta-shifted factorials defined as [4, Eq. (11.2.5)]
$$(a; q, p)_n = \prod_{i=0}^{n-1} \theta(aq^i; p) = \prod_{i=0}^{n-1} \prod_{j=0}^{\infty} \big(1 - ap^j q^i\big)\big(1 - p^{j+1} q^{-i}/a\big).$$
As a natural extension of the Borwein Conjecture, consider
$$(q; q^3, p)_n (q^2; q^3, p)_n,$$

TABLE 1. Apparent values of $N_{m,k}$, for $m = 1, 2, \ldots, 5$ and $k = 0, 1, \ldots, 15$

$m \backslash k$	0	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15
1	0	0	0	0	0	2	2	2	2	3	3	3	3	4	4	4
2	0	0	0	5	5	8	8	11	12	14	15	17	18	20	21	23
3	0	0	0	5	5	8	8	11	12	14	15	17	18	20	21	23
4	0	0	0	5	5	8	8	11	12	14	15	17	18	20	21	23
5	0	0	0	5	5	8	8	11	12	14	15	17	18	20	21	23

or,

$$\prod_{i=0}^{n-1}\prod_{j=0}^{\infty}\left(1-p^jq^{3i+1}\right)\left(1-p^jq^{3i+2}\right)\left(1-p^{j+1}q^{-3i-1}\right)\left(1-p^{j+1}q^{-3i-2}\right).$$

The product in Conjecture 2.1 should now be transparent. It is obtained by truncating the infinite products indexed by j. Indeed, one can try even more general ways to truncate the products.

Conjecture 2.2. *Let m_1, m_2, n_1, n_2, n_3, and k be non-negative integers. Let the Laurent polynomials $A(q) = A_{m_1,m_2,n_1,n_2,n_3,k}(q)$, $B(q) = B_{m_1,m_2,n_1,n_2,n_3,k}(q)$ and $C(q) = C_{m_1,m_2,n_1,n_2,n_3,k}(q)$ be defined by*

$$\begin{aligned}(q, q^2; q^3)_{n_1} \prod_{j=1}^{m_1}(p^jq, p^jq^2; q^3)_{n_2} \prod_{j=1}^{m_2}(p^jq^{-1}, p^jq^{-2}; q^{-3})_{n_3}\\ = \sum_{k\geq 0} p^k \left[A(q^3) - qB(q^3) - q^2C(q^3)\right].\end{aligned} \tag{2.2}$$

For given k, if $m_1, m_2 \geq 1$, and n_1, n_2 and n_3 are large enough, then the polynomials $A(q)$, $B(q)$, and $C(q)$ have non-negative coefficients.

Notes

1. Borwein's second conjecture [1, Eq. (1.3)] states that

$$(q, q^2; q^3)_n^2$$

satisfies the Borwein $+--$ condition. If we take $m_1 = 1$, $m_2 = 0$, $n_2 = n_1$, $p = 1$, and ignore the condition $m_1, m_2 \geq 1$, then the statement of Conjecture 2.2, reduces to Borwein's second conjecture.
2. Andrews' refinement of Borwein's first two conjectures [1, eq. (1.5), $x = p$] states that for each k, the coefficient of p^k in

$$(q, q^2; q^3)_{n_1}(pq, pq^2; q^3)_{n_2}$$

satisfies the Borwein $+--$ condition. Ae Ja Yee kindly informed us (private communication, January 2019), that Andrews' refinement does not hold. For example, it fails for $n_1 = 1$, $n_2 = 40$, and $k = 40$. Again, if we take $m_1 = 1$ and $m_2 = 0$, the statement of Conjecture 2.2 reduces to Andrews' refinement of Borwein's first two conjectures.
3. Our numerical experiments suggest that we must have $m_1, m_2 \geq 1$ in Conjecture 2.2. But the data we generated do not contradict Borwein's second conjecture. Further, it may still be true that Andrews' refinement of Borwein's conjectures is true for large enough values of n_1 and n_2.
4. It appears that Table 1 is relevant to Conjecture 2.2 too. We observed the following from the data we generated. Let k be fixed, and $m_1, m_2 \geq 2$. Let $n = \min\{n_1, n_2, n_3\}$. Now if $n \geq N_k$, where $N_k \equiv N_{2,k}$ is taken from Table 1, the coefficients of p^k in the expansion of the products in question satisfy the Borwein $+--$ condition.

Next, on the suggestion of Dennis Stanton, we examine a conjecture due to Ismail, Kim and Stanton [5, Conjecture 1] (see also Stanton [11, Conjecture 3]), who considered

$$(q^a, q^{K-a}; q^K)_n = \sum_{m=0}^{\infty} a_m q^m,$$

where a and K are relatively prime integers with $a < K/2$. These authors conjectured:

If K is odd, then

$$a_m \geq 0 \text{ if } m \equiv \pm aj \mod K, \text{ for some non-negative even integer } j < K/2,$$

and,

$$a_m \leq 0 \text{ if } m \equiv \pm aj \mod K, \text{ for some positive odd integer } j < K/2.$$

In [11], this conjecture is followed by the statement: If K is even, then $(-1)^m a_m \geq 0$. The unfortunate placement of this statement suggests that it is part of the conjecture. In fact, it is easy to prove. Since a is relatively prime to K, and K is even, both a and $K-a$ are odd. Thus all the factors in the product are of the form $(1-q^{\text{odd}})$. Now to obtain a term q^m with m even, we will need to multiply an even number of monomials of the form $(-q^{\text{odd}})$, so the sign will be positive. Similarly, if m is odd, the sign will be negative.

As in Conjecture 2.2, we consider the formal expression

$$(q^a; q^K, p)_n (q^{K-a}; q^K, p)_n,$$

truncate the infinite products, and check whether the coefficients satisfy a similar sign pattern. For K even, it is easy to see that an analogous statement holds for the coefficient of p^k for all non-negative integers k.

For K odd, we found that the sign pattern is the same as mentioned above, but only when $a = \lfloor K/2 \rfloor$. In this case, the pattern is an elegant extension of Borwein's $+--$. When K is of the form $4l+1$ or $4l+3$, the sign pattern is as follows:

$$K = 4l+1: \quad \underbrace{++\cdots+}_{l+1}\underbrace{--\cdots-}_{2l}\underbrace{++\cdots+}_{l}$$

$$K = 4l+3: \quad \underbrace{++\cdots+}_{l+1}\underbrace{--\cdots-}_{2l+2}\underbrace{++\cdots+}_{l}$$

For example, when $K=5$, then the pattern is $++--+$, and when $K=7$, then the pattern is $++----+$. (As before, the $+$ sign represents a non-negative, and the $-$ sign represents a non-positive coefficient.)

In what follows, we have replaced K by $2K+1$; we consider only the odd powers of the base q.

Conjecture 2.3. *Let m_1, m_2, n_1, n_2, n_3, and k be non-negative integers. Let K be any positive number. Let the Laurent polynomials $A_k(q) = A_{m_1,m_2,n_1,n_2,n_3,k,K}(q)$ be defined by*

$$(q^K, q^{K+1}; q^{2K+1})_{n_1} \prod_{j=1}^{m_1} (p^j q^K, p^j q^{K+1}; q^{2K+1})_{n_2}$$
$$\times \prod_{j=1}^{m_2} (p^j q^{-K}, p^j q^{-K-1}; q^{-2K-1})_{n_3} = \sum_{k \geq 0} p^k A_k(q), \tag{2.3}$$

where $A_k(q)$ is a Laurent polynomial of the form

$$A_k(q) = \sum_M a_{M,k} q^M.$$

Let $l = \lfloor \frac{2K+1}{4} \rfloor$. For given k and K, if $m_1, m_2 \geq 1$, and n_1, n_2 and n_3 are large enough, then the coefficients $a_{M,k}$ satisfy the following sign pattern:

$$a_{M,k} = \begin{cases} \geq 0, & \text{if } M \equiv 0, \pm i \mod 2K+1, \text{ for } i = 1, 2, \ldots, l, \\ \leq 0, & \text{otherwise.} \end{cases}$$

Notes

1. If $m_1 = 0 = m_2$, then the products on the left-hand side of (2.3) are a special case of those considered in [5, Conjecture 1].
2. When $K = 1$, Conjecture 2.3 reduces to Conjecture 2.2.
3. We gathered data for the following values of the variables systematically:
$$m_1, m_2 \in \{2, 3\},$$
$$n_1, n_2, n_3 \in \{1, 2, \ldots, 5\},$$
$$k \in \{1, 2, \ldots, 10\},$$
$$K \in \{2, 3, 4, \ldots, 14\}.$$
In addition, we considered many random values, with
$$m_1, m_2, n_1, n_2, n_3 \in \{0, 1, \ldots, 10\},$$
$$k \in \{0, 1, \ldots, 30\},$$
$$K \in \{1, 2, 3, 4, \ldots, 20\}.$$
In case we obtained a set of values that did not satisfy the required sign pattern, we performed further computations with larger values of n_1, n_2 or n_3.
4. In our experiments, we found only a few values where the predicted sign pattern does not hold, even for large values of n_1, n_2 and n_3. All of these were with either $m_1 = 0$ or $m_2 = 0$. For example, when $m_1 = 4, m_2 = 0, K = 3, k = 18$. In particular the coefficient of $p^{18} q^{26}$ is predicted to be negative, but is in fact 1, when n_1 and n_2 are large. This is the reason for the condition $m_1, m_2 \geq 1$ in the statements of Conjectures 2.2 and 2.3.

3. Multiple Series Representations

In this section we extend Andrews' explicit expressions for the polynomials $A_n(q)$, $B_n(q)$ and $C_n(q)$ of (1.1) appearing in the first Borwein conjecture. Andrews [1, Eqs. (3.4)–(3.6)] showed that

$$A_n(q) = \sum_{\lambda=-\infty}^{\infty} (-1)^\lambda q^{\lambda(9\lambda+1)/2} \begin{bmatrix} 2n \\ n+3\lambda \end{bmatrix}, \tag{3.1a}$$

$$B_n(q) = \sum_{\lambda=-\infty}^{\infty} (-1)^\lambda q^{\lambda(9\lambda-5)/2} \begin{bmatrix} 2n \\ n+3\lambda-1 \end{bmatrix}, \tag{3.1b}$$

$$C_n(q) = \sum_{\lambda=-\infty}^{\infty} (-1)^\lambda q^{\lambda(9\lambda+7)/2} \begin{bmatrix} 2n \\ n+3\lambda+1 \end{bmatrix}, \tag{3.1c}$$

where

$$\begin{bmatrix} m \\ j \end{bmatrix} = \begin{cases} 0, & \text{if } j<0 \text{ or } j>m, \\ \dfrac{(q;q)_m}{(q;q)_j (q;q)_{m-j}}, & \text{otherwise,} \end{cases}$$

denotes the q-binomial coefficient. We use a result of Kaneko [7] from the theory of basic hypergeometric series with Macdonald polynomial argument (see [6,8]) to give analogous expressions for the functions involved in Conjecture 2.1.

Let $F_{m,n}(p,q)$ denote the left-hand side of (2.1). We first dissect it as follows:

$$F_{m,n}(p,q) = F^0_{m,n}(p,q^3) - qF^1_{m,n}(p,q^3) - q^2 F^2_{m,n}(p,q^3).$$

Thus, we have the definitions:

$$F^0_{m,n}(p,q) = \sum_{k=0}^{2m(m+1)n} p^k A_{m,n,k}(q),$$

$$F^1_{m,n}(p,q) = \sum_{k=0}^{2m(m+1)n} p^k B_{m,n,k}(q),$$

$$F^2_{m,n}(p,q) = \sum_{k=0}^{2m(m+1)n} p^k C_{m,n,k}(q).$$

We extend Andrews' identities by writing each $F^l_{m,n}(p,q)$ (for $l=0,1,2$) as a $(2m+1)$-fold sum.

In the following, λ is an integer partition. That is, λ is any sequence

$$\lambda = (\lambda_1, \lambda_2, \dots, \lambda_n, \dots)$$

of non-negative integers such that $\lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_n \geq \cdots$, and contains only finitely many non-zero terms, called the parts of λ. We use the symbol $|\lambda| = \lambda_1 + \lambda_2 + \cdots$ and say λ is a partition of $|\lambda|$. In slight misuse of notation we shall also use λ to denote finite non-increasing sequences of integers which are not necessarily all non-negative. For such sequences λ the symbol $|\lambda|$ is understood to denote the sum of the elements of λ, as one would expect.

Theorem 3.1. *For $l = 0, 1, 2$ we have*

$$F^l_{m,n}(p,q) = (-1)^{\binom{l+1}{2}} p^{m(m+1)n} q^{-mn^2}$$
$$\times \sum_{\substack{n \geq \lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_{2m+1} \geq -n \\ |\lambda| \equiv -l \pmod 3}} \Bigg(\prod_{1 \leq i < j \leq 2m+1} \frac{(1 - p^{j-i} q^{\lambda_i - \lambda_j})(p^{j-i+1};q)_{\lambda_i - \lambda_j}}{(1-p^{j-i})(p^{j-i-1}q;q)_{\lambda_i - \lambda_j}}$$
$$\times \prod_{i=1}^{2m+1} \frac{(p^{i-1}q;q)_{2n}}{(p^{i-1}q;q)_{n-\lambda_i}(p^{2m+1-i}q;q)_{n+\lambda_i}}$$
$$\times (-1)^{|\lambda|} p^{\sum_{i=1}^{2m+1}(i-1-m)\lambda_i} \times q^{\binom{\lambda_1+1}{2} + \cdots + \binom{\lambda_{2m+1}+1}{2} - \frac{|\lambda|+l}{3}} \Bigg).$$

Remark 3.2. From the expression in Theorem 3.1, it is not obvious that the functions $F^l_{m,n}(p,q)$ are actually polynomials in p of degree $2m(m+1)n$.

Before proving the theorem, we outline some background information from the theory of basic hypergeometric series with Macdonald polynomial argument. For the definition of the Macdonald polynomials $P_\lambda(x_1, \ldots, x_n; q, t)$ together with their most essential properties, we refer to Macdonald's book [9].

In particular, the $P_\lambda(x_1, \ldots, x_n; q, t)$ are homogenous in $x_1, \ldots, x_n$ of degree $|\lambda|$; we have, after scaling each x_i by z,

$$P_\lambda(zx_1, \ldots, zx_n; q, t) = z^{|\lambda|} P_\lambda(x_1, \ldots, x_n; q, t). \tag{3.2}$$

We also make use of the principal specialization formula [9, p. 343, Ex. 5]: Let

$$P_\lambda(1, t, \ldots, t^{n-1}; q, t) = t^{n(\lambda)} \prod_{1 \leq i < j \leq n} \frac{(t^{j-i+1};q)_{\lambda_i - \lambda_j}}{(t^{j-i};q)_{\lambda_i - \lambda_j}}, \tag{3.3}$$

where λ has at most n parts, and $n(\lambda) = \sum_{i=1}^n (i-1)\lambda_i$.

We require the following lemma.

Lemma 3.3. *Let N be a non-negative integer. Then*

$$\prod_{i=1}^n (zt^{1-i}, z^{-1}qt^{i-1}; q)_N$$
$$= \sum_{N \geq \lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_n \geq -N} \Bigg(\prod_{1 \leq i < j \leq n} \frac{(1 - q^{\lambda_i - \lambda_j} t^{j-i})(t^{j-i+1};q)_{\lambda_i - \lambda_j}}{(1-t^{j-i})(qt^{j-i-1};q)_{\lambda_i - \lambda_j}}$$
$$\times \prod_{i=1}^n \frac{(qt^{i-1};q)_{2N}}{(qt^{i-1};q)_{N-\lambda_i}(qt^{n-i};q)_{N+\lambda_i}}$$
$$\times q^{\binom{\lambda_1+1}{2} + \cdots + \binom{\lambda_n+1}{2}} t^{\sum_{i=1}^n (i-1)\lambda_i} (-z^{-1})^{|\lambda|} \Bigg).$$

Proof. We use a reformulation of a result by Kaneko [7, Lemma 2]. Let N be a non-negative integer. Then

$$\begin{aligned}
&\prod_{i=1}^{n}(-x_i q, -x_i^{-1}; q)_N \\
&\quad = \sum_{N\geq\lambda_1\geq\lambda_2\geq\cdots\geq\lambda_n\geq -N}\left(\prod_{1\leq i<j\leq n}\frac{(qt^{j-i};q)_{\lambda_i-\lambda_j}}{(qt^{j-i-1};q)_{\lambda_i-\lambda_j}}\right. \\
&\qquad \times\prod_{i=1}^{n}\frac{(qt^{i-1};q)_{2N}}{(qt^{i-1};q)_{N-\lambda_i}(qt^{n-i};q)_{N+\lambda_i}} \\
&\qquad \times q^{\binom{\lambda_1+1}{2}+\cdots+\binom{\lambda_n+1}{2}} \\
&\qquad \left.\times (x_1\cdots x_n)^{\lambda_n}P_{\lambda-\lambda_n}(x_1,\ldots,x_n;q,t)\right),
\end{aligned}$$

where $\lambda-\lambda_n$ stands for the partition $(\lambda_1-\lambda_n,\ldots,\lambda_n-\lambda_n)$.

In Kaneko's identity, we take $x_i=-z^{-1}t^{i-1}$, for $1\leq i\leq n$, and make use of the homogeneity (3.2) and the principal specialization in (3.3), to obtain the lemma. □

Proof of Theorem 3.1. We first observe that the product on the left-hand side of (2.1) can be written as

$$\begin{aligned}
&\prod_{j=0}^{m}(p^jq,p^jq^2;q^3)_n\prod_{j=1}^{m}(p^jq^{-1},p^jq^{-2};q^{-3})_n \\
&\quad = p^{m(m+1)n}q^{-3mn^2}\prod_{i=1}^{2m+1}(p^{-m+i-1}q^2,p^{m-i+1}q;q^3)_n.
\end{aligned}$$

Next, we apply the $(n,N,z,q,t)\mapsto(2m+1,n,p^mq,q^3,p)$ case of Lemma 3.3 to arrive at

$$\begin{aligned}
&\prod_{j=0}^{m}(p^jq,p^jq^2;q^3)_n\prod_{j=1}^{m}(p^jq^{-1},p^jq^{-2};q^{-3})_n \\
&\quad = p^{m(m+1)n}q^{-3mn^2}\sum_{n\geq\lambda_1\geq\lambda_2\geq\cdots\geq\lambda_{2m+1}\geq -n} \\
&\qquad \times\left(\prod_{1\leq i<j\leq 2m+1}\frac{(1-p^{j-i}q^{3\lambda_i-3\lambda_j})(p^{j-i+1};q^3)_{\lambda_i-\lambda_j}}{(1-p^{j-i})(p^{j-i-1}q^3;q^3)_{\lambda_i-\lambda_j}}\right. \\
&\qquad \times\prod_{i=1}^{2m+1}\frac{(p^{i-1}q^3;q^3)_{2n}}{(p^{i-1}q^3;q^3)_{n-\lambda_i}(p^{2m+1-i}q^3;q^3)_{n+\lambda_i}} \\
&\qquad \times(-1)^{|\lambda|}p^{\sum_{i=1}^{2m+1}(i-1-m)\lambda_i} \\
&\qquad \left.\times q^{3\binom{\lambda_1+1}{2}+\cdots+3\binom{\lambda_{2m+1}+1}{2}-|\lambda|}\right).
\end{aligned}$$

By picking the coefficients of q^l with l belonging to a residue class modulo 3, we obtain the theorem. □

Remark 3.4. We can obtain a more general multiseries expression for the products

$$\prod_{j=0}^{m}(p^j q^a, p^j q^{2K+1-a}; q^{2K+1})_n \prod_{j=1}^{m}(p^j q^{-a}, p^j q^{a-1-2K}; q^{-2K-1})_n$$

by following a similar analysis as carried out in the proof of Theorem 3.1, where we apply the $(n, N, z, q, t) \mapsto (2m+1, n, p^m q^a, q^{2K+1}, p)$ case of Lemma 3.3. The case $a = K$ gives the products on the left-hand side of (2.3), with $n = n_1 = n_2 = n_3$ and $m = m_1 = m_2$.

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Gaurav Bhatnagar and Michael J. Schlosser
Fakultät für Mathematik
Universität Wien
Oskar-Morgenstern-Platz 1
1090 Vienna
Austria
e-mail: michael.schlosser@univie.ac.at

Present Address
Gaurav Bhatnagar
School of Physical Sciences (SPS)
Jawaharlal Nehru University
Delhi 110067
India
e-mail: bhatnagarg@gmail.com

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A Note on Andrews' Partitions with Parts Separated by Parity

Dedicated to George Andrews in honor of his 80th birthday

Kathrin Bringmann and Chris Jennings-Shaffer

Abstract. In this note, we give three identities for partitions with parts separated by parity, which were recently introduced by Andrews.

Mathematics Subject Classification. Primary 11P81, 11P84.

Keywords. Number theory, Partitions, Parity, Modular forms, Mock theta functions.

1. Introduction

Recently, Andrews [1] studied integer partitions in which all parts of a given parity are smaller than those of the opposite parity. Furthermore, he considered eight subcases based on the parity of the smaller parts and parts of a given parity appearing at most once or an unlimited number of times. Following Andrews, we use "ed" for evens distinct, "eu" for evens unlimited, "od" for odds distinct, and "ou" for odds unlimited. With "zw" and "xy" from the four choices above, we let $F_{\mathrm{xy}}^{\mathrm{zw}}(q)$ denote the generating function of partitions where zw specifies the parity and condition of the larger parts and xy specifies the parity and condition of the smaller parts.

The eight relevant generating functions are:

$$F_{\mathrm{eu}}^{\mathrm{ou}}(q) := \sum_{n=0}^{\infty} \frac{q^{2n}}{(q^2;q^2)_n\, (q^{2n+1};q^2)_\infty},$$

$$F_{\mathrm{eu}}^{\mathrm{od}}(q) := \sum_{n=0}^{\infty} \frac{q^{2n}\left(-q^{2n+1};q^2\right)_\infty}{(q^2;q^2)_n},$$

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$$
\begin{aligned}
F_{\mathrm{ed}}^{\mathrm{ou}}(q) &:= \sum_{n=0}^{\infty} \frac{\left(-q^2;q^2\right)_n q^{2n+2}}{\left(q^{2n+3};q^2\right)_\infty}, \\
F_{\mathrm{ed}}^{\mathrm{od}}(q) &:= \sum_{n=0}^{\infty} q^{2n+2}\left(-q^2;q^2\right)_n \left(-q^{2n+3};q^2\right)_\infty, \\
F_{\mathrm{ou}}^{\mathrm{eu}}(q) &:= \sum_{n=0}^{\infty} \frac{q^{2n+1}}{\left(q;q^2\right)_{n+1}\left(q^{2n+2};q^2\right)_\infty}, \\
F_{\mathrm{ou}}^{\mathrm{ed}}(q) &:= \sum_{n=0}^{\infty} \frac{q^{2n+1}\left(-q^{2n+2};q^2\right)_\infty}{\left(q;q^2\right)_{n+1}}, \\
F_{\mathrm{od}}^{\mathrm{eu}}(q) &:= \sum_{n=0}^{\infty} \frac{q^{2n+1}\left(-q;q^2\right)_n}{\left(q^{2n+2};q^2\right)_\infty}, \\
F_{\mathrm{od}}^{\mathrm{ed}}(q) &:= \sum_{n=0}^{\infty} q^{2n+1}\left(-q;q^2\right)_n \left(-q^{2n+2};q^2\right)_\infty.
\end{aligned}
$$

Here, we are using the standard product notation:

$$
(a;q)_n := \prod_{j=0}^{n-1}\left(1-aq^j\right)
$$

for $n \in \mathbb{N}_0 \cup \{\infty\}$. We note that with the exception of $F_{\mathrm{eu}}^{\mathrm{ou}}(q)$ and $F_{\mathrm{eu}}^{\mathrm{od}}(q)$, we do not allow the subpartition consisting of the smaller parts to be empty.

Andrews' identities (after minor corrections) can be stated as:

$$
\begin{aligned}
F_{\mathrm{eu}}^{\mathrm{ou}}(q) &= \frac{1}{(1-q)\left(q^2;q^2\right)_\infty}, \\
F_{\mathrm{eu}}^{\mathrm{od}}(q) &= \frac{1}{2}\left(\frac{1}{\left(q^2;q^2\right)_\infty} + \left(-q;q^2\right)_\infty^2\right), \\
F_{\mathrm{ed}}^{\mathrm{ou}}(-q) &= \frac{1}{2\left(-q;q^2\right)_\infty}\left((-q;q)_\infty - 1 - \sum_{n=0}^{\infty} q^{\frac{n(3n-1)}{2}}\left(1-q^n\right)\right), \\
F_{\mathrm{ou}}^{\mathrm{eu}}(q) &= \frac{1}{1-q}\left(\frac{1}{\left(q;q^2\right)_\infty} - \frac{1}{\left(q^2;q^2\right)_\infty}\right), \\
F_{\mathrm{ou}}^{\mathrm{ed}}(-q) &= -\frac{\left(-q^2;q^2\right)_\infty}{2}\left(2 - \frac{1}{(-q;q)_\infty} - \sum_{n=0}^{\infty} \frac{q^{n^2+n}}{(-q;q)_n^2\left(1+q^{n+1}\right)}\right), \\
F_{\mathrm{od}}^{\mathrm{eu}}(-q) &= -\frac{1}{\left(q^2;q^2\right)_\infty}\sum_{j=1}^{\infty}\sum_{n=j}^{\infty}(-1)^{n+j} q^{\frac{n(3n+1)}{2}-j^2}\left(1-q^{2n+1}\right).
\end{aligned}
$$

Surprisingly, these identities are derived with little more than the q-binomial theorem, Heine's transformation, and the Rogers–Fine identity. In the following theorem, we give new identities for $F_{\mathrm{ed}}^{\mathrm{od}}(q)$, $F_{\mathrm{od}}^{\mathrm{ed}}(q)$, and $F_{\mathrm{ou}}^{\mathrm{ed}}(-q)$.

Theorem 1.1. *The following identities hold:*

$$F_{\rm ed}^{\rm od}(q) = \frac{q\left(-q;q^2\right)_\infty}{1-q}\left(1-\frac{(-q^2;q^2)_\infty}{(-q;q^2)_\infty}\right), \tag{1.1}$$

$$F_{\rm od}^{\rm ed}(q) = \frac{q(-q^2;q^2)_\infty}{1-q}\left(2-\frac{(-q;q^2)_\infty}{(-q^2;q^2)_\infty}\right), \tag{1.2}$$

$$\begin{aligned} F_{\rm ou}^{\rm ed}(-q) = -\frac{\left(-q^2;q^2\right)_\infty}{2}\Bigg(2 &- \frac{1}{(-q;q)_\infty} \\ &- \frac{2}{(q;q)_\infty}\sum_{n\in\mathbb{Z}}\frac{(-1)^n q^{\frac{3n(n+1)}{2}}}{1+q^n}\Bigg). \end{aligned} \tag{1.3}$$

Remark 1.2. The functions $F_{\rm ed}^{\rm od}(q)$ and $F_{\rm od}^{\rm ed}(q)$ are basically modular functions. Also we find that $F_{\rm ou}^{\rm ed}(-q)$ is related to Ramanujan's third-order mock theta function $f(q)$, as:

$$\begin{aligned} f(q) := \sum_{n=0}^\infty \frac{q^{n^2}}{(-q;q)_n^2} &= \frac{2}{(q;q)_\infty}\sum_{n\in\mathbb{Z}}\frac{(-1)^n q^{\frac{n(3n+1)}{2}}}{1+q^n} \\ &= 2-\frac{2}{(q;q)_\infty}\sum_{n\in\mathbb{Z}}\frac{(-1)^n q^{\frac{3n(n+1)}{2}}}{1+q^n}, \end{aligned}$$

where the final equality uses Euler's pentagonal number theorem.

2. Proof of Theorem 1.1

To prove Eqs. (1.1) and (1.2), we require the following q-series identity:

$$\sum_{n=0}^\infty \frac{(x;q)_n q^n}{(y;q)_n} = \frac{q(x;q)_\infty}{y(y;q)_\infty\left(1-\frac{xq}{y}\right)} + \frac{\left(1-\frac{q}{y}\right)}{\left(1-\frac{xq}{y}\right)}. \tag{2.1}$$

We note that (2.1) is (4.1) from [3] and was proved with Heine's transformation [4, p. 241, (III.2)]. To prove Eq. (1.3), we require the concept of a Bailey pair and Bailey's Lemma, which are described in [2, Chapter 3]. A pair of sequences (α,β) is called a *Bailey pair relative* to $a=q$ if:

$$\beta_n = \sum_{j=0}^n \frac{\alpha_j}{(q;q)_{n-j}(q^2;q)_{n+j}}.$$

A limiting form of Bailey's Lemma states that if (α_n,β_n) is a Bailey pair relative to q, then

$$\sum_{n=0}^\infty q^{n^2+n}\beta_n = \frac{1}{(q^2;q)_\infty}\sum_{n=0}^\infty q^{n^2+n}\alpha_n. \tag{2.2}$$

The Bailey pair that we use is given by:

$$\beta_n' := \frac{1}{(-q;q)_n^2(1+q^{n+1})}, \qquad \alpha_n' := \frac{2(-1)^n q^{\frac{n(n+1)}{2}}(1-q^{2n+1})}{(1-q)(1+q^n)(1+q^{n+1})}, \tag{2.3}$$

which follows from taking the Bailey pair from Theorem 8 of [5] with $a \to q$, $b = -1$, $c = -q$, and $d = -1$ and dividing both α_n and β_n by $(1+q)$.

Proof of Theorem 1.1. We find that:

$$\begin{aligned} F_{\mathrm{ed}}^{\mathrm{od}}(q) &= \left(-q;q^2\right)_\infty \sum_{n=1}^{\infty} \frac{\left(-q^2;q^2\right)_{n-1} q^{2n}}{\left(-q;q^2\right)_n} \\ &= \frac{\left(-q;q^2\right)_\infty}{2} \left(-1 + \sum_{n=0}^{\infty} \frac{\left(-1;q^2\right)_n q^{2n}}{\left(-q;q^2\right)_n}\right). \end{aligned}$$

With $q \mapsto q^2$, $x = -1$, and $y = -q$, Eq. (2.1) implies that:

$$\sum_{n=0}^{\infty} \frac{(-1;q^2)q^{2n}}{(-q;q^2)_n} = -\frac{q\left(-1;q^2\right)_\infty}{\left(-q;q^2\right)_\infty (1-q)} + \frac{1+q}{1-q}.$$

Equation (1.1) then follows after elementary simplifications.

Similarly, we have that:

$$F_{\mathrm{od}}^{\mathrm{ed}}(q) = \left(-q^2;q^2\right)_\infty \sum_{n=0}^{\infty} \frac{\left(-q;q^2\right)_n q^{2n+1}}{\left(-q^2;q^2\right)_n}.$$

By applying (2.1) with $q \mapsto q^2$, $x = -q$, and $y = -q^2$, we find that

$$\sum_{n=0}^{\infty} \frac{\left(-q;q^2\right) q^{2n}}{\left(-q^2;q^2\right)_n} = -\frac{\left(-q;q^2\right)_\infty}{\left(-q^2;q^2\right)_\infty (1-q)} + \frac{2}{1-q},$$

and (1.2) follows.

For $F_{\mathrm{ou}}^{\mathrm{ed}}(q)$, we begin with Andrews' identity [1]:

$$F_{\mathrm{ou}}^{\mathrm{ed}}(-q) = -\frac{\left(-q^2;q^2\right)_\infty}{2} \left(2 - \frac{1}{(-q;q)_\infty} - \sum_{n=0}^{\infty} \frac{q^{n^2+n}}{(-q;q)_n^2 \left(1+q^{n+1}\right)}\right).$$

By applying (2.2) to the Bailey pair (α', β') in (2.3), we have that:

$$\sum_{n=0}^{\infty} \frac{q^{n^2+n}}{(-q;q)_n^2 \left(1+q^{n+1}\right)} = \frac{2}{(q;q)_\infty} \sum_{n=0}^{\infty} \frac{(-1)^n q^{\frac{3n(n+1)}{2}} \left(1-q^{2n+1}\right)}{\left(1+q^n\right)\left(1+q^{n+1}\right)}.$$

We use the partial fraction decomposition:

$$\frac{1-q^{2n+1}}{\left(1+q^n\right)\left(1+q^{n+1}\right)} = \frac{1}{1+q^n} - \frac{q^{n+1}}{1+q^{n+1}},$$

to deduce that

$$\begin{aligned} \sum_{n=0}^{\infty} \frac{(-1)^n q^{\frac{3n(n+1)}{2}} \left(1-q^{2n+1}\right)}{\left(1+q^n\right)\left(1+q^{n+1}\right)} &= \sum_{n=0}^{\infty} (-1)^n q^{\frac{3n(n+1)}{2}} \left(\frac{1}{1+q^n} - \frac{q^{n+1}}{1+q^{n+1}}\right) \\ &= \sum_{n\in\mathbb{Z}} \frac{(-1)^n q^{\frac{3n(n+1)}{2}}}{1+q^n}. \end{aligned}$$

Altogether, this implies Eq. (1.3). $\square$

By applying Theorem 1.1 part 3 of [6] to the Bailey pair $E(3)$ of [7], we find that:

$$F_{\mathrm{od}}^{\mathrm{ed}}(-q) = -\frac{q\,(q;q)_\infty \left(-q^2;q^2\right)_\infty}{\left(q^2;q^2\right)_\infty^2} \times \sum_{n=0}^{\infty}\sum_{m=0}^{\infty} (-1)^m\, q^{\frac{n(n+3)}{2}+2nm+2m^2+2m} \left(1+q^{2m+1}\right).$$

As such, we have that

$$\left(\sum_{n,m\geq 0} - \sum_{n,m<0}\right) (-1)^m q^{\frac{n(n+3)}{2}+2nm+2m(m+1)} = \frac{2\left(q^2;q^2\right)_\infty}{(1+q)\left(q;q^2\right)_\infty} - \frac{\left(q^2;q^2\right)_\infty}{(1+q)\left(-q^2;q^2\right)_\infty}.$$

We note that the corresponding quadratic form is degenerate, and so a priori, the modularity properties of this theta function are unclear. More generally, one can prove directly that, for $c \in \mathbb{N}$:

$$\sum_{n,m\geq 0} z^n w^m q^{n^2+2cnm+c^2m^2} = \frac{1}{1-\frac{w}{z^c}} \sum_{k=0}^{c-1}\sum_{n=0}^{\infty} z^{cn+k} q^{(cn+k)^2} \left(1-\frac{w^{n+1}}{z^{cn+c}}\right).$$

The above is a sum of partial theta functions, which sometimes combine to give a modular form.

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Kathrin Bringmann and Chris Jennings-Shaffer
Mathematical Institute
University of Cologne
Weyertal 86-90
50931 Cologne
Germany

Kathrin Bringmann
e-mail: kbringma@math.uni-koeln.de

Chris Jennings-Shaffer
e-mail: cjenning@math.uni-koeln.de

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A Bijective Proof of a False Theta Function Identity from Ramanujan's Lost Notebook

Dedicated to Professor George Andrews on the occasion of his eightieth birthday

Hannah E. Burson

Abstract. In his lost notebook, Ramanujan listed five identities related to the false theta function:

$$f(q) = \sum_{n=0}^{\infty} (-1)^n q^{n(n+1)/2}.$$

A new combinatorial interpretation and a proof of one of these identities are given. The methods of the proof allow for new multivariate generalizations of this identity. Additionally, the same technique can be used to obtain a combinatorial interpretation of another one of the identities.

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Keywords. Partitions, Overpartitions, False theta functions.

1. Introduction

Rogers [16] introduced false theta functions, which are series that would be classical theta functions except for changes in signs of an infinite number of terms. In his notebooks [14] and the lost notebook [15], Ramanujan recorded many false theta function identities that he discovered. However, in Ramanujan's last letter to Hardy in 1920, Ramanujan introduced mock theta functions and shifted his focus away from false theta functions. The mathematical community followed Ramanujan's lead and largely ignored false theta functions for the next several decades.

In recent times, there has been an increase in interest in false theta functions. G.E. Andrews devoted a section of [3] to partition theoretic applications of false theta functions. More recently, such as in [1,2,9,10,13], researchers have found combinatorial proofs of some of Ramanujan's false theta function identities.

In his lost notebook [15] (cf. [5, p. 227]), Ramanujan stated five identities related to the false theta function:

$$f(q) = \sum_{n=0}^{\infty} (-1)^n q^{n(n+1)/2}, \quad |q| < 1. \tag{1.1}$$

These identities were first proved by Andrews in [4], using identities such as the Rogers–Fine identity and Heine's transformation. Other analytic proofs have been given in [7,11,17]. There are no previously known bijective proofs of any of these identities.

In this paper, we adopt the standard q-series notation:

$$(a;q)_n = \prod_{j=0}^{n-1} (1 - aq^j), \quad |q| < 1, n \in \{0, 1, 2 \ldots\}.$$

We will focus on a bijective proof of Ramanujan's identity for $f(q^4)$.

Theorem 1.1. (Ramanujan) *If $f(q)$ is defined by (1.1), then for $|q| < 1$:*

$$\sum_{n=0}^{\infty} \frac{(q;q^2)_n \, q^n}{(-q;q^2)_{n+1}} = f(q^4).$$

The combinatorial interpretation of this identity has several similarities to the combinatorial version of Euler's pentagonal number theorem, which was bijectively proved by Franklin [12]. Since the function in the pentagonal number theorem is a theta function, these similarities are not obvious analytically.

In Sect. 2, we explain the necessary background on partitions. Then, in Sect. 3, we introduce a new combinatorial analog of Theorem 1.1 and give its bijective proof in Sect. 4. In Sect. 5, we introduce new identities that arise from generalizing the proof in Sect. 4. Finally, in Sect. 6, we give a similar combinatorial interpretation of another one of Ramanujan's identities.

2. Background

We use several tools from the theory of partitions. Recall that a partition of n is a non-increasing sequence of integers $(\pi_1, \pi_2, \ldots, \pi_k)$, where $\pi_1 + \pi_2 + \cdots + \pi_k = n$. An overpartition of n is a partition of n where the first appearance of a part of any size may be overlined. For example, $(7, 6, \overline{5}, 5, 5, \overline{3}, 2, 2, 2)$ is an overpartition of 37. We can create a graphical representation of a partition, called a Ferrers diagram, by making an array of boxes whose ith row has as many boxes as the ith part of the partition. There is a variation of a Ferrers diagram called an m-modular diagram (also called a MacMahon diagram) where the part $mj + r$ with $0 \leq r < m$ is represented by a row made of j boxes containing an m following one box containing an r.

For this paper, we create an analog of a two-modular diagram called a boxed two-modular diagram, which is a graphical representation of a pair (k, π) where k is a non-negative integer and π is a partition. To obtain the boxed two-modular diagram, we represent π as a two-modular diagram and k

as a row of one 0 and k 1s at the top of the diagram. For example, the figure below is a boxed two-modular diagram for the pair $(3,(8,7,5,5,3))$:

0	1	1	1
2	2	2	2
1	2	2	2
1	2	2	
1	2	2	
1	2		

Note that, if π is a partition into odd parts with the largest part no greater than $2k+1$, the boxed 2-modular diagram will have the shape of a partition and the boxes in the first column will not contain any 2s. In this case, our boxed two-modular diagram is exactly the odd Ferrers graph introduced in [6] and named in [8].

To represent an overpartition, we add a shaded box to the end of any overlined part. Note that, to maintain the general shape of a Ferrers diagram, we must require that any overlined part be no larger than 2k − 1. For example, the figure below is a boxed 2-modular diagram for the pair (3, (8, 7, $\bar{5}$, 5, 3)) :

0	1	1	1
2	2	2	2
1	2	2	2
1	2	2	
1	2	2	
1	2		

We use the following notation when discussing pairs $(k,\overline{\pi})$.

- $\mathcal{P}_n$ is the set of pairs $(k,\overline{\pi})$ where k is a non-negative integer and $\overline{\pi}$ is an overpartition of $n-k$ into odd parts of size no greater than $2k+1$ and with all overlined parts of size no greater than $2k-1$.
- $\nu(\overline{\pi})$ is the number of parts of the overpartition $\overline{\pi}$.
- $s(\overline{\pi})$ is the size of the smallest part of $\overline{\pi}$.
- $\nu_\ell(k,\overline{\pi})$ is the number of parts of size $2k+1$ in $\overline{\pi}$.
- $\nu_s(\overline{\pi})$ is the number of times the smallest part appears in $\overline{\pi}$.

3. Combinatorial Interpretation

In this section, we interpret Theorem 1.1 in terms of pairs $(k,\overline{\pi})\in\mathcal{P}_n$. We count each pair $(k,\overline{\pi})\in\mathcal{P}_n$ with weight $(-1)^{\nu(\overline{\pi})}$.

Theorem 3.1. *Let $\overline{p}_o(n)$ (resp. $\overline{p}_e(n)$) be the number of pairs $(k,\overline{\pi})$, where k is a non-negative integer and $\overline{\pi}$ is an overpartition of $n-k$ into an odd number (resp. even number) of odd parts of size not exceeding $2k+1$, where all overlined parts must have size $<2k+1$. Then, for $n\geq 0$:*

$$\overline{p}_e(n)-\overline{p}_o(n)=\begin{cases}(-1)^k, & \text{if } n=2k(k+1),\\ 0, & \text{otherwise.}\end{cases}$$

Theorem 3.2. *Theorems 1.1 and 3.1 are equivalent.*

Proof. The equivalence of the right sides is trivial, so we focus on the left sides. Note that $(q;q^2)_k$ generates partitions into distinct odd parts of size no greater than $2k-1$, where each partition into ν parts has weight $(-1)^\nu$. Similarly, $\frac{1}{(-q;q^2)_{k+1}}$ generates partitions into odd parts of size no greater than $2k+1$, where each partition into ν parts has weight $(-1)^\nu$. Thus, if we let the parts coming from $(q;q^2)_k$ be overlined, we find that $\frac{(q;q^2)_k}{(-q;q^2)_{k+1}}$ generates overpartitions into odd parts of size no greater than $2k+1$, where all overlined parts are no larger than $2k-1$, and each overpartition into ν parts is counted with weight $(-1)^\nu$. Additionally, q^k generates the integer k. Therefore

$$\sum_{k=0}^{\infty} \frac{(q;q^2)_k q^k}{(-q;q^2)_{k+1}} = \sum_{n=0}^{\infty} [\overline{p}_e(n) - \overline{p}_o(n)] q^n,$$

where $\overline{p}_e(n)$ and $\overline{p}_o(n)$ are as defined in Theorem 3.1. $\square$

4. Proof of the Main Theorem

We devote this section to proving Theorem 3.1 combinatorially. To obtain the bijection, we split $\mathcal{P}_n$ into cases. First, we show that conjugation is a sign-reversing involution on the case where $k + \nu(\overline{\pi}) \equiv 1 \pmod 2$. Then, for the case where $k + \nu(\overline{\pi}) \equiv 0 \pmod 2$, we further divide this subset of $\mathcal{P}_n$ into cases depending on the relative sizes of the last row and the last column of the boxed 2-modular diagram and introduce variations of the conjugation that provide sign-reversing bijections and involutions on these cases.

4.1. Conjugation

For an ordinary partition π, the conjugate partition π' is defined to be the partition created by reflecting the Ferrers diagram of π about the line $y = -x$. Similarly, for a pair (k, π), where k is a non-negative integer and π is a partition into odd parts of size $\leq 2k+1$, we can reflect our boxed 2-modular diagram about the line $y = -x$ to get the conjugate pair $(k', \overline{\pi}')$.

Example 4.1. The conjugate of $(4, (9, 9, 7, 7, 5, 5, 3))$ is $(7, (15, 13, 9, 5))$:

$$\begin{array}{ccccc} 0&1&1&1&1\\ 1&2&2&2&2\\ 1&2&2&2&2\\ 1&2&2&2& \\ 1&2&2&2& \\ 1&2&2& & \\ 1&2&2& & \\ 1&2& & & \end{array} \quad \rightarrow \quad \begin{array}{cccccccc} 0&1&1&1&1&1&1&1\\ 1&2&2&2&2&2&2&2\\ 1&2&2&2&2&2&2& \\ 1&2&2&2&2& & & \\ 1&2&2& & & & & \end{array}$$

Furthermore, we note that conjugation generalizes to the overpartition case. Specifically, we let $(k', \overline{\pi}')$ be the conjugate partition of the pair $(k, \overline{\pi}) \in \mathcal{P}_n$. Then, for every j, where a part of size $2j+1$ is overlined in $\overline{\pi}$, the $(j+1)^{\text{st}}$ part in $\overline{\pi}'$ must be overlined.

Example 4.2. The conjugate of $(3, (7, \overline{5}, 5, 5, \overline{3}, 1, 1, 1))$ is $(8, (11, \overline{9}, \overline{3}))$:

$$
\begin{array}{|c|c|c|c|}
\hline
0 & 1 & 1 & 1 \\ \hline
1 & 2 & 2 & 2 \\ \hline
1 & 2 & 2 & \blacksquare \\ \hline
1 & 2 & 2 \\ \cline{1-3}
1 & 2 & 2 \\ \cline{1-3}
1 & 2 & \blacksquare \\ \cline{1-3}
1 \\ \cline{1-1}
1 \\ \cline{1-1}
1 \\ \cline{1-1}
\end{array}
\quad \longrightarrow \quad
\begin{array}{|c|c|c|c|c|c|c|c|c|}
\hline
0 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 \\ \hline
1 & 2 & 2 & 2 & 2 & 2 \\ \cline{1-6}
1 & 2 & 2 & 2 & 2 & \blacksquare \\ \cline{1-6}
1 & 2 & \blacksquare \\ \cline{1-3}
\end{array}
$$

Note that, because conjugation swaps rows and columns and preserves the boxes in the diagram, $k' = \nu(\overline{\pi})$, $\nu(\overline{\pi}') = k$, and $k' + |\overline{\pi}'| = k + |\overline{\pi}|$. Furthermore, since conjugation is its own inverse, we obtain the following lemma.

Lemma 4.3. *Let $\mathcal{S}_{n,k,\ell}$ be the set of pairs $(k, \overline{\pi})$, where k is a non-negative integer and $\overline{\pi}$ is an overpartition of $n-k$ into ℓ odd parts of size $\leq 2k+1$, with all overlined parts no larger than $2k-1$. Then, $|\mathcal{S}_{n,k,\ell}| = |\mathcal{S}_{n,\ell,k}|$.*

When $k + \nu(\overline{\pi}) \equiv 1 \pmod 2$, conjugation is sign-reversing, which leads to the next lemma.

Lemma 4.4. *Conjugation is a sign-reversing involution on pairs $(k, \overline{\pi}) \in \mathcal{P}_n$ counted with weight $(-1)^{\nu(\overline{\pi})}$, when $k + \nu(\overline{\pi}) \equiv 1 \pmod 2$.*

Proof. This follows from Lemma 4.3 and the fact that $k \not\equiv \nu(\overline{\pi}) \pmod 2$, so conjugation must be sign-reversing. □

4.2. Variations

For the case $k + \nu(\overline{\pi}) \equiv 0 \pmod 2$, we consider two variations of conjugation. First, we define $\phi_s(k, \overline{\pi})$ by fixing the last row of the boxed two-modular diagram for $(k, \overline{\pi})$ and conjugating the remainder of the diagram.

Example 4.5. $\phi_s(4, (9, 9, 9, \overline{7}, 7, \overline{5})) = (5, (11, 11, 11, \overline{7}, \overline{5}))$

$$
\begin{array}{|c|c|c|c|c|}
\hline
0 & 1 & 1 & 1 & 1 \\ \hline
1 & 2 & 2 & 2 & 2 \\ \hline
1 & 2 & 2 & 2 & 2 \\ \hline
1 & 2 & 2 & 2 & 2 \\ \hline
1 & 2 & 2 & 2 & \blacksquare \\ \hline
1 & 2 & 2 & 2 \\ \cline{1-4}
1 & 2 & 2 & \blacksquare \\ \cline{1-4}
\end{array}
\quad \xrightarrow{\phi_s} \quad
\begin{array}{|c|c|c|c|c|c|}
\hline
0 & 1 & 1 & 1 & 1 & 1 \\ \hline
1 & 2 & 2 & 2 & 2 & 2 \\ \hline
1 & 2 & 2 & 2 & 2 & 2 \\ \hline
1 & 2 & 2 & 2 & 2 & 2 \\ \hline
1 & 2 & 2 & 2 & \blacksquare \\ \cline{1-5}
1 & 2 & 2 & \blacksquare \\ \cline{1-4}
\end{array}
$$

Note that ϕ_s is well defined for pairs $(k, \overline{\pi}) \in \mathcal{P}_n$ where the last row of the boxed two-modular diagram is shorter than the last column. Equivalently, ϕ_s is well defined when $\frac{s(\overline{\pi})-1}{2} < \nu_\ell(k, \overline{\pi})$. Furthermore, ϕ_s is also well defined when $\frac{s(\overline{\pi})-1}{2} = \nu_\ell(k, \overline{\pi})$, $s(\overline{\pi}) < 2k+1$, and the last part of $\overline{\pi}$ is not overlined. The last condition is necessary to maintain the restriction on overpartitions that only the first part of any size may be overlined. If we define $(k_s, \overline{\pi}_s) = \phi_s((k, \overline{\pi}))$, we can note that $k_s = \nu(\overline{\pi}) - 1$ and $\nu(\overline{\pi}_s) = k + 1$. Moreover, the size of the penultimate part of $\overline{\pi}$ determines $\nu_\ell(\phi_s(k, \overline{\pi}))$, so we consider separately the cases where $\nu_s(\overline{\pi}) = 1$ and $\nu_s(\overline{\pi}) > 1$. Then, we have the following lemmas.

Lemma 4.6. *The map ϕ_s is a sign-reversing involution on the set* $\{(k,\overline{\pi}) \in \mathcal{P}_n \colon k+\nu(\overline{\pi}) \equiv 0 \pmod 2, s(\overline{\pi}) < 2\nu_\ell(k,\overline{\pi})+1,$ *and* $\nu_s(\overline{\pi}) = 1\}$.

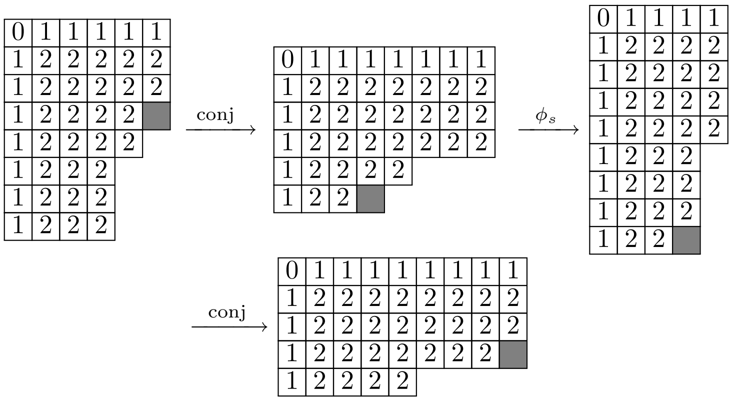
Proof. Let $(k,\overline{\pi}) \in \mathcal{P}_n$, such that $k+\nu(\overline{\pi}) \equiv 0 \pmod 2, s(\overline{\pi}) < 2\nu_\ell(k,\overline{\pi})+1,$ and $\nu_s(\overline{\pi}) = 1$. Let $(k_s, \overline{\pi}_s) = \phi_s(k,\overline{\pi})$. Since $\nu_s(\overline{\pi}) = 1$, the second smallest part of $\overline{\pi}$, which determines $\nu_\ell(k_s,\overline{\pi}_s)$, will be larger than $s(\overline{\pi}) = s(\overline{\pi}_s)$, so $s(\overline{\pi}_s) < 2\nu_\ell(k_s,\overline{\pi}_s)+1$. Moreover, since $s(\overline{\pi}) < 2\nu_\ell(k,\overline{\pi})+1$, $\nu_s(\overline{\pi}_s) = 1$. Thus, $(k_s,\overline{\pi}_s) \in \mathcal{P}_n$, $k_s+\nu(\overline{\pi}_s) \equiv 0 \pmod 2, s(\overline{\pi}_s) < 2\nu_\ell(k_s,\overline{\pi}_s)+1,$ and $\nu_s(\overline{\pi}_s) = 1$. Finally, since $\nu(\overline{\pi}) \equiv k \not\equiv k+1 \pmod 2$ and $\nu(\overline{\pi}_s) = k+1$, $\nu(\overline{\pi}_s) \not\equiv \nu(\overline{\pi}) \pmod 2$, so the map is sign-reversing. □

Lemma 4.7. *The map ϕ_s is a sign-reversing involution on the set* $\{(k,\overline{\pi}) \in \mathcal{P}_n \colon k+\nu(\overline{\pi}) \equiv 0 \pmod 2, s(\overline{\pi}) = 2\nu_\ell(k,\overline{\pi})+1, s(\overline{\pi}) \neq 2k+1,$ *and* $\nu_s(\overline{\pi}) > 1\}$.

Proof. Let $(k,\overline{\pi}) \in \mathcal{P}_n$, such that $k+\nu(\overline{\pi}) \equiv 0 \pmod 2$, $s(\overline{\pi}) = 2\nu_\ell(k,\overline{\pi})+1$, $s(\overline{\pi}) \neq 2k+1$ and $\nu_s(\overline{\pi}) > 1$. Since $s(\overline{\pi}) \neq 2k+1$, ϕ_s is well defined and we can let $(k_s,\overline{\pi}_s) = \phi_s(k,\overline{\pi})$. Since $\nu_s(\overline{\pi}) > 1$, $2\nu_\ell(k_s,\overline{\pi}_s)+1 = s(\overline{\pi}) = s(\overline{\pi}_s)$. Furthermore, because $s(\overline{\pi}) \neq 2k+1$ and $\nu_s(\overline{\pi}) > 1$, $\nu_\ell(k,\overline{\pi}) < \nu(\overline{\pi}) - 1$, we have $s(\overline{\pi}_s) = s(\overline{\pi}) = 2\nu_\ell(k,\overline{\pi})+1 < 2\nu(\overline{\pi}) - 1 = 2k_s+1$. Moreover, since $s(\overline{\pi}) = 2\nu_\ell(k,\overline{\pi})+1$, $\nu_s(\overline{\pi}_s) > 1$. Thus, $(k_s,\overline{\pi}_s) \in \mathcal{P}_n$, $k_s+\nu(\overline{\pi}_s) \equiv 0 \pmod 2$, $s(\overline{\pi}_s) = 2\nu_\ell(k_s,\overline{\pi}_s)+1$, $s(\overline{\pi}_s) \neq 2k+1,$ and $\nu_s(\overline{\pi}_s) > 1$. Finally, as explained above, the map is sign-reversing, because $\nu(\overline{\pi}_s) \not\equiv \nu(\overline{\pi}) \pmod 2$. □

Another variation of conjugation is ϕ_r, defined as $\phi_r(k,\overline{\pi}) = \mathrm{conj} \circ \phi_s \circ \mathrm{conj}(k,\overline{\pi})$, where conj is the conjugation map described in Sect. 4.1. Note that this is the same as fixing the right-most column of the boxed 2-modular diagram, and conjugating the remainder.

Example 4.8. We have $\phi_r(5,(11,11,9,9,\overline{7},7,7)) = (8,(17,17,15,\overline{9}))$.



Note that ϕ_r is well defined for pairs $(k,\overline{\pi}) \in \mathcal{P}_n$ where the last row of the boxed 2-modular diagram is longer than the last column. Equivalently, ϕ_r is well defined when $\frac{s(\overline{\pi})-1}{2} > \nu_\ell(k,\overline{\pi})$. Thus, we obtain the following lemma.

Lemma 4.9. *The map ϕ_r is a sign-reversing involution on the set* $\{(k,\overline{\pi}) \in \mathcal{P}_n \colon k+\nu(\overline{\pi}) \equiv 0 \pmod 2, s(\overline{\pi}) > 2\nu_\ell(k,\overline{\pi})+1,$ *and $\overline{\pi}$ has a part of size* $2k-1\}$.

Proof. Let $(k, \overline{\pi}) \in \mathcal{P}_n$, such that $k + \nu(\overline{\pi}) \equiv 0 \pmod 2), s(\overline{\pi}) > 2\nu_\ell(k, \overline{\pi}) + 1$, and $\overline{\pi}$ has a part of size $2k-1$. Let $(k', \overline{\pi}')$ be the conjugate of $(k, \overline{\pi})$. Then, $s(\overline{\pi}') < 2\nu_\ell(k', \overline{\pi}') + 1$ and $\nu_s(\overline{\pi}) = 1$, so we can apply Lemma 4.3. □

After applying Lemmas 4.4, 4.6, 4.7, and 4.9, we are left with four cases, all of which have $k + \nu(\overline{\pi}) \equiv 0 \pmod 2$.

- Case 1: The smallest part of $\overline{\pi}$ appears once and is equal to $2\nu_\ell(k, \overline{\pi})+1 \neq 2k + 1$.
- Case 2: The smallest part of $\overline{\pi}$ appears multiple times and is smaller than $2\nu_\ell(k, \overline{\pi}) + 1$.
- Case 3: The smallest part of $\overline{\pi}$ is greater than $2\nu_\ell(k, \overline{\pi}) + 1$ and $\overline{\pi}$ has no part of size $2k - 1$.
- Case 4: $s(\overline{\pi}) = 2\nu_\ell(k, \overline{\pi}) + 1 = 2k + 1$.

Note that applying ϕ_s to a pair $(k, \overline{\pi})$ in Case 1 reduces the number of distinct parts by one. Since the number of overpartitions of a given shape depends on the number of distinct parts, reducing the number of distinct parts by one requires us to restrict which parts of $\overline{\pi}$ may be overlined. The next two lemmas provide the details of dividing Case 1 into two halves by considering whether or not the smallest part is overlined.

Lemma 4.10. *There is a sign-reversing bijection between the pairs in Case 1 where the smallest part is not overlined and the pairs in Case 2.*

Proof. Let $(k, \overline{\pi})$ be a pair in Case 1 where the smallest part of $\overline{\pi}$ is not overlined. Since the smallest part of $\overline{\pi}$ is not overlined, ϕ_s is well defined. Thus, let $(k_s, \overline{\pi}_s) = \phi_s(k, \overline{\pi})$. Since $\nu_s(\overline{\pi}) = 1$, $s(\overline{\pi}_s) < 2\nu_\ell(k_s, \overline{\pi}_s) + 1$. Moreover, because $s(\overline{\pi}) = 2\nu_\ell(k, \overline{\pi}) + 1$, $\nu_s(\overline{\pi}_s) > 1$. Therefore, $(k_s, \overline{\pi}_s)$ is in Case 2.

Since ϕ_s is its own inverse, we can take a pair $(k_2, \overline{\pi}_2)$ in Case 2 and apply ϕ_s to find a pair in Case 1. Because $\overline{\pi}_2$ has more than one appearance of the smallest part, the last part will not be overlined, so the smallest part of $\phi_s(k_2, \overline{\pi}_2)$ will not be overlined. □

Lemma 4.11. *There is a sign-reversing bijection between the pairs in Case 1, where the smallest part is overlined, and the pairs in Case 3.*

Proof. Note that conjugation is a sign-preserving bijection between the pairs in Case 2 and the pairs in Case 3. Thus, we can remove the overline on the smallest part of $\overline{\pi}$, apply ϕ_s, and take the conjugate to obtain a sign-reversing bijection between the pairs in Case 1, where the smallest part is overlined, and the pairs in Case 3. □

Now, the only pairs left are those in Case 4. These occur exactly when $n = k + |\overline{\pi}| = 2k(k + 1)$, proving Theorem 3.1.

5. Generalizations

First, we note that all of our maps preserve the number of boxes containing a 1 in our diagrams. Furthermore, this number of 1s is exactly $k + \nu(\overline{\pi})$. Thus, if we let z count the number of 1s in the diagram, we obtain a generalization:

Theorem 5.1.

$$\sum_{n=0}^{\infty} \frac{(zq;q^2)_n z^n q^n}{(-zq;q^2)_{n+1}} = \sum_{n=0}^{\infty} (-1)^n z^{2n} q^{2n(n+1)}.$$

Theorem 1.1 is the case $z = 1$ of this generalization. Additionally, we can generalize boxed two-modular diagrams as boxed m-modular diagrams by replacing the 2s in the diagram with m's and all 1s with r's to allow parts of size $r \pmod{m}$ for some fixed $0 \leq r < m$. Then, we obtain the following generalization:

Theorem 5.2.

$$\sum_{n=0}^{\infty} \frac{(zq^r;q^m)_n z^n q^{rn}}{(-zq^r;q^m)_{n+1}} = \sum_{n=0}^{\infty} (-1)^n z^{2n} q^{n(mn+2r)},$$

Theorem 5.2 yields Theorem 1.1 when $z = 1$, $m = 2$, and $r = 1$. Analytically, Theorem 5.2 follows from the Rogers–Fine identity.

6. Further Work

This work allows us to obtain a similar combinatorial interpretation for another one of Ramanujan's identities.

Theorem 6.1. (Ramanujan) *If $f(q)$ is defined by (1.1), then for $|q| < 1$:*

$$\sum_{k=0}^{\infty} \frac{q^k (q;q^2)_k}{(-q;q)_{2k+1}} = f(q^3).$$

We can interpret Theorem 6.1 in terms of pairs $(k, \overline{\pi}) \in \mathcal{P}'_n$ where $\mathcal{P}'_n$ contains all pairs $(k, \overline{\pi})$ where $k \in \mathbb{Z}_{\geq 0}$, $\overline{\pi}$ is an overpartition into parts of size $\leq 2k+1$ where all overlined parts are odd and of size $\leq 2k-1$, and $k + |\overline{\pi}| = n$. We count each pair with weight $(-1)^{\nu(\overline{\pi})}$.

Theorem 6.2. *Let $\overline{p}'_0(n)$ (resp. $\overline{p}'_e(n)$) be the number of pairs $(k, \overline{\pi}) \in \mathcal{P}'_n$ where $\overline{\pi}$ has an odd number (resp. even number) of parts. Then, for $n \geq 0$:*

$$\overline{p}'_e(n) - \overline{p}'_o(n) = \begin{cases} (-1)^k, & \textit{if } n = \frac{3k(k+1)}{2}, \\ 0, & \textit{otherwise}. \end{cases}$$

Due to the presence of even parts in the partition, a bijective proof of Theorem 6.2 appears to be more difficult than the proof of Theorem 3.1 and would be a welcome contribution. We suspect that the involution necessary for a bijective proof of Theorem 6.2 will fix pairs $(k, (2k + 2k - 1 + \cdots + (k+1)))$.

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Hannah E. Burson
Department of Mathematics
University of Illinois
1409 W. Green St.
Urbana
IL 61801
USA
e-mail: hburso2@illinois.edu

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Quasipolynomials and Maximal Coefficients of Gaussian Polynomials

Dedicated to George Andrews on the occasion of his 80th birthday

Angelica Castillo, Stephanie Flores, Anabel Hernandez, Brandt Kronholm, Acadia Larsen and Arturo Martinez

Abstract. We establish an algorithm for producing formulas for $p(n, m, N)$, the function enumerating partitions of n into at most m parts with no part larger than N. Recent combinatorial results of H. Hahn et al. on a collection of partition identities for $p(n, 3, N)$ are considered. We offer direct proofs of these identities and then place them in a larger context of the unimodality of Gaussian polynomials $\begin{bmatrix} N+m \\ m \end{bmatrix}$ whose coefficients are precisely $p(n, m, N)$. We give complete characterizations of the maximal coefficients of $\begin{bmatrix} M \\ 3 \end{bmatrix}$ and $\begin{bmatrix} M \\ 4 \end{bmatrix}$. Furthermore, we prove a general theorem on the period of quasipolynomials for central/maximal coefficients of Gaussian polynomials. We place some of Hahn's identities into the context of some known results on differences of partitions into at most m parts, $p(n, m)$, which we then extend to $p(n, m, N)$.

Mathematics Subject Classification. Primary 11P81; Secondary 05A15.

Keywords. Integer partition, Gaussian polynomial, Quasipolynomial, q-Series.

1. Prologue

There is no question that George Andrews knows his mathematics. He also knows his mathematicians, his students in particular. The origins of this paper begin with the following message sent by George Andrews to the fourth author.

> *It occurred to me that you might think about applying your methods to the coefficients in the Gaussian polynomial (i.e. p(j,k,n)). Given the success of Hahn et al., this seems like a likely venture.*
>
> *Best wishes, George*
> *email, 6/7/2017*

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As happy as we are that Professor Andrews knows us, we are ever more thankful that we know him.

2. Introduction

Along with the celebrated Hardy–Ramanujan [19] asymptotic formula for the unrestricted partition function, given by

$$p(n) \sim \frac{\mathrm{e}^{\pi\sqrt{\frac{2n}{3}}}}{4n\sqrt{3}}$$

as $n \to \infty$, formulas for partition functions remain at the leading edge of research. In this paper, we introduce a very natural way of producing formulas for $p(n, m, N)$, the function that enumerates partitions of n into at most m parts with no part larger than N. For example, following the methods in this paper, the formula for $p(12k+2, 4, 12j+5)$ obtained after a few straightforward computations:

$$\begin{aligned} p(12k+2, 4, 12j+5) = {} & 2\tbinom{k+3}{3} + 39\tbinom{k+2}{3} + 30\tbinom{k+1}{3} + \tbinom{k}{3} - 41\tbinom{k+2-j}{3} \\ & - 194\tbinom{k+1-j}{3} - 53\tbinom{k-j}{3} + 2\tbinom{k+2-2j}{3} + 160\tbinom{k+1-2j}{3} \\ & + 250\tbinom{k-2j}{3} + 20\tbinom{k-1-2j}{3} - 16\tbinom{k+1-3j}{3} - 173\tbinom{k-3j}{3} \\ & - 98\tbinom{k-1-3j}{3} - \tbinom{k-2-3j}{3} + 15\tbinom{k-4j}{3} \\ & + 48\tbinom{k-1-4j}{3} + 9\tbinom{k-2-4j}{3}. \end{aligned}$$

With a computer algebra program, one might compute $p(420k+297, 7, 420j)$, which begins with $248893190\tbinom{k+6}{6} + 40291579602\tbinom{k+5}{6} + \cdots$ and concludes approximately 50 terms later with $\cdots - 8212122234\tbinom{k+1-7j}{6} - 1805085\tbinom{k-7j}{6}$.

The techniques employed in this paper lead us to a complete characterization of the maximal coefficients of all Gaussian polynomials of the form $\left[{N+3 \atop 3}\right]$ in Sect. 4. This characterization is complete in terms of quantity and quality. For example, we prove that for $\ell \geq 0$, there are exactly three largest coefficients of $\left[{4\ell+1 \atop 3}\right]$, and they are $p(6\ell-4, 3, 4\ell-2)$, $p(6\ell-3, 3, 4\ell-2)$ and $p(6\ell-2, 3, 4\ell-2)$ and that they are equal to $2\ell^2$. We go on to characterize the maximal coefficients of $\left[{N+4 \atop 4}\right]$ and then establish a general theorem on the period of quasipolynomials for maximal coefficients of Gaussian polynomials in Sect. 5. Section 6 details a general result on the difference $p(n, m, N) - p(n-1, m, N-1)$.

Despite the fact that the generating function for $p(n, m, N)$ is the well-known Gaussian polynomial denoted by $\left[{N+m \atop m}\right]$, the authors could find very little information on formulas for $p(n, m, N)$. However, a Hardy–Ramanujan formula can be found in [1] and formulas based on Sylvester's "waves" can be found in [24]. Further asymptotic behavior of the coefficients of Gaussian polynomials can be found in [26]. Although Sylvester [27] was the first to prove the unimodality of Gaussian polynomials, we make special mention of Kathleen O'Hara's celebrated constructive proof of unimodality [22]. Recent results on strict unimodality may be found in [8] and [23].

In this paper, we will also encounter the function $p(n, m)$ which enumerates partitions of n into at most m parts. Unlike $p(n, m, N)$, formulas for $p(n, m)$ are plentiful. Many formulas for specific small values of m are catalogued in [12, 13, 15, 21]. Andrews [3] notes several varieties of formulas for $p(n, 4)$. The simplest among these may be the 1896 result of Glösel [14]:

$$p(n,4) = \left\lfloor \left\lfloor \frac{n+4}{2} \right\rfloor^2 \left(3\left\lfloor \frac{n+9}{2} \right\rfloor - \left\lfloor \frac{n+10}{2} \right\rfloor \right) \frac{1}{36} \right\rceil, \tag{2.1}$$

where $\lfloor \cdot \rceil$ is the nearest integer function. In [25], one will find a computer algebra (Maple) package that "completely automatically discovers, and then proves, explicit expressions ... for $p(n, m)$ for any desired m" in similar fashion to (2.1).

The methods in this paper can also be applied to compute formulas for $p(n, m)$, although the resulting formulas are of a different nature than (2.1) and those in the previously cited references. Our methods come from a novel manipulation of generating functions and q-series initially informed by the arithmetic of E. Ehrhart's polyhedral combinatorics [9–11]. In short, we create a piece-wise polynomial, otherwise known as a quasipolynomial, and our results follow after a bit of arithmetic.

Definition 2.1. A function $f(n)$ is a *quasipolynomial* if there exist d polynomials $f_0(n), \ldots, f_{d-1}(n)$, such that

$$f(n) = \begin{cases} f_0(n), & \text{if } n \equiv 0 \pmod{d}, \\ f_1(n), & \text{if } n \equiv 1 \pmod{d}, \\ \vdots & \vdots \\ f_{d-1}(n), & \text{if } n \equiv d-1 \pmod{d}, \end{cases}$$

for all $n \in \mathbb{Z}$. The polynomials f_i are called the *constituents* of f and the number of them, d, is the *period* of f.

The formulas which we present for $p(n, 3, N)$ and $p(n, 3)$ will be displayed following the format of Definition 2.1.

We will express these formulas in a binomial basis $\binom{a}{b}$ and/or a monomial basis $\alpha k^n + \beta k^{n-1} + \cdots + \omega k + z$, as appropriate, because there is geometric and combinatorial meaning inferred from each format. The accompanying Ehrhart Theory and polyhedral geometry will not be considered beyond just a few scattered comments in this paper. However, the geometric implications could be of significant interest and may be considered in future research.

Breuer's paper [6] is a very nice and comprehensive introduction to Ehrhart's ideas and [7] features an exploration of the deeper geometric, algebraic, and combinatorial mathematics related to partitions.

3. Background Information

The generating functions for the partition functions $p(n,m)$ and $p(n,m,N)$ are well known:

$$\sum_{n=0}^{\infty} p(n,m)q^n = \frac{1}{(1-q)(1-q^2)\cdots(1-q^m)} = \frac{1}{(q;q)_m}, \tag{3.1}$$

$$\sum_{n=0}^{mN} p(n,m,N)q^n = \frac{(q;q)_{N+m}}{(q;q)_m(q;q)_N} = \frac{(q^{N+1};q)_m}{(q;q)_m} = \begin{bmatrix} N+m \\ m \end{bmatrix}. \tag{3.2}$$

For $n < 0$, we agree that $p(n,m) = p(n,m,N) = 0$ and whenever $n > Nm$, $p(n,m,N) = 0$.

Gaussian polynomials $\left[\begin{smallmatrix} N+m \\ m \end{smallmatrix}\right]$ are q-analogues of binomial coefficients. A proof that $\left[\begin{smallmatrix} N+m \\ m \end{smallmatrix}\right]$ is a polynomial can be found in Chapter 3 of [2]. Also, in Chapter 14 of [2], one can find a small table of coefficients for a selection of Gaussian polynomials.

We require the following definition.

Definition 3.1. Let $\mathrm{lcm}(m)$ denote the least common multiple of the numbers $1, 2, 3, \ldots, m$.

For example, $\mathrm{lcm}(3) = 6$, $\mathrm{lcm}(4) = 12$, $\mathrm{lcm}(5) = 60$, etc.

3.1. Quasipolynomials for $p(n,m)$

Because it is somewhat simpler, we begin with $p(n,m)$. Our procedure is to recast the generating function for $p(n,m)$ as the product of a polynomial and a generating function for binomial coefficients:

$$\begin{aligned}
\sum_{n=0}^{\infty} p(n,m)q^n = \frac{1}{(q;q)_m} &= \frac{1}{(q;q)_m} \times \frac{\left(\frac{(1-q^{\mathrm{lcm}(m)})^m}{(q;q)_m}\right)}{\left(\frac{(1-q^{\mathrm{lcm}(m)})^m}{(q;q)_m}\right)} \\
&= \frac{\prod_{j=1}^{m} \sum_{i=0}^{\frac{\mathrm{lcm}(m)-j}{j}} q^{ij}}{(1-q^{\mathrm{lcm}(m)})^m} \\
&= \prod_{j=1}^{m} \sum_{i=0}^{\frac{\mathrm{lcm}(m)-j}{j}} q^{ij} \times \sum_{k=0}^{\infty} \binom{k+m-1}{m-1} q^{\mathrm{lcm}(m)k} \\
&= E_m(q) \times \sum_{k=0}^{\infty} \binom{k+m-1}{m-1} q^{\mathrm{lcm}(m)k}.
\end{aligned} \tag{3.3}$$

The penultimate equality in (3.3) comes from the generating function for binomial coefficients:

$$\frac{1}{(1-q)^b} = \sum_{a \ge 0} \binom{a+b-1}{b-1} q^a,$$

and the final equality is simply introducing some notation:

$$E_m(q) = \prod_{j=1}^{m} \sum_{i=0}^{\frac{\mathrm{lcm}(m)-j}{j}} q^{ij}. \tag{3.4}$$

To establish a quasipolynomial and obtain formulas for $p(n,m)$, one fixes m and then proceeds to multiply and collect like terms in the far right side of (3.3). With the exponent on q being $\mathrm{lcm}(m)k$ in the far right side of (3.3), it is natural to set $n = \mathrm{lcm}(m)k + r$ for $0 \le r < \mathrm{lcm}(m)$ and expect a quasipolynomial of period $\mathrm{lcm}(m)$.

As a worked example and because we will use this information later in Sect. 6.2, we compute the quasipolynomial for $p(n,3)$:

$$\sum_{n=0}^{\infty} p(n,3)q^n = \frac{1}{(q;q)_3} = E_3(q) \times \sum_{k=0} \binom{k+2}{2} q^{6k}.$$

Noting that

$$E_3(q)=1+q+2q^2+3q^3+4q^4+5q^5+4q^6+5q^7+4q^8+3q^9+2q^{10}+q^{11}+q^{12},$$

we compute for instance, the formula for $p(6k+1,3)$:

$$\begin{aligned}\sum_{k=0}^{\infty} p(6k+1,3)q^{6k+1} &= \left(q+5q^7\right) \times \sum_{k=0}^{\infty} \binom{k+2}{2} q^{6k}\\ &= \sum_{k=0}^{\infty} \left(\binom{k+2}{2} + 5\binom{k+1}{2}\right) q^{6k+1}.\end{aligned} \tag{3.5}$$

Hence

$$p(6k+1,3) = \binom{k+2}{2} + 5\binom{k+1}{2} = 3k^2+4k+1.$$

Multiplying and collecting like terms five more times give us the complete quasipolynomial for $p(n,3)$ displayed below:

$$p(n,3) = \begin{cases} p(6k,3) & = 1\binom{k+2}{2} + 4\binom{k+1}{2} + 1\binom{k}{2} = 3k^2+3k+1,\\ p(6k+1,3) & = 1\binom{k+2}{2} + 5\binom{k+1}{2} = 3k^2+4k+1,\\ p(6k+2,3) & = 2\binom{k+2}{2} + 4\binom{k+1}{2} = 3k^2+5k+2,\\ p(6k+3,3) & = 3\binom{k+2}{2} + 3\binom{k+1}{2} = 3k^2+6k+3,\\ p(6k+4,3) & = 4\binom{k+2}{2} + 2\binom{k+1}{2} = 3k^2+7k+4,\\ p(6k+5,3) & = 5\binom{k+2}{2} + 1\binom{k+1}{2} = 3k^2+8k+5. \end{cases} \tag{3.6}$$

3.2. Quasipolynomials for $p(n,m,N)$

Following the notation $E_m(q)$ in (3.4), for Gaussian polynomials, we set

$$(q^{N+1};q)_m = G_{m,N}(q) \tag{3.7}$$

and write

$$\begin{bmatrix} N+m \\ m \end{bmatrix} = \sum_{n=0}^{mN} p(n,m,N)q^n$$
$$= G_{m,N}(q) \times E_m(q) \times \sum_{k=0}^{\infty} \binom{k+m-1}{m-1} q^{\mathrm{lcm}(m)k}. \qquad (3.8)$$

Despite the fact that we are expressing the Gaussian polynomial as the product of a polynomial and a power series, the procedure for computing a quasipolynomial for $p(n,m,N)$ is the same as it was for $p(n,m)$. After setting

$$n = \mathrm{lcm}(m)k + r$$

for $0 \le r < \mathrm{lcm}(m)$ and

$$N = \mathrm{lcm}(m)j + t$$

for $0 \le t < \mathrm{lcm}(m)$, we multiply and collect like terms in the right side of (3.8). Because there are $\mathrm{lcm}(m)$ choices for N in $G_{m,N}$ and there are $\mathrm{lcm}(m)$ constituents part and parcel with $E_m(q)$, the natural quasipolynomial in two variables for $p(n,m,N)$ consists of $\mathrm{lcm}(m)^2$ constituents.

We compute the 36 constituents of the quasipolynomial for $p(n,3,N)$:

$$\begin{bmatrix} N+3 \\ 3 \end{bmatrix} = \sum_{n=0}^{3N} p(n,3,N)q^n = \frac{(q;q)_{N+3}}{(q;q)_3 (q;q)_N}$$
$$= G_{3,N}(q) \times E_3(q) \times \sum_{k=0}^{\infty} \binom{k+2}{2} q^{6k}. \qquad (3.9)$$

We expand $G_{3,N}(q) \times E_3(q)$ below:

$$\begin{aligned} &G_{3,N}(q) \times E_3(q) \\ &\quad = 1 + q + 2q^2 + 3q^3 + 4q^4 + 5q^5 + 4q^6 + 5q^7 + 4q^8 + 3q^9 + 2q^{10} + q^{11} + q^{12} \\ &\qquad - q^{1+N} - 2q^{2+N} - 4q^{3+N} - 6q^{4+N} - 9q^{5+N} - 12q^{6+N} - 13q^{7+N} \\ &\qquad - 14q^{8+N} - 13q^{9+N} - 12q^{10+N} - 9q^{11+N} - 6q^{12+N} - 4q^{13+N} - 2q^{14+N} \\ &\qquad - q^{15+N} + q^{3+2N} + 2q^{4+2N} + 4q^{5+2N} + 6q^{6+2N} + 9q^{7+2N} + 12q^{8+2N} \\ &\qquad + 13q^{9+2N} + 14q^{10+2N} + 13q^{11+2N} + 12q^{12+2N} + 9q^{13+2N} + 6q^{14+2N} \\ &\qquad + 4q^{15+2N} + 2q^{16+2N} + q^{17+2N} - q^{6+3N} - q^{7+3N} - 2q^{8+3N} - 3q^{9+3N} \\ &\qquad - 4q^{10+3N} - 5q^{11+3N} - 4q^{12+3N} - 5q^{13+3N} - 4q^{14+3N} - 3q^{15+3N} \\ &\qquad - 2q^{16+3N} - q^{17+3N} - q^{18+3N}. \end{aligned} \qquad (3.10)$$

Depending on N modulo 6 in (3.10), we establish the quasipolynomial for $p(n,3,N)$. The 36 constituents for this quasipolynomial appear in Appendix A and are referenced from there throughout the remainder of the paper.

We display one constituent followed by a definition as fuel for discussion.

Example 3.2. Replacing N with $6j+4$ in (3.10), expanding $G_{3,6j+4}(q)\times E_3(q)$, and selecting exponents of the form $n = 6k+3$ as one multiplies and collects like terms, we obtain the following constituent:

$$\begin{aligned} p(6k+3,3,6j+4) &= 3\binom{k+2}{2} + 3\binom{k+1}{2} - 9\binom{k+1-j}{2} - 9\binom{k-j}{2} + 9\binom{k-2j}{2} \\ &\quad + 9\binom{k-1-2j}{2} - 3\binom{k-1-3j}{2} - 3\binom{k-2-3j}{2}. \end{aligned} \tag{3.11}$$

Regarding the constituents of $p(n,m,N)$, we adhere to a strict interpretation of binomial terms.

Definition 3.3. Let a and b be natural numbers.

- Whenever $a < b$, then $\binom{a}{b} = 0$.
- Whenever $a \geq b$, we allow the usual translation to the monomial basis: $\binom{a}{b} = \frac{a!}{b!(a-b)!}$.

The immediate reason for this comes from our q-series arithmetic where we recast the standard generating function for $p(n,m,N)$ from an index of n to a new index of k for $k \geq 0$ and the fact that N is non-negative. Another reason for this interpretation comes from the polyhedral geometry in which a term like $9\binom{k-2j}{2}$ in (3.11) indicates a certain "tiling" of a slice of a certain partition cone. See [6,7] for more information. There are partition theoretic interpretations for some negative variables when it comes to theorems of combinatorial reciprocity [5].

4. A Characterization of the Maximal Coefficients of $\begin{bmatrix} N+3 \\ 3 \end{bmatrix}$

4.1. Motivation from Recent Results

We are motivated by the following results of Hahn et al. in [17] and [18].

Theorem 4.1 ([17]). *For any integer $\ell \geq 1$, one has:*

$$p(6\ell-3,3,4\ell-2) - p(6\ell-4,3,4\ell-2) = 0, \tag{4.1}$$

$$p(6\ell,3,4\ell) - p(6\ell-1,3,4\ell) = 1, \tag{4.2}$$

$$p(6\ell-3,3,4\ell-1) - p(6\ell-4,3,4\ell-1) = 1, \tag{4.3}$$

$$p(6\ell,3,4\ell+1) - p(6\ell-1,3,4\ell+1) = 1. \tag{4.4}$$

Theorem 4.2 ([18]). *For any integer $\ell \geq 1$, one has:*

$$p(6\ell,3,4\ell) - p(6\ell-3,3,4\ell-1) = \ell+1, \tag{4.5}$$

$$p(6\ell-1,3,4\ell) - p(6\ell-4,3,4\ell-1) = \ell+1, \tag{4.6}$$

$$p(6\ell+3,3,4\ell+2) - p(6\ell,3,4\ell+1) = \ell+1, \tag{4.7}$$

$$p(6\ell+2,3,4\ell+2) - p(6\ell-1,3,4\ell+1) = \ell+2. \tag{4.8}$$

Theorem 4.1 was first established in [17] while working on detection of subgroups of GL_n by representations and are part of Langlands' beyond endoscopy proposal [16]. Combinatorial proofs of both theorems appear in [18].

Notice that both Theorem 4.1 and Theorem 4.2 deal with coefficients near or at the middle of Gaussian polynomials $\begin{bmatrix} N+3 \\ 3 \end{bmatrix}$. It is possible to directly verify each line of Theorem 4.1 simply by referring to the relevant constituents from the quasipolynomial for $p(n,3,N)$ in Appendix A. Instead, we prove a result that places Theorem 4.1 in the context of the unimodality of all Gaussian polynomials of the form $\begin{bmatrix} N+3 \\ 3 \end{bmatrix}$. The first part of the characterization is established by Theorem 4.6 which, depending on N, describes how many maximal coefficients for $\begin{bmatrix} N+3 \\ 3 \end{bmatrix}$ which one should expect. The second part of this characterization is found in Corollary 4.8 in which we create a quasipolynomial of period 4 for the maximal coefficients of $\begin{bmatrix} N+3 \\ 3 \end{bmatrix}$. We state analogous results for $\begin{bmatrix} N+4 \\ 4 \end{bmatrix}$ in Sect. 4.4.

4.2. Enumerating the Maximal Coefficients of $\begin{bmatrix} N+3 \\ 3 \end{bmatrix}$

We require the following definitions before we can prove our results.

Definition 4.3. A polynomial

$$P(q) = a_0 + a_1 q + a_2 q^2 + \cdots + a_d q^d$$

of degree d is said to be reciprocal if for each i, one has $a_i = a_{d-i}$.

Definition 4.4. A polynomial

$$P(q) = a_0 + a_1 q + a_2 q^2 + \cdots + a_d q^d$$

of degree d is called unimodal if there exists m, such that

$$a_0 \le a_1 \le a_2 \le \cdots \le a_m \ge a_{m+1} \ge a_{m+2} \ge \cdots \ge a_d.$$

Because the coefficients of Gaussian polynomials are positive, unimodal and reciprocal, it follows that the value of the central coefficient(s) of a Gaussian polynomial will be greater than or equal to all the other coefficients.

Remark 4.5. Whenever $\deg \begin{bmatrix} N+m \\ m \end{bmatrix}$ is odd, there will be a pair of central coefficients. Otherwise, there will be a single central coefficient. As such, $p\left(\lfloor \frac{mN}{2} \rceil, m, N\right)$ is one of the central and hence one of the maximal coefficient(s) of $\begin{bmatrix} N+m \\ m \end{bmatrix}$, where $\lfloor \cdot \rceil$ is the nearest integer function.

Theorem 4.6. *There are at most four but never exactly two maximal coefficients for any Gaussian polynomial of the form* $\begin{bmatrix} N+3 \\ 3 \end{bmatrix}$.

Proof. We will prove Theorem 4.6 in four cases. We will show that for $k \ge 0$,

Case 1. There is exactly one largest coefficient for Gaussian polynomials of the form $\begin{bmatrix} 4k+3 \\ 3 \end{bmatrix}$.

Case 2. There are exactly four largest coefficients for Gaussian polynomials of the form $\begin{bmatrix} 4k+4 \\ 3 \end{bmatrix}$.

Case 3. There are exactly three largest coefficients for Gaussian polynomials of the form $\begin{bmatrix} 4k+5 \\ 3 \end{bmatrix}$.

Case 4. There are exactly four largest coefficients for Gaussian polynomials of the form $\begin{bmatrix} 4k+6 \\ 3 \end{bmatrix}$.

Case 1 is proved entirely from (4.2) of Theorem 4.1. We will make use of (4.1), from Theorem 4.1, and the quasipolynomial for $p(n,3,N)$ in Appendix

A to prove Case 3. The remaining Cases 2 and 4 are verified in the same way as Case 3 and are left to the reader.

We now treat Case 1. With $\deg \begin{bmatrix} 4k+3 \\ 3 \end{bmatrix} = 12k$, there are exactly $12k+1$ terms in these Gaussian polynomials, the central of which is $p(6k,3,4k)q^{6k}$. For $k \geq 0$, we will show:

$$p(6k-1,3,4k) < p(6k,3,4k) > p(6k+1,3,4k).$$

We note that Case 1 is satisfied in the instance of $k=0$ where we have $\begin{bmatrix} 3 \\ 3 \end{bmatrix} = 1$. Now, by (4.2) of Theorem 4.1 and the reciprocity of Gaussian polynomials, we immediately obtain:

$$\begin{aligned} p(6\ell,3,4\ell) - p(6\ell-1,3,4\ell) &= 1, \\ p(6\ell,3,4\ell) - p(6\ell+1,3,4\ell) &= 1. \end{aligned} \tag{4.9}$$

Replacing ℓ with k in (4.9), we observe that

$$p(6k-1,3,4k) < p(6k,3,4k) > p(6k+1,3,4k).$$

Hence, Case 1 is proved.

We prove Case 3 by showing

$$\begin{aligned} p(6k+1,3,4k+2) &< p(6k+2,3,4k+2) = p(6k+3,3,4k+2) \\ &= p(6k+4,3,4k+2) > p(6k+5,3,4k+2). \end{aligned}$$

Since $\deg \begin{bmatrix} 4k+5 \\ 3 \end{bmatrix} = 12k+6$, we find that the central coefficient of $\begin{bmatrix} 4k+5 \\ 3 \end{bmatrix}$ is:

$$p(6k+3,3,4k+2). \tag{4.10}$$

Because Gaussian polynomials are reciprocal, we have:

$$p(6k+2,3,4k+2) = p(6k+4,3,4k+2). \tag{4.11}$$

Setting $k=\ell-1$ in both (4.10) and (4.11) and taking the result of (4.1) from Theorem 4.1, we obtain:

$$p(6\ell-4,3,4\ell-2) = p(6\ell-3,3,4\ell-2) = p(6\ell-2,3,4\ell-2),$$

and hence

$$p(6k+2,3,4k+2) = p(6k+3,3,4k+2) = p(6k+4,3,4k+2). \tag{4.12}$$

We now show that these three coefficients are maximal by proving:

$$p(6k+4,3,4k+2) - p(6k+5,3,4k+2) = 1, \tag{4.13}$$

which, because of the reciprocity of the coefficients of Gaussian polynomials, will simultaneously show:

$$p(6k+2,3,4k+2) - p(6k+1,3,4k+2) = 1. \tag{4.14}$$

We prove (4.13), hence (4.14), by making use of the quasipolynomial for $p(n,3,N)$ in Appendix A. We have three cases to consider in (4.13): $4k+2 = 6j, 6j+2$ and $6j+4$. For the case $4k+2=6j$, we set $k=3f+1$, so that we may rewrite (4.13) for $f \geq 0$ as:

$$p(6(3f+1)+4,3,6(2f+1)) - p(6(3f+1)+5,3,6(2f+1)). \tag{4.15}$$

Turning to (A.5) and (A.6), we compute (4.15):

$$\begin{aligned}
&p(6(3f+1)+4,3,6(2f+1)) - p(6(3f+1)+5,3,6(2f+1))\\
&\quad = 4\tbinom{3f+3}{2} + 2\tbinom{3f+2}{2} - 6\tbinom{f+2}{2} - 12\tbinom{f+1}{2} + 2\tbinom{-f+1}{2} + 14\tbinom{-f}{2}\\
&\qquad + \tbinom{-f-1}{2} - 4\tbinom{-3f-1}{2} - 2\tbinom{-3f-2}{2} - 5\tbinom{3f+3}{2} - \tbinom{3f+2}{2} + 9\tbinom{f+2}{2}\\
&\qquad + 9\tbinom{f+1}{2} - 4\tbinom{-f+1}{2} - 13\tbinom{-f}{2} - \tbinom{-f-1}{2} + 5\tbinom{-3f-1}{2} - \tbinom{-3f-2}{2}\\
&\quad = -\tbinom{3f+3}{2} + \tbinom{3f+2}{2} + 3\tbinom{f+2}{2} - 3\tbinom{f+1}{2} - 0\\
&\quad = 1. \qquad (4.16)
\end{aligned}$$

For the case $4k+2 = 6j+2$, we set $k = 3f$ and rewrite (4.13) with the relevant constituents (A.17) and (A.18), so that for $f \geq 0$, we compute

$$p(6(3f)+4,3,6(2f)+2) - p(6(3f)+5,3,6(2f)+2) = 1. \qquad (4.17)$$

For the case $4k+2 = 6j+4$, we set $k = 3f+2$ in (A.29) and (A.30) and following (4.17), we compute

$$p(6(3f+2)+4,3,6(2f+1)+4) - p(6(3f+2)+5,3,6(2f+1)+4) = 1. \qquad (4.18)$$

Hence, (4.16), (4.17) and (4.18) establish (4.13) and, simultaneously, by reciprocity of coefficients, (4.14). Hence, combining (4.12), (4.13) and (4.14), we obtain:

$$\begin{aligned}
p(6\ell-5,3,4\ell-2) < p(6\ell-4,3,4\ell-2) &= p(6\ell-3,3,4\ell-2)\\
&= p(6\ell-2,3,4\ell-2) > p(6\ell-1,3,4\ell-2).
\end{aligned}$$

Thus, Case 3 is settled.

Case 2 and Case 4 are similar to Case 3 and can be verified by the reader to complete the proof of Theorem 4.6. □

4.3. Formulas for the Maximal Coefficients of $\left[{N+3 \atop 3}\right]$

In this section, we establish a quasipolynomial for the maximal coefficients of Gaussian polynomials $\left[{N+3 \atop 3}\right]$.

Definition 4.7. For non-negative integers m and N, let $M_m(N)$ denote the function whose value is equal to the maximal coefficient(s) of the Gaussian polynomial $\left[{N+m \atop m}\right]$. Note that for all N, $M_0(N) = 1$.

This is a corollary to Theorem 4.6.

Corollary 4.8. *For $\ell \geq 0$, the quasipolynomial for $M_3(N)$ has period 4 and is given by:*

$$M_3(4\ell-2) = \left\{\begin{array}{l} p(6\ell-4,3,4\ell-2)\\ p(6\ell-3,3,4\ell-2)\\ p(6\ell-2,3,4\ell-2)\end{array}\right\} = 2\binom{\ell+1}{2} + 2\binom{\ell}{2} = 2\ell^2, \qquad (4.19)$$

$$M_3(4\ell-1) = \left\{\begin{array}{l} p(6\ell-3,3,4\ell-1)\\ p(6\ell-2,3,4\ell-1)\\ p(6\ell-1,3,4\ell-1)\\ p(6\ell,3,4\ell-1)\end{array}\right\} = 3\binom{\ell+1}{2} + \binom{\ell}{2} = 2\ell^2+\ell, \qquad (4.20)$$

$$M_3(4\ell) = p(6\ell,3,4\ell) = \binom{\ell+2}{2} + 2\binom{\ell+1}{2} + \binom{\ell}{2} = 2\ell^2+2\ell+1, \tag{4.21}$$

$$M_3(4\ell+1) = \left\{\begin{matrix} p(6\ell,3,4\ell+1) \\ p(6\ell+1,3,4\ell+1) \\ p(6\ell+2,3,4\ell+1) \\ p(6\ell+3,3,4\ell+1) \end{matrix}\right\} = \binom{\ell+2}{2} + 3\binom{\ell+1}{2} = 2\ell^2+3\ell+1. \tag{4.22}$$

Corollary 4.8 allows for a straightforward proof of Theorem 4.2 which is completed later in Sect. 6.1.

We prove only (4.21) from Corollary 4.8. With the multiplicity of the maximal coefficients established by Theorem 4.6, the remainder of Corollary 4.8 is proved simply by referring to the quasipolynomial and is left to the reader.

Proof. For (4.21), we have the following three cases: $4\ell = 6j, 6j+2$ and $6j+4$. For the case $4\ell = 6j$, we set $\ell = 3f$, so that we may rewrite $p(6\ell,3,4\ell)$ as $p(6(3f),3,6(2f))$. We now refer to (A.1) in the quasipolynomial for $p(n,3,N)$ and compute:

$$\begin{aligned} p(6(3f),3,6(2f)) &= \tbinom{3f+2}{2} + 4\tbinom{3f+1}{2} + \tbinom{3f}{2} - 12\tbinom{f+1}{2} - 6\tbinom{f}{2} + 6\tbinom{-f+1}{2} \\ &\quad + 12\tbinom{-f}{2} - \tbinom{-3f+1}{2} - 4\tbinom{-3f}{2} - \tbinom{-3f-1}{2} \\ &= 18f^2+6f+1. \end{aligned}$$

Since $\ell = 3f$, we replace f with $\ell/3$ to arrive at:

$$p(6\ell,3,4\ell) = 2\ell^2+2\ell+1. \tag{4.23}$$

For the case $4\ell = 6j+2$, we set $\ell = 3f+2$, so that we may rewrite $p(6\ell,3,4\ell)$ as $p(6(3f+2),3,6(2f+1)+2)$ and working from the constituent for $p(6k,3,6j+2)$, we compute:

$$\begin{aligned} &p(6(3f+2),3,6(2f+1)+2) \\ &\quad = \tbinom{3f+4}{2} + 4\tbinom{3f+3}{2} + \tbinom{3f+2}{2} - 6\tbinom{f+2}{2} - 12\tbinom{f+1}{2} \\ &\qquad + 12\tbinom{-f}{2} + 6\tbinom{-f-1}{2} - \tbinom{-3f-2}{2} - 4\tbinom{-3f-3}{2} - \tbinom{-3f-4}{2} \\ &\quad = 18f^2+30f+13. \end{aligned}$$

Since $\ell = 3f+2$, we replace f with $(\ell-2)/3$ to arrive at:

$$p(6\ell,3,4\ell) = 2\ell^2+2\ell+1. \tag{4.24}$$

We finish the proof of (4.21) by considering the case $4\ell = 6j+4$. We set $\ell = 3f+1$, so that we may rewrite $p(6\ell,3,4\ell)$ as $p(6(3f+1),3,6(2f)+4)$ and working from the constituent for $p(6k,3,6j+4)$, we compute:

$$\begin{aligned} &p(6(3f+1),3,6(2f)+4) \\ &\quad = \tbinom{3f+3}{2} + 4\tbinom{3f+2}{2} + \tbinom{3f+1}{2} - 2\tbinom{f+2}{2} - 14\tbinom{f+1}{2} - 2\tbinom{f}{2} \end{aligned}$$

$$+ 2\binom{-f+1}{2} + 14\binom{-f}{2} + 2\binom{-f-1}{2} - \binom{-3f}{2} - 4\binom{-3f-1}{2} - \binom{-3f-2}{2}$$
$$= 18f^2 + 18f + 5.$$

Since $\ell = 3f + 1$, we replace f with $(\ell - 1)/3$ to arrive at:

$$p(6\ell, 3, 4\ell) = 2\ell^2 + 2\ell + 1. \tag{4.25}$$

Since (4.23), (4.24), and (4.25) are equal, it follows that for $\ell \geq 0$, the maximal coefficient of the Gaussian polynomial $\begin{bmatrix} 4\ell+3 \\ 3 \end{bmatrix}$ is $p(6\ell, 3, 4\ell)$ and is equal to $2\ell^2 + 2\ell + 1$ which proves (4.21) of Corollary 4.8. □

As an after-the-fact observation, we note that the quasipolynomial for $M_3(N)$ goes hand-in-hand with the following generating function

$$\sum_{N=0}^{\infty} M_3(N-2)q^N = \frac{q^2(1+q^3)}{(1-q)(1-q^2)(1-q^4)}. \tag{4.26}$$

After applying the arithmetic from Sect. 3.1, (4.26) yields the same constituents as Corollary 4.8 but without the full context of Theorem 4.6. Nevertheless, this would appear to be a very fruitful area of further inquiry.

4.4. A Characterization of the Maximal Coefficients of $\begin{bmatrix} N+4 \\ 4 \end{bmatrix}$

We offer the following results on maximal coefficients of $\begin{bmatrix} N+4 \\ 4 \end{bmatrix}$. Proofs are omitted as they are done similarly to Theorem 4.6 and Corollary 4.8.

Theorem 4.9. *The maximal coefficient of Gaussian polynomials of the form* $\begin{bmatrix} N+4 \\ 4 \end{bmatrix}$ *is unique except when* $N = 1$ *in which case there are exactly 5 coefficients, each of which is 1.*

Corollary 4.10. *For* $\ell \geq 0$*, the quasipolynomial for* $M_4(N)$ *has period 6 and is given by:*

$$\begin{aligned}
M_4(6\ell) = p(12\ell, 4, 6\ell) &= \binom{\ell+3}{3} + 14\binom{\ell+2}{3} + 20\binom{\ell+1}{3} + \binom{\ell}{3} \\
&= 6\ell^3 + \frac{15\ell^2}{2} + \frac{7\ell}{2} + 1, \\
M_4(6\ell+1) = p(12\ell+2, 4, 6\ell+1) &= \binom{\ell+3}{3} + 20\binom{\ell+2}{3} + 14\binom{\ell+1}{3} + \binom{\ell}{3} \\
&= 6\ell^3 + \frac{21\ell^2}{2} + \frac{13\ell}{2} + 1, \\
M_4(6\ell+2) = p(12\ell+4, 4, 6\ell+2) &= 3\binom{\ell+3}{3} + 21\binom{\ell+2}{3} + 12\binom{\ell+1}{3} \\
&= 6\ell^3 + \frac{27\ell^2}{2} + \frac{21\ell}{2} + 3, \\
M_4(6\ell+3) = p(12\ell+6, 4, 6\ell+3) &= 5\binom{\ell+3}{3} + 23\binom{\ell+2}{3} + 8\binom{\ell+1}{3} \\
&= 6\ell^3 + \frac{33\ell^2}{2} + \frac{31\ell}{2} + 5, \\
M_4(6\ell+4) = p(12\ell+8, 4, 6\ell+4) &= 8\binom{\ell+3}{3} + 23\binom{\ell+2}{3} + 5\binom{\ell+1}{3} \\
&= 6\ell^3 + \frac{39\ell^2}{2} + \frac{43\ell}{2} + 8,
\end{aligned}$$

$$\begin{aligned} M_4(6\ell+5) = p(12\ell+10,4,6\ell+5) &= 12\tbinom{\ell+3}{3} + 21\tbinom{\ell+2}{3} + 3\tbinom{\ell+1}{3}, \\ &= 6\ell^3 + \frac{45\ell^2}{2} + \frac{57\ell}{2} + 12. \end{aligned}$$

One cannot help, but notice that

$$p(12\ell+6,4,6\ell+2) \equiv p(12\ell+10,4,6\ell+5) \equiv 0 \pmod 3.$$

And, similar to (4.26), we have:

$$\sum_{N=0}^{\infty} M_4(N)q^n = \frac{(1+q^3)}{(1-q)(1-q^2)^2(1-q^3)}. \tag{4.27}$$

5. On the Period of Quasipolynomials for Maximal Coefficients of $\begin{bmatrix} N+m \\ m \end{bmatrix}$

One might expect that the period of the corresponding quasipolynomial for $M_m(N)$ would naturally be $\mathrm{lcm}(m)$. However, Corollary 4.8 and Corollary 4.10 lead us to the following result.

Theorem 5.1. *The quasipolynomial for $M_m(N)$ has period: $\frac{2\cdot\mathrm{lcm}(m)}{m}$.*

We note that $\frac{2\cdot\mathrm{lcm}(m)}{m} \leq \mathrm{lcm}(m)$ for $m \geq 2$, with equality for $m = 2$.

Proof. Beginning with Remark 4.5, we set $N = \mathrm{lcm}(m)j + b$ for $0 \leq b \leq \mathrm{lcm}(m) - 1$ and write:

$$p\left(\left\lfloor \frac{mN}{2} \right\rceil, m, N\right) = p\left(\frac{m\mathrm{lcm}(m)j}{2} + \left\lfloor \frac{mN}{2} \right\rceil, m, \mathrm{lcm}(m)j + b\right). \tag{5.1}$$

Set

$$\frac{m\mathrm{lcm}(m)j}{2} + \left\lfloor \frac{mN}{2} \right\rceil = \mathrm{lcm}(m)k + a$$

for $0 \leq a \leq \mathrm{lcm}(m) - 1$ and solve for j which depends on k to obtain:

$$j = \frac{2\left(\mathrm{lcm}(m)k + a - \left\lfloor \frac{mN}{2} \right\rceil\right)}{m\mathrm{lcm}(m)}. \tag{5.2}$$

Now, set $k = \frac{\mathrm{lcm}(m)f}{2} + i$, for those i with $|i| < \frac{\mathrm{lcm}(m)}{4}$, such that

$$\mathrm{lcm}(m)i + a - \left\lfloor \frac{mN}{2} \right\rceil = \mathrm{lcm}(m)t$$

for some integer $t \geq 0$. We are interested in only these k.

Setting

$$j = \frac{2\left(\frac{\mathrm{lcm}(m)f}{2} + t\right)}{m},$$

we rewrite the left side of (5.1) as:

$$p\left(\frac{m\operatorname{lcm}(m)\left(\frac{2\left(\frac{\operatorname{lcm}(m)f}{2}+t\right)}{m}\right)}{2}+\left\lfloor\frac{mN}{2}\right\rceil, m, \operatorname{lcm}(m)\left(\frac{2\left(\frac{\operatorname{lcm}(m)f}{2}+t\right)}{m}\right)+b\right). \tag{5.3}$$

Writing

$$\left\lfloor\frac{mN}{2}\right\rceil = x\operatorname{lcm}(m)+r_1$$

for $0 \leq r_1 \leq \operatorname{lcm}(m)-1$ and

$$b = x\frac{2\left(\frac{\operatorname{lcm}(m)f}{2}+t\right)}{m}+r_2$$

for $0 \leq r_2 \leq \frac{2\cdot\operatorname{lcm}(m)}{m}-1$, we rewrite (5.3) as:

$$p\left(\operatorname{lcm}(m)\left(\frac{\operatorname{lcm}(m)f}{2}+t+x\right)+r_1, m, \frac{2\cdot\operatorname{lcm}(m)}{m}\left(\frac{\operatorname{lcm}(m)f}{2}+t+x\right)+r_2\right). \tag{5.4}$$

Finally, allowing

$$\frac{\operatorname{lcm}(m)f}{2}+t+x=\ell,$$

we have

$$p\left(\left\lfloor\frac{mN}{2}\right\rceil, m, N\right) = p\left(\operatorname{lcm}(m)\ell+r_1, m, \frac{2\cdot\operatorname{lcm}(m)}{m}\ell+r_2\right). \tag{5.5}$$

Hence, quasipolynomials for $M_m(N)$ have a period of length $\frac{2\cdot\operatorname{lcm}(m)}{m}$. $\square$

It can be shown that the quasipolynomial for $M_5(N)$ has period 24, and curiously, for $M_6(N)$, the period is even shorter with length 20.

The $\frac{2\cdot\operatorname{lcm}(m)}{m}$ period of $M_m(N)$ appears to extend well beyond the collection of maximal/central coefficients of Gaussian polynomials. For example, just as the period of $M_4(N)$ in Corollary 4.10 is 6, it can be shown that the quasipolynomials for the coefficients that are one and two terms away from the center of $\begin{bmatrix}N+4\\4\end{bmatrix}$ also have a period of 6.

6. On the Difference $p(n, m, N) - p(n-1, m, N-1)$

In this section, we provide a direct proof of Theorem 4.2, which, having already completed the relevant computations from the quasipolynomial in Appendix A, is straightforward.

We are further motivated to extend known theorems on first differences of $p(n, 3)$ and $p(n, 4)$ to $p(n, 3, N)$ and $p(n, 4, N)$ respectively after establishing a generalization on first differences of $p(n, m, N)$.

6.1. Proof of Theorem 4.2

Proof. We first prove (4.5) of Theorem 4.2. Since both $p(6\ell, 3, 4\ell)$ and $p(6\ell - 3, 3, 4\ell - 1)$ from (4.5) are maximal coefficients of their respective Gaussian polynomials, we use (4.21) and (4.20) from Corollary 4.8 to compute:

$$p(6\ell, 3, 4\ell) - p(6\ell - 3, 3, 4\ell - 1) = 2\ell^2 + 2\ell + 1 - (2\ell^2 + \ell) = \ell + 1, \quad (6.1)$$

which verifies (4.5).

The proof of (4.6) is slightly different, because $p(6\ell - 1, 3, 4\ell)$ and $p(6\ell - 4, 3, 4\ell - 1)$ are not central coefficients. However, (4.2) and (4.3) tell us respectively, that $p(6\ell, 3, 4\ell) - 1 = p(6\ell - 1, 3, 4\ell)$ and $p(6\ell - 3, 3, 4\ell - 1) - 1 = p(6\ell - 4, 3, 4\ell - 1)$, which together with (6.1) yields (4.6).

To prove (4.7), we replace ℓ with $\ell + 1$ in $p(6\ell - 3, 3, 4\ell - 2)$ in (4.19) to obtain:

$$p(6\ell + 3, 3, 4\ell + 2) = 2\ell^2 + 4\ell + 2. \quad (6.2)$$

Computing the difference between (6.2) and (4.22) proves (4.7).

The proof of (4.8) is done similarly to (4.7) and completes the proof of Theorem 4.2. □

Corollary 4.8 lets us expand Theorem 4.2 slightly. For example, with (4.5) in mind, one has:

$$p(6\ell, 3, 4\ell) - \left\{\begin{array}{l} p(6\ell - 3, 3, 4\ell - 1) \\ p(6\ell - 2, 3, 4\ell - 1) \\ p(6\ell - 1, 3, 4\ell - 1) \\ p(6\ell, 3, 4\ell - 1) \end{array}\right\} = \ell + 1.$$

6.2. Known Results on First Differences of $p(n, 3)$ and $p(n, 4)$ are Extended to $p(n, 3, N)$ and $p(n, 4, N)$

Theorem 4.2 describes the differences of Gaussian polynomial coefficients which are near the middle of their respective generating polynomials. What can be said of related differences that are away from the middle of their generating polynomials? To answer this question, we consider Theorem 6.1 and Theorem 6.2 on differences of partitions into three and four parts, respectively. These theorems are proved combinatorially in [4]. Quasipolynomials allow for direct proofs.

Theorem 6.1 ([4]). *For $k \geq 0$, one has:*

$$\left.\begin{array}{l} p(6k + 1, 3) - p(6k, 3) \\ p(6k - 1, 3) - p(6k - 2, 3) \\ p(6k - 2, 3) - p(6k - 3, 3) \end{array}\right\} = p(2k - 1, 2),$$

$$\left.\begin{array}{l} p(6k + 3, 3) - p(6k + 2, 3) \\ p(6k + 2, 3) - p(6k + 1, 3) \\ p(6k, 3) - p(6k - 1, 3) \end{array}\right\} = p(2k, 2).$$

Theorem 6.2 ([4]). *For $k \geq 0$, one has:*

$$p(2k - 3, 4) - p(2k - 4, 4) = p(k - 3, 3),$$
$$p(2k - 4, 4) - p(2k - 5, 4) = p(k - 2, 3).$$

We provide a new and very direct proof of Theorem 6.1 below.

Proof. Referring to the six constituents of the quasipolynomial for $p(n,3)$ in (3.6), one computes:

$$\left.\begin{array}{r}p(6k+1,3)-p(6k,3)\ =3k^2+4k+1-(3k^2+3k+1)\\ p(6k-1,3)-p(6k-2,3)=3k^2+2k-(3k^2+k)\\ p(6k-2,3)-p(6k-3,3)=3k^2+k-(3k^2)\end{array}\right\}=k=p(2k-1,2)$$

and

$$\left.\begin{array}{r}p(6k+3,3)-p(6k+2,3)=3k^2+6k+3-(3k^2+5k+2)\\ p(6k+2,3)-p(6k+1,3)=3k^2+5k+2-(3k^2+4k+1)\\ p(6k,3)-p(6k-1,3)=3k^2+4k+1-(3k^2+2k)\end{array}\right\}=k+1=p(2k,2).$$

□

The direct proof of Theorem 6.2 follows from the 12 constituents of the quasipolynomial for $p(n,4)$ and is left to the reader. A forthcoming paper by the fifth author [20] will generalize the difference $p(n,m)-p(n-1,m)$ in terms of $p(n,m-1)$.

Motivated by Theorem 4.2, we extend Theorem 6.1 and Theorem 6.2 to the coefficients of Gaussian polynomials. The extensions are corollaries of a general theorem, Theorem 6.3 below, that we state and prove and which requires the q-binomial theorem:

$$(z;q)_m=\sum_{h=0}^{m}\begin{bmatrix}m\\h\end{bmatrix}(-1)^h q^{\frac{h(h-1)}{2}}z^h. \tag{6.3}$$

Theorem 6.3. *For $n<2N+2$:*

$$p(n,m,N)-p(n-1,m,N-1)=p(n,m)-p(n-1,m).$$

Proof. We manipulate a difference of generating functions to show

$$p(n,m,N)-p(n-1,m,N-1)=p(n,m)-p(n-1,m)$$

for all $n<2N+2$:

$$\begin{aligned}\sum_{n=0}^{\infty}&(p(n,m,N)-p(n-1,m,N-1))\,q^n\\ &=\frac{(q^{N+1};q)_m}{(q;q)_m}-q\frac{(q^N;q)_m}{(q;q)_m}\\ &=\frac{(q^{N+1};q)_{m-1}((1-q^{m+N})-q(1-q^N))}{(q;q)_m}\\ &=\frac{(q^{N+1};q)_{m-1}(1-q)\left(\sum_{i=0}^{m+N-1}q^i-\sum_{j=1}^{N}q^j\right)}{(q;q)_m}\\ &=\left(\sum_{n=0}^{\infty}(p(n,m)-p(n-1,m))\,q^n\right)(q^{N+1};q)_{m-1}\left(1+\sum_{i=1}^{m-1}q^{N+i}\right).\end{aligned} \tag{6.4}$$

We apply the q-binomial theorem (6.3) to re-express $(q^{N+1};q)_{m-1}$ in (6.4):

$$\begin{aligned}
&(q^{N+1};q)_{m-1}\\
&=\sum_{h=0}^{m-1}\begin{bmatrix}m-1\\h\end{bmatrix}q^{\frac{h(h-1)}{2}+h(N+1)}\\
&=\sum_{h=0}^{m-1}\sum_{i=0}^{h(m-1-h)}(-1)^h p(i,h,m-1-h)q^{\frac{h(h-1)}{2}+h(N+1)+i}\\
&=1-\sum_{j=0}^{m-2}q^{N+1+j}+\sum_{h=2}^{m-1}\sum_{i=0}^{h(m-1-h)}(-1)^h p(i,h,m-1-h)q^{\frac{h(h-1)}{2}+h(N+1)+i}.
\end{aligned}\tag{6.5}$$

Note that the following polynomial from (6.5), denoted by $A(q)$, has degree strictly greater than $2N+2$:

$$A(q)=\sum_{h=2}^{m-1}\sum_{i=0}^{h(m-1-h)}(-1)^h p(i,h,m-1-h)q^{\frac{h(h-1)}{2}+h(N+1)+i}.$$

Now, we have:

$$\begin{aligned}
&(q^{N+1};q)_{m-1}\left(1+\sum_{i=1}^{m-1}q^{N+i}\right)\\
&=\left(1-q^{N+1}\sum_{j=0}^{m-2}q^j+A(q)\right)\left(1+q^{N+1}\sum_{i=0}^{m-2}q^i\right)\\
&=1-q^{2N+2}\left(\sum_{j=0}^{m-2}q^j\right)^2+A(q)+q^{N+1}\sum_{i=0}^{m-2}q^iA(q).
\end{aligned}\tag{6.6}$$

Therefore, the smallest non-zero power of q in (6.6) is $2N+2$, and Theorem 6.3 is proved. $\square$

The cases for $m=3,4$ echo Theorem 6.1 and Theorem 6.2 and are left to the reader to verify.

Corollary 6.4. *For $k<\frac{2N-3}{6}$, one has:*

$$\left.\begin{array}{l}p(6k+1,3,N)-p(6k,3,N-1)\\p(6k-1,3,N)-p(6k-1,3,N-1)\\p(6k-2,3,N)-p(6k-3,3,N-1)\end{array}\right\}=k=p(2k-1,2),$$

$$\left.\begin{array}{l}p(6k+3,3,N)-p(6k+2,3,N-1)\\p(6k+2,3,N)-p(6k+1,3,N-1)\\p(6k,3,N)-p(6k-1,3,N-1)\end{array}\right\}=k+1=p(2k,2).$$

Corollary 6.5. *For $k<\frac{2N-9}{12}$, one has:*

$$\begin{aligned}
p(2k-3,4,N)-p(2k-4,4,N-1)&=p(k-3,3),\\
p(2k-4,4,N)-p(2k-5,4,N-1)&=p(k-2,3).
\end{aligned}$$

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$$\sum_{k=0}^{\infty}\binom{k+m-1}{m-1}q^{\mathrm{lcm}(m)k}=A_m(q)$$

and consider generalizations to $p(n,m,N)$ and $\left[{N+m \atop m}\right]$ by further inquiry into

$$\begin{bmatrix}N+m\\ m\end{bmatrix}=G_m(q)E_m(q)A_m(q). \tag{6.7}$$

It is clear that GEA in (6.7) is a great mathematical object rich with wisdom and generosity and worthy of our attention and admiration.

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Appendix A. The 36 Constituents of the Quasipolynomial for $p(n,3,N)$ Arranged by $N \pmod 6$

$N=6j$

$$\begin{aligned}p(6k,3,6j)&=\tbinom{k+2}{2}+4\tbinom{k+1}{2}+\tbinom{k}{2}-12\tbinom{k+1-j}{2}-6\tbinom{k-j}{2}+6\tbinom{k+1-2j}{2}\\&\quad+12\tbinom{k-2j}{2}-\tbinom{k+1-3j}{2}-4\tbinom{k-3j}{2}-\tbinom{k-1-3j}{2}\end{aligned}\tag{A.1}$$

$$\begin{aligned}p(6k+1,3,6j)&=\tbinom{k+2}{2}+5\tbinom{k+1}{2}-\tbinom{k+2-j}{2}-13\tbinom{k+1-j}{2}-4\tbinom{k-j}{2}\\&\quad+9\tbinom{k+1-2j}{2}+9\tbinom{k-2j}{2}-\tbinom{k+1-3j}{2}-5\tbinom{k-3j}{2}\end{aligned}\tag{A.2}$$

$$\begin{aligned}p(6k+2,3,6j)&=2\tbinom{k+2}{2}+4\tbinom{k+1}{2}-2\tbinom{k+2-j}{2}-14\tbinom{k+1-j}{2}-2\tbinom{k-j}{2}\\&\quad+12\tbinom{k+1-2j}{2}+6\tbinom{k-2j}{2}-2\tbinom{k+1-3j}{2}-4\tbinom{k-3j}{2}\end{aligned}\tag{A.3}$$

$$\begin{aligned}p(6k+3,3,6j)&=3\tbinom{k+2}{2}+3\tbinom{k+1}{2}-4\tbinom{k+2-j}{2}-13\tbinom{k+1-j}{2}-\tbinom{k-j}{2}+\tbinom{k+2-2j}{2}\\&\quad+13\tbinom{k+1-2j}{2}+4\tbinom{k-2j}{2}-3\tbinom{k+1-3j}{2}-3\tbinom{k-3j}{2}\end{aligned}\tag{A.4}$$

$$\begin{aligned}p(6k+4,3,6j)&=4\tbinom{k+2}{2}+2\tbinom{k+1}{2}-6\tbinom{k+2-j}{2}-12\tbinom{k+1-j}{2}+2\tbinom{k+2-2j}{2}\\&\quad+14\tbinom{k+1-2j}{2}+2\tbinom{k-2j}{2}-4\tbinom{k+1-3j}{2}-2\tbinom{k-3j}{2}\end{aligned}\tag{A.5}$$

$$\begin{aligned}p(6k+5,3,6j)&=5\tbinom{k+2}{2}+\tbinom{k+1}{2}-9\tbinom{k+2-j}{2}-9\tbinom{k+1-j}{2}+4\tbinom{k+2-2j}{2}\\&\quad+13\tbinom{k+1-2j}{2}+\tbinom{k-2j}{2}-5\tbinom{k+1-3j}{2}-\tbinom{k-3j}{2}\end{aligned}\tag{A.6}$$

$$N = 6j + 1$$

$$\begin{aligned} p(6k, 3, 6j+1) &= \tbinom{k+2}{2} + 4\tbinom{k+1}{2} + \tbinom{k}{2} - 9\tbinom{k+1-j}{2} - 9\tbinom{k-j}{2} + 2\tbinom{k+1-2j}{2} \\ &\quad + 14\tbinom{k-2j}{2} + 2\tbinom{k-1-2j}{2} - 3\tbinom{k-3j}{2} - 3\tbinom{k-1-3j}{2} \end{aligned} \tag{A.7}$$

$$\begin{aligned} p(6k+1, 3, 6j+1) &= \tbinom{k+2}{2} + 5\tbinom{k+1}{2} - 12\tbinom{k+1-j}{2} - 6\tbinom{k-j}{2} + 4\tbinom{k+1-2j}{2} \\ &\quad + 13\tbinom{k-2j}{2} + \tbinom{k-1-2j}{2} - 4\tbinom{k-3j}{2} - 2\tbinom{k-1-3j}{2} \end{aligned} \tag{A.8}$$

$$\begin{aligned} p(6k+2, 3, 6j+1) &= 2\tbinom{k+2}{2} + 4\tbinom{k+1}{2} - \tbinom{k+2-j}{2} - 13\tbinom{k+1-j}{2} - 4\tbinom{k-j}{2} \\ &\quad + 6\tbinom{k+1-2j}{2} + 12\tbinom{k-2j}{2} - 5\tbinom{k-3j}{2} - \tbinom{k-1-3j}{2} \end{aligned} \tag{A.9}$$

$$\begin{aligned} p(6k+3, 3, 6j+1) &= 3\tbinom{k+2}{2} + 3\tbinom{k+1}{2} - 2\tbinom{k+2-j}{2} - 14\tbinom{k+1-j}{2} - 2\tbinom{k-j}{2} \\ &\quad + 9\tbinom{k+1-2j}{2} + 9\tbinom{k-2j}{2} - \tbinom{k+1-3j}{2} - 4\tbinom{k-3j}{2} - \tbinom{k-1-3j}{2} \end{aligned} \tag{A.10}$$

$$\begin{aligned} p(6k+4, 3, 6j+1) &= 4\tbinom{k+2}{2} + 2\tbinom{k+1}{2} - 4\tbinom{k+2-j}{2} - 13\tbinom{k+1-j}{2} - \tbinom{k-j}{2} \\ &\quad + 12\tbinom{k+1-2j}{2} + 6\tbinom{k-2j}{2} - \tbinom{k+1-3j}{2} - 5\tbinom{k-3j}{2} \end{aligned} \tag{A.11}$$

$$\begin{aligned} p(6k+5, 3, 6j+1) &= 5\tbinom{k+2}{2} + \tbinom{k+1}{2} - 6\tbinom{k+2-j}{2} - 12\tbinom{k+1-j}{2} + \tbinom{k+2-2j}{2} \\ &\quad + 13\tbinom{k+1-2j}{2} + 4\tbinom{k-2j}{2} - 2\tbinom{k+1-3j}{2} - 4\tbinom{k-3j}{2} \end{aligned} \tag{A.12}$$

$$N = 6j + 2$$

$$\begin{aligned} p(6k, 3, 6j+2) &= \tbinom{k+2}{2} + 4\tbinom{k+1}{2} + \tbinom{k}{2} - 6\tbinom{k+1-j}{2} - 12\tbinom{k-j}{2} + 12\tbinom{k-2j}{2} \\ &\quad + 6\tbinom{k-1-2j}{2} - \tbinom{k-1-3j}{2} - 4\tbinom{k-2-3j}{2} - \tbinom{k-3-3j}{2} \end{aligned} \tag{A.13}$$

$$\begin{aligned} p(6k+1, 3, 6j+2) &= \tbinom{k+2}{2} + 5\tbinom{k+1}{2} - 9\tbinom{k+1-j}{2} - 9\tbinom{k-j}{2} + \tbinom{k+1-2j}{2} \\ &\quad + 13\tbinom{k-2j}{2} + 4\tbinom{k-1-2j}{2} - \tbinom{k-3j}{2} - 5\tbinom{k-1-3j}{2} \end{aligned} \tag{A.14}$$

$$\begin{aligned} p(6k+2, 3, 6j+2) &= 2\tbinom{k+2}{2} + 4\tbinom{k+1}{2} - 12\tbinom{k+1-j}{2} - 6\tbinom{k-j}{2} + 2\tbinom{k+1-2j}{2} \\ &\quad + 14\tbinom{k-2j}{2} + 2\tbinom{k-1-2j}{2} - 2\tbinom{k-3j}{2} - 4\tbinom{k-1-3j}{2} \end{aligned} \tag{A.15}$$

$$\begin{aligned} p(6k+3, 3, 6j+2) &= 3\tbinom{k+2}{2} + 3\tbinom{k+1}{2} - \tbinom{k+2-j}{2} - 13\tbinom{k+1-j}{2} - 4\tbinom{k-j}{2} + 4\tbinom{k+1-2j}{2} \\ &\quad + 13\tbinom{k-2j}{2} + \tbinom{k-1-2j}{2} - 3\tbinom{k-3j}{2} - 3\tbinom{k-1-3j}{2} \end{aligned} \tag{A.16}$$

$$\begin{aligned} p(6k+4, 3, 6j+2) &= 4\tbinom{k+2}{2} + 2\tbinom{k+1}{2} - 2\tbinom{k+2-j}{2} - 14\tbinom{k+1-j}{2} - 2\tbinom{k-j}{2} \\ &\quad + 6\tbinom{k+1-2j}{2} + 12\tbinom{k-2j}{2} - 4\tbinom{k-3j}{2} - 2\tbinom{k-1-3j}{2} \end{aligned} \tag{A.17}$$

$$\begin{aligned} p(6k+5, 3, 6j+2) &= 5\tbinom{k+2}{2} + \tbinom{k+1}{2} - 4\tbinom{k+2-j}{2} - 13\tbinom{k+1-j}{2} - \tbinom{k-j}{2} \\ &\quad + 9\tbinom{k+1-2j}{2} + 9\tbinom{k-2j}{2} - 5\tbinom{k-3j}{2} - \tbinom{k-1-3j}{2} \end{aligned} \tag{A.18}$$

$$N = 6j + 3$$

$$\begin{aligned} p(6k, 3, 6j+3) &= \tbinom{k+2}{2} + 4\tbinom{k+1}{2} + \tbinom{k}{2} - 4\tbinom{k+1-j}{2} - 13\tbinom{k-j}{2} - \tbinom{k-1-j}{2} \\ &\quad + 6\tbinom{k-2j}{2} + 12\tbinom{k-1-2j}{2} - 3\tbinom{k-1-3j}{2} - 3\tbinom{k-2-3j}{2} \end{aligned} \tag{A.19}$$

$$\begin{aligned} p(6k+1,3,6j+3) &= \tbinom{k+2}{2} + 5\tbinom{k+1}{2} - 6\tbinom{k+1-j}{2} - 12\tbinom{k-j}{2} + 9\tbinom{k-2j}{2} \\ &\quad + 9\tbinom{k-1-2j}{2} - 4\tbinom{k-1-3j}{2} - 2\tbinom{k-2-3j}{2} \end{aligned} \tag{A.20}$$

$$\begin{aligned} p(6k+2,3,6j+3) &= 2\tbinom{k+2}{2} + 4\tbinom{k+1}{2} - 9\tbinom{k+1-j}{2} - 9\tbinom{k-j}{2} + 12\tbinom{k-2j}{2} \\ &\quad + 6\tbinom{k-1-2j}{2} - 5\tbinom{k-1-3j}{2} - \tbinom{k-2-3j}{2} \end{aligned} \tag{A.21}$$

$$\begin{aligned} p(6k+3,3,6j+3) &= 3\tbinom{k+2}{2} + 3\tbinom{k+1}{2} - 12\tbinom{k+1-j}{2} - 6\tbinom{k-j}{2} + \tbinom{k+1-2j}{2} + 13\tbinom{k-2j}{2} \\ &\quad + 4\tbinom{k-1-2j}{2} - \tbinom{k-3j}{2} - 4\tbinom{k-1-3j}{2} - \tbinom{k-2-3j}{2} \end{aligned} \tag{A.22}$$

$$\begin{aligned} p(6k+4,3,6j+3) &= 4\tbinom{k+2}{2} + 2\tbinom{k+1}{2} - \tbinom{k+2-j}{2} - 13\tbinom{k+1-j}{2} - 4\tbinom{k-j}{2} + 2\tbinom{k+1-2j}{2} \\ &\quad + 14\tbinom{k-2j}{2} + 2\tbinom{k-1-2j}{2} - \tbinom{k-3j}{2} - 5\tbinom{k-1-3j}{2} \end{aligned} \tag{A.23}$$

$$\begin{aligned} p(6k+5,3,6j+3) &= 5\tbinom{k+2}{2} + \tbinom{k+1}{2} - 2\tbinom{k+2-j}{2} - 14\tbinom{k+1-j}{2} - 2\tbinom{k-j}{2} + 4\tbinom{k+1-2j}{2} \\ &\quad + 13\tbinom{k-2j}{2} + \tbinom{k-1-2j}{2} - 2\tbinom{k-3j}{2} - 4\tbinom{k-1-3j}{2} \end{aligned} \tag{A.24}$$

$$N = 6j+4$$

$$\begin{aligned} p(6k,3,6j+4) &= \tbinom{k+2}{2} + 4\tbinom{k+1}{2} + \tbinom{k}{2} - 2\tbinom{k+1-j}{2} - 14\tbinom{k-j}{2} \\ &\quad - 2\tbinom{k-1-j}{2} + 2\tbinom{k-2j}{2} + 14\tbinom{k-1-2j}{2} + 2\tbinom{k-2-2j}{2} \\ &\quad - \tbinom{k-1-3j}{2} - 4\tbinom{k-2-3j}{2} - \tbinom{k-3-3j}{2} \end{aligned} \tag{A.25}$$

$$\begin{aligned} p(6k+1,3,6j+4) &= \tbinom{k+2}{2} + 5\tbinom{k+1}{2} - 4\tbinom{k+1-j}{2} - 13\tbinom{k-j}{2} - \tbinom{k-1-j}{2} + 4\tbinom{k-2j}{2} \\ &\quad + 13\tbinom{k-1-2j}{2} + \tbinom{k-2-2j}{2} - \tbinom{k-1-3j}{2} - 5\tbinom{k-2-3j}{2} \end{aligned} \tag{A.26}$$

$$\begin{aligned} p(6k+2,3,6j+4) &= 2\tbinom{k+2}{2} + 4\tbinom{k+1}{2} - 6\tbinom{k+1-j}{2} - 12\tbinom{k-j}{2} + 6\tbinom{k-2j}{2} \\ &\quad + 12\tbinom{k-1-2j}{2} - 2\tbinom{k-1-3j}{2} - 4\tbinom{k-2-3j}{2} \end{aligned} \tag{A.27}$$

$$\begin{aligned} p(6k+3,3,6j+4) &= 3\tbinom{k+2}{2} + 3\tbinom{k+1}{2} - 9\tbinom{k+1-j}{2} - 9\tbinom{k-j}{2} + 9\tbinom{k-2j}{2} \\ &\quad + 9\tbinom{k-1-2j}{2} - 3\tbinom{k-1-3j}{2} - 3\tbinom{k-2-3j}{2} \end{aligned} \tag{A.28}$$

$$\begin{aligned} p(6k+4,3,6j+4) &= 4\tbinom{k+2}{2} + 2\tbinom{k+1}{2} - 12\tbinom{k+1-j}{2} - 6\tbinom{k-j}{2} + 12\tbinom{k-2j}{2} \\ &\quad + 6\tbinom{k-1-2j}{2} - 4\tbinom{k-1-3j}{2} - 2\tbinom{k-2-3j}{2} \end{aligned} \tag{A.29}$$

$$\begin{aligned} p(6k+5,3,6j+4) &= 5\tbinom{k+2}{2} + \tbinom{k+1}{2} - \tbinom{k+2-j}{2} - 13\tbinom{k+1-j}{2} - 4\tbinom{k-j}{2} + \tbinom{k+1-2j}{2} \\ &\quad + 13\tbinom{k-2j}{2} + 4\tbinom{k-1-2j}{2} - 5\tbinom{k-1-3j}{2} - \tbinom{k-2-3j}{2} \end{aligned} \tag{A.30}$$

$$N = 6j+5$$

$$\begin{aligned} p(6k,3,6j+5) &= \tbinom{k+2}{2} + 4\tbinom{k+1}{2} + \tbinom{k}{2} - \tbinom{k+1-j}{2} - 13\tbinom{k-j}{2} - 4\tbinom{k-1-j}{2} \\ &\quad + 12\tbinom{k-1-2j}{2} + 6\tbinom{k-2-2j}{2} - 3\tbinom{k-2-3j}{2} - 3\tbinom{k-3-3j}{2} \end{aligned} \tag{A.31}$$

$$\begin{aligned} p(6k+1,3,6j+5) &= \tbinom{k+2}{2} + 5\tbinom{k+1}{2} - 2\tbinom{k+1-j}{2} - 14\tbinom{k-j}{2} \\ &\quad - 2\tbinom{k-1-j}{2} + \tbinom{k-2j}{2} + 13\tbinom{k-1-2j}{2} \end{aligned}$$

$$+4\binom{k-2-2j}{2}-4\binom{k-2-3j}{2}-2\binom{k-3-3j}{2} \tag{A.32}$$

$$\begin{aligned} p(6k+2,3,6j+5) &= 2\binom{k+2}{2}+4\binom{k+1}{2}-4\binom{k+1-j}{2}-13\binom{k-j}{2}-\binom{k-1-j}{2} \\ &\quad +2\binom{k-2j}{2}+14\binom{k-1-2j}{2}+2\binom{k-2-2j}{2} \\ &\quad -5\binom{k-2-3j}{2}-\binom{k-3-3j}{2} \end{aligned} \tag{A.33}$$

$$\begin{aligned} p(6k+3,3,6j+5) &= 3\binom{k+2}{2}+3\binom{k+1}{2}-6\binom{k+1-j}{2}-12\binom{k-j}{2}+4\binom{k-2j}{2} \\ &\quad +13\binom{k-1-2j}{2}+\binom{k-2-2j}{2}-\binom{k-1-3j}{2} \\ &\quad -4\binom{k-2-3j}{2}-\binom{k-3-3j}{2} \end{aligned} \tag{A.34}$$

$$\begin{aligned} p(6k+4,3,6j+5) &= 4\binom{k+2}{2}+2\binom{k+1}{2}-9\binom{k+1-j}{2}-9\binom{k-j}{2}+6\binom{k-2j}{2} \\ &\quad +12\binom{k-1-2j}{2}-\binom{k-1-3j}{2}-5\binom{k-2-3j}{2} \end{aligned} \tag{A.35}$$

$$\begin{aligned} p(6k+5,3,6j+5) &= 5\binom{k+2}{2}+\binom{k+1}{2}-12\binom{k+1-j}{2}-6\binom{k-j}{2}+9\binom{k-2j}{2} \\ &\quad +9\binom{k-1-2j}{2}-2\binom{k-1-3j}{2}-4\binom{k-2-3j}{2}-4\binom{k-2-3j}{2} \end{aligned} \tag{A.36}$$

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Angelica Castillo, Stephanie Flores, Brandt Kronholm, Acadia Larsen and Arturo Martinez
School of Mathematical and Statistical Sciences
University of Texas Rio Grande Valley
Edinburg
TX 7839-2999
USA

Brandt Kronholm
e-mail: brandt.kronholm@utrgv.edu

Angelica Castillo
e-mail: Angelica.castillo01@utrgv.edu

Stephanie Flores
e-mail: stephanie.flores01@utrgv.edu

Acadia Larsen
e-mail: acadia.larsen01@utrgv.edu

Arturo Martinez
e-mail: arturo.j.martinez01@utrgv.edu

Anabel Hernandez
College of Engineering and Computer Science
University of Texas Rio Grande Valley
Edinburg TX 7839-2999
USA
e-mail: anabel.hernandez01@utrgv.edu

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Finding Modular Functions for Ramanujan-Type Identities

Dedicated to Professor George E. Andrews on the occasion of his 80th birthday

William Y. C. Chen, Julia Q. D. Du and Jack C. D. Zhao

Abstract. This paper is concerned with a class of partition functions $a(n)$ introduced by Radu and defined in terms of eta-quotients. By utilizing the transformation laws of Newman, Schoeneberg and Robins, and Radu's algorithms, we present an algorithm to find Ramanujan-type identities for $a(mn+t)$. While this algorithm is not guaranteed to succeed, it applies to many cases. For example, we deduce a witness identity for $p(11n+6)$ with integer coefficients. Our algorithm also leads to Ramanujan-type identities for the overpartition functions $\overline{p}(5n+2)$ and $\overline{p}(5n+3)$ and Andrews–Paule's broken 2-diamond partition functions $\triangle_2(25n+14)$ and $\triangle_2(25n+24)$. It can also be extended to derive Ramanujan-type identities on a more general class of partition functions. For example, it yields the Ramanujan-type identities on Andrews' singular overpartition functions $\overline{Q}_{3,1}(9n+3)$ and $\overline{Q}_{3,1}(9n+6)$ due to Shen, the 2-dissection formulas of Ramanujan, and the 8-dissection formulas due to Hirschhorn.

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Keywords. Ramanujan-type identities, Modular functions, Generalized eta-functions, Partition functions.

1. Introduction

Throughout this paper, we follow the standard q-series notation in [16]:

$$(a;q)_\infty = \prod_{n=0}^{\infty}(1-aq^n) \quad \text{and} \quad (a_1,a_2,\ldots,a_m;q)_\infty = \prod_{j=1}^{m}(a_j;q)_\infty,$$

where $|q|<1$. In the study of congruence properties and identities on the partition functions, Radu [35–37] defined a class of partition functions $a(n)$ by

$$\sum_{n=0}^{\infty} a(n)q^n = \prod_{\delta|M} (q^\delta; q^\delta)_\infty^{r_\delta}, \tag{1.1}$$

where M is a positive integer and r_δ are integers. Many partition functions fall into the framework of the above definition of $a(n)$, such as the partition function $p(n)$, the overpartition function $\overline{p}(n)$ [11], the Ramanujan τ-function [18,19,39], the k-colored partition functions, the t-core partition functions, the 2-colored Frobenius partition functions, and the broken k-diamond partition functions $\Delta_k(n)$ [4].

In this paper, we aim to present an algorithm to compute the generating function:

$$\sum_{n=0}^{\infty} a(mn+t)q^n, \tag{1.2}$$

for fixed $m > 0$ and $0 \le t \le m-1$ by finding suitable modular functions for $\Gamma_1(N)$. When $M = 1$ and $r_1 = -1$, $a(n)$ specializes to the partition function $p(n)$. Kolberg [26] proved that for any prime m, and $0 \le t \le m-1$:

$$\sum_{n=0}^{\infty} p(mn+t)q^{mn+t} = (-1)^{(m-1)t} \frac{(q^{m^2}; q^{m^2})_\infty}{(q^m; q^m)_\infty^{m+1}} \det M_t, \tag{1.3}$$

where $M_t = (g_{-t-i+j})_{(m-1)\times(m-1)}$:

$$g_t = \sum_{\frac{1}{2}n(3n+1)\equiv t (\mathrm{mod}\ m)} (-1)^n q^{\frac{1}{2}n(3n+1)},$$

and $g_t = g_s$ when $t \equiv s \pmod{m}$. In view of (1.3), he derived some identities on $p(n)$, for example:

$$\sum_{n=0}^{\infty} p(5n)q^n = \frac{(q^5; q^5)_\infty}{(q; q)_\infty^2 (q, q^4; q^5)_\infty^8} - 3q \frac{(q^5; q^5)_\infty^6 (q, q^4; q^5)_\infty^2}{(q; q)_\infty^7} \tag{1.4}$$

and

$$\left(\sum_{n=0}^{\infty} p(5n)q^n\right)\left(\sum_{n=0}^{\infty} p(5n+3)q^n\right) = 3\frac{(q^5; q^5)_\infty^4}{(q; q)_\infty^6} + 25q \frac{(q^5; q^5)_\infty^{10}}{(q; q)_\infty^{12}}.$$

Atkin and Swinnerton-Dyer [5] have shown that g_t can always be expressed by certain infinite products for $m > 3$. Then, the left-hand side of (1.3) can be expressed in terms of certain infinite products. Kolberg pointed out that when $m > 5$, this becomes much more complicated. For $m = 11, 13$, Bilgici and Ekin [7,8] used the method of Kolberg to compute the generating function:

$$\sum_{n=0}^{\infty} p(mn+t)q^{mn+t}$$

for all $0 \le t \le m-1$.

Based on the ideas of Rademacher [33], Newman [28,29], and Kolberg [26], Radu [37] developed an algorithm to verify the congruences

$$a(mn+t) \equiv 0 \pmod{u}, \tag{1.5}$$

for any given m, t and u, and for all $n \geq 0$, where $a(n)$ is defined in (1.1). Moreover, Radu [35] developed an algorithm, called the Ramanujan–Kolberg algorithm, to derive identities on the generating functions of $a(mn+t)$ using modular functions for $\Gamma_0(N)$. A description of the Ramanujan–Kolberg algorithm can be found in Paule and Radu [32]. Smoot [46] developed a Mathematica package `RaduRK` to implement Radu's algorithm. It should be mentioned that Eichhorn [13] extended the technique in [14,15] to partition functions $a(n)$ defined by

$$\sum_{n=0}^{\infty} a(n)q^n = \prod_{j=1}^{L} (q^j;q^j)_{\infty}^{e_j}, \tag{1.6}$$

where L is a positive integer and e_j are integers, and reduced the verification of the congruences (1.5) to a finite number of cases. It is easy to see that the defining relations (1.1) and (1.6) are equivalent to each other. In this paper, we shall adopt the form of (1.1) in accordance with the notation of eta-quotients.

Recall that the Dedekind eta-function $\eta(\tau)$ is defined by

$$\eta(\tau) = q^{\frac{1}{24}} \prod_{n=1}^{\infty} (1-q^n),$$

where $q = e^{2\pi i\tau}$, $\tau \in \mathbb{H} = \{\tau \in \mathbb{C} \colon \operatorname{Im}(\tau) > 0\}$. An eta-quotient is a function of the form

$$\prod_{\delta | M} \eta^{r_\delta}(\delta\tau),$$

where $M \geq 1$ and each r_δ is an integer.

The Ramanujan–Kolberg algorithm leads to verifications of some identities on $p(n)$ due to Ramanujan [38], Zuckerman [50], and Kolberg [26]; for example:

$$\sum_{n=0}^{\infty} p(5n+4)q^n = 5\frac{(q^5;q^5)_{\infty}^5}{(q;q)_{\infty}^6},$$

see [38, eq. (18)]. It should be noted that there are some Ramanujan-type identities that are not covered by the Ramanujan–Kolberg algorithm, such as the identity (1.4).

In this paper, we develop an algorithm to derive Ramanujan-type identities for $a(mn+t)$ for $m > 0$ and $0 \leq t \leq m-1$, which is essentially a modified version of Radu's algorithm. We first find a necessary and sufficient condition for a product of a generalized eta-quotient and the generating function (1.2) to be a modular function for $\Gamma_1(N)$ up to a power of q. Then, we try to express this modular function as a linear combination of generalized eta-quotients over $\mathbb{Q}$.

For example, our algorithm leads to a verification of (1.4) for $p(5n)$. Moreover, we obtain Ramanujan-type identities for the overpartition functions $\overline{p}(5n+2)$ and $\overline{p}(5n+3)$ and the broken 2-diamond partition functions $\Delta_2(25n+14)$ and $\Delta_2(25n+24)$. We also obtain the following witness identity with integer coefficients for $p(11n+6)$.

Theorem 1.1. *We have*

$$\begin{aligned} z_0 &\sum_{n=0}^{\infty} p(11n+6)q^n \\ &= 11z^{10} + 121z^8e + 330z^9 - 484z^7e - 990z^8 + 484z^6e + 792z^7 \\ &\quad - 484z^5e + 44z^6 + 1089z^4e - 132z^5 - 1452z^3e - 451z^4 \\ &\quad + 968z^2e + 748z^3 - 242ze - 429z^2 + 77z + 11, \end{aligned} \tag{1.7}$$

where

$$z_0 = \frac{(q;q)_\infty^{24}}{q^{20}(q^{11};q^{11})_\infty^{23}(q,q^{10};q^{11})_\infty^{28}(q^2,q^9;q^{11})_\infty^{16}(q^3,q^8;q^{11})_\infty^{12}(q^4,q^7;q^{11})_\infty^{4}},$$

$$z = \frac{(q;q)_\infty}{q^2(q^{11};q^{11})_\infty(q,q^{10};q^{11})_\infty^3(q^2,q^9;q^{11})_\infty^2}, \tag{1.8}$$

$$e = \frac{(q;q)_\infty^3}{q^3(q^{11};q^{11})_\infty^3(q,q^{10};q^{11})_\infty^5(q^2,q^9;q^{11})_\infty^5(q^3,q^8;q^{11})_\infty^4(q^4,q^7;q^{11})_\infty}. \tag{1.9}$$

Bilgici and Ekin [8] deduced a witness identity for $p(11n+6)$ with integer coefficients using the method of Kolberg. Radu [35] obtained a witness identity for $p(11n+6)$ using the Ramanujan–Kolberg algorithm. Hemmecke [20] generalized Radu's algorithm and derived a witness identity for $p(11n+6)$. Paule and Radu [31] found a polynomial relation on the generating function of $p(11n+6)$, which can also be viewed as a witness identity. Moreover, Paule and Radu [30] found a witness identity for $p(11n+6)$ in terms of eta-quotients and the U_2-operator acting on eta-quotients.

Our algorithm can be extended to a more general class of partition functions $b(n)$ defined by

$$\sum_{n=0}^{\infty} b(n)q^n = \prod_{\delta|M}(q^\delta;q^\delta)_\infty^{r_\delta} \prod_{\substack{\delta|M \\ 0<g<\delta}} (q^g,q^{\delta-g};q^\delta)_\infty^{r_{\delta,g}}, \tag{1.10}$$

where M is a positive integer, and r_δ and $r_{\delta,g}$ are integers. Notice that (1.10) is a generalized eta-quotient up to a power of q.

Recall that for a positive integer δ and a residue class $g \pmod{\delta}$, the generalized Dedekind eta-function $\eta_{\delta,g}(\tau)$ is defined by

$$\eta_{\delta,g}(\tau) = q^{\frac{\delta}{2}P_2\left(\frac{g}{\delta}\right)} \prod_{\substack{n>0 \\ n\equiv g(\mathrm{mod}\,\delta)}} (1-q^n) \prod_{\substack{n>0 \\ n\equiv -g(\mathrm{mod}\,\delta)}} (1-q^n), \tag{1.11}$$

where

$$P_2(t) = \{t\}^2 - \{t\} + \frac{1}{6}$$

is the second Bernoulli function and $\{t\}$ is the fractional part of t; see, for example, [41,43]. Note that

$$\eta_{\delta,0}(\tau)=\eta^2(\delta\tau) \quad \text{and} \quad \eta_{\delta,\frac{\delta}{2}}(\tau)=\frac{\eta^2(\frac{\delta}{2}\tau)}{\eta^2(\delta\tau)}. \tag{1.12}$$

A generalized eta-quotient is a function of the form

$$\prod_{\substack{\delta|M\\ 0\leq g<\delta}} \eta_{\delta,g}^{r_{\delta,g}}(\tau), \tag{1.13}$$

where $M\geq 1$ and

$$r_{\delta,g}\in\begin{cases}\frac{1}{2}\mathbb{Z}, & \text{if } g=0 \text{ or } g=\frac{\delta}{2},\\ \mathbb{Z}, & \text{otherwise,}\end{cases}$$

see, for example, Robins [41]. In view of (1.12), when $g=0$ or $g=\frac{\delta}{2}$, if $r_{\delta,g}\in\frac{1}{2}\mathbb{Z}$, then the powers of the eta-functions in (1.13) are integers.

For partition functions $b(n)$ as defined in (1.10), our algorithm can be extended to derive Ramanujan-type identities on $b(mn+t)$ for $m>0$ and $0\leq t\leq m-1$, such as the Ramanujan-type identities on Andrews' singular overpartition functions $\overline{Q}_{3,1}(9n+3)$ and $\overline{Q}_{3,1}(9n+6)$ due to Shen [45]. The extended algorithm can also be employed to derive dissection formulas, such as the 2-dissection formulas of Ramanujan, first proved by Andrews [2], and the 8-dissection formulas due to Hirschhorn [22].

2. Finding Modular Functions for $\Gamma_1(N)$

For the partition functions $a(n)$ as defined by (1.1), namely,

$$\sum_{n=0}^{\infty} a(n)q^n=\prod_{\delta|M}(q^\delta;q^\delta)_\infty^{r_\delta},$$

where M is a positive integer and r_δ are integers, Radu [37] defined

$$g_{m,t}(\tau)=q^{\frac{t-\ell}{m}}\sum_{n=0}^{\infty} a(mn+t)q^n, \tag{2.1}$$

where

$$\ell=-\frac{1}{24}\sum_{\delta|M}\delta r_\delta.$$

Let $\phi(\tau)$ be a generalized eta-quotient, and let $F(\tau)=\phi(\tau)g_{m,t}(\tau)$. The objective of this section is to give a criterion for $F(\tau)$ to be a modular function for $\Gamma_1(N)$. We find that the transformation formula for $g_{m,t}(\tau)$ under $\Gamma_1(N)^*$ is analogous to the transformation formula of Radu [37, Lemma 2.14] with respect to $\Gamma_0(N)^*$. Then, we utilize the transformation laws of Newman [29] and Robins [41] to obtain the transformation formula of $F(\tau)$. With the aid of the Laurent expansions of $\phi(\tau)$ and $g_{m,t}(\tau)$, we obtain a necessary and sufficient condition for $F(\tau)$ to be a modular function for $\Gamma_1(N)$.

We first state the conditions on N. In fact, we make the following changes on the conditions on N given by Definition 34 and Definition 35 in [35]: change the condition $\delta|mN$ for every $\delta|M$ with $r_\delta \neq 0$ to $M|N$, and add the following condition 7. For completeness, we list all the conditions on N. Let $\kappa = \gcd(m^2-1, 24)$. Assume that N satisfies the following conditions:

1. $M|N$.
2. $p|N$ for any prime $p|m$.
3. $\kappa N \sum_{\delta|M} r_\delta \equiv 0 \pmod 8$.
4. $\kappa m N^2 \sum_{\delta|M} \frac{r_\delta}{\delta} \equiv 0 \pmod{24}$.
5. $\frac{24m}{\gcd(\kappa\alpha(t), 24m)} \,|N$, where $\alpha(t) = -\sum_{\delta|M} \delta r_\delta - 24t$.
6. Let $\prod_{\delta|M} \delta^{|r_\delta|} = 2^z j$, where $z \in \mathbb{N}$ and j is odd. If $2|m$, then $\kappa N \equiv 0 \pmod 4$ and $Nz \equiv 0 \pmod 8$, or $z \equiv 0 \pmod 2$ and $N(j-1) \equiv 0 \pmod 8$.
7. Let $\mathbb{S}_n = \{j^2 \pmod n \colon j \in \mathbb{Z}_n,\ \gcd(j,n) = 1,\ j \equiv 1 \pmod N\}$. For any $s \in \mathbb{S}_{24mM}$,
$$\frac{s-1}{24}\sum_{\delta|M} \delta r_\delta + ts \equiv t \pmod m.$$

Note that there always exists N satisfying the above conditions, because $N = 24mM$ would make a feasible choice. From now on, we denote by γ the matrix $\left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right)$.

Theorem 2.1. *For a given partition function $a(n)$ as defined by (1.1), and for given integers m and t, suppose that N is a positive integer satisfying the conditions 1–7. Let*
$$F(\tau) = \phi(\tau)\, g_{m,t}(\tau),$$
where
$$\phi(\tau) = \prod_{\delta|N} \eta^{a_\delta}(\delta\tau) \prod_{\substack{\delta|N \\ 0<g\le\lfloor\delta/2\rfloor}} \eta_{\delta,g}^{a_{\delta,g}}(\tau), \tag{2.2}$$
and a_δ and $a_{\delta,g}$ are integers. Then, $F(\tau)$ is a modular function with respect to $\Gamma_1(N)$ if and only if a_δ and $a_{\delta,g}$ satisfy the following conditions:

(1) $\sum_{\delta|N} a_\delta + \sum_{\delta|M} r_\delta = 0$.

(2) $N\sum_{\delta|N} \frac{a_\delta}{\delta} + 2N \sum_{\substack{\delta|N \\ 0<g\le\lfloor\delta/2\rfloor}} \frac{a_{\delta,g}}{\delta} + Nm\sum_{\delta|M}\frac{r_\delta}{\delta} \equiv 0 \pmod{24}$.

(3) $\sum_{\delta|N} \delta a_\delta + 12\sum_{\substack{\delta|N \\ 0<g\le\lfloor\delta/2\rfloor}} \delta P_2\left(\frac{g}{\delta}\right) a_{\delta,g} + m\sum_{\delta|M}\delta r_\delta + \frac{(m^2-1)\alpha(t)}{m} \equiv 0 \pmod{24}$.

(4) *For any integer $0 < a < 12N$ with $\gcd(a,6) = 1$ and $a \equiv 1 \pmod N$:*
$$\prod_{\delta|N}\left(\frac{\delta}{a}\right)^{|a_\delta|}\prod_{\delta|M}\left(\frac{m\delta}{a}\right)^{|r_\delta|} e^{\sum_{\delta|N}\sum_{g=1}^{\lfloor\delta/2\rfloor}\pi i\left(\frac{g}{\delta}-\frac{1}{2}\right)(a-1)a_{\delta,g}} = 1.$$

For example, consider the overpartition function $\overline{p}(n)$. Recall that an overpartition of a positive integer n is a partition of n where the first occurrence of each distinct part may be overlined, and the number of overpartitions of n

is denoted by $\overline{p}(n)$ for $n \geq 1$ and $\overline{p}(0) = 1$. As noted by Corteel and Lovejoy [11], the generating function of $\overline{p}(n)$ is given by:

$$\sum_{n=0}^{\infty} \overline{p}(n) q^n = \frac{(q^2;q^2)_\infty}{(q;q)_\infty^2}.$$

For the overpartition function $\overline{p}(5n+2)$, we see that $N = 10$ satisfies the conditions 1–7. Next, we proceed to find a generalized eta-quotient $\phi(\tau)$, such that $\phi(\tau) g_{5,2}(\tau)$ is a modular function for $\Gamma_1(10)$. By the above theorem, the function

$$\prod_{\delta|10} \eta^{a_\delta}(\delta\tau) \prod_{\substack{\delta|10 \\ 0<g\leq\lfloor\delta/2\rfloor}} \eta_{\delta,g}^{a_{\delta,g}}(\tau)\, g_{5,2}(\tau)$$

is a modular function for $\Gamma_1(10)$ if and only if a_δ and $a_{\delta,g}$ fulfill the following conditions:

$$\begin{cases} a_1 + a_2 + a_5 + a_{10} - 1 = 0, \\ 10a_1 + 5a_2 + 10a_{2,1} + 2a_5 + 4a_{5,1} + 4a_{5,2} + a_{10} + 2a_{10,1} \\ \qquad\qquad +2a_{10,2} + 2a_{10,3} + 2a_{10,4} + 2a_{10,5} - 3 \equiv 0 \pmod{24}, \\ a_1 + 2a_2 - 2a_{2,1} + 5a_5 + \frac{2a_{5,1}}{5} - \frac{22a_{5,2}}{5} + 10a_{10} + \frac{46a_{10,1}}{5} \\ \qquad\qquad + \frac{4a_{10,2}}{5} - \frac{26a_{10,3}}{5} - \frac{44a_{10,4}}{5} - 10a_{10,5} + \frac{48}{5} \equiv 0 \pmod{24}, \\ \left(\frac{10}{a}\right) \prod_{\delta|10} \left(\frac{\delta}{a}\right)^{|a_\delta|} e^{\sum_{\delta|10}\sum_{g=1}^{\lfloor\delta/2\rfloor} \pi i \left(\frac{g}{\delta}-\frac{1}{2}\right)(a-1)a_{\delta,g}} = 1, \end{cases} \tag{2.3}$$

for any $0 < a < 120$ with $\gcd(a,6) = 1$ and $a \equiv 1 \pmod{10}$. We find that

$$\begin{aligned}&(a_1, a_2, a_{2,1}, a_5, a_{5,1}, a_{5,2}, a_{10}, a_{10,1}, a_{10,2}, a_{10,3}, a_{10,4}, a_{10,5}) \\ &\quad = (0,0,0,0,0,0,1,0,0,0,-8,9)\end{aligned}$$

is an integer solution of (2.3). Let

$$\phi(\tau) = \frac{\eta(10\tau)\eta_{10,5}^9(\tau)}{\eta_{10,4}^8(\tau)}.$$

Since

$$g_{5,2}(\tau) = q^{\frac{2}{5}} \sum_{n=0}^{\infty} \overline{p}(5n+2) q^n,$$

we find that

$$F(\tau) = q^{\frac{2}{5}} \phi(\tau) \sum_{n=0}^{\infty} \overline{p}(5n+2) q^n \tag{2.4}$$

is a modular function with respect to $\Gamma_1(10)$.

Let

$$\Gamma_1(N)^* = \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \Gamma_1(N) \colon \gcd(a,6) = 1,\ ac > 0 \right\}.$$

The following lemma asserts that the invariance of the function $f(\tau)$ under $\Gamma_1(N)$ is equivalent to the invariance under $\Gamma_1(N)^*$.

Lemma 2.2. *Let k be an integer, N be a positive integer, and $f\colon \mathbb{H} \to \mathbb{C}$ be a function, such that*

$$f(\gamma\tau) = (c\tau + d)^k f(\tau) \tag{2.5}$$

for any $\gamma \in \Gamma_1(N)^*$. Then, f is weight-k invariant under $\Gamma_1(N)$.

Proof. Let

$$A = \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \Gamma_1(N)\colon\ \gcd(a, 6) = 1 \right\}.$$

By Lemma 3 of Newman [29], we know that $\Gamma_1(N)$ is generated by A. Hence, it suffices to show that

$$f(\gamma\tau) = (c\tau + d)^k f(\tau)$$

for any $\gamma \in A$. By the condition of Lemma 2.2, we may restrict our attention only to two cases. (1) $\gamma \in A$, $a > 0$ and $c \le 0$. (2) $\gamma \in A$, $a < 0$ and $c \ge 0$. Here, we only consider the first case, and the second case can be justified in the same manner. For the first case, since $a > 0$ and $c \le 0$, there exists a positive integer x, such that $ax + \frac{c}{N} > 0$. Let

$$\gamma_1 = \begin{pmatrix} 1 & 0 \\ Nx & 1 \end{pmatrix} \quad \text{and} \quad \gamma_2 = \begin{pmatrix} a & b \\ Nax + c & Nbx + d \end{pmatrix}.$$

Then, $\gamma_2 = \gamma_1\gamma$ and $\gamma_1 \in \Gamma_1(N)^*$. Therefore,

$$f(\gamma_2\tau) = f(\gamma_1(\gamma\tau)) = (Nx(\gamma\tau) + 1)^k f(\gamma\tau). \tag{2.6}$$

Since $\gamma \in A$, we have $\gcd(a, 6) = 1$, and so, $\gamma_2 \in \Gamma_1(N)^*$. Applying (2.5) with $\gamma_2 \in \Gamma_1(N)^*$, we get

$$f(\gamma_2\tau) = ((Nax + c)\tau + (Nbx + d))^k f(\tau). \tag{2.7}$$

Combining (2.6) and (2.7), we deduce that

$$f(\gamma\tau) = (c\tau + d)^k f(\tau),$$

as claimed. □

The following transformation formula for $g_{m,t}(\tau)$ under $\Gamma_1(N)^*$ is analogous to the transformation formula of Radu [37, Lemma 2.14] with respect to $\Gamma_0(N)^*$. The proof parallels that of Radu, and hence, it is omitted.

Lemma 2.3. *For a given partition function $a(n)$ as defined by (1.1), and for given integers m and t, let N be a positive integer satisfying the above conditions 1–7. For any $\gamma \in \Gamma_1(N)^*$, we have*

$$g_{m,t}(\gamma\tau) = (c\tau + d)^{\frac{1}{2}\sum_{\delta|M} r_\delta} e^{\pi i \zeta(\gamma)} \prod_{\delta|M} L(m\delta c, a)^{|r_\delta|} g_{m,t}(\tau),$$

where

$$L(c, a) = \begin{cases} \left(\frac{c}{a}\right), & \text{if } a > 0, \\ \left(\frac{-c}{-a}\right), & \text{otherwise,} \end{cases}$$

$(-)$ *is the Jacobi symbol,*

$$\zeta(\gamma) = \frac{ab(m^2-1)\alpha(t)}{12m} + \frac{abm}{12}\sum_{\delta|M}\delta r_\delta - \frac{acm}{12}\sum_{\delta|M}\frac{r_\delta}{\delta} + \frac{\operatorname{sgn}(c)\,(a-1)}{4}\sum_{\delta|M} r_\delta,$$

and $\alpha(t)$ *is defined as in the condition 5.*

Next, we derive a transformation formula for $F(\tau)$ under $\Gamma_1(N)^*$. Recall the notation of Schoeneberg [43]:

$$\eta_{g,h}^{(s)}(\tau) = \alpha_0(h) e^{\pi i P_2(\frac{g}{\delta})\tau} \prod_{\substack{m>0 \\ m\equiv g (\bmod \delta)}} \left(1-\zeta_\delta^h e^{\frac{2\pi i\tau}{\delta}m}\right) \prod_{\substack{m>0 \\ m\equiv -g (\bmod \delta)}} \left(1-\zeta_\delta^{-h} e^{\frac{2\pi i\tau}{\delta}m}\right), \tag{2.8}$$

where ζ_δ is a primitive δth root of unity,

$$\alpha_0(h) = \begin{cases} (1-\zeta_\delta^{-h})e^{\pi i P_1\left(\frac{h}{\delta}\right)}, & \text{if } g \equiv 0 \pmod{\delta} \text{ and } h \not\equiv 0 \pmod{\delta}, \\ 1, & \text{otherwise,} \end{cases}$$

the first Bernoulli function $P_1(x)$ is given by

$$P_1(x) = \begin{cases} x - \lfloor x \rfloor - \frac{1}{2}, & \text{if } x \notin \mathbb{Z}, \\ 0, & \text{otherwise,} \end{cases}$$

and $\lfloor x \rfloor$ is the greatest integer less than or equal to x. Since

$$\eta_{\delta,g}(\tau) = q^{\frac{\delta}{2}P_2\left(\frac{g}{\delta}\right)} \prod_{\substack{n>0 \\ n\equiv g (\bmod \delta)}} (1-q^n) \prod_{\substack{n>0 \\ n\equiv -g (\bmod \delta)}} (1-q^n),$$

we have

$$\eta_{\delta,g}(\tau) = \eta_{g,0}^{(s)}(\delta\tau). \tag{2.9}$$

Lemma 2.4. *For a given partition function* $a(n)$ *as defined by* (1.1), *and for given integers* m *and* t, *let* N *be a positive integer satisfying the conditions 1–7, and*

$$F(\tau) = \prod_{\delta|N} \eta^{a_\delta}(\delta\tau) \prod_{\substack{\delta|N \\ 0<g\le\lfloor\delta/2\rfloor}} \eta_{\delta,g}^{a_{\delta,g}}(\tau)\, g_{m,t}(\tau), \tag{2.10}$$

where a_δ *and* $a_{\delta,g}$ *are integers. Then, for any* $\gamma \in \Gamma_1(N)^*$,

$$F(\gamma\tau) = \prod_{\delta|N} L\left(\frac{c}{\delta}, a\right)^{a_\delta} \prod_{\delta|M} L(m\delta c, a)^{|r_\delta|} e^{\pi i(\nu(\gamma)+\xi(\gamma))}$$
$$\times\ (c\tau+d)^{\frac{1}{2}\left(\sum\limits_{\delta|N} a_\delta + \sum\limits_{\delta|M} r_\delta\right)} F(\tau), \tag{2.11}$$

where

$$\nu(\gamma) = \sum_{\substack{\delta|N \\ 0<g\le\lfloor\delta/2\rfloor}} \left(\frac{g}{\delta} - \frac{1}{2}\right)(a-1)a_{\delta,g} \tag{2.12}$$

and

$$\xi(\gamma) = \frac{a-1}{4}\operatorname{sgn}(c)\left(\sum_{\delta|N} a_\delta + \sum_{\delta|M} r_\delta\right)$$
$$- ac\left(\sum_{\delta|N}\frac{a_\delta}{12\delta} + \sum_{\substack{\delta|N\\0<g\le\lfloor\delta/2\rfloor}}\frac{a_{\delta,g}}{6\delta} + \sum_{\delta|M}\frac{mr_\delta}{12\delta}\right)$$
$$+ ab\left(\sum_{\delta|N}\frac{\delta a_\delta}{12} + \sum_{\substack{\delta|N\\0<g\le\lfloor\delta/2\rfloor}}\delta P_2\left(\frac{g}{\delta}\right)a_{\delta,g} + \sum_{\delta|M}\frac{m\delta r_\delta}{12} + \frac{(m^2-1)\alpha(t)}{12m}\right). \tag{2.13}$$

Proof. For any $\gamma = \left(\begin{smallmatrix} a & b\\ c & d\end{smallmatrix}\right) \in \Gamma_1(N)^*$, we have

$$F(\gamma\tau) = \prod_{\delta|N}\eta^{a_\delta}(\delta\gamma\tau)\prod_{\substack{\delta|N\\0<g\le\lfloor\delta/2\rfloor}}\eta_{\delta,g}^{a_{\delta,g}}(\gamma\tau)\,g_{m,t}(\gamma\tau). \tag{2.14}$$

For any $\delta|N$, let $\gamma'_\delta = \left(\begin{smallmatrix} a & \delta b\\ \frac{c}{\delta} & d\end{smallmatrix}\right)$. Since $\gamma \in \Gamma_1(N)^*$, we have $N|c$ and so $\delta|c$ for any $\delta|N$. It follows that $\gamma'_\delta \in \Gamma$. Thus, (2.14) can be written as:

$$F(\gamma\tau) = \prod_{\delta|N}\eta^{a_\delta}(\gamma'_\delta(\delta\tau))\prod_{\substack{\delta|N\\0<g\le\lfloor\delta/2\rfloor}}\eta_{\delta,g}^{a_{\delta,g}}(\gamma\tau)\,g_{m,t}(\gamma\tau). \tag{2.15}$$

The transformation formula of Newman [29, Lemma 2] states that for any $\gamma \in \Gamma$ with $a > 0$, $c > 0$, and $\gcd(a,6) = 1$,

$$\eta(\gamma\tau) = \left(\frac{c}{a}\right)e^{-\frac{a\pi i}{12}(c-b-3)}(-i(c\tau+d))^{\frac{1}{2}}\eta(\tau).$$

Therefore, for any $\gamma \in \Gamma$ with $ac > 0$ and $\gcd(a,6) = 1$, we have

$$\eta(\gamma\tau) = L(c,a)\,e^{\pi i\left(\frac{a}{12}(-c+b)+\frac{a-1}{4}\operatorname{sgn}(c)\right)}(c\tau+d)^{\frac{1}{2}}\eta(\tau). \tag{2.16}$$

Since $\gamma \in \Gamma_1(N)^*$, we see that $\gcd(a,6) = 1$ and $ac > 0$. Applying the transformation formula (2.16) to each γ'_δ, we deduce that

$$\prod_{\delta|N}\eta^{a_\delta}(\gamma'_\delta(\delta\tau)) = \prod_{\delta|N}L\left(\frac{c}{\delta},a\right)^{a_\delta}e^{\pi i\left(\frac{a}{12}\left(-\frac{c}{\delta}+\delta b\right)+\frac{a-1}{4}\operatorname{sgn}(c)\right)a_\delta}(c\tau+d)^{\frac{a_\delta}{2}}\eta^{a_\delta}(\delta\tau). \tag{2.17}$$

Using the transformation formula of Robins [41, Theorem 2]:

$$\eta_{\delta,g}(\gamma\tau) = e^{\pi i\left(\delta abP_2\left(\frac{g}{\delta}\right)-\frac{ac}{6\delta}+(a-1)\left(\frac{g}{\delta}-\frac{1}{2}\right)\right)}\eta_{\delta,g}(\tau),$$

we find that

$$\prod_{\substack{\delta|N\\0<g\le\lfloor\delta/2\rfloor}}\eta_{\delta,g}^{a_{\delta,g}}(\gamma\tau) = \prod_{\substack{\delta|N\\0<g\le\lfloor\delta/2\rfloor}}e^{\pi i\left(\delta abP_2\left(\frac{g}{\delta}\right)-\frac{ac}{6\delta}+(a-1)\left(\frac{g}{\delta}-\frac{1}{2}\right)\right)a_{\delta,g}}\eta_{\delta,g}^{a_{\delta,g}}(\tau). \tag{2.18}$$

Substituting the transformation formulas in (2.17), (2.18) and Lemma 2.3 into (2.15), we reach the transformation formula (2.11). $\square$

To prove Theorem 2.1, we need the Laurent expansions of $g_{m,t}(\gamma\tau)$ and $\phi(\gamma\tau)$ for $\gamma \in \Gamma$. Let us recall the two maps $p\colon \Gamma \times \mathbb{Z}_m \to \mathbb{Q}$ and $p\colon \Gamma \to \mathbb{Q}$ defined by Radu [37], namely, for $\gamma \in \Gamma$ and $\lambda \in \mathbb{Z}_m$:

$$p(\gamma, \lambda) = \frac{1}{24} \sum_{\delta | M} \frac{\gcd^2(\delta(a + \kappa\lambda c), mc)}{\delta m} r_\delta \tag{2.19}$$

and for $\gamma \in \Gamma$:

$$p(\gamma) = \min\{p(\gamma, \lambda)\colon \lambda = 0, 1, \ldots, m-1\}. \tag{2.20}$$

The parabolic subgroup of Γ is defined by

$$\Gamma_\infty = \left\{ \pm \begin{pmatrix} 1 & b \\ 0 & 1 \end{pmatrix} : b \in \mathbb{Z} \right\}.$$

For any $\gamma \in \Gamma$, the $(\Gamma_1(N), \Gamma_\infty)$-double coset of γ is given by

$$\Gamma_1(N)\gamma\Gamma_\infty = \{\gamma_N \gamma \gamma_\infty \colon \gamma_N \in \Gamma_1(N),\ \gamma_\infty \in \Gamma_\infty\}.$$

Assume that Γ has the following disjoint decomposition:

$$\Gamma = \bigcup_{i=1}^{\epsilon} \Gamma_1(N)\gamma_i\Gamma_\infty, \tag{2.21}$$

where $R = \{\gamma_1, \gamma_2, \ldots, \gamma_\epsilon\} \subseteq \Gamma$. Denote the set of $(\Gamma_1(N), \Gamma_\infty)$-double cosets in Γ by $\Gamma_1(N)\backslash\Gamma/\Gamma_\infty$. Then, (2.21) can be written as:

$$\Gamma_1(N)\backslash\Gamma/\Gamma_\infty = \{\Gamma_1(N)\gamma\Gamma_\infty \colon \gamma \in R\}.$$

We say that R is a complete set of representatives of the double cosets $\Gamma_1(N)\backslash \Gamma/\Gamma_\infty$.

The following lemma gives a Laurent expansion of $g_{m,t}(\gamma\tau)$, and the proof is similar to that of Lemma 3.4 in Radu [37], and hence, it is omitted.

Lemma 2.5. *For a given partition function $a(n)$ as defined by (1.1), and for given integers m and t, let N be a positive integer satisfying the conditions 1–7, and $R = \{\gamma_1, \gamma_2, \ldots, \gamma_\epsilon\}$ be a complete set of representatives of the double cosets $\Gamma_1(N)\backslash\Gamma/\Gamma_\infty$. For any $\gamma \in \Gamma$, assume that $\gamma \in \Gamma_1(N)\gamma_i\Gamma_\infty$ for some $1 \le i \le \epsilon$. Then, there exist a positive integer w and a Taylor series $h(q)$ in powers of $q^{\frac{1}{w}}$, such that*

$$g_{m,t}(\gamma\tau) = (c\tau + d)^{\frac{1}{2}\sum_{\delta|M} r_\delta} q^{p(\gamma_i)} h(q).$$

The following lemma gives a Laurent expansion of $\phi(\gamma\tau)$ for any $\gamma \in \Gamma$.

Lemma 2.6. *Let*

$$\phi(\tau) = \prod_{\delta|N} \eta^{a_\delta}(\delta\tau) \prod_{\substack{\delta|N \\ 0<g\le\lfloor\delta/2\rfloor}} \eta_{\delta,g}^{a_{\delta,g}}(\tau),$$

where a_δ and $a_{\delta,g}$ are integers. For any $\gamma \in \Gamma$, there exist a positive integer w and a Taylor series $h^*(q)$ in powers of $q^{\frac{1}{w}}$, such that

$$\phi(\gamma\tau) = (c\tau + d)^{\frac{1}{2}\sum_{\delta|N} a_\delta} q^{p^*(\gamma)} h^*(q),$$

where

$$p^*(\gamma) = \frac{1}{24}\sum_{\delta|N}\frac{\gcd^2(\delta,c)}{\delta}a_\delta + \frac{1}{2}\sum_{\substack{\delta|N\\0<g\le\lfloor\delta/2\rfloor}}\frac{\gcd^2(\delta,c)}{\delta}P_2\left(\frac{ag}{\gcd(\delta,c)}\right)a_{\delta,g}.$$

Furthermore, for any $\gamma_1\in\Gamma$ and $\gamma_2\in\Gamma_1(N)\gamma_1\Gamma_\infty$, we have $p^*(\gamma_1)=p^*(\gamma_2)$.

Proof. For any $\gamma=\left(\begin{smallmatrix}a&b\\c&d\end{smallmatrix}\right)\in\Gamma$, we have

$$\phi(\gamma\tau)=\prod_{\delta|N}\eta^{a_\delta}(\delta\gamma\tau)\prod_{\substack{\delta|N\\0<g\le\lfloor\delta/2\rfloor}}\eta_{\delta,g}^{a_{\delta,g}}(\gamma\tau).$$

It follows from (2.9) that

$$\phi(\gamma\tau)=\prod_{\delta|N}\eta^{a_\delta}(\delta\gamma\tau)\prod_{\substack{\delta|N\\0<g\le\lfloor\delta/2\rfloor}}{\eta_{g,0}^{(s)}}^{a_{\delta,g}}(\delta\gamma\tau). \tag{2.22}$$

Since $\gcd(a,c)=1$, for any $\delta|N$, there exist integers x_δ and y_δ, such that

$$\delta a x_\delta + c y_\delta = \gcd(\delta a,c)=\gcd(\delta,c),$$

and hence

$$\begin{pmatrix}\delta a & \delta b\\ c & d\end{pmatrix}=\begin{pmatrix}\frac{\delta a}{\gcd(\delta,c)} & -y_\delta\\ \frac{c}{\gcd(\delta,c)} & x_\delta\end{pmatrix}\begin{pmatrix}\gcd(\delta,c) & \delta b x_\delta + d y_\delta\\ 0 & \frac{\delta}{\gcd(\delta,c)}\end{pmatrix}. \tag{2.23}$$

Set

$$\gamma_\delta=\begin{pmatrix}\frac{\delta a}{\gcd(\delta,c)} & -y_\delta\\ \frac{c}{\gcd(\delta,c)} & x_\delta\end{pmatrix}\quad\text{and}\quad T_\delta=\begin{pmatrix}\gcd(\delta,c) & \delta b x_\delta + d y_\delta\\ 0 & \frac{\delta}{\gcd(\delta,c)}\end{pmatrix}.$$

Note that $\gamma_\delta\in\Gamma$. Combining (2.22) and (2.23), we deduce that

$$\phi(\gamma\tau)=\prod_{\delta|N}\eta^{a_\delta}(\gamma_\delta T_\delta\tau)\prod_{\substack{\delta|N\\0<g\le\lfloor\delta/2\rfloor}}{\eta_{g,0}^{(s)}}^{a_{\delta,g}}(\gamma_\delta T_\delta\tau). \tag{2.24}$$

By the transformation law for $\eta(\tau)$ under Γ [34, p. 145], namely, there exists a map $\varepsilon'\colon\Gamma\to\mathbb{C}$, such that for any $\gamma\in\Gamma$:

$$\eta(\gamma\tau)=\varepsilon'(\gamma)(c\tau+d)^{\frac{1}{2}}\eta(\tau),$$

and the transformation formula for $\eta_{g,h}^{(s)}(\tau)$ under Γ in [43, p. 199 (30)], namely, when $0<g<\delta$, there exists a map $\epsilon_1\colon\Gamma\to\mathbb{C}$, such that for any $\gamma\in\Gamma$:

$$\eta_{g,h}^{(s)}(\gamma\tau)=\epsilon_1(\gamma)\,\eta_{g',h'}^{(s)}(\tau),$$

where $g'=ag+ch$, $h'=bg+dh$, it follows from (2.24) that there is a map $\varepsilon\colon\Gamma\to\mathbb{C}$, such that for any $\gamma\in\Gamma$:

$$\phi(\gamma\tau)=\varepsilon(\gamma)(c\tau+d)^{\frac{1}{2}\sum_{\delta|N}a_\delta}\prod_{\delta|N}\eta^{a_\delta}(T_\delta\tau)\prod_{\substack{\delta|N\\0<g\le\lfloor\delta/2\rfloor}}{\eta^{(s)}_{\frac{\delta a}{\gcd(\delta,c)}g,-y_\delta g}}^{a_{\delta,g}}(T_\delta\tau). \tag{2.25}$$

Substituting the q-expansions of the eta-function and the generalized eta-function into (2.25), we see that there exist a positive integer w and a Taylor series $h^*(q)$ in powers of $q^{\frac{1}{w}}$, such that

$$\phi(\gamma\tau) = (c\tau+d)^{\frac{1}{2}\sum_{\delta|N} a_\delta} q^{p^*(\gamma)} h^*(q).$$

Next, we aim to show that $p^*(\gamma_1) = p^*(\gamma_2)$ for any $\gamma_1 \in \Gamma$ and $\gamma_2 \in \Gamma_1(N)\gamma_1\Gamma_\infty$. Under the assumption that $\gamma_2 \in \Gamma_1(N)\gamma_1\Gamma_\infty$, there exist $\gamma_3 \in \Gamma_1(N)$ and $\gamma_4 \in \Gamma_\infty$, such that

$$\gamma_2 = \gamma_3\gamma_1\gamma_4. \tag{2.26}$$

Write

$$\gamma_1 = \begin{pmatrix} a_1 & b_1 \\ c_1 & d_1 \end{pmatrix}, \quad \gamma_2 = \begin{pmatrix} a_2 & b_2 \\ c_2 & d_2 \end{pmatrix}, \quad \gamma_3 = \begin{pmatrix} a_3 & b_3 \\ c_3 & d_3 \end{pmatrix}, \quad \gamma_4 = \begin{pmatrix} \pm 1 & b_4 \\ 0 & \pm 1 \end{pmatrix}.$$

Owing to (2.26), we find that

$$a_2 = \pm(a_1a_3 + b_3c_1) \tag{2.27}$$

and

$$c_2 = \pm(a_1c_3 + c_1d_3). \tag{2.28}$$

For any $\delta|N$, since $\gamma_3 \in \Gamma_1(N)$, we see that $a_3 \equiv 1 \pmod{\delta}$, $\delta|c_3$ and $\gcd(\delta, d_3) = 1$. Using (2.27), it can be verified that

$$a_2g \equiv \pm a_1g \pmod{\gcd(\delta, c_1)}. \tag{2.29}$$

In view of (2.28), we obtain that

$$\gcd(\delta, c_2) = \gcd(\delta, c_1). \tag{2.30}$$

Combining (2.29) and (2.30), we arrive at

$$P_2\left(\frac{a_1g}{\gcd(\delta, c_1)}\right) = P_2\left(\frac{a_2g}{\gcd(\delta, c_2)}\right); \tag{2.31}$$

here, we have used the fact that $P_2(-\alpha) = P_2(\alpha)$ for any $\alpha \in \mathbb{R}$. Combining (2.30) and (2.31), we conclude that $p^*(\gamma_1) = p^*(\gamma_2)$, as claimed. □

We are now ready to complete the proof of Theorem 2.1.

Proof of Theorem 2.1. Assume that

$$F(\tau) = \prod_{\delta|N} \eta^{a_\delta}(\delta\tau) \prod_{\substack{\delta|N \\ 0<g\leq\lfloor\delta/2\rfloor}} \eta_{\delta,g}^{a_{\delta,g}}(\tau)\, g_{m,t}(\tau) \tag{2.32}$$

is a modular function with respect to $\Gamma_1(N)$, where a_δ and $a_{\delta,g}$ are integers. We proceed to show that the conditions (1)–(4) are fulfilled by the integers a_δ and $a_{\delta,g}$.

Since $\Gamma_1(N)^* \subseteq \Gamma_1(N)$ and $F(\tau)$ is a modular function for $\Gamma_1(N)$, for any $\gamma \in \Gamma_1(N)^*$, we have

$$F(\gamma\tau) = F(\tau). \tag{2.33}$$

To compute $F(\gamma\tau)$, we need the transformation formula for $F(\tau)$ under $\Gamma_1(N)^*$ as given in Lemma 2.4, that is, for any $\gamma \in \Gamma_1(N)^*$:

$$F(\gamma\tau) = \prod_{\delta|N} L\left(\frac{c}{\delta}, a\right)^{a_\delta} \prod_{\delta|M} L(m\delta c, a)^{|r_\delta|} e^{\pi i(\nu(\gamma)+\xi(\gamma))} \\ \times\ (c\tau + d)^{\frac{1}{2}\left(\sum_{\delta|N} a_\delta + \sum_{\delta|M} r_\delta\right)} F(\tau), \tag{2.34}$$

where $\nu(\gamma)$ and $\xi(\gamma)$ are defined in (2.12) and (2.13). Combining (2.33) and (2.34), we see that

$$\sum_{\delta|N} a_\delta + \sum_{\delta|M} r_\delta = 0,$$

and thus, (1) is satisfied. Consequently, $\xi(\gamma)$ reduces to

$$-\ ac\left(\sum_{\delta|N} \frac{a_\delta}{12\delta} + \sum_{\substack{\delta|N \\ 0<g\le\lfloor\delta/2\rfloor}} \frac{a_{\delta,g}}{6\delta} + \sum_{\delta|M} \frac{mr_\delta}{12\delta}\right) \\ +\ ab\left(\sum_{\delta|N} \frac{\delta a_\delta}{12} + \sum_{\substack{\delta|N \\ 0<g\le\lfloor\delta/2\rfloor}} \delta P_2\left(\frac{g}{\delta}\right) a_{\delta,g} + \sum_{\delta|M} \frac{m\delta r_\delta}{12} + \frac{(m^2-1)\alpha(t)}{12m}\right).$$

To prove (2), consider the matrix $\gamma = \left(\begin{smallmatrix} 1 & 0 \\ N & 1 \end{smallmatrix}\right) \in \Gamma_1(N)^*$. In this case, (2.34) becomes

$$F(\gamma\tau) = e^{-\pi i N\left(\sum_{\delta|N} \frac{a_\delta}{12\delta} + \sum_{\delta|N} \sum_{g=1}^{\lfloor\delta/2\rfloor} \frac{a_{\delta,g}}{6\delta} + \sum_{\delta|M} \frac{mr_\delta}{12\delta}\right)} F(\tau). \tag{2.35}$$

Hence, (2) follows from (2.33) and (2.35). Setting $\gamma = \left(\begin{smallmatrix} 1 & 1 \\ N & N+1 \end{smallmatrix}\right) \in \Gamma_1(N)^*$, (2.34) becomes

$$F(\gamma\tau) = e^{\pi i\left(\sum_{\delta|N} \frac{\delta a_\delta}{12} + \sum_{\delta|N} \sum_{g=1}^{\lfloor\delta/2\rfloor} \delta P_2\left(\frac{g}{\delta}\right) a_{\delta,g} + \sum_{\delta|M} \frac{m\delta r_\delta}{12} + \frac{(m^2-1)\alpha(t)}{12m}\right)} F(\tau),$$

which, together with (2.33), implies (3). Using the conditions (1)–(3), it can be checked that $\xi(\gamma) \equiv 0 \pmod 2$ for any $\gamma \in \Gamma_1(N)^*$. It follows that:

$$e^{\pi i \xi(\gamma)} = 1,$$

and so, (2.34) reduces to

$$F(\gamma\tau) = \prod_{\delta|N} L\left(\frac{c}{\delta}, a\right)^{a_\delta} \prod_{\delta|M} L(m\delta c, a)^{|r_\delta|} e^{\pi i \nu(\gamma)} (c\tau + d)^{\frac{1}{2}\left(\sum_{\delta|N} a_\delta + \sum_{\delta|M} r_\delta\right)} F(\tau). \tag{2.36}$$

By the definition of L, we find that for any $\delta|N$,

$$L\left(\frac{c}{\delta}, a\right) = L(\delta c, a) = \left(\frac{\delta|c|}{|a|}\right),$$

and for any $\delta|M$,

$$L(m\delta c, a) = \left(\frac{m\delta|c|}{|a|}\right).$$

Hence, (2.36) is equivalent to

$$F(\gamma\tau)=\prod_{\delta|N}\left(\frac{\delta|c|}{|a|}\right)^{|a_\delta|}\prod_{\delta|M}\left(\frac{m\delta|c|}{|a|}\right)^{|r_\delta|}e^{\pi i\nu(\gamma)}\,(c\tau+d)^{\frac{1}{2}\left(\sum_{\delta|N}a_\delta+\sum_{\delta|M}r_\delta\right)}F(\tau). \tag{2.37}$$

In view of the condition (1), it is easily verified that

$$(c\tau+d)^{\frac{1}{2}\left(\sum_{\delta|N}a_\delta+\sum_{\delta|M}r_\delta\right)}=1 \tag{2.38}$$

and

$$\prod_{\delta|N}\left(\frac{|c|}{|a|}\right)^{|a_\delta|}\prod_{\delta|M}\left(\frac{|c|}{|a|}\right)^{|r_\delta|}=1. \tag{2.39}$$

Substituting (2.38) and (2.39) into (2.37) yields:

$$F(\gamma\tau)=\prod_{\delta|N}\left(\frac{\delta}{|a|}\right)^{|a_\delta|}\prod_{\delta|M}\left(\frac{m\delta}{|a|}\right)^{|r_\delta|}e^{\pi i\nu(\gamma)}\,F(\tau). \tag{2.40}$$

Comparing (2.33) with (2.40), we deduce that

$$\prod_{\delta|N}\left(\frac{\delta}{|a|}\right)^{|a_\delta|}\prod_{\delta|M}\left(\frac{m\delta}{|a|}\right)^{|r_\delta|}e^{\pi i\nu(\gamma)}=1 \tag{2.41}$$

for all integers a with $\gcd(a,6)=1$ and $a\equiv 1 \pmod N$. Invoking the interpretation of the Jacobi symbol, we conclude that (2.41) holds for all integers $0<a<12N$ with $\gcd(a,6)=1$ and $a\equiv 1 \pmod N$. This confirms (4).

Conversely, assume that the integers a_δ, $a_{\delta,g}$ ($\delta|N, 0<g\le\lfloor\delta/2\rfloor$) satisfy the conditions (1)–(4). We proceed to show that

$$F(\tau)=\prod_{\delta|N}\eta^{a_\delta}(\delta\tau)\prod_{\substack{\delta|N\\0<g\le\lfloor\delta/2\rfloor}}\eta_{\delta,g}^{a_{\delta,g}}(\tau)\,g_{m,t}(\tau)$$

is a modular function for $\Gamma_1(N)$. It is clear that $F(\tau)$ is holomorphic on $\mathbb{H}$.

Based on the conditions (1)–(3), it follows from Lemma 2.4 that the transformation formula (2.40) for $F(\tau)$ holds for any $\gamma\in\Gamma_1(N)^*$. Given the condition (4), we see that (2.41) holds for all integers a with $\gcd(a,6)=1$ and $a\equiv 1 \pmod N$. Combining (2.40) and (2.41), we find that for any $\gamma\in\Gamma_1(N)^*$,

$$F(\gamma\tau)=F(\tau).$$

In view of Lemma 2.2, we conclude that $F(\gamma\tau)=F(\tau)$ for any $\gamma\in\Gamma_1(N)$.

It remains to show that for any $\gamma\in\Gamma$, $F(\gamma\tau)$ has a Laurent expansion with a finite principal part in powers of $q^{\frac{1}{N}}$. Let $\gamma\in\Gamma$ and $R=\{\gamma_1,\gamma_2,\ldots,\gamma_\epsilon\}$ be a complete set of representatives of the double cosets $\Gamma_1(N)\backslash\Gamma/\Gamma_\infty$. By the decomposition of Γ in (2.21), there exist an integer $1\le i\le\epsilon$ and matrices $\gamma_N\in\Gamma_1(N)$, $\gamma_\infty\in\Gamma_\infty$, such that $\gamma=\gamma_N\gamma_i\gamma_\infty$. By Lemmas 2.5 and 2.6, there exist a positive integer w and Taylor series $h(q)$ and $h^*(q)$ in powers of $q^{\frac{1}{w}}$, such that

$$F(\gamma\tau) = (c\tau+d)^{\frac{1}{2}\left(\sum_{\delta|N} a_\delta + \sum_{\delta|M} r_\delta\right)} q^{p(\gamma_i)+p^*(\gamma_i)} h(q)\, h^*(q). \tag{2.42}$$

In view of the condition (1), (2.42) reduces to

$$F(\gamma\tau) = q^{p(\gamma_i)+p^*(\gamma_i)} h(q)\, h^*(q), \tag{2.43}$$

which implies that there exists a positive integer k, such that $F(\gamma\tau)$ has the Laurent expansion with a finite principal part in powers of $q^{\frac{1}{k}}$. Since we have shown that $F(\tau)$ is invariant under $\Gamma_1(N)$, by Lemma 1.14 in [48], we obtain that for any $\gamma \in \Gamma$, $F(\gamma\tau)$ is invariant under $\gamma^{-1}\Gamma_1(N)\gamma$. Notice that $\left(\begin{smallmatrix} 1 & N \\ 0 & 1 \end{smallmatrix}\right) \in \gamma^{-1}\Gamma_1(N)\gamma$. Therefore, $F(\gamma\tau)$ has period N, namely,

$$F(\gamma(\tau+N)) = F(\gamma\tau).$$

Thus, $F(\gamma\tau)$ has a Laurent expansion in powers of $q^{\frac{1}{N}}$. By (2.43), we see that this Laurent expansion has at most finitely many negative terms. Therefore, we reach the assertion that $F(\tau)$ is a modular function for $\Gamma_1(N)$. $\square$

Given a generating function of $a(n)$ as defined in (1.1) and integers m and t, we can find an integer N satisfying the conditions 1–7. If we are lucky, we may use Theorem 2.1 to find integers a_δ, $a_{\delta,g}$ $(\delta|N, 0 < g \le \lfloor \delta/2 \rfloor)$ satisfying the conditions (1)–(4), which lead to a generalized eta-quotient:

$$\phi(\tau) = \prod_{\delta|N} \eta^{a_\delta}(\delta\tau) \prod_{\substack{\delta|N \\ 0<g\le\lfloor\delta/2\rfloor}} \eta_{\delta,g}^{a_{\delta,g}}(\tau),$$

such that

$$F(\tau) = \phi(\tau)\, g_{m,t}(\tau) \tag{2.44}$$

is a modular function. It should be noted that such a modular function $F(\tau)$ may be not unique. To derive a Ramanujan-type identity for $a(mn+t)$, we aim to express $F(\tau)$ as a linear combination of generalized eta-quotients over $\mathbb{Q}$. To this end, we first investigate the behavior of $F(\tau)$ at each cusp of $\Gamma_1(N)$. Let us recall some terminology of modular functions, see, for example [12, 48]. For $\gamma = \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \in \Gamma$, the width w_γ of $\frac{a}{c}$ relative to $\Gamma_1(N)$ is the minimal positive integer h, such that

$$\begin{pmatrix} 1 & h \\ 0 & 1 \end{pmatrix} \in \gamma^{-1}\Gamma_1(N)\gamma.$$

Let $f(\tau)$ be a modular function for $\Gamma_1(N)$. It is known that $f(\gamma\tau)$ is invariant under $\gamma^{-1}\Gamma_1(N)\gamma$, see [48, Lemma 1.14]. Therefore, $f(\gamma\tau)$ has period w_γ, which implies that $f(\gamma\tau)$ has a Laurent expansion in powers of q^{1/w_γ}. Since $f(\tau)$ is a modular function, this Laurent expansion has at most finitely many negative terms. Write

$$f(\gamma\tau) = \sum_{n=-\infty}^{\infty} b_n q^{n/w_\gamma}, \tag{2.45}$$

where $b_n = 0$ for almost all negative integers n. Let n_γ be the smallest integer, such that $b_{n_\gamma} \neq 0$. We call n_γ the γ-order of f at $\frac{a}{c}$, denoted by $\mathrm{ord}_\gamma(f)$. Denote the smallest exponent of q on the right-hand side of (2.45) by v_γ, so that

$$\mathrm{ord}_\gamma(f) = v_\gamma w_\gamma. \tag{2.46}$$

Furthermore, the order of f at the cusp $\frac{a}{c} \in \mathbb{Q} \cup \{\infty\}$ is defined by

$$\mathrm{ord}_{a/c}(f) = \mathrm{ord}_\gamma(f) \tag{2.47}$$

for some $\gamma \in \Gamma$, such that $\gamma\infty = \frac{a}{c}$. It is known that $\mathrm{ord}_{a/c}(f)$ is well defined (see [12, p. 72]).

The following theorem gives estimates of the orders of $F(\tau)$ at cusps of $\Gamma_1(N)$.

Theorem 2.7. *For a given partition function $a(n)$ as defined by (1.1), and for given integers m and t, let*

$$F(\tau) = \phi(\tau)\, g_{m,t}(\tau),$$

where

$$\phi(\tau) = \prod_{\delta|N} \eta^{a_\delta}(\delta\tau) \prod_{\substack{\delta|N \\ 0<g\leq\lfloor\delta/2\rfloor}} \eta_{\delta,g}^{a_{\delta,g}}(\tau),$$

a_δ and $a_{\delta,g}$ are integers. Assume that $F(\tau)$ is a modular function for $\Gamma_1(N)$. Let $\{s_1, s_2, \ldots, s_\epsilon\}$ be a complete set of inequivalent cusps of $\Gamma_1(N)$, and for each $1 \leq i \leq \epsilon$, let $\alpha_i \in \Gamma$ be such that $\alpha_i\infty = s_i$. Then

$$\mathrm{ord}_{s_i}(F(\tau)) \geq w_{\alpha_i}\,(p(\alpha_i) + p^*(\alpha_i)), \tag{2.48}$$

where $p(\gamma)$ is given by (2.20) and $p^*(\gamma)$ is defined in Lemma 2.6.

To compute the right-hand side of (2.48), we need the following formula due to Cho, Koo and Park [10]:

$$w_\gamma = \begin{cases} 1, & \text{if } N = 4 \text{ and } \gcd(c,4) = 2, \\ \frac{N}{\gcd(c,N)}, & \text{otherwise,} \end{cases} \tag{2.49}$$

where $\gamma = \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \in \Gamma$. For example, consider the modular function

$$F(\tau) = q^{\frac{2}{5}} \frac{\eta(10\tau)\eta_{10,5}^9(\tau)}{\eta_{10,4}^8(\tau)} \sum_{n=0}^{\infty} \overline{p}(5n+2)q^n$$

for $\Gamma_1(10)$ as given in (2.4). A complete set $\mathcal{S}(N)$ of inequivalent cusps of $\Gamma_1(N)$ has been found in [10, Corollary 4]. In particular, for $N = 10$, we have

$$\mathcal{S}(10) = \left\{0,\ \frac{1}{5},\ \frac{1}{4},\ \frac{3}{10},\ \frac{1}{3},\ \frac{3}{5},\ \frac{1}{2},\ \infty\right\}. \tag{2.50}$$

Employing Theorem 2.7, we obtain the following lower bounds of the orders of $F(\tau)$ at cusps of $\Gamma_1(10)$:

$$\begin{aligned}&\mathrm{ord}_0(F(\tau)) \geq -3,\ \mathrm{ord}_{1/5}(F(\tau)) \geq \frac{19}{5},\ \mathrm{ord}_{1/4}(F(\tau)) \geq -2,\\&\mathrm{ord}_{3/10}(F(\tau)) \geq -\frac{18}{5},\ \mathrm{ord}_{1/3}(F(\tau)) \geq -3,\ \mathrm{ord}_{3/5}(F(\tau)) \geq \frac{27}{5},\\&\mathrm{ord}_{1/2}(F(\tau)) \geq -2,\ \mathrm{ord}_{\infty}(F(\tau)) \geq -\frac{2}{5}.\end{aligned}$$

Notice that $F(\tau)$ may have poles at some cusps not equivalent to infinity.

We are now ready to prove Theorem 2.7.

Proof of Theorem 2.7. It is known that there exists a bijection from the set of all inequivalent cusps of $\Gamma_1(N)$ to the double coset space $\Gamma_1(N)\backslash\Gamma/\Gamma_\infty$, as given by

$$\Gamma_1(N)(a/c) \mapsto \Gamma_1(N)\begin{pmatrix} a & b \\ c & d \end{pmatrix}\Gamma_\infty,$$

see [12, Proposition 3.8.5]. Since $\{s_1, s_2, \ldots, s_\epsilon\}$ is a complete set of inequivalent cusps of $\Gamma_1(N)$ and $\alpha_i\infty = s_i$ for $1 \leq i \leq \epsilon$, we see that $\{\alpha_1, \alpha_2, \ldots, \alpha_\epsilon\}$ is a complete set of representatives of $\Gamma_1(N)\backslash\Gamma/\Gamma_\infty$. Applying Lemma 2.5 with $\gamma_i = \alpha_i$, we find that there exist a positive integer w_1 and a Taylor series $h(q)$ in powers of $q^{\frac{1}{w_1}}$, such that

$$g_{m,t}(\alpha_i\tau) = (c\tau + d)^{\frac{1}{2}\sum_{\delta|M} r_\delta} q^{p(\alpha_i)} h(q). \tag{2.51}$$

By Lemma 2.6, there exist a positive integer w_2 and a Taylor series $h^*(q)$ in powers of $q^{\frac{1}{w_2}}$, such that

$$\phi(\alpha_i\tau) = (c\tau + d)^{\frac{1}{2}\sum_{\delta|N} a_\delta} q^{p^*(\alpha_i)} h^*(q). \tag{2.52}$$

Combining (2.51) and (2.52), we get

$$F(\alpha_i\tau) = (c\tau + d)^{\frac{1}{2}\left(\sum_{\delta|N} a_\delta + \sum_{\delta|M} r_\delta\right)} q^{p(\alpha_i)+p^*(\alpha_i)}\, h(q)\, h^*(q). \tag{2.53}$$

Since $F(\tau)$ is a modular function for $\Gamma_1(N)$, using the condition (1) in Theorem 2.1, (2.53) reduces to

$$F(\alpha_i\tau) = q^{p(\alpha_i)+p^*(\alpha_i)}\, h(q)\, h^*(q). \tag{2.54}$$

Let v_{α_i} denote the smallest exponent of q on the right-hand side of (2.54). The relation $\mathrm{ord}_\gamma(f) = v_\gamma w_\gamma$ as given in (2.46) yields

$$v_{\alpha_i} = \frac{\mathrm{ord}_{\alpha_i}(F(\tau))}{w_{\alpha_i}}. \tag{2.55}$$

Since $h(q)$ and $h^*(q)$ are Taylor series, it follows from (2.54) that

$$v_{\alpha_i} \geq p(\alpha_i) + p^*(\alpha_i). \tag{2.56}$$

Combining (2.55) and (2.56), we conclude that

$$\mathrm{ord}_{\alpha_i}(F(\tau)) \geq w_{\alpha_i}\,(p(\alpha_i) + p^*(\alpha_i)). \tag{2.57}$$

By the definition (2.47), we have

$$\mathrm{ord}_{s_i}(F(\tau)) = \mathrm{ord}_{\alpha_i}(F(\tau)). \tag{2.58}$$

Thus, the estimate (2.48) follows from (2.57) and (2.58). □

3. Sketch of the Algorithm

In this section, we give a sketch of our algorithm. Given a generating function of $a(n)$ as defined in (1.1) and integers m and t, we can find an integer N satisfying the conditions 1–7. Assume that we have found a generalized eta-quotient $\phi(\tau)$, such that

$$F(\tau) = \phi(\tau)\, g_{m,t}(\tau) \tag{3.1}$$

is a modular function for $\Gamma_1(N)$. To derive an expression of $F(\tau)$, we consider a class of modular functions: the set of generalized eta-quotients which are modular functions for $\Gamma_1(N)$ with poles only at infinity, denoted by $GE^\infty(N)$. Note that the notation $E^\infty(N)$ is used by Radu [35] to denote the set of modular eta-quotients with poles only at infinity for $\Gamma_0(N)$. Our goal is to derive an expression of $F(\tau)$ in terms of the generators of $GE^\infty(N)$. Then, we are led to a Ramanujan-type identity for $a(mn+t)$.

Our algorithm consists of the following steps:

Step 1 Use Theorem 2.1 to find a generalized eta-quotient $\phi(\tau)$ for which $F(\tau)$ in (3.1) is a modular function for $\Gamma_1(N)$.

Step 2 Find a finite set $\{z_1, z_2, \ldots, z_k\}$ of generators of $GE^\infty(N)$ by utilizing a formula of Robins and the theory of Diophantine inequalities.

Step 3 Let $\langle GE^\infty(N)\rangle_{\mathbb{Q}}$ be the vector space over $\mathbb{Q}$ generated by generalized eta-quotients in $GE^\infty(N)$. Employ the Algorithm AB of Radu for $\Gamma_1(N)$ on $\{z_1, z_2, \ldots, z_k\}$ to generate a modular function z and a $\mathbb{Q}[z]$-module basis $1, e_1, \ldots, e_w$ of $\langle GE^\infty(N)\rangle_{\mathbb{Q}}$.

Step 4 Find a generalized eta-quotient h in terms of generators of $GE^\infty(N)$ for which the modular function hF has a pole only at infinity. Theorem 2.7 can be used to compute the lower bounds of the orders of hF at all cusps of $\Gamma_1(N)$.

Step 5 Determine whether hF is in $\langle GE^\infty(N)\rangle_{\mathbb{Q}}$ by applying the Algorithm MW of Radu to hF, z, and $1, e_1, \ldots, e_w$. If this goal can be achieved, then F can be expressed as a linear combination of generalized eta-quotients over $\mathbb{Q}$.

For example, let us consider the overpartition function $\overline{p}(5n+2)$. In Sect. 2, we found that $N = 10$ satisfies the conditions 1–7.

Step 1 As shown in (2.4):

$$F(\tau) = q^{\frac{2}{5}} \frac{\eta(10\tau)\eta_{10,5}^9(\tau)}{\eta_{10,4}^8(\tau)} \sum_{n=0}^{\infty} \overline{p}(5n+2)q^n$$

is a modular function with respect to $\Gamma_1(10)$.

Step 2 We obtain the following generators of $GE^\infty(10)$:

$$\begin{aligned}
z &= \frac{\eta(\tau)\eta(5\tau)}{\eta_{5,1}^2(\tau)\eta^2(10\tau)\eta_{10,1}(\tau)}, \quad z_1 = \frac{\eta^2(2\tau)\eta(5\tau)\eta_{5,1}^2(\tau)}{\eta(\tau)\eta^2(10\tau)\eta_{10,1}^4(\tau)},\\
z_2 &= \frac{\eta^3(5\tau)\eta_{5,1}^4(\tau)}{\eta(\tau)\eta(2\tau)\eta(10\tau)\eta_{10,1}^3(\tau)}, \quad z_3 = \frac{\eta(\tau)\eta_{5,1}^2(\tau)\eta^2(10\tau)}{\eta^2(2\tau)\eta(5\tau)\eta_{10,1}^4(\tau)},\\
z_4 &= \frac{\eta^4(\tau)\eta_{5,1}^2(\tau)}{\eta^3(2\tau)\eta(10\tau)\eta_{10,1}^4(\tau)}.
\end{aligned} \tag{3.2}$$

Step 3 Applying the Algorithm AB of Radu to $\{z, z_1, z_2, z_3, z_4\}$, we find that 1 is a z-module basis of $\langle GE^\infty(10)\rangle_{\mathbb{Q}}$. Thus

$$\langle GE^\infty(10)\rangle_{\mathbb{Q}} = \langle 1\rangle_{\mathbb{Q}[z]}\,. \tag{3.3}$$

Step 4 We obtain that

$$h = \frac{z_1^2 z_3^3 z_4^3}{z^6 z_2^4} = \frac{\eta^{11}(\tau)\eta_{5,1}^{12}(\tau)\eta^{15}(10\tau)}{\eta^7(2\tau)\eta^{19}(5\tau)\eta_{10,1}^{14}(\tau)}, \tag{3.4}$$

for which hF has a pole only at infinity.

Step 5 Applying Radu's Algorithm MW to hF, z, and 1, we see that $hF \in \langle GE^\infty(10)\rangle_{\mathbb{Q}}$ and

$$hF = 4z^3 + 4z^2 - 32z + 32. \tag{3.5}$$

The relation (3.5) can be restated as the following theorem. The implementations of the above steps will be described in the subsequent sections.

Theorem 3.1. *We have*

$$y\sum_{n=0}^{\infty} \overline{p}(5n+2)q^n = 4z^3 + 4z^2 - 32z + 32, \tag{3.6}$$

where

$$\begin{aligned}
y &= \frac{(q;q)_\infty^{11}(q^{10};q^{10})_\infty^{16}(q,q^4;q^5)_\infty^{12}(q^5;q^{10})_\infty^{18}}{q^3(q^2;q^2)_\infty^7(q^5;q^5)_\infty^{19}(q,q^9;q^{10})_\infty^{14}(q^4,q^6;q^{10})_\infty^8},\\
z &= \frac{(q;q)_\infty(q^5;q^5)_\infty}{q(q,q^4;q^5)_\infty^2(q^{10};q^{10})_\infty^2(q,q^9;q^{10})_\infty}.
\end{aligned}$$

4. Generators of $GE^\infty(N)$

In this section, we show how to implement Step 1 as in the sketch of the previous section, that is, finding a finite set of generators of $GE^\infty(N)$.

In light of the symmetry

$$\eta_{\delta,g}(\tau) = \eta_{\delta,\delta-g}(\tau),$$

for any $\delta > 0$ and $\lfloor\delta/2\rfloor < g \le \delta$, we may rewrite the generalized eta-quotient $h(\tau)$ in $GE^\infty(N)$ in the following form:

$$\prod_{\substack{\delta\mid N\\ 0\le g\le\lfloor\delta/2\rfloor}} \eta_{\delta,g}^{a_{\delta,g}}(\tau), \tag{4.1}$$

where

$$a_{\delta,g} \in \begin{cases} \frac{1}{2}\mathbb{Z}, & \text{if } g = 0 \text{ or } g = \frac{\delta}{2}, \\ \mathbb{Z}, & \text{otherwise}. \end{cases} \tag{4.2}$$

Throughout this section, we assume that the generalized eta-quotients are of the form (4.1).

To find a set of generators of $GE^\infty(N)$, we first give a characterization of generalized eta-quotients $h(\tau)$ in $GE^\infty(N)$, which involves the orders of $h(\tau)$ at all cusps of $\Gamma_1(N)$. For any cusp s of $\Gamma_1(N)$, to apply a formula of Robins [41, Theorem 4] to compute the order of $h(\tau)$ at a cusp s, we need to find a cusp of the form $\frac{\lambda}{\mu\varepsilon}$ that is equivalent to s, where $\varepsilon|N$ and

$$\gcd(\lambda, N) = \gcd(\lambda, \mu) = \gcd(\mu, N) = 1. \tag{4.3}$$

The existence of such a cusp in the above form is ensured by Corollary 4 of Cho, Koo, and Park [10].

The following theorem gives a characterization of generalized eta-quotients in $GE^\infty(N)$.

Theorem 4.1. *Let*

$$\mathcal{S}(N) = \{s_1, s_2, \ldots, s_\epsilon\}$$

be a complete set of inequivalent cusps of $\Gamma_1(N)$ *and* $s_\epsilon = \infty$. *Assume that for any* $1 \le i \le \epsilon$, s_i *is equivalent to* $\frac{\lambda_i}{\mu_i\varepsilon_i}$, *where* $\varepsilon_i|N$ *and*

$$\gcd(\lambda_i, N) = \gcd(\lambda_i, \mu_i) = \gcd(\mu_i, N) = 1. \tag{4.4}$$

Then, a generalized eta-quotient $h(\tau)$ *in the form of* (4.1) *is in* $GE^\infty(N)$ *if and only if the following conditions hold:*

$$\begin{cases} \sum_{\delta|N} a_{\delta,0} = 0, \\ \frac{N}{2} \sum_{\substack{\delta|N \\ 0\le g\le \lfloor \delta/2 \rfloor}} \frac{\gcd^2(\delta,\varepsilon_1)}{\delta\varepsilon_1} P_2\left(\frac{\lambda_1 g}{\gcd(\delta,\varepsilon_1)}\right) a_{\delta,g} \in \mathbb{N}, \\ \quad \vdots \\ \frac{N}{2} \sum_{\substack{\delta|N \\ 0\le g\le \lfloor \delta/2 \rfloor}} \frac{\gcd^2(\delta,\varepsilon_{\epsilon-1})}{\delta\varepsilon_{\epsilon-1}} P_2\left(\frac{\lambda_{\epsilon-1} g}{\gcd(\delta,\varepsilon_{\epsilon-1})}\right) a_{\delta,g} \in \mathbb{N}, \\ \frac{N}{2} \sum_{\substack{\delta|N \\ 0\le g\le \lfloor \delta/2 \rfloor}} \frac{\gcd^2(\delta,\varepsilon_\epsilon)}{\delta\varepsilon_\epsilon} P_2\left(\frac{\lambda_\epsilon g}{\gcd(\delta,\varepsilon_\epsilon)}\right) a_{\delta,g} \in \mathbb{Z}. \end{cases} \tag{4.5}$$

Proof. Assume that the generalized eta-quotient $h(\tau)$ as given by (4.1) is in $GE^\infty(N)$. By the transformation formula of Schoeneberg [43, p. 199 (30)] for $\eta_{g,h}^{(s)}(\tau)$, we have

$$\sum_{\delta|N} a_{\delta,0} = 0, \tag{4.6}$$

and so, the first condition in (4.5) is satisfied. To show that the remaining conditions in (4.5) are satisfied, we proceed to compute the order of $h(\tau)$ at each cusp in $\mathcal{S}(N)$. Since $h(\tau) \in GE^\infty(N)$, for all $1 \le i \le \epsilon - 1$,

$$\mathrm{ord}_{s_i}(h(\tau)) \in \mathbb{N} \tag{4.7}$$

and

$$\mathrm{ord}_{s_\epsilon}(h(\tau)) \in \mathbb{Z}. \tag{4.8}$$

For any $1 \leq i \leq \epsilon$, since s_i is equivalent to $\frac{\lambda_i}{\mu_i \varepsilon_i}$, we get

$$\mathrm{ord}_{s_i}(h(\tau)) = \mathrm{ord}_{\lambda_i/\mu_i\varepsilon_i}(h(\tau)).$$

Using the formula of Robins [41, Theorem 4] for the order of $h(\tau)$ at the cusp $\lambda_i/\mu_i\varepsilon_i$, namely,

$$\mathrm{ord}_{\lambda_i/\mu_i\varepsilon_i}(h(\tau)) = \frac{N}{2} \sum_{\substack{\delta|N \\ 0 \leq g \leq \lfloor \delta/2 \rfloor}} \frac{\gcd^2(\delta, \varepsilon_i)}{\delta\varepsilon_i} P_2\left(\frac{\lambda_i g}{\gcd(\delta, \varepsilon_i)}\right) a_{\delta,g},$$

we find that

$$\mathrm{ord}_{s_i}(h(\tau)) = \frac{N}{2} \sum_{\substack{\delta|N \\ 0 \leq g \leq \lfloor \delta/2 \rfloor}} \frac{\gcd^2(\delta, \varepsilon_i)}{\delta\varepsilon_i} P_2\left(\frac{\lambda_i g}{\gcd(\delta, \varepsilon_i)}\right) a_{\delta,g}. \tag{4.9}$$

For $1 \leq i \leq \epsilon - 1$, combining (4.7) and (4.9), we obtain that

$$\frac{N}{2} \sum_{\substack{\delta|N \\ 0 \leq g \leq \lfloor \delta/2 \rfloor}} \frac{\gcd^2(\delta, \varepsilon_i)}{\delta\varepsilon_i} P_2\left(\frac{\lambda_i g}{\gcd(\delta, \varepsilon_i)}\right) a_{\delta,g} \in \mathbb{N}. \tag{4.10}$$

Setting $i = \epsilon$ in (4.9), it follows from (4.8) that

$$\frac{N}{2} \sum_{\substack{\delta|N \\ 0 \leq g \leq \lfloor \delta/2 \rfloor}} \frac{\gcd^2(\delta, \varepsilon_\epsilon)}{\delta\varepsilon_\epsilon} P_2\left(\frac{\lambda_\epsilon g}{\gcd(\delta, \varepsilon_\epsilon)}\right) a_{\delta,g} \in \mathbb{Z}. \tag{4.11}$$

Combining (4.6), (4.10), and (4.11), we are led to (4.5).

Conversely, assume that the conditions in (4.5) are satisfied. From (4.5) and (4.9), we see that

$$\mathrm{ord}_0(h(\tau)) \in \mathbb{Z} \quad \text{and} \quad \mathrm{ord}_\infty(h(\tau)) \in \mathbb{Z}. \tag{4.12}$$

The first condition of (4.5) says that

$$\sum_{\delta|N} a_{\delta,0} = 0. \tag{4.13}$$

Robins [41] showed that if a generalized eta-quotient $h(\tau)$ satisfies (4.12) and (4.13), then for any $\gamma \in \Gamma_1(N)$:

$$h(\gamma\tau) = h(\tau). \tag{4.14}$$

By (4.9) and the conditions in (4.5), we see that for any $s \in \mathcal{S}(N) \setminus \{\infty\}$,

$$\mathrm{ord}_s(h(\tau)) \in \mathbb{N}. \tag{4.15}$$

Combining (4.14) and (4.15), we conclude that $h(\tau) \in GE^\infty(N)$. □

Based on the above theorem, the generalized eta-quotients in $GE^\infty(N)$ are determined by the solutions of (4.5). Next, we show that (4.5) can be solved by transforming the conditions in (4.5) to a system of Diophantine inequalities, so that we can obtain a finite set of generators of $GE^\infty(N)$.

Set

$$y_i = \frac{N}{2} \sum_{\substack{\delta | N \\ 0 \le g \le \lfloor \delta/2 \rfloor}} \frac{\gcd^2(\delta, \varepsilon_i)}{\delta \varepsilon_i} P_2\left(\frac{\lambda_i g}{\gcd(\delta, \varepsilon_i)}\right) a_{\delta,g}$$

for $1 \le i \le \epsilon$. It follows from (4.5) that $y_i \in \mathbb{N}$ for $1 \le i \le \epsilon - 1$ and $y_\epsilon \in \mathbb{Z}$. Let

$$\chi_\delta(g) = \begin{cases} 2, & \text{if } g = 0 \text{ or } g = \frac{\delta}{2}, \\ 1, & \text{otherwise,} \end{cases}$$

and $a'_{\delta,g} = \chi_\delta(g)\, a_{\delta,g}$ for any $\delta | N$ and $0 \le g \le \lfloor \delta/2 \rfloor$. By (4.2), it can be easily checked that each $a'_{\delta,g}$ is an integer. Then, by Theorem 4.1, $h(\tau) \in GE^\infty(N)$ if and only if $a'_{\delta,g}$ $(\delta | N, 0 \le g \le \lfloor \delta/2 \rfloor)$ and y_i $(1 \le i \le \epsilon)$ is an integer solution of the following Diophantine inequalities:

$$\begin{cases} \sum_{\delta | N} a'_{\delta,0} = 0, \\ \frac{N}{2} \sum_{\substack{\delta | N \\ 0 \le g \le \lfloor \delta/2 \rfloor}} \frac{\gcd^2(\delta,\varepsilon_1)}{\delta\varepsilon_1} P_2\left(\frac{\lambda_1 g}{\gcd(\delta,\varepsilon_1)}\right) \frac{a'_{\delta,g}}{\chi_\delta(g)} - y_1 = 0, \\ \quad \vdots \\ \frac{N}{2} \sum_{\substack{\delta | N \\ 0 \le g \le \lfloor \delta/2 \rfloor}} \frac{\gcd^2(\delta,\varepsilon_{\epsilon-1})}{\delta\varepsilon_{\epsilon-1}} P_2\left(\frac{\lambda_{\epsilon-1} g}{\gcd(\delta,\varepsilon_{\epsilon-1})}\right) \frac{a'_{\delta,g}}{\chi_\delta(g)} - y_{\epsilon-1} = 0, \\ \frac{N}{2} \sum_{\substack{\delta | N \\ 0 \le g \le \lfloor \delta/2 \rfloor}} \frac{\gcd^2(\delta,\varepsilon_\epsilon)}{\delta\varepsilon_\epsilon} P_2\left(\frac{\lambda_\epsilon g}{\gcd(\delta,\varepsilon_\epsilon)}\right) \frac{a'_{\delta,g}}{\chi_\delta(g)} - y_\epsilon = 0, \\ y_1 \ge 0, \\ \quad \vdots \\ y_{\epsilon-1} \ge 0. \end{cases} \tag{4.16}$$

Notice that different cusps may have the same order for $h(\tau)$, there may exist redundant relations in above system of relations. More precisely, if for two cusps $s_i, s_j \in \mathcal{S}(N) \backslash \{\infty\}$,

$$\mathrm{ord}_{s_i}(h(\tau)) = \mathrm{ord}_{s_j}(h(\tau)),$$

then we may ignore the relations contributed by s_j. We now assume that after the elimination of redundant relations, the remaining relations are still in the same form as in (4.16). It is known that there exist integral vectors $\alpha_1, \ldots, \alpha_k$, such that the set of integer solutions of (4.16) is given by

$$\{u_1\alpha_1 + \cdots + u_k\alpha_k \colon u_1, \ldots, u_k \in \mathbb{N}\},$$

see [44, p. 234], which implies that $GE^\infty(N)$ has a finite set of generators $z_1, \ldots, z_k$. One can use the package `4ti2` [1] in SAGE to find such a set of integral vectors $\alpha_1, \ldots, \alpha_k$.

Let us consider the case $N = 10$ as an example. Notice that for any generalized eta-quotient $h(\tau)$:

$$\mathrm{ord}_{1/4}(h(\tau)) = \mathrm{ord}_{1/2}(h(\tau))$$

and

$$\mathrm{ord}_0(h(\tau)) = \mathrm{ord}_{1/3}(h(\tau)).$$

By (4.16), we obtain the following Diophantine inequalities after eliminating the relations contributed by the cusps 1/2 and 1/3:

$$\begin{cases} a'_{1,0} + a'_{2,0} + a'_{5,0} + a'_{10,0} = 0, \\ \frac{5\,a'_{1,0}}{12} + \frac{5\,a'_{2,0}}{24} + \frac{5\,a'_{2,1}}{24} + \frac{a'_{5,0}}{12} + \frac{a_{5,1}}{6} + \frac{a_{5,2}}{6} \\ \quad + \frac{a'_{10,0}}{24} + \frac{a_{10,1}}{12} + \frac{a_{10,2}}{12} + \frac{a_{10,3}}{12} + \frac{a_{10,4}}{12} + \frac{a'_{10,5}}{24} - y_1 = 0, \\ \quad \vdots \\ \frac{5\,a'_{1,0}}{24} + \frac{5\,a'_{2,0}}{12} - \frac{5\,a'_{2,1}}{24} + \frac{a'_{5,0}}{24} + \frac{a'_{5,1}}{12} + \frac{a'_{5,2}}{12} \\ \quad + \frac{a'_{10,0}}{12} - \frac{a'_{10,1}}{12} + \frac{a'_{10,2}}{6} - \frac{a'_{10,3}}{12} + \frac{a'_{10,4}}{6} - \frac{a'_{10,5}}{24} - y_5 = 0, \\ \frac{a'_{1,0}}{24} + \frac{a'_{2,0}}{12} - \frac{a'_{2,1}}{24} + \frac{5\,a'_{5,0}}{24} + \frac{a'_{5,1}}{60} - \frac{11\,a'_{5,2}}{60} \\ \quad + \frac{5\,a'_{10,0}}{12} + \frac{23\,a'_{10,1}}{60} + \frac{a'_{10,2}}{30} - \frac{13\,a'_{10,3}}{60} - \frac{11\,a'_{10,4}}{30} - \frac{5\,a'_{10,5}}{24} - y_6 = 0 \\ y_1 \geq 0, \\ \quad \vdots \\ y_5 \geq 0. \end{cases} \tag{4.17}$$

Each solution $(a'_{1,0}, \ldots, a'_{10,5}, y_1, \ldots, y_6)$ of (4.17) can be expressed as:

$$\sum_{i=1}^{5} c_i \alpha_i + \sum_{i=1}^{6} d_i \beta_i, \tag{4.18}$$

where $c_1, \ldots, c_5$ are nonnegative integers, $d_1, \ldots, d_6$ are integers, and

$$\begin{aligned}
\alpha_1 &= (-1, 2, 0, 1, 2, 0, -2, -4, 0, 0, 0, 0, 0, 1, 0, 0, 0, -2), \\
\alpha_2 &= (-1, -1, 0, 3, 4, 0, -1, -3, 0, 0, 0, 0, 0, 0, 1, 0, 0, -1), \\
\alpha_3 &= (1, -2, 0, -1, 2, 0, 2, -4, 0, 0, 0, 0, 0, 0, 0, 0, 1, -1), \\
\alpha_4 &= (1, 0, 0, 1, -2, 0, -2, -1, 0, 0, 0, 0, 0, 0, 0, 1, 0, -1), \\
\alpha_5 &= (4, -3, 0, 0, 2, 0, -1, -4, 0, 0, 0, 0, 1, 0, 0, 0, 0, -2), \\
\beta_1 &= (0, 0, 0, -1, 0, 0, 1, 0, 0, 0, 0, 1, 0, 0, 0, 0, 0, 0), \\
\beta_2 &= (-1, 1, 1, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0), \\
\beta_3 &= (-1, 0, 0, 1, 1, 1, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0), \\
\beta_4 &= (0, -1, 0, 0, 1, 0, 1, -1, 1, 0, 0, 0, 0, 0, 0, 0, 0, 0), \\
\beta_5 &= (-1, 1, 0, 1, 0, 0, -1, 1, 0, 1, 0, 0, 0, 0, 0, 0, 0, 0), \\
\beta_6 &= (0, 0, 0, 0, -1, 0, 0, 1, 0, 0, 1, 0, 0, 0, 0, 0, 0, 0).
\end{aligned}$$

Since $a_{\delta,g} = a'_{\delta,g}/\chi_\delta(g)$, we obtain 11 generalized eta-quotients. It can be checked that the generalized eta-quotients corresponding to $\beta_1, \ldots, \beta_6$ are equal to 1. For example, the generalized eta-quotient corresponding to β_1 is given by

$$h(\tau) = \frac{\eta_{10,0}^{\frac{1}{2}}(\tau)\eta_{10,5}^{\frac{1}{2}}(\tau)}{\eta_{5,0}^{\frac{1}{2}}(\tau)}. \tag{4.19}$$

Invoking (1.12), namely,

$$\eta_{\delta,0}(\tau) = \eta^2(\delta\tau) \quad \text{and} \quad \eta_{\delta,\frac{\delta}{2}}(\tau) = \frac{\eta^2(\frac{\delta}{2}\tau)}{\eta^2(\delta\tau)},$$

we obtain that $h(\tau) = 1$. The generalized eta-quotients corresponding to $\alpha_1, \ldots, \alpha_5$ are the generators z_1, z_2, z_3, z, z_4 as given in (3.2).

5. Radu's Algorithm AB

In the previous section, it was shown that $GE^\infty(N)$ admits a finite set of generators $z_1, \ldots, z_k$. Radu [37] developed the Algorithm AB to produce a module basis of $\langle E^\infty(N)\rangle_{\mathbb{Q}}$, based on a finite set of generators of $E^\infty(N)$. In this section, we demonstrate how to apply Radu's Algorithm AB to a finite set of generators of $GE^\infty(N)$ to derive a modular function z and a module basis $1, e_1, \ldots, e_w$ of the $\mathbb{Q}[z]$-module $\langle GE^\infty(N)\rangle_{\mathbb{Q}}$.

We first give an overview of Radu's Algorithm AB. Given modular functions $z_1, \ldots, z_k$ for $\Gamma_0(N)$ with poles only at infinity, Radu's Algorithm AB aims to produce a modular function $z \in \mathbb{Q}[z_1, \ldots, z_k]$ and a z-reduced sequence $e_1, \ldots, e_w \in \mathbb{Q}[z_1, \ldots, z_k]$, such that

$$\mathbb{Q}[z_1, \ldots, z_k] = \mathbb{Q}[z] + \mathbb{Q}[z]e_1 + \cdots + \mathbb{Q}[z]e_w. \tag{5.1}$$

The condition on a z-reduced sequence ensures that $1, e_1, \ldots, e_w$ form a $\mathbb{Q}[z]$-module basis of $\mathbb{Q}[z_1, \ldots, z_k]$. The right-hand side of (5.1) is denoted by $\langle 1, e_1, \ldots, e_w\rangle_{\mathbb{Q}[z]}$.

Let $\langle E^\infty(N)\rangle_{\mathbb{Q}}$ denote the vector space over $\mathbb{Q}$ generated by $E^\infty(N)$. As pointed out by Radu [35], $\langle E^\infty(N)\rangle_{\mathbb{Q}}$ does not have a finite basis as a vector space over $\mathbb{Q}$, but it has a finite basis when considered as a $\mathbb{Q}[z]$-module for some z in $\langle E^\infty(N)\rangle_{\mathbb{Q}}$. To obtain such a modular function z and a $\mathbb{Q}[z]$-module basis, Radu applied the Algorithm AB to the generators $z_1, \ldots, z_k$ of $E^\infty(N)$ and then obtained a z-module basis $1, e_1, \ldots, e_w$ of the $\mathbb{Q}[z]$-module $\langle E^\infty(N)\rangle_{\mathbb{Q}}$ for some $z \in \langle E^\infty(N)\rangle_{\mathbb{Q}}$.

As will be seen, Radu's Algorithm AB can be adapted to $\Gamma_1(N)$. The output of Algorithm AB consists of a modular function $z \in \mathbb{Q}[z_1, \ldots, z_k]$ and a z-reduced sequence $e_1, \ldots, e_w$. The output of the Algorithm AB will be carried over to the Algorithm MC and the Algorithm MW, which require the input of a z-reduced sequence. Thus, for the purpose of this paper, we do not need to elaborate on the definition of a z-reduced sequence, which can be found in [35].

It is known that if f is a modular function for $\Gamma_0(N)$ such that $\mathrm{ord}_{a/c}(f) \geq 0$ for every cusp a/c of $\Gamma_0(N)$, then f is a constant, see Newman [28, Section, Proof of Lemma 3], Knopp [25, Chapter 2, Theorem 7], and Radu [35, Lemma 5]. Notice that this assertion also holds for $\Gamma_1(N)$. Thus, the Algorithm AB applies to modular functions with poles only at infinity for $\Gamma_1(N)$. It is worth mentioning that the Algorithm AB is based on the algorithms MC, VB, and MB, which are also valid for modular functions with poles only at infinity for $\Gamma_1(N)$. Since the Algorithm MW of Radu is a refinement of the Algorithm MC, it also works for $\Gamma_1(N)$.

We proceed to find a modular function z and a module basis of $\mathbb{Q}[z]$-module $\langle GE^\infty(N)\rangle_\mathbb{Q}$. Let $\{z_1, \dots, z_k\}$ be a finite set of generators of $GE^\infty(N)$. Note that

$$\langle GE^\infty(N)\rangle_\mathbb{Q} = \mathbb{Q}[z_1, \dots, z_k]. \tag{5.2}$$

Applying the Algorithm AB to $z_1, z_2, \dots, z_k$, we obtain a modular function $z \in \langle GE^\infty(N)\rangle_\mathbb{Q}$ and a z-reduced sequence $e_1, \dots, e_w \in \langle GE^\infty(N)\rangle_\mathbb{Q}$, such that

$$\mathbb{Q}[z_1, \dots, z_k] = \langle 1, e_1, \dots, e_w\rangle_{\mathbb{Q}[z]}. \tag{5.3}$$

Combining (5.2) and (5.3), we find that

$$\langle GE^\infty(N)\rangle_\mathbb{Q} = \langle 1, e_1, \dots, e_w\rangle_{\mathbb{Q}[z]}.$$

Using the property that $e_1, e_2, \dots, e_w$ form a z-reduced sequence, we deduce that $1, e_1, \dots, e_w$ constitute a $\mathbb{Q}[z]$-module basis of $\langle GE^\infty(N)\rangle_\mathbb{Q}$.

For example, applying the Algorithm AB for $\Gamma_1(N)$ to the generators z, z_1, z_2, z_3, z_4 of $GE^\infty(10)$ given by (3.2), we obtain that

$$\langle GE^\infty(10)\rangle_\mathbb{Q} = \mathbb{Q}[z]. \tag{5.4}$$

6. Finding a Generalized Eta-Quotient

In this section, we present an implementation of Step 4 in the algorithm outlined in Sect. 3. Assume that $\{z_1, z_2, \dots, z_k\}$ is a set of generators of $GE^\infty(N)$ and $F(\tau)$ is a modular function for $\Gamma_1(N)$ as given in (3.1). Our objective is to find a generalized eta-quotient $h(\tau)$ of the form:

$$h(\tau) = \prod_{j=1}^{k} z_j^{t_j}, \tag{6.1}$$

such that the modular function hF has a pole only at infinity, that is, for any cusp $s \neq \infty$,

$$\mathrm{ord}_s(hF) \geq 0, \tag{6.2}$$

where t_j are integers. To find the integers t_j for which the relation (6.2) holds, we shall establish a system of linear inequalities any solution of which leads to a desired generalized eta-quotient h. The linear inequalities are derived by the lower bounds of $\mathrm{ord}_s(hF)$ for all cusps $s \neq \infty$.

Now, we utilize Theorem 2.7 to obtain the lower bound of $\mathrm{ord}_s(hF)$. Let

$$\mathcal{S}(N) = \{s_1, s_2, \dots, s_\epsilon\}$$

be a complete set of inequivalent cusps of $\Gamma_1(N)$ and $s_\epsilon = \infty$. For any $1 \le i \le \epsilon$ and $1 \le j \le k$, denote $\mathrm{ord}_{s_i} z_j$ by b_{ij}. By the definition (6.1), we have for each cusp s_i,

$$\mathrm{ord}_{s_i}(hF) = \sum_{j=1}^{k} t_j b_{ij} + \mathrm{ord}_{s_i}(F). \tag{6.3}$$

By Theorem 2.7, we see that for any $1 \le i \le \epsilon$,

$$\mathrm{ord}_{s_i}(F(\tau)) \ge d_i, \tag{6.4}$$

where

$$d_i = w_{\alpha_i}\,(p(\alpha_i) + p^*(\alpha_i)),$$

and α_i is defined in Theorem 2.7. Combining (6.3) and (6.4), we get

$$\mathrm{ord}_{s_i}(hF) \ge \sum_{j=1}^{k} t_j b_{ij} + d_i. \tag{6.5}$$

Consider the Diophantine inequalities

$$\begin{cases} \sum\limits_{j=1}^{k} t_j b_{1j} + d_1 > -1, \\ \quad\vdots \\ \sum\limits_{j=1}^{k} t_j b_{(\epsilon-1)j} + d_{\epsilon-1} > -1. \end{cases} \tag{6.6}$$

Now, if we can find integers $t_1, \ldots, t_k$, such that (6.6) holds, then (6.5) implies that the generalized eta-quotient $h(\tau)$ determined by $z_1, z_2, \ldots, z_k$ and $t_1, t_2, \ldots, t_k$ satisfies (6.2). Hence, we deduce that any integer solution of (6.6) leads to a generalized eta-quotient $h(\tau)$, such that hF has a pole only at infinity.

We note that different generalized eta-quotients h may lead to different expressions for F. To get a relatively simple expression for F, we impose a further condition that the order of hF at infinity is as large as possible. While we cannot rigorously describe what a simple expression means, intuitively speaking, the above condition appears to play a role in getting a relatively simple expression for F. Next, we state how to find such a generalized eta-quotient $h(\tau)$.

It is known that there exist integral vectors $\alpha_1, \ldots, \alpha_w, \beta_1, \ldots, \beta_l$, such that the set of integer solutions of (6.6) is given by

$$\{\alpha_i + v_1\beta_1 + \cdots + v_l\beta_l \colon 1 \le i \le w \text{ and } v_1, \ldots, v_l \in \mathbb{N}\}, \tag{6.7}$$

see [44, p. 234].

The following theorem shows how to find a generalized eta-quotient h, such that $\mathrm{ord}_\infty(hF)$ attains the maximum value among all the h satisfying (6.6).

Theorem 6.1. *For $1 \le i \le w$, let*

$$\alpha_i = (\alpha_{i1}, \alpha_{i2}, \ldots, \alpha_{ik}),$$

as given in (6.7). Let h_i be the generalized eta-quotient determined by $z_1, z_2, \ldots, z_k$ and α_i, that is,

$$h_i(\tau) = \prod_{j=1}^{k} z_j^{\alpha_{ij}}. \tag{6.8}$$

Assume that

$$\mathrm{ord}_\infty(h_1 F) \ge \mathrm{ord}_\infty(h_i F) \tag{6.9}$$

for $2 \le i \le w$. For any integer solution $\mu = (\mu_1, \mu_2, \ldots, \mu_k)$ of (6.6), let g be the generalized eta-quotient

$$g(\tau) = \prod_{j=1}^{k} z_j^{\mu_j}. \tag{6.10}$$

Then, we have

$$\mathrm{ord}_\infty(h_1 F) \ge \mathrm{ord}_\infty(g F).$$

Proof. By (6.7), there exist an integer $1 \le i \le w$, and nonnegative integers $v_1, \ldots, v_l$, such that

$$\mu = \alpha_i + v_1\beta_1 + \cdots + v_l\beta_l. \tag{6.11}$$

For $1 \le j \le l$, let

$$\beta_j = (\beta_{j1}, \beta_{j2}, \ldots, \beta_{jk}),$$

and let f_j be the generalized eta-quotient defined by

$$f_j(\tau) = \prod_{i=1}^{k} z_i^{\beta_{ji}}. \tag{6.12}$$

Combining (6.8), (6.11), and (6.12), we obtain that

$$g(\tau) = h_i \prod_{j=1}^{l} f_j^{v_j}.$$

Thus:

$$\mathrm{ord}_\infty(gF) = \mathrm{ord}_\infty(h_i F) + \sum_{j=1}^{l} v_j \mathrm{ord}_\infty(f_j). \tag{6.13}$$

Under the condition (6.9), it follows from (6.13) that

$$\mathrm{ord}_\infty(gF) \le \mathrm{ord}_\infty(h_1 F) + \sum_{j=1}^{l} v_j \mathrm{ord}_\infty(f_j). \tag{6.14}$$

We claim that for each $1 \le j \le l$,

$$\mathrm{ord}_\infty(f_j) \le 0. \tag{6.15}$$

There are two cases.

Case 1. If $f_j(\tau)$ is a constant, then $\mathrm{ord}_\infty(f_j) = 0$.

Case 2. If $f_j(\tau)$ is not a constant, we shall show that $\mathrm{ord}_\infty(f_j) < 0$. Assume to the contrary that $\mathrm{ord}_\infty(f_j) \geq 0$. Since $f_j(\tau)$ is not a constant, there exists a cusp $s \neq \infty$, such that $\mathrm{ord}_s(f_j) < 0$. By the assumption (6.2), we have $\mathrm{ord}_s(h_1 F) \geq 0$. Let $d = \mathrm{ord}_s(h_1 F)$. By (6.7), we see that $\alpha_1 + (d+1)\beta_j$ is a solution of (6.6). It follows that the generalized eta-quotient $f_j^{d+1} h_1$ satisfies (6.2), and so

$$\mathrm{ord}_s(f_j^{d+1} h_1 F) \geq 0. \tag{6.16}$$

However, since $\mathrm{ord}_s(f_j) < 0$, we have

$$\mathrm{ord}_s(f_j^{d+1} h_1 F) = (d+1)\mathrm{ord}_s(f_j) + d < 0,$$

which contradicts (6.16). Thus, we deduce that $\mathrm{ord}_\infty(f_j) < 0$, as claimed. Combining the above two cases, we find that (6.15) holds for each $1 \leq j \leq l$. In view of (6.14), we conclude that

$$\mathrm{ord}_\infty(gF) \leq \mathrm{ord}_\infty(h_1 F), \tag{6.17}$$

and this completes the proof. $\square$

For the overpartition function $\overline{p}(5n+2)$, we have found a modular function $F(\tau)$ for $\Gamma_1(10)$ as given in (2.4). For the generators z, z_1, z_2, z_3, z_4 of $GE^\infty(10)$ as given in (3.2), we obtain the following system of linear inequalities (6.6):

$$\begin{cases} t_5 - 3 > -1, \\ t_3 + \frac{19}{5} > -1, \\ t_2 - 2 > -1, \\ t_4 - \frac{18}{5} > -1, \\ t_1 + \frac{27}{5} > -1. \end{cases} \tag{6.18}$$

Each integer solution $(t_1, t_2, t_3, t_4, t_5)$ of (6.18) can be expressed as

$$\alpha_1 + \sum_{i=1}^{5} v_i \beta_i, \tag{6.19}$$

where $v_1, \ldots, v_5$ are nonnegative integers, and

$$\begin{aligned} \alpha_1 &= (-6, 2, -4, 3, 3), \\ \beta_1 &= (1, 0, 0, 0, 0), \\ \beta_2 &= (0, 1, 0, 0, 0), \\ \beta_3 &= (0, 0, 1, 0, 0), \\ \beta_4 &= (0, 0, 0, 1, 0), \\ \beta_5 &= (0, 0, 0, 0, 1). \end{aligned}$$

The generalized eta-quotient corresponding to α_1 is

$$h = \frac{z_1^2 z_3^3 z_4^3}{z^6 z_2^4} = \frac{\eta^{11}(\tau)\eta_{5,1}^{12}(\tau)\eta^{15}(10\tau)}{\eta^7(2\tau)\eta^{19}(5\tau)\eta_{10,1}^{14}(\tau)}, \tag{6.20}$$

and hF has a pole only at infinity. Consider a different solution $\mu = \alpha_1 + 2\beta_2 = (6, 4, -4, 3, 3)$ of (6.18), we get a generalized eta-quotient:

$$h' = \frac{z_1^4 z_3^3 z_4^3}{z^6 z_2^4} = \frac{\eta^9(\tau)\eta_{5,1}^{16}(\tau)\eta^{11}(10\tau)}{\eta^3(2\tau)\eta^{17}(5\tau)\eta_{10,1}^{22}(\tau)}, \tag{6.21}$$

and $h'F$ has a pole only at infinity. The orders of hF and $h'F$ at infinity are -3 and -7, respectively. As will be seen in the next section, the Ramanujan-type identity derived from hF takes a simpler form than that derived from $h'F$.

7. Ramanujan-Type Identities

Given a partition function $a(n)$ as defined by (1.1), and integers m and t, let

$$F(\tau) = \phi(\tau)\, g_{m,t}(\tau) \tag{7.1}$$

be a modular function as given in (3.1), where $\phi(\tau)$ is a generalized eta-quotient of the form (2.2), and

$$g_{m,t}(\tau) = q^{\frac{t-\ell}{m}} \sum_{n=0}^{\infty} a(mn+t)q^n,$$

as given in (2.1).

Assume that we have found a generalized eta-quotient $h(\tau)$, such that hF has a pole only at infinity. In Sect. 5, we derived a modular function $z \in \langle GE^{\infty}(N)\rangle_{\mathbb{Q}}$ and a z-reduced sequence $e_1, \ldots, e_w$, such that

$$\langle GE^{\infty}(N)\rangle_{\mathbb{Q}} = \mathbb{Q}[z] + \mathbb{Q}[z]e_1 + \cdots + \mathbb{Q}[z]e_w.$$

In this section, we aim to derive an expression for hF in terms of z and the module basis $1, e_1, \ldots, e_w$. This leads to a Ramanujan-type identity for $a(mn+t)$.

We first adapt Radu's Algorithm MC, original designed for $\Gamma_0(N)$, to $\Gamma_1(N)$, and apply it to hF, z, and $e_1, \ldots, e_w$ to determine whether hF belongs to $\langle GE^{\infty}(N)\rangle_{\mathbb{Q}}$. By Radu [35, Lemma 5], the Algorithm MC requires the non-positive parts of the q-expansion of hF, and finite parts of the q-expansions of z, and $e_1, \ldots, e_w$. More precisely, by (7.1), the non-positive parts of the q-expansion of hF can be computed via the generating function (1.1) of $a(n)$ and the q-expansions of $h(\tau)$ and $\phi(\tau)$. If the algorithm confirms that $hF \in \langle GE^{\infty}(N)\rangle_{\mathbb{Q}}$, then we may utilize the $\Gamma_1(N)$ version of Algorithm MW to express hF as

$$hF = p_0(z) + p_1(z)e_1 + \cdots + p_w(z)e_w, \tag{7.2}$$

where $p_i(z) \in \mathbb{Q}[z]$ for $0 \le i \le w$.

To this end, we first utilize the Radu's Algorithm MC for $\Gamma_1(N)$ to determine whether hF belongs to $\langle GE^{\infty}(N)\rangle_{\mathbb{Q}}$. Once we have confirmed that $hF \in \langle GE^{\infty}(N)\rangle_{\mathbb{Q}}$, we may utilize the Algorithm MW of Radu for $\Gamma_1(N)$ to derive a Ramanujan-type identity for $a(mn+t)$.

We now give an algorithmic derivation of the Ramanujan-type identity for $\overline{p}(5n+2)$, as stated in Theorem 3.1.

For F, z, and h given in (2.4), (3.2), and (6.20), we have

$$hF = \frac{4}{q^3} + \frac{28}{q^2} + \frac{56}{q} + 140 + O(q),$$
$$z = \frac{1}{q} + 2 + 2q + q^2 + O(q^3).$$

Applying Radu's Algorithm MC to hF and z, we deduce that

$$hF \in \langle GE^\infty(10)\rangle_{\mathbb{Q}}.$$

With the input hF and z, the Algorithm MW yields

$$hF = 4z^3 + 4z^2 - 32z + 32. \tag{7.3}$$

Substituting F, z, and h into (7.3), we obtain the Ramanujan-type identity in Theorem 3.1. However, if we take h' as given in (6.21), then we get

$$h'F = 4z^7 - 4z^6 - 44z^5 + 100z^4 - 20z^3 - 92z^2 + 32z + 32. \tag{7.4}$$

In the same vain, we obtain a Ramanujan-type identity for $\overline{p}(5n+3)$.

Theorem 7.1. *We have*

$$y\sum_{n=0}^{\infty} \overline{p}(5n+3)q^n = 8z^3 - 12z^2 + 16z - 16,$$

where z is given in Theorem 3.1 and

$$y = \frac{(q;q)_\infty^{12}(q^5;q^5)_\infty^{12}(q,q^9;q^{10})_\infty^2(q^4,q^6;q^{10})_\infty^8}{q^3(q^2;q^2)_\infty^7(q,q^4;q^5)_\infty^6(q^{10};q^{10})_\infty^{16}(q^5;q^{10})_\infty^{14}}.$$

Notice that Theorems 3.1 and 7.1 can be considered as witness identities for the following congruences of Hirschhorn and Sellers [24]:

$$\overline{p}(5n+2) \equiv 0 \pmod 4,$$
$$\overline{p}(5n+3) \equiv 0 \pmod 4.$$

8. A Witness Identity for $p(11n+6)$

In this section, we demonstrate how our algorithm gives rise to a witness identity for $p(11n+6)$. We begin with an overview of the witness identities due to Bilgici and Ekin [8], Radu [35], and Hemmecke [20]. Bilgici and Ekin [8] used the method of Kolberg to deduce the generating functions of $p(11n+t)$ for all $0 \le t \le 10$. In particular, they obtained the following witness identity:

$$\begin{aligned}\sum_{n=0}^{\infty} p(11n+6)q^n =& 11x(-x_1^3x_4 - x_2^3x_5 - x_4^3x_2 - x_3^3x_1 - x_5^3x_3 - 14x_1^2x_4 \\ &- 14x_2^2x_5 - 14x_4^2x_2 - 14x_3^2x_1 - 14x_5^2x_3 - 29x_1x_4 \\ &- 29x_2x_5 - 29x_2x_4 - 29x_1x_3 - 29x_3x_5 + 106),\end{aligned} \tag{8.1}$$

where

$$
\begin{aligned}
x &= \frac{q^4(q^{11};q^{11})_\infty^{11}}{(q;q)_\infty^{12}},\\
x_1 &= -\frac{(q^4,q^7;q^{11})_\infty^2(q,q^{10};q^{11})_\infty}{(q^2,q^9;q^{11})_\infty^2(q^5,q^6;q^{11})_\infty}\\
x_2 &= -\frac{(q^2,q^9;q^{11})_\infty^2(q^5,q^6;q^{11})_\infty}{q(q,q^{10};q^{11})_\infty^2(q^3,q^8;q^{11})_\infty},\\
x_3 &= \frac{q^2(q,q^{10};q^{11})_\infty^2(q^3,q^8;q^{11})_\infty}{(q^4,q^7;q^{11})_\infty(q^5,q^6;q^{11})_\infty^2},\\
x_4 &= \frac{(q^4,q^7;q^{11})_\infty(q^5,q^6;q^{11})_\infty^2}{q(q^2,q^9;q^{11})_\infty(q^3,q^8;q^{11})_\infty^2},\\
x_5 &= -\frac{(q^2,q^9;q^{11})_\infty(q^3,q^8;q^{11})_\infty^2}{(q^4,q^7;q^{11})_\infty^2(q,q^{10};q^{11})_\infty}.
\end{aligned}
$$

Using the Ramanujan–Kolberg algorithm, Radu [35] derived a witness identity for $p(11n+6)$. A set $\{M_1, M_2, \dots, M_7\}$ of generators of $E^\infty(22)$ can be found in [35]. For example:

$$M_1 = \frac{\eta^7(\tau)\eta^3(11\tau)}{\eta^3(2\tau)\eta^7(22\tau)}.$$

Let

$$F = \frac{(q;q)_\infty^{10}(q^2;q^2)_\infty^2(q^{11};q^{11})_\infty^{11}}{q^{14}(q^{22};q^{22})_\infty^{22}}\sum_{n=0}^\infty p(11n+6)q^n.$$

Radu showed that

$$
\begin{aligned}
F &= 11(98t^4 + 1263t^3 + 2877t^2 + 1019t - 1997)\\
&\quad + 11z_1(17t^2 + 490t^2 + 54t - 871)\\
&\quad + 11z_2(t^3 + 251t^2 + 488t - 614),
\end{aligned}
\tag{8.2}
$$

where

$$
\begin{aligned}
t &= \frac{3}{88}M_1 + \frac{1}{11}M_2 - \frac{1}{8}M_4,\\
z_1 &= -\frac{5}{88}M_1 + \frac{2}{11}M_2 - \frac{1}{8}M_4 - 3,\\
z_2 &= \frac{1}{44}M_1 - \frac{3}{11}M_2 + \frac{5}{4}M_4.
\end{aligned}
$$

Noting that $(1-q^n)^{11} \equiv 1-q^{11n} \pmod{11}$ and $(1-q^n)^8 \equiv (1-q^{2n})^4 \pmod{8}$, we see that (8.2) implies the Ramanujan congruence for $p(11n+6)$. Hemmecke [20] generalized Radu's algorithm and derived the following witness identity:

$$\begin{aligned} F = {} & 11^2 \cdot 3068 M_7 + 11^2 \cdot (3M_1 + 4236) M_6 \\ & + 11 \cdot (285 M_1 + 11 \cdot 5972) M_5 + 11(1867 M_1 + 11 \cdot 2476) M_2 \\ & - \frac{11}{8}(M_1^3 + 1011\, M_1^2 + 11 \cdot 6588 M_1 + 11^2 \cdot 10880) \\ & + \frac{11}{8}(M_1^2 + 11 \cdot 4497\, M_1 + 11^2 \cdot 3156) M_4. \end{aligned} \tag{8.3}$$

We are now ready to give an algorithmic derivation of the identity for $p(11n+6)$ as stated in Theorem 1.1.

Proof of Theorem 1.1. Notice that $N = 11$ satisfies all the conditions 1–7. We proceed with the following steps.

Step 1 By Theorem 2.1, we find that

$$F(\tau) = q(q^{11}; q^{11})_\infty \sum_{n=0}^{\infty} p(11n+6) q^n$$

is a modular function for $\Gamma_1(11)$.

Step 2 Solving the system of Diophantine inequalities (4.16) for $N = 11$, we obtain a set of 27 generators of $GE^\infty(11)$ including z and e as given in (1.8) and (1.9).

Step 3 Applying Radu's Algorithm AB, we deduce that

$$\langle GE^\infty(11) \rangle_{\mathbb{Q}} = \langle 1, e \rangle_{\mathbb{Q}[z]}.$$

Step 4 By virtue of Theorems 2.7 and 6.1, we get

$$h = \frac{\eta^{24}(\tau)}{\eta^{24}(11\tau)\eta_{11,1}^{28}(\tau)\eta_{11,2}^{16}(\tau)\eta_{11,3}^{12}(\tau)\eta_{11,4}^{4}(\tau)}$$

for which hF has a pole only at infinity.

Step 5 Employing Radu's Algorithm MC and Algorithm MW, we deduce that $hF \in \langle GE^\infty(11) \rangle_{\mathbb{Q}}$ and

$$\begin{aligned} hF = {} & 11z^{10} + 121z^8 e + 330z^9 - 484z^7 e - 990z^8 + 484z^6 e + 792z^7 \\ & - 484z^5 e + 44z^6 + 1089z^4 e - 132z^5 - 1452z^3 e - 451z^4 \\ & + 968z^2 e + 748z^3 - 242ze - 429z^2 + 77z + 11. \end{aligned}$$

This completes the proof. □

9. Further Examples

In this section, we derive Ramanujan-type identities on the broken 2-diamond partition function. The notion of the broken k-diamond partitions was introduced by Andrews and Paule [4] in their study of MacMahon's partition analysis. The number of broken k-diamond partitions of n is denoted by $\Delta_k(n)$. They showed that the generating function of $\Delta_k(n)$ is given by

$$\sum_{n=0}^{\infty} \Delta_k(n) q^n = \frac{(q^2; q^2)_\infty (q^{2k+1}; q^{2k+1})_\infty}{(q; q)_\infty^3 (q^{4k+2}; q^{4k+2})_\infty}.$$

Andrews and Paule conjectured that

$$\Delta_2(25n+14) \equiv 0 \pmod 5. \tag{9.1}$$

Chan [9] proved this conjecture and also showed that

$$\Delta_2(25n+24) \equiv 0 \pmod 5. \tag{9.2}$$

Define $a(n)$ by

$$\sum_{n=0}^{\infty} a(n)q^n = \frac{(q;q)_\infty^2 (q^2;q^2)_\infty}{(q^{10};q^{10})_\infty}.$$

Since $(1-q^n)^5 \equiv 1-q^{5n} \pmod 5$, we see that $\Delta_2(n) \equiv a(n) \pmod 5$. By the Ramanujan–Kolberg algorithm, Radu [35] obtained the following identity:

$$\begin{aligned} &\frac{(q^2;q^2)_\infty^{12}(q^5;q^5)_\infty^{10}}{q^4(q;q)_\infty^6(q^{10};q^{10})_\infty^{20}} \left(\sum_{n=0}^{\infty} a(25n+14)q^n\right)\left(\sum_{n=0}^{\infty} a(25n+24)q^n\right) \\ &\quad = 25\left(2t^4+28t^3+155t^2+400t+400\right), \end{aligned} \tag{9.3}$$

where

$$t = \frac{(q;q)_\infty^3 (q^5;q^5)_\infty}{q(q^2;q^2)_\infty (q^{10};q^{10})_\infty^3}.$$

The congruences (9.1) and (9.2) are easy consequences of (9.3). Let

$$z = \frac{(q^2;q^2)_\infty (q^5;q^5)_\infty^5}{q(q;q)_\infty (q^{10};q^{10})_\infty^5}.$$

Using the package `RaduRK`, Smoot [46] deduced that

$$\frac{(q;q)_\infty^{126}(q^5;q^5)_\infty^{70}}{q^{58}(q^2;q^2)_\infty^2(q^{10};q^{10})_\infty^{190}} \left(\sum_{n=0}^{\infty} \Delta_2(25n+14)q^n\right)\left(\sum_{n=0}^{\infty} \Delta_2(25n+24)q^n\right)$$

is a polynomial in z of degree 58 with integer coefficients divisible by 25. It is not hard to see that the above relation implies the congruences (9.1) and (9.2).

Our algorithm provides the following witness identities for $\Delta_2(25n+14)$ and $\Delta_2(25n+24)$.

Theorem 9.1. *Let*

$$z = \frac{(q;q)_\infty (q^5;q^5)_\infty}{q(q,q^4;q^5)_\infty^2 (q^{10};q^{10})_\infty^2 (q,q^9;q^{10})_\infty}.$$

Then

$$\frac{(q;q)_\infty^{92}(q^5;q^5)_\infty^{14}(q,q^4;q^5)_\infty^{52}(q^4,q^6;q^{10})_\infty^4}{q^{57}(q^2;q^2)_\infty^{58}(q^{10};q^{10})_\infty^{46}(q,q^9;q^{10})_\infty^{109}(q^5;q^{10})_\infty^{10}} \sum_{n=0}^{\infty} \Delta_2(25n+14)q^n \tag{9.4}$$

and

$$\frac{(q;q)_\infty^{92}(q,q^4;q^5)_\infty^{62}(q^5;q^{10})_\infty^6}{q^{57}(q^2;q^2)_\infty^{59}(q^5;q^5)_\infty^2(q^{10};q^{10})_\infty^{29}(q,q^9;q^{10})_\infty^{119}(q^4,q^6;q^{10})_\infty^4} \sum_{n=0}^{\infty} \Delta_2(25n+24)q^n \tag{9.5}$$

are both polynomials in z of degree 57 with integer coefficients divisible by 5.

More precisely, (9.4) equals

$$
\begin{aligned}
&10445z^{57} + 65072505z^{56} + 29885191700z^{55} + 2909565072375z^{54}\\
&\quad + 58232762317950z^{53} - 771909964270635z^{52} - 8976196273201590z^{51}\\
&\quad + 168096305999838525z^{50} - 552704071429548750z^{49}\\
&\quad - 6285133254753356625z^{48} + 76077164750182724400z^{47}\\
&\quad - 350853605818104040400z^{46} + 430844106211910184000z^{45}\\
&\quad + 4332665789140456020000z^{44} - 31965516977695010144000z^{43}\\
&\quad + 116598487085627561478400z^{42} - 254498980254624708134400z^{41}\\
&\quad + 226239786150985106784000z^{40} + 630144010340120712320000z^{39}\\
&\quad - 3270835930300215379968000z^{38} + 7873377561448743273881600z^{37}\\
&\quad - 12188753588700934348185600z^{36} + 11409105186984502777856000z^{35}\\
&\quad - 1853370295840331059200000z^{34} - 12922596637778941349888000z^{33}\\
&\quad + 19993842975085327602810880z^{32} - 4136695001339260651438080z^{31}\\
&\quad - 40585258593920366687027200z^{30} + 107607975413970670190592000z^{29}\\
&\quad - 189170246667253453894451200z^{28} + 290673733377906514130370560z^{27}\\
&\quad - 429481500981884772899880960z^{26} + 614653426107799377123737600z^{25}\\
&\quad - 825958110337598656348160000z^{24} + 1014095417844181497806848000z^{23}\\
&\quad - 1125028176866670548300595200z^{22} + 1129311459482608004707123200z^{21}\\
&\quad - 1033623338399676468559872000z^{20} + 869136778177466010173440000z^{19}\\
&\quad - 672028063551221072396288000z^{18} + 473438441949368700161228800z^{17}\\
&\quad - 299190013959544777788620800z^{16} + 167798468337926970277888000z^{15}\\
&\quad - 84223564508812395151360000z^{14} + 39006701101726128144384000z^{13}\\
&\quad - 16949659707832925998284800z^{12} + 6525804102142065953996800z^{11}\\
&\quad - 1953358789335809261568000z^{10} + 408567853900785254400000z^{9}\\
&\quad - 90672379909684330496000z^{8} + 43132985715615837716480z^{7}\\
&\quad - 13837533253868380487680z^{6} + 786549936580722688000z^{5}\\
&\quad + 776840149395832832000z^{4} - 48290550691921920 0z^{3}\\
&\quad - 31960428074332323840z^{2} - 1612499772831170560z\\
&\quad - 7036874417766400.
\end{aligned}
$$

The explicit expression for (9.5) is omitted.

We end this section by noting that our algorithmic approach can be used to derive dissection formulas on quotients in the form of (1.1), that is,

$$\prod_{\delta \mid M} \left(q^{\delta}; q^{\delta}\right)_{\infty}^{r_\delta}, \tag{9.6}$$

where M is a positive integer and r_δ, $r_{\delta,g}$ are integers. Let $a(n)$ be the partition function defined by (1.1), and let m be a positive integer. If our algorithm can be utilized to find a formula for the generating function of $a(mn+t)$ for each $0 \le t \le m-1$, then we are led to an m-dissection formula on the quotient (9.6). For example, the algorithm is valid to produce the 5-dissection formulas for $(q;q)_\infty$ and $\frac{1}{(q;q)_\infty}$, see Berndt [6, p. 165].

10. More General Partition Functions

While many partition functions $a(n)$ are of the form (1.1), there are partition functions that do not seem to fall into this framework, such as Andrews' (k,i)-singular overpartition function $\overline{Q}_{k,i}(n)$. Andrews [3] derived the generating function:

$$\sum_{n=0}^{\infty} \overline{Q}_{k,i}(n) q^n = \frac{(q^k, -q^i, -q^{k-i}; q^k)_\infty}{(q;q)_\infty}. \tag{10.1}$$

In general, it is not always the case that a quotient on the right-hand side of (10.1) can be expressed in the form of (1.1).

The objective of this section is to extend our algorithm to partition functions $b(n)$ defined by

$$\sum_{n=0}^{\infty} b(n) q^n = \prod_{\delta|M} (q^\delta; q^\delta)_\infty^{r_\delta} \prod_{\substack{\delta|M \\ 0<g<\delta}} (q^g, q^{\delta-g}; q^\delta)_\infty^{r_{\delta,g}}, \tag{10.2}$$

where M is a positive integer and r_δ, $r_{\delta,g}$ are integers. In fact, for any k and $1 \le i < \frac{k}{2}$, (10.1) can be written in the form of (10.2):

$$\sum_{n=0}^{\infty} \overline{Q}_{k,i}(n) q^n = \frac{(q^k;q^k)_\infty (q^{2i}, q^{2k-2i}; q^{2k})_\infty}{(q;q)_\infty (q^i, q^{k-i}; q^k)_\infty}, \tag{10.3}$$

where $M = 2k$:

$$r_\delta = \begin{cases} -1, & \delta = 1, \\ 1, & \delta = k, \\ 0, & \text{otherwise}, \end{cases} \quad \text{and} \quad r_{\delta,g} = \begin{cases} -1, & \delta = k, g = i, \\ 1, & \delta = 2k, g = 2i, \\ 0, & \text{otherwise}. \end{cases}$$

Analogous to the generating function $g_{m,t}(\tau)$ in Sect. 2 as given by Radu [37], we adopt the same notation for the generating function of $b(mn+t)$:

$$g_{m,t}(\tau) = q^{\frac{t-\ell}{m}} \sum_{n=0}^{\infty} b(mn+t) q^n, \tag{10.4}$$

where

$$\ell = -\frac{1}{24} \sum_{\delta|M} \delta r_\delta - \sum_{\substack{\delta|M \\ 0<g<\delta}} \frac{\delta}{2} P_2\left(\frac{g}{\delta}\right) r_{\delta,g}.$$

As before

$$P_2(t) = \{t\}^2 - \{t\} + \frac{1}{6},$$

and $\{t\}$ is the fractional part of t.

To derive a Ramanujan-type identity for $b(mn+t)$, we follow the same procedure as in Sect. 3. There are only a few modifications that should be taken into account to extend Theorems 2.1 and 2.7 to the generating function $g_{m,t}(\tau)$ in (10.4). The proofs are similar to those of Theorems 2.1 and 2.7 and hence are omitted.

Let $\phi(\tau)$ be a generalized eta-quotient and $F = \phi(\tau)g_{m,t}(\tau)$. Similar to Theorem 2.1, we give a criterion for $F(\tau)$ to be a modular function for $\Gamma_1(N)$. Let $\kappa = \gcd(m^2-1, 24)$. First, we assume that N satisfies the following conditions:

1. $M|N$.
2. $p|N$ for any prime $p|m$.
3. $\kappa N \sum_{\substack{\delta|M \\ 0<g<\delta}} \frac{g}{\delta} r_{\delta,g} \equiv 0 \pmod 2$.
4. $\kappa N \sum_{\substack{\delta|M \\ 0<g<\delta}} r_{\delta,g} \equiv 0 \pmod 4$.
5. $\kappa m N^2 \sum_{\substack{\delta|M \\ 0<g<\delta}} \frac{r_{\delta,g}}{\delta} \equiv 0 \pmod{12}$.
6. $\kappa N \sum_{\delta|M} r_\delta \equiv 0 \pmod 8$.
7. $\kappa m N^2 \sum_{\delta|M} \frac{r_\delta}{\delta} \equiv 0 \pmod{24}$.
8. $\frac{24mM}{\gcd(\kappa\alpha(t), 24mM)} |N$, where
$$\alpha(t) = -M \sum_{\delta|M} \delta r_\delta - 12M \sum_{\substack{\delta|M \\ 0<g<\delta}} \delta P_2\left(\frac{g}{\delta}\right) r_{\delta,g} - 24Mt.$$
9. Let $\prod_{\delta|M} \delta^{|r_\delta|} = 2^z j$, where $z \in \mathbb{N}$ and j is odd. If $2|m$, then $\kappa N \equiv 0 \pmod 4$ and $Nz \equiv 0 \pmod 8$, or $z \equiv 0 \pmod 2$ and $N(j-1) \equiv 0 \pmod 8$.
10. Let $\mathbb{S}_n = \{j^2 \pmod n \colon j \in \mathbb{Z}_n,\ \gcd(j,n) = 1,\ j \equiv 1 \pmod N\}$. For any $s \in \mathbb{S}_{24mM}$:
$$\frac{s-1}{24} \sum_{\delta|M} \delta r_\delta + (s-1) \sum_{\substack{\delta|M \\ 0<g<\delta}} \frac{\delta}{2} P_2\left(\frac{g}{\delta}\right) r_{\delta,g} + ts \equiv t \pmod m.$$

For a given partition function $b(n)$, and given integers m and t, such a positive integer N always exists, because $N = 24mM$ satisfies the conditions 1–10. For example, for Andrews' (3,1)-singular overpartition function $\overline{Q}_{3,1}(n)$, and for $m = 9$ and $t = 3$, we have $N = 6$. Compared with the conditions in Sect. 2, the conditions 3–5 are required to deal with the generalized eta-quotients.

Theorem 10.1. *For a given partition function $b(n)$ as defined by (10.2), and for given integers m and t, suppose that N is a positive integer satisfying the conditions 1–10. Let*

$$F(\tau) = \phi(\tau)\, g_{m,t}(\tau),$$

where

$$\phi(\tau) = \prod_{\delta|N} \eta^{a_\delta}(\delta\tau) \prod_{\substack{\delta|N \\ 0<g\leq\lfloor\delta/2\rfloor}} \eta_{\delta,g}^{a_{\delta,g}}(\tau),$$

and a_δ and $a_{\delta,g}$ are integers. Then $F(\tau)$ is a modular function with respect to $\Gamma_1(N)$ if and only if a_δ and $a_{\delta,g}$ satisfy the following conditions:

(1) $\sum_{\delta|N} a_\delta + \sum_{\delta|M} r_\delta = 0$,

(2) $N \sum_{\delta|N} \frac{a_\delta}{\delta} + 2N \sum_{\substack{\delta|N \\ 0<g\leq\lfloor\delta/2\rfloor}} \frac{a_{\delta,g}}{\delta} + Nm \sum_{\delta|M} \frac{r_\delta}{\delta} + 2Nm \sum_{\substack{\delta|M \\ 0<g<\delta}} \frac{r_{\delta,g}}{\delta} \equiv 0 \pmod{24}$,

(3)

$$\sum_{\delta|N} \delta a_\delta + 12 \sum_{\substack{\delta|N \\ 0<g\leq\lfloor\delta/2\rfloor}} \delta P_2\left(\frac{g}{\delta}\right) a_{\delta,g} + m \sum_{\delta|M} \delta r_\delta$$
$$+12m \sum_{\substack{\delta|M \\ 0<g<\delta}} \delta P_2\left(\frac{g}{\delta}\right) r_{\delta,g} + \frac{(m^2-1)\alpha(t)}{mM} \equiv 0 \pmod{24},$$

where

$$\alpha(t) = -M \sum_{\delta|M} \delta r_\delta - 12M \sum_{\substack{\delta|M \\ 0<g<\delta}} \delta P_2\left(\frac{g}{\delta}\right) r_{\delta,g} - 24Mt,$$

(4) *For any integer* $0 < a < 12N$ *with* $\gcd(a,6) = 1$ *and* $a \equiv 1 \pmod{N}$*:*

$$\prod_{\delta|N} \left(\frac{\delta}{a}\right)^{|a_\delta|} \prod_{\delta|M} \left(\frac{m\delta}{a}\right)^{|r_\delta|} e^{\sum_{\delta|N} \sum_{g=1}^{\lfloor\delta/2\rfloor} \pi i \left(\frac{g}{\delta}-\frac{1}{2}\right)(a-1)a_{\delta,g} + \sum_{\delta|M} \sum_{g=1}^{\delta-1} \pi i \left(\frac{g}{\delta}-\frac{1}{2}\right)(a-1)r_{\delta,g}} = 1.$$

In the notation $p(\gamma,\lambda)$ and $p(\gamma)$ in (2.19) and (2.20), we define the map $p\colon \Gamma \times \mathbb{Z}_m \to \mathbb{Q}$ by

$$p(\gamma,\lambda) = \frac{1}{24} \sum_{\delta|M} \frac{\gcd^2(\delta(a+\kappa\lambda c), mc)}{\delta m} r_\delta$$
$$+ \frac{1}{2} \sum_{\substack{\delta|M \\ 0<g<\delta}} \frac{\gcd^2(\delta(a+\kappa\lambda c), mc)}{\delta m} P_2\left(\frac{(a+\kappa\lambda c)g}{\gcd(\delta(a+\kappa\lambda c), mc)}\right) r_{\delta,g},$$

and define $p(\gamma)$ by

$$p(\gamma) = \min\{p(\gamma,\lambda) : \lambda = 0, 1, \ldots, m-1\}. \tag{10.5}$$

Parallel to Theorem 2.7, we obtain lower bounds of the orders of $F(\tau)$ at cusps of $\Gamma_1(N)$.

Theorem 10.2. *For a given partition function $b(n)$ as defined by (10.2), and for given integers m and t, let*

$$F(\tau) = \phi(\tau)\, g_{m,t}(\tau),$$

where

$$\phi(\tau)=\prod_{\delta|N}\eta^{a_\delta}(\delta\tau)\prod_{\substack{\delta|N\\0<g\leq\lfloor\delta/2\rfloor}}\eta_{\delta,g}^{a_{\delta,g}}(\tau),$$

a_δ and $a_{\delta,g}$ are integers. Assume that $F(\tau)$ is a modular function for $\Gamma_1(N)$. Let $\{s_1,s_2,\ldots,s_\epsilon\}$ be a complete set of inequivalent cusps of $\Gamma_1(N)$, and for each $1\leq i\leq\epsilon$, let $\alpha_i\in\Gamma$ be such that $\alpha_i\infty=s_i$. Then

$$\operatorname{ord}_{s_i}(F(\tau))\geq w_{\alpha_i}\left(p(\alpha_i)+p^*(\alpha_i)\right),\tag{10.6}$$

where $p(\gamma)$ is given by (10.5) and $p^*(\gamma)$ is defined in Lemma 2.6.

For a given partition function $b(n)$, and given integers m and t, assume that we have found a generalized eta-quotient $\phi(\tau)$, such that

$$F(\tau)=\phi(\tau)\,g_{m,t}(\tau)\tag{10.7}$$

is a modular function for $\Gamma_1(N)$. Utilizing the algorithm in Sect. 3, we try to express $F(\tau)$ as a linear combination of generalized eta-quotients with level N. If we succeed, then we obtain a Ramanujan-type identity for $b(mn+t)$. Note that Theorem 10.2 is needed to find a generalized eta-quotient $h(\tau)$, such that hF has a pole only at infinity.

For example, we can derive Ramanujan-type identities on the singular overpartition function introduced by Andrews [3]. The number of (k,i)-singular overpartitions of n is denoted by $\overline{Q}_{k,i}(n)$ $(1\leq i<\frac{k}{2})$. For $k=3$ and $i=1$, (10.3) specializes to

$$\sum_{n=0}^{\infty}\overline{Q}_{3,1}(n)q^n=\frac{(q^3;q^3)_\infty(q^2,q^4;q^6)_\infty}{(q;q)_\infty(q,q^2;q^3)_\infty}.$$

When applied to the above generating function, our algorithm produces the Ramanujan-type identities on $\overline{Q}_{3,1}(9n+3)$ and $\overline{Q}_{3,1}(9n+6)$ due to Shen [45].

Theorem 10.3. *We have*

$$\frac{(q;q)_\infty^{14}}{q(q^2;q^2)_\infty^5(q^3;q^3)_\infty^6(q^6;q^6)_\infty^3}\sum_{n=0}^{\infty}\overline{Q}_{3,1}(9n+3)q^n=6z+96,$$

and

$$\frac{(q;q)_\infty^{13}}{q(q^2;q^2)_\infty^4(q^3;q^3)_\infty^3(q^6;q^6)_\infty^6}\sum_{n=0}^{\infty}\overline{Q}_{3,1}(9n+6)q^n=24z+96,$$

where

$$z=\frac{(q^2;q^2)_\infty^3(q^3;q^3)_\infty^9}{q(q;q)_\infty^3(q^6;q^6)_\infty^9}.$$

Our extended algorithm can also be used to derive dissection formulas on the quotients in the form of (10.2), that is,

$$\prod_{\delta|M}(q^\delta;q^\delta)_\infty^{r_\delta}\prod_{\substack{\delta|M\\0<g<\delta}}(q^g,q^{\delta-g};q^\delta)_\infty^{r_{\delta,g}},\tag{10.8}$$

where M is a positive integer and r_δ, $r_{\delta,g}$ are integers. Let $b(n)$ be the partition function defined by (10.2), and let m be a positive integer. If our algorithm can be utilized to find a formula for the generating function of $b(mn+t)$ for each $0 \le t \le m-1$, then we are led to an m-dissection formula on the quotient in (10.8). For example, we get the 2- and 4-dissections of the Rogers–Ramanujan continued fraction [2,21,27,40], the 8-dissections of the Gordon's continued fraction [22,49] and the 2-, 3-, 4-, 6-dissections of Ramanujan's cubic continued fraction [23,47].

We now demonstrate how to deduce the 2-dissection formula for the Rogers–Ramanujan continued fraction:

$$R(q) = \frac{1}{1} + \frac{q}{1} + \frac{q^2}{1} + \frac{q^3}{1} + \cdots .$$

Rogers [42, p. 329] showed that

$$R(q) = \frac{(q^2, q^3; q^5)_\infty}{(q, q^4; q^5)_\infty}. \tag{10.9}$$

The following 2-dissection formulas of Ramanujan [40, p. 50] were first proved by Andrews [2]. With respect to the quotient in (10.9), we have to count on the extended algorithm, because (10.9) cannot be expressed in the form of (1.1).

Theorem 10.4. *We have*

$$R(q) = \frac{(q^8, q^{12}; q^{20})_\infty^2}{(q^6, q^{14}; q^{20})_\infty (q^{10}, q^{10}; q^{20})_\infty} + q\frac{(q^2, q^{18}; q^{20})_\infty (q^8, q^{12}; q^{20})_\infty}{(q^4, q^{16}; q^{20})_\infty (q^{10}, q^{10}; q^{20})_\infty} \tag{10.10}$$

and

$$R(q)^{-1} = \frac{(q^4, q^{16}; q^{20})_\infty^2}{(q^2, q^{18}; q^{20})_\infty (q^{10}, q^{10}; q^{20})_\infty} - q\frac{(q^4, q^{16}; q^{20})_\infty (q^6, q^{14}; q^{20})_\infty}{(q^8, q^{12}; q^{20})_\infty (q^{10}, q^{10}; q^{20})_\infty}. \tag{10.11}$$

Proof. As far as (10.9) is concerned, we have $M = 5$, $r_{5,1} = -1$, and $r_{5,2} = 1$. We find that $N = 10$ satisfies the conditions 1–10. Let $r(n)$ be defined by

$$R(q) = \sum_{n=0}^{\infty} r(n) q^n.$$

Employing our algorithm, we obtain that

$$\sum_{n=0}^{\infty} r(2n) q^n = \frac{z_1 z_3}{z_2 z^2} \cdot \frac{\eta_{10,5}^2(\tau)}{\eta_{10,4}^2(\tau)}$$

and

$$\sum_{n=0}^{\infty} r(2n+1) q^n = \frac{z_2^3 z^4}{z_1^2 z_3^3} \cdot \frac{\eta_{10,4}^8(\tau)}{\eta_{10,5}^8(\tau)},$$

where z, z_1, z_2, and z_3 are given in (3.2). A direct computation yields (10.10). Similarly, we get (10.11). This completes the proof. □

Gordon [17] showed that

$$1+q+\frac{q^2}{1+q^3}+\frac{q^4}{1+q^5}+\frac{q^6}{1+q^7}+\cdots=\frac{(q^3,q^5;q^8)_\infty}{(q,q^7;q^8)_\infty}. \tag{10.12}$$

Using our algorithm, we deduce the following 8-dissection formulas of Hirschhorn for (10.12) and its reciprocal, see [22, pp. 373–374].

Theorem 10.5. (Hirschhorn [22]) *We have*

$$\begin{aligned}
\frac{(q^3,q^5;q^8)_\infty}{(q,q^7;q^8)_\infty} &= \frac{(-q^{24},-q^{32},-q^{32},-q^{40},q^{64},q^{64};q^{64})_\infty}{(q^8,q^{16},q^{16},q^{24},q^{32},q^{32};q^{32})_\infty}\\
&\quad + q\,\frac{(-q^{16},-q^{24},-q^{40},-q^{48},q^{64},q^{64};q^{64})_\infty}{(q^{16},q^{8},q^{24},q^{24},q^{48},q^{64},q^{64};q^{64})_\infty}\\
&\quad + q^2\,\frac{(-q^{16},-q^{24},-q^{40},-q^{48},q^{64},q^{64};q^{64})_\infty}{(q^8,q^{16},q^{16},q^{24},q^{32},q^{64};q^{64};q^{64})_\infty}\\
&\quad - 2q^{12}\,\frac{(-q^{8},-q^{16},-q^{64},-q^{64},q^{64},q^{64};q^{64})_\infty}{(q^8,q^{16},q^{16},q^{24},q^{32},q^{32};q^{32})_\infty}\\
&\quad - q^5\,\frac{(-q^{8},-q^{16},-q^{48},-q^{56},q^{64},q^{64};q^{64})_\infty}{(q^8,q^{8},q^{24},q^{24},q^{32},q^{32};q^{32})_\infty}\\
&\quad - q^6\,\frac{(-q^{8},-q^{16},-q^{48},-q^{56},q^{64},q^{64};q^{64})_\infty}{(q^8,q^{16},q^{16},q^{24},q^{32},q^{32};q^{32})_\infty},\\
\frac{(q,q^7;q^8)_\infty}{(q^3,q^5;q^8)_\infty} &= \frac{(-q^{16},-q^{24},-q^{40},-q^{48},q^{64},q^{64};q^{64})_\infty}{(q^8,q^{8},q^{24},q^{24},q^{32},q^{32};q^{32})_\infty}\\
&\quad - q\,\frac{(-q^{16},-q^{24},-q^{40},-q^{48},q^{64},q^{64};q^{64})_\infty}{(q^8,q^{16},q^{16},q^{24},q^{32},q^{32};q^{32})_\infty}\\
&\quad + q^3\,\frac{(-q^{8},-q^{32},-q^{32},-q^{56},q^{64},q^{64};q^{64})_\infty}{(q^8,q^{16},q^{16},q^{24},q^{32},q^{32};q^{32})_\infty}\\
&\quad - q^4\,\frac{(-q^{8},-q^{16},-q^{48},-q^{56},q^{64},q^{64};q^{64})_\infty}{(q^8,q^{8},q^{24},q^{24},q^{32},q^{32};q^{32})_\infty}\\
&\quad + q^5\,\frac{(-q^{8},-q^{16},-q^{48},-q^{56},q^{64},q^{64};q^{64})_\infty}{(q^8,q^{16},q^{16},q^{24},q^{32},q^{32};q^{32})_\infty}\\
&\quad - 2q^7\,\frac{(-q^{24},-q^{40},-q^{64},-q^{64},q^{64},q^{64};q^{64})_\infty}{(q^8,q^{16},q^{16},q^{24},q^{32},q^{32};q^{32})_\infty}.
\end{aligned}$$

Ramanujan's cubic continued fraction is defined by

$$\frac{1}{1}+\frac{q+q^2}{1}+\frac{q^2+q^4}{1}+\cdots,$$

which equals

$$\frac{(q,q^5;q^6)_\infty}{(q^3,q^3;q^6)_\infty}, \tag{10.13}$$

see [40, p. 44]. Applying our algorithm to (10.13) and its reciprocal, we are led to the 2-, 3-, 4-, and 6-dissection formulas in Theorem 1.1–Theorem 1.4 in [23].

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William Y. C. Chen and Julia Q. D. Du
Center for Applied Mathematics
Tianjin University
Tianjin 300072
People's Republic of China

William Y. C. Chen
e-mail: chenyc@tju.edu.cn

Julia Q. D. Du
e-mail: qddu@tju.edu.cn

Jack C. D. Zhao
Center for Combinatorics
Nankai University
Tianjin 300071
People's Republic of China
e-mail: cdzhao@mail.nankai.edu.cn

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Partitions into Distinct Parts Modulo Powers of 5

Dedicated to George E. Andrews on the occasion of his 80th Birthday

Shane Chern and Michael D. Hirschhorn

Abstract. If $p_D(n)$ denotes the number of partitions of n into distinct parts, it is known that for $\alpha \geq 1$ and $n \geq 0$,

$$p_D\left(5^{2\alpha+1}n + \frac{5^{2\alpha+2}-1}{24}\right) \equiv 0 \pmod{5^\alpha}.$$

We give a completely elementary proof of this fact.

Mathematics Subject Classification. Primary 11P83; Secondary 05A17.

Keywords. Distinct parts, Congruences, Powers of 5.

1. Introduction

Let $p_D(n)$ denote the number of partitions of n into distinct parts. Then

$$\sum_{n\geq 0} p_D(n)q^n = (-q;q)_\infty = \frac{E(q^2)}{E(q)},$$

where

$$E(q) = (q;q)_\infty.$$

Baruah and Begum [1] proved the following results:

$$\sum_{n\geq 0} p_D(5n+1)q^n = \frac{E(q^2)^2E(q^5)^3}{E(q)^4E(q^{10})}, \tag{1.1}$$

$$\sum_{n\geq 0} p_D(25n+1)q^n = \frac{E(q^2)^3E(q^5)^4}{E(q)^5E(q^{10})^2}$$

$$\times \left(1 + 160q\left(\frac{E(q^2)E(q^{10})^3}{E(q)^3E(q^5)}\right) + 2800q^2\left(\frac{E(q^2)E(q^{10})^3}{E(q)^3E(q^5)}\right)^2\right.$$

$$+16000q^3\left(\frac{E(q^2)E(q^{10})^3}{E(q)^3E(q^5)}\right)^3+32000q^4\left(\frac{E(q^2)E(q^{10})^3}{E(q)^3E(q^5)}\right)^4\Bigg), \quad (1.2)$$

as well as the corresponding result for $\sum_{n\geq 0} p_D(125n+26)q^n$.

Inspired by their work, we prove the following general result.

Theorem 1.1. *For $\alpha \geq 1$,*

$$\sum_{n\geq 0} p_D\left(5^{2\alpha-1}n+\frac{5^{2\alpha}-1}{24}\right)q^n=\gamma\sum_{i=1}^{(5^{2\alpha}-1)/24} x_{2\alpha-1,i}\zeta^{i-1}, \quad (1.3)$$

$$\sum_{n\geq 0} p_D\left(5^{2\alpha}n+\frac{5^{2\alpha}-1}{24}\right)q^n=\delta\sum_{i=1}^{(5^{2\alpha+1}-5)/24} x_{2\alpha,i}\zeta^{i-1}, \quad (1.4)$$

where

$$\gamma=\frac{E(q^2)^2E(q^5)^3}{E(q)^4E(q^{10})}, \quad \delta=\frac{E(q^2)^3E(q^5)^4}{E(q)^5E(q^{10})^2}, \quad \zeta=q\frac{E(q^2)E(q^{10})^3}{E(q)^3E(q^5)}$$

and where the coefficient vectors $\mathbf{x}_\alpha=(x_{\alpha,1}, x_{\alpha,2}, \dots)$ are given recursively by

$$\mathbf{x}_1=(1, 0, \dots),$$

and for $\alpha\geq 1$,

$$\mathbf{x}_{2\alpha}=\mathbf{x}_{2\alpha-1}A$$

and

$$\mathbf{x}_{2\alpha+1}=\mathbf{x}_{2\alpha}B,$$

where A is the matrix $(\alpha_{i,j})_{i,j\geq 1}$ and B is the matrix $(\beta_{i,j})_{i,j\geq 1}$, where the $\alpha_{i,j}$ and $\beta_{i,j}$ are given by

$$\sum_{i,j\geq 1}\alpha_{i,j}x^iy^j=\frac{N_\alpha}{D'}$$

and

$$\sum_{i,j\geq 1}\beta_{i,j}x^iy^j=\frac{N_\beta}{D'},$$

where

$$\begin{aligned} N_\alpha &= (y+160y^2+2800y^3+16000y^4+32000y^5)x \\ &\quad +(180y^2+3000y^3+16800y^4+32000y^5)x^2 \\ &\quad +(75y^2+1215y^3+6600y^4+12000y^5)x^3 \\ &\quad +(14y^2+220y^3+1150y^4+2000y^5)x^4 \\ &\quad +(y^2+15y^3+75y^4+125y^5)x^5, \\ N_\beta &= (5y+660y^2+14400y^3+120000y^4+448000y^5+640000y^6)x \\ &\quad +(y+680y^2+14900y^3+123200y^4+456000y^5+640000y^6)x^2 \end{aligned}$$

$$
\begin{aligned}
&+(265y^2+5785y^3+47500y^4+174000y^5+240000y^6)x^3\\
&+(46y^2+1000y^3+8150y^4+29500y^5+40000y^6)x^4\\
&+(3y^2+65y^3+525y^4+1875y^5+2500y^6)x^5
\end{aligned}
$$

and

$$
\begin{aligned}
D' = 1 &-(205y+4300y^2+34000y^3+120000y^4+160000y^5)x\\
&-(215y+4475y^2+35000y^3+122000y^4+160000y^5)x^2\\
&-(85y+1750y^2+13525y^3+46000y^4+60000y^5)x^3\\
&-(15y+305y^2+2325y^3+7875y^4+10000y^5)x^4\\
&-(y+20y^2+150y^3+500y^4+625y^5)x^5.
\end{aligned}
$$

Furthermore, for $\alpha \geq 1$*,*

$$
\begin{aligned}
x_{2\alpha+1,i} &\equiv 0 \pmod{5^\alpha},\\
x_{2\alpha+2,i} &\equiv 0 \pmod{5^\alpha},
\end{aligned}
$$

from which it follows that for $\alpha \geq 1$*,*

$$p_D\left(5^{2\alpha+1}n+\frac{5^{2\alpha+2}-1}{24}\right) \equiv 0 \pmod{5^\alpha}, \tag{1.5}$$

$$p_D\left(5^{2\alpha+2}n+\frac{5^{2\alpha+2}-1}{24}\right) \equiv 0 \pmod{5^\alpha}. \tag{1.6}$$

(Of course, (1.6) *is a special case of* (1.5)*.)*

This result is due to Rødseth [6] and independently to Gordon and Hughes [3]. See also Lovejoy [5].

2. Preliminaries

Let

$$R(q)=\left(\begin{matrix}q,q^4\\q^2,q^3\end{matrix};q^5\right)_\infty, \qquad \chi(-q)=(q;q^2)_\infty=\frac{E(q)}{E(q^2)}.$$

Then ([4, (8.1.1)])

$$E(q)=E(q^{25})\left(\frac{1}{R(q^5)}-q-q^2R(q^5)\right),$$

([4, (8.4.4)])

$$
\begin{aligned}
\frac{1}{E(q)}=\frac{E(q^{25})^5}{E(q^5)^6}\Bigg(&\frac{1}{R(q^5)^4}+\frac{q}{R(q^5)^3}+\frac{2q^2}{R(q^5)^2}+\frac{3q^3}{R(q^5)}+5q^4\\
&-3q^5R(q^5)+2q^6R(q^5)^2-q^7R(q^5)^3+q^8R(q^5)^4\Bigg),
\end{aligned}
$$

([4, (40.2.3)])

$$R(q^2)-R(q)^2=2q\left(\begin{matrix}q,\ q,\ q^9,q^9\\q^3,q^5,q^5,q^7\end{matrix};q^{10}\right)_\infty, \tag{2.1}$$

([4, (40.2.4)])

$$R(q^2)+R(q)^2=2\begin{pmatrix} q,\ q^4,q^6,q^9 \\ q^2,q^5,q^5,q^8 \end{pmatrix}_{\infty} ; q^{10}, \tag{2.2}$$

([4, (41.1.3)])

$$1-qR(q)R(q^2)^2=\begin{pmatrix} q,\ q^4,q^5,q^5,q^6,q^9 \\ q^2,q^3,q^3,q^7,q^7,q^8 \end{pmatrix}_{\infty} ; q^{10}, \tag{2.3}$$

([4, (41.1.2)])

$$1+qR(q)R(q^2)^2=\begin{pmatrix} q^2,q^2,q^5,q^5,q^8,q^8 \\ q,\ q^4,q^4,q^6,q^6,q^9 \end{pmatrix}_{\infty} ; q^{10}, \tag{2.4}$$

([4, (34.8.4)])

$$\frac{E(q^2)^4}{E(q)^2}-q\frac{E(q^{10})^4}{E(q^5)^2}=\frac{E(q^2)E(q^5)^3}{E(q)E(q^{10})}$$

and ([4, (34.8.3)])

$$\frac{E(q^5)^4}{E(q^{10})^2}-\frac{E(q)^4}{E(q^2)^2}=4q\frac{E(q)E(q^{10})^3}{E(q^2)E(q^5)}.$$

We require the following results.

Lemma 2.1.

$$\frac{R(q^2)}{R(q)^2}-\frac{R(q)^2}{R(q^2)}=4q\frac{\chi(-q)}{\chi(-q^5)^5}, \tag{2.5}$$

$$\frac{R(q^2)-R(q)^2}{R(q^2)+R(q)^2}=qR(q)R(q^2)^2, \tag{2.6}$$

$$\frac{1}{R(q)R(q^2)^2}-q^2R(q)R(q^2)^2=\frac{\chi(-q^5)^5}{\chi(-q)}, \tag{2.7}$$

$$\frac{1-qR(q)R(q^2)^2}{1+qR(q)R(q^2)^2}=\frac{R(q)^2}{R(q^2)}, \tag{2.8}$$

$$\frac{R(q)}{R(q^2)^3}+q^2\frac{R(q^2)^3}{R(q)}=\frac{\chi(-q^5)^5}{\chi(-q)}-2q+4q^2\frac{\chi(-q)}{\chi(-q^5)^5}, \tag{2.9}$$

$$\frac{1}{R(q)^3R(q^2)}+q^2R(q)^3R(q^2)=\frac{\chi(-q^5)^5}{\chi(-q)}+2q+4q^2\frac{\chi(-q)}{\chi(-q^5)^5}, \tag{2.10}$$

$$\frac{\chi(-q^5)^5}{\chi(-q)}+q=\frac{E(q^2)^4E(q^5)^2}{E(q)^2E(q^{10})^4} \tag{2.11}$$

and

$$1-4q\frac{\chi(-q)}{\chi(-q^5)^5}=\frac{E(q)^4E(q^{10})^2}{E(q^2)^2E(q^5)^4}. \tag{2.12}$$

Proof of (2.5). If we multiply (2.1) by (2.2) and divide by $R(q)^2R(q^2)$, we find that

$$\frac{R(q^2)}{R(q)^2}-\frac{R(q)^2}{R(q^2)}=\frac{(R(q^2)-R(q)^2)(R(q^2)+R(q)^2)}{R(q)^2R(q^2)}$$

$$
\begin{aligned}
&= \frac{2q\begin{pmatrix} q,\, q,\, q^9, q^9 \\ q^3, q^5, q^5, q^7 \end{pmatrix}; q^{10}\Big)_\infty \cdot 2\begin{pmatrix} q,\, q^4, q^6, q^9 \\ q^2, q^5, q^5, q^8 \end{pmatrix}; q^{10}\Big)_\infty}{\left(\begin{matrix} q,\, q,\, q^4, q^4, q^6, q^6, q^9, q^9, q^2, q^8 \\ q^2, q^2, q^3, q^3, q^7, q^7, q^8, q^8, q^4, q^6 \end{matrix}; q^{10}\right)_\infty} \\
&= 4q\left(\begin{matrix} q,\, q^3, q^5, q^7, q^9 \\ q^5, q^5, q^5, q^5, q^5 \end{matrix}; q^{10}\right)_\infty \\
&= 4q\frac{(q;q^2)_\infty}{(q^5;q^{10})_\infty^5} \\
&= 4q\frac{\chi(-q)}{\chi(-q^5)^5}.
\end{aligned}
$$

□

Proof of (2.6). If we divide (2.1) by (2.2), we obtain

$$
\begin{aligned}
\frac{R(q^2)-R(q)^2}{R(q^2)+R(q)^2} &= q\frac{\left(\begin{matrix} q,\, q,\, q^9, q^9 \\ q^3, q^5, q^5, q^7 \end{matrix}; q^{10}\right)_\infty}{\left(\begin{matrix} q,\, q^4, q^6, q^9 \\ q^2, q^5, q^5, q^8 \end{matrix}; q^{10}\right)_\infty} \\
&= q\left(\begin{matrix} q,\, q^2, q^8, q^9 \\ q^3, q^4, q^6, q^7 \end{matrix}; q^{10}\right)_\infty \\
&= q\left(\begin{matrix} q,\, q^4, q^6, q^9, q^2, q^2, q^8, q^8 \\ q^2, q^3, q^7, q^8, q^4, q^4, q^6, q^6 \end{matrix}; q^{10}\right)_\infty \\
&= qR(q)R(q^2)^2.
\end{aligned}
$$

□

Proof of (2.7). If we multiply (2.3) by (2.4) and divide by $R(q)R(q^2)^2$, we find

$$
\begin{aligned}
&\frac{1}{R(q)R(q^2)^2} - q^2R(q)R(q^2)^2 = \frac{(1-qR(q)R(q^2)^2)(1+qR(q)R(q^2)^2)}{R(q)R(q^2)^2} \\
&= \frac{\left(\begin{matrix} q,\, q^4, q^5, q^5, q^6, q^9 \\ q^2, q^3, q^3, q^7, q^7, q^8 \end{matrix}; q^{10}\right)_\infty \left(\begin{matrix} q^2, q^2, q^5, q^5, q^8, q^8 \\ q,\, q^4, q^4, q^6, q^6, q^9 \end{matrix}; q^{10}\right)_\infty}{\left(\begin{matrix} q,\, q^4, q^6, q^9 \\ q^2, q^3, q^7, q^8 \end{matrix}; q^{10}\right)_\infty \left(\begin{matrix} q^2, q^2, q^8, q^8 \\ q^4, q^4, q^6, q^6 \end{matrix}; q^{10}\right)_\infty} \\
&= \left(\begin{matrix} q^5, q^5, q^5, q^5, q^5 \\ q,\, q^3, q^5, q^7, q^9 \end{matrix}; q^{10}\right)_\infty \\
&= \frac{\chi(-q^5)^5}{\chi(-q)}.
\end{aligned}
$$

□

Proof of (2.8). If we divide (2.3) by (2.4), we obtain

$$
\frac{1-qR(q)R(q^2)^2}{1+qR(q)R(q^2)^2} = \frac{\left(\begin{matrix} q,\, q^4, q^5, q^5, q^6, q^9 \\ q^2, q^3, q^3, q^7, q^7, q^9 \end{matrix}; q^{10}\right)_\infty}{\left(\begin{matrix} q^2, q^2, q^5, q^5, q^8, q^8 \\ q,\, q^4, q^4, q^6, q^6, q^9 \end{matrix}; q^{10}\right)_\infty}
$$

$$= \left(\begin{matrix} q,\ q,\ q^4, q^4, q^6, q^6, q^9, q^9, q^4, q^6 \\ q^2, q^2, q^3, q^3, q^7, q^7, q^8, q^8, q^2, q^8 \end{matrix} ; q^{10} \right)_\infty$$
$$= \frac{R(q)^2}{R(q^2)}.$$

□

Proof of (2.9). Note that (2.6) is equivalent to (2.8), because they are both equivalent to

$$R(q^2) - R(q)^2 = qR(q)^3 R(q^2)^2 + qR(q)R(q^2)^3. \tag{2.13}$$

If we divide (2.13) by $R(q)R(q^2)^3$ and rearrange, we find that

$$\frac{R(q)}{R(q^2)^3} = \frac{1}{R(q)R(q^2)^2} - q\frac{R(q)^2}{R(q^2)} - q, \tag{2.14}$$

while if we divide (2.13) by $R(q)^2$, rearrange and multiply by q, we obtain

$$q^2 \frac{R(q^2)^3}{R(q)} = -q^2 R(q)R(q^2)^2 + q\frac{R(q^2)}{R(q)^2} - q. \tag{2.15}$$

If we add (2.14) and (2.15), we obtain

$$\begin{aligned} \frac{R(q)}{R(q^2)^3} + q^2 \frac{R(q^2)^3}{R(q)} &= \left(\frac{1}{R(q)R(q^2)^2} - q^2 R(q)R(q^2)^2 \right) - 2q \\ &\quad + q \left(\frac{R(q^2)}{R(q)^2} - \frac{R(q)^2}{R(q^2)} \right) \\ &= \frac{\chi(-q^5)^5}{\chi(-q)} - 2q + 4q^2 \frac{\chi(-q)}{\chi(-q^5)^5}. \end{aligned}$$

□

Proof of (2.10). If we multiply (2.5) by (2.7) and add (2.9), we find that

$$\begin{aligned} &\frac{1}{R(q)^3 R(q^2)} + q^2 R(q)^3 R(q^2) \\ &= \left(\frac{R(q^2)}{R(q)^2} - \frac{R(q)^2}{R(q^2)} \right) \left(\frac{1}{R(q)R(q^2)^2} - q^2 R(q)R(q^2)^2 \right) \\ &\quad + \left(\frac{R(q)}{R(q^2)^3} + q^2 \frac{R(q^2)^3}{R(q)} \right) \\ &= 4q\frac{\chi(-q)}{\chi(-q^5)^5} \cdot \frac{\chi(-q^5)^5}{\chi(-q)} + \left(\frac{\chi(-q^5)^5}{\chi(-q)} - 2q + 4q^2 \frac{\chi(-q)}{\chi(-q^5)^5} \right) \\ &= \frac{\chi(-q^5)^5}{\chi(-q)} + 2q + 4q^2 \frac{\chi(-q)}{\chi(-q^5)^5}. \end{aligned}$$

□

Proof of (2.11).

$$\frac{\chi(-q^5)^5}{\chi(-q)} + q = \frac{E(q^2)E(q^5)^5}{E(q)E(q^{10})^5} + q$$

$$= \frac{E(q^5)^2}{E(q^{10})^4}\left(\frac{E(q^2)E(q^5)^3}{E(q)E(q^{10})} + q\frac{E(q^{10})^4}{E(q^5)^2}\right)$$
$$= \frac{E(q^5)^2}{E(q^{10})^4}\cdot\frac{E(q^2)^4}{E(q)^2}.$$

□

Proof of (2.12).

$$\begin{aligned}1 - 4q\frac{\chi(-q)}{\chi(-q^5)^5} &= 1 - 4q\frac{E(q)E(q^{10})^5}{E(q^2)E(q^5)^5}\\
&= \frac{E(q^{10})^2}{E(q^5)^4}\left(\frac{E(q^5)^4}{E(q^{10})^2} - 4q\frac{E(q)E(q^{10})^3}{E(q^2)E(q^5)}\right)\\
&= \frac{E(q^{10})^2}{E(q^5)^4}\cdot\frac{E(q)^4}{E(q^2)^2}.\end{aligned}$$

□

3. Proof of (1.1)

In this section, we effectively reproduce the proof of Baruah and Begum [1].

We have

$$\begin{aligned}\sum_{n\geq 0} p_D(n)q^n = (-q;q)_\infty = \frac{E(q^2)}{E(q)}\\
&= \frac{E(q^{25})^5}{E(q^5)^6}\left(\frac{1}{R(q^5)^4} + \frac{q}{R(q^5)^3} + \frac{2q^2}{R(q^5)^2} + \frac{3q^3}{R(q^5)} + 5q^4\right.\\
&\quad \left. -3q^5R(q^5) + 2q^6R(q^5)^2 - q^7R(q^5)^3 + q^8R(q^5)^4\right)\\
&\quad \times E(q^{50})\left(\frac{1}{R(q^{10})} - q^2 - q^4R(q^{10})\right).\end{aligned}$$

It follows that

$$\begin{aligned}&\sum_{n\geq 0} p_D(5n+1)q^n\\
&= \frac{E(q^5)^5E(q^{10})}{E(q)^6}\\
&\quad\times\left(\left(\frac{1}{R(q)^3R(q^2)} + q^2R(q)^3R(q^2)\right) - 5q - 2q\left(\frac{R(q^2)}{R(q)^2} - \frac{R(q)^2}{R(q^2)}\right)\right)\\
&= \frac{E(q^5)^5E(q^{10})}{E(q)^6}\\
&\quad\times\left(\left(\frac{\chi(-q^5)^5}{\chi(-q)} + 2q + 4q^2\frac{\chi(-q)}{\chi(-q^5)^5}\right) - 5q - 2q\cdot 4q\frac{\chi(-q)}{\chi(-q^5)^5}\right)\\
&= \frac{E(q^5)^5E(q^{10})}{E(q)^6}\left(\frac{\chi(-q^5)^5}{\chi(-q)} - 3q - 4q^2\frac{\chi(-q)}{\chi(-q^5)^5}\right)\end{aligned}$$

$$
\begin{aligned}
&= \frac{E(q^5)^5E(q^{10})}{E(q)^6}\left(\frac{\chi(-q^5)^5}{\chi(-q)}+q\right)\left(1-4q\frac{\chi(-q)}{\chi(-q^5)^5}\right)\\
&= \left(\frac{E(q^5)^5E(q^{10})}{E(q)^6}\right)\left(\frac{E(q^2)^4E(q^5)^2}{E(q)^2E(q^{10})^4}\right)\left(\frac{E(q)^4E(q^{10})^2}{E(q^2)^2E(q^5)^4}\right)\\
&= \frac{E(q^2)^2E(q^5)^3}{E(q)^4E(q^{10})}.
\end{aligned}
$$

□

Note that (1.1) is the case $\alpha = 1$ of (1.3).

4. The Modular Equation

We obtain the modular equation for ζ.

Let $\zeta(q^5) = Z$.

Theorem 4.1.

$$
\begin{aligned}
&\zeta^5 - (205Z + 4300Z^2 + 34000Z^3 + 120000Z^4 + 160000Z^5)\zeta^4\\
&\quad - (215Z + 4475Z^2 + 35000Z^3 + 122000Z^4 + 160000Z^5)\zeta^3\\
&\quad - (85Z + 1750Z^2 + 13525Z^3 + 46500Z^4 + 60000Z^5)\zeta^2\\
&\quad - (15Z + 305Z^2 + 2325Z^3 + 7875Z^4 + 10000Z^5)\zeta\\
&\quad - (Z + 20Z^2 + 150Z^3 + 500Z^4 + 625Z^5) = 0.
\end{aligned}
\tag{4.1}
$$

Proof. Let H be the huffing operator, given by

$$H\left(\sum_n a(n)q^n\right) = \sum_n a(5n)q^{5n}.$$

We can show, using extremely lengthy but elementary calculations (see Sect. 9 "Appendix"), that

$$H(\zeta) = 41Z + 860Z^2 + 6800Z^3 + 24000Z^4 + 32000Z^5, \tag{4.2}$$

$$
\begin{aligned}
H(\zeta^2) &= 86Z + 10195Z^2 + 366600Z^3 + 6534800Z^4 + 68384000Z^5\\
&\quad + 450720000Z^6 + 1907200000Z^7 + 5056000000Z^8\\
&\quad + 7680000000Z^9 + 5120000000Z^{10},
\end{aligned}
\tag{4.3}
$$

$$
\begin{aligned}
H(\zeta^3) &= 51Z + 27495Z^2 + 2836265Z^3 + 128688900Z^4 + 3343692000Z^5\\
&\quad + 56283680000Z^6 + 656205600000Z^7 + 5502096000000Z^8\\
&\quad + 33821312000000Z^9 + 153192960000000Z^{10}\\
&\quad + 506956800000000Z^{11} + 1195008000000000Z^{12}\\
&\quad + 1904640000000000Z^{13} + 1843200000000000Z^{14}\\
&\quad + 819200000000000Z^{15},
\end{aligned}
\tag{4.4}
$$

$$
\begin{aligned}
H(\zeta^4) &= 12Z + 32674Z^2 + 8579260Z^3 + 831492275Z^4 + 42958434000Z^5\\
&\quad + 1396773180000Z^6 + 31314949600000Z^7 + 511802288800000Z^8
\end{aligned}
$$

$$
\begin{aligned}
&+ 6319880448000000Z^{9} + 60349364480000000Z^{10}\\
&+ 452174745600000000Z^{11} + 2679038592000000000Z^{12}\\
&+ 12574269440000000000Z^{13} + 46561935360000000000Z^{14}\\
&+ 134544588800000000000Z^{15} + 297365504000000000000Z^{16}\\
&+ 485949440000000000000Z^{17} + 553779200000000000000Z^{18}\\
&+ 393216000000000000000Z^{19} + 131072000000000000000Z^{20} \qquad (4.5)
\end{aligned}
$$

and

$$
\begin{aligned}
H(\zeta^5) = {} & Z + 21370Z^2 + 13932050Z^3 + 2684902125Z^4 + 251131688125Z^5\\
&+ 14097638650000Z^6 + 532547945100000Z^7 + 14515766554000000Z^8\\
&+ 298883447380000000Z^9 + 4797842366000000000Z^{10}\\
&+ 61395781800000000000Z^{11} + 636255683040000000000Z^{12}\\
&+ 5398601306880000000000Z^{13} + 37772239436800000000000Z^{14}\\
&+ 218755840000000000000000Z^{15} + 1049457704960000000000000Z^{16}\\
&+ 4160657715200000000000000Z^{17} + 13552680960000000000000000Z^{18}\\
&+ 35909189632000000000000000Z^{19} + 76195266560000000000000000Z^{20}\\
&+ 126438604800000000000000000Z^{21} + 158138368000000000000000000Z^{22}\\
&+ 140247040000000000000000000Z^{23} + 78643200000000000000000000Z^{24}\\
&+ 20971520000000000000000000Z^{25}. \qquad (4.6)
\end{aligned}
$$

Let η be a fifth root of unity other than 1, and for $i = 0, 1, 2, 3, 4$ define

$$\zeta_i = \zeta(\eta^i q).$$

Then the power sums $\pi_1, \ldots, \pi_5$ of the ζ_i are given by

$$
\begin{aligned}
\pi_1 &= \zeta_0 + \cdots + \zeta_4 = 5H(\zeta),\\
\pi_2 &= \zeta_0^2 + \cdots + \zeta_4^2 = 5H(\zeta^2),\\
&\cdots\\
\pi_5 &= \zeta_0^5 + \cdots + \zeta_4^5 = 5H(\zeta^5). \qquad (4.7)
\end{aligned}
$$

From (4.7) we obtain the symmetric functions $\sigma_1, \ldots, \sigma_5$ of the ζ_i,

$$
\begin{aligned}
\sigma_1 &= \sum_i \zeta_i = \pi_1\\
&= 205Z + 4300Z^2 + 34000Z^3 + 120000Z^4 + 160000Z^5,\\
\sigma_2 &= \sum_{i<j} \zeta_i\zeta_j = \frac{1}{2}(\pi_1\sigma_1 - \pi_2)\\
&= -215Z - 4475Z^2 - 35000Z^3 - 122000Z^4 - 160000Z^5,\\
\sigma_3 &= \sum_{i<j<k} \zeta_i\zeta_j\zeta_k = \frac{1}{3}(\pi_1\sigma_2 - \pi_2\sigma_1 + \pi_3)
\end{aligned}
$$

$$
\begin{aligned}
&= 85Z + 1750Z^2 + 13525Z^3 + 46000Z^4 + 60000Z^5, \\
\sigma_4 &= \sum_{i<j<k<l} \zeta_i\zeta_j\zeta_k\zeta_l = \frac{1}{4}(\pi_1\sigma_3 - \pi_2\sigma_2 + \pi_3\sigma_1 - \pi_4) \\
&= -15Z - 305Z^2 - 2325Z^3 - 7875Z^4 - 10000Z^5, \\
\sigma_5 &= \zeta_0\zeta_1\cdots\zeta_4 = \frac{1}{5}(\pi_1\sigma_4 - \pi_2\sigma_3 + \pi_3\sigma_2 - \pi_4\sigma_1 + \pi_5) \\
&= Z + 20Z^2 + 150Z^3 + 500Z^4 + 625Z^5.
\end{aligned}
$$

Now, $\zeta_0, \ldots, \zeta_4$ are the roots of

$$
\begin{aligned}
&(X-\zeta_0)(X-\zeta_1)(X-\zeta_2)(X-\zeta_3)(X-\zeta_4) \\
&\quad = X^5 - \sigma_1 X^4 + \sigma_2 X^3 - \sigma_3 X^2 + \sigma_4 X - \sigma_5 = 0,
\end{aligned}
$$

or,

$$
\begin{aligned}
X^5 &- (205Z + 4300Z^2 + 34000Z^3 + 120000Z^4 + 160000Z^5)X^4 \\
&- (215Z + 4475Z^2 + 35000Z^3 + 122000Z^4 + 160000Z^5)X^3 \\
&- (85Z + 1750Z^2 + 13525Z^3 + 46500Z^4 + 60000Z^5)X^2 \\
&- (15Z + 305Z^2 + 2325Z^3 + 7875Z^4 + 10000Z^5)X \\
&- (Z + 20Z^2 + 150Z^3 + 500Z^4 + 625Z^5) = 0.
\end{aligned}
$$

In particular, ζ is a root, and we obtain (4.1). □

Remark 4.2. It is truly remarkable, amazing even, that although $\pi_1, \ldots, \pi_5$ are polynomials of degree up to 25, $\sigma_1, \ldots, \sigma_5$ are of degree 5.

5. Some Important Recurrences and Generating Functions

Let U be the unitizing operator, given by

$$
U\left(\sum_n a(n)q^n\right) = \sum_n a(5n)q^n.
$$

It follows from (4.1) that for $i \geq 6$, $u_i = U(\zeta^i)$ satisfies the recurrence

$$
\begin{aligned}
u_i = &(205\zeta + 4300\zeta^2 + 34000\zeta^3 + 120000\zeta^4 + 160000\zeta^5)u_{i-1} \\
&+ (215\zeta + 4475\zeta^2 + 35000\zeta^3 + 122000\zeta^4 + 160000\zeta^5)u_{i-2} \\
&+ (85\zeta + 1750\zeta^2 + 13525\zeta^3 + 46500\zeta^4 + 60000\zeta^5)u_{i-3} \\
&+ (15\zeta + 305\zeta^2 + 2325\zeta^3 + 7875\zeta^4 + 10000\zeta^5)u_{i-4} \\
&+ (\zeta + 20\zeta^2 + 150\zeta^3 + 500\zeta^4 + 625\zeta^5)u_{i-5}.
\end{aligned}
\tag{5.1}
$$

The recurrence (5.1), together with the five initial values $u_1, u_2, \ldots, u_5$, which can be read off from (4.2)–(4.6) by replacing Z by ζ, gives

$$
\sum_{i\geq 1} u_i x^i = \frac{N}{D}, \tag{5.2}
$$

where

$$\begin{aligned} N = {} & (41\zeta + 860\zeta^2 + 6800\zeta^3 + 24000\zeta^4 + 32000\zeta^5)x \\ & + (86\zeta + 1790\zeta^2 + 14000\zeta^3 + 48800\zeta^4 + 64000\zeta^5)x^2 \\ & + (51\zeta + 1050\zeta^2 + 8115\zeta^3 + 27900\zeta^4 + 36000\zeta^5)x^3 \\ & + (12\zeta + 244\zeta^2 + 1869\zeta^3 + 6300\zeta^4 + 8000\zeta^5)x^4 \\ & + (\zeta + 20\zeta^2 + 150\zeta^3 + 500\zeta^4 + 625\zeta^5)x^5 \end{aligned} \tag{5.3}$$

and

$$\begin{aligned} D = 1 & - (205\zeta + 4300\zeta^2 + 34000\zeta^3 + 120000\zeta^4 + 160000\zeta^5)x \\ & - (215\zeta + 4475\zeta^2 + 35000\zeta^3 + 122000\zeta^4 + 160000\zeta^5)x^2 \\ & - (85\zeta + 1750\zeta^2 + 13525\zeta^3 + 46500\zeta^4 + 60000\zeta^5)x^3 \\ & - (15\zeta + 305\zeta^2 + 2325\zeta^3 + 7875\zeta^4 + 10000\zeta^5)x^4 \\ & - (\zeta + 20\zeta^2 + 150\zeta^3 + 500\zeta^4 + 625\zeta^5)x^5. \end{aligned} \tag{5.4}$$

From (5.2)–(5.4), we deduce that for $i \geq 1$,

$$U(\zeta^i) = u_i = \sum_{j=1}^{5i} \mu_{i,j}\zeta^j,$$

where the $\mu_{i,j}$ are given by

$$\sum_{i=1}^{\infty}\sum_{j=1}^{5i} \mu_{i,j}x^i y^j = \frac{N'}{D'},$$

where

$$\begin{aligned} N' = {} & (41y + 860y^2 + 6800y^3 + 24000y^4 + 32000y^5)x \\ & + (86y + 1790y^2 + 14000y^3 + 48800y^4 + 64000y^5)x^2 \\ & + (51y + 1050y^2 + 8115y^3 + 27900y^4 + 36000y^5)x^3 \\ & + (12y + 244y^2 + 1869y^3 + 6300y^4 + 8000y^5)x^4 \\ & + (y + 20y^2 + 150y^3 + 500y^4 + 625y^5)x^5 \end{aligned}$$

and

$$\begin{aligned} D' = 1 & - (205y + 4300y^2 + 34000y^3 + 120000y^4 + 160000y^5)x \\ & - (215y + 4475y^2 + 35000y^3 + 122000y^4 + 160000y^5)x^2 \\ & - (85y + 1750y^2 + 13525y^3 + 46500y^4 + 60000y^5)x^3 \\ & - (15y + 305y^2 + 2325y^3 + 7875y^4 + 10000y^5)x^4 \\ & - (y + 20y^2 + 150y^3 + 500y^4 + 625y^5)x^5. \end{aligned} \tag{5.5}$$

More importantly, if we multiply (4.1) by γ and apply the operator U, we see that $v_i = U(\gamma\zeta^{i-1})$ satisfy the recurrence (5.1) (with v for u).

Also, using the same sort of calculations as in Sect. 4 (see Sect. 9 “Appendix”),

$$v_1 = U(\gamma) = \delta(1 + 160\zeta + 2800\zeta^2 + 16000\zeta^3 + 32000\zeta^4), \tag{5.6}$$

$$\begin{aligned} v_2 = U(\gamma\zeta) = \delta(&385\zeta + 40100\zeta^2 + 1312800\zeta^3 + 20912000\zeta^4 + 189920000\zeta^5 \\ &+ 1043200000\zeta^6 + 3456000000\zeta^7 + 6400000000\zeta^8 + 5120000000\zeta^9), \end{aligned} \tag{5.7}$$

$$\begin{aligned} v_3 = U(\gamma\zeta^2) = \delta(&290\zeta + 119015\zeta^2 + 11235600\zeta^3 + 476348000\zeta^4 \\ &+ 11537760000\zeta^5 + 179434400000\zeta^6 + 1908992000000\zeta^7 \\ &+ 14377472000000\zeta^8 + 77783040000000\zeta^9 + 301644800000000\zeta^{10} \\ &+ 821248000000000\zeta^{11} + 1495040000000000\zeta^{12} \\ &+ 1638400000000000\zeta^{13} + 819200000000000\zeta^{14}), \end{aligned} \tag{5.8}$$

$$\begin{aligned} v_4 = U(\gamma\zeta^3) = \delta(&99\zeta + 157795\zeta^2 + 36522125\zeta^3 + 3308569500\zeta^4 \\ &+ 161943150000\zeta^5 + 4995603800000\zeta^6 + 105933588800000\zeta^7 \\ &+ 1628976896000000\zeta^8 + 18797435520000000\zeta^9 \\ &+ 166360908800000000\zeta^{10} + 1143762304000000000\zeta^{11} \\ &+ 6142300160000000000\zeta^{12} + 25729781760000000000\zeta^{13} \\ &+ 83330457600000000000\zeta^{14} + 204857344000000000000\zeta^{15} \\ &+ 370032640000000000000\zeta^{16} + 463667200000000000000\zeta^{17} \\ &+ 360448000000000000000\zeta^{18} + 131072000000000000000\zeta^{19}) \end{aligned} \tag{5.9}$$

and

$$\begin{aligned} v_5 = U(\gamma\zeta^4) = \delta(&16\zeta + 118090\zeta^2 + 63835100\zeta^3 + 11315760375\zeta^4 \\ &+ 1002222145000\zeta^5 + 53778439200000\zeta^6 + 1946392973200000\zeta^7 \\ &+ 50789296612000000\zeta^8 + 998696483520000000\zeta^9 \\ &+ 15256932894400000000\zeta^{10} + 185007570368000000000\zeta^{11} \\ &+ 180767148928000000000\zeta^{12} + 1437629353984000000000\zeta^{13} \\ &+ 9363034552320000000000\zeta^{14} + 50063652249600000000000\zeta^{15} \\ &+ 219558209536000000000000\zeta^{16} + 786078842880000000000000\zeta^{17} \\ &+ 2276812390400000000000000\zeta^{18} + 5256465612800000000000000\zeta^{19} \\ &+ 9452257280000000000000000\zeta^{20} + 12766412800000000000000000\zeta^{21} \\ &+ 12189696000000000000000000\zeta^{22} + 7340032000000000000000000\zeta^{23} \\ &+ 20971520000000000000000000\zeta^{24}). \end{aligned} \tag{5.10}$$

It follows that for $i \geq 1$,

$$U(\gamma\zeta^{i-1}) = \delta\sum_{j=1}^{5i} \alpha_{i,j}\zeta^{j-1}, \tag{5.11}$$

where

$$\sum_{i=1}^{\infty}\sum_{j=1}^{5i}\alpha_{i,j}x^i y^j = \frac{N_\alpha}{D'},$$

and where

$$\begin{aligned}N_\alpha = {} & (y + 160y^2 + 2800y^3 + 16000y^4 + 32000y^5)x \\ & + (180y^2 + 3000y^3 + 16800y^4 + 32000y^5)x^2 \\ & + (75y^2 + 1215y^3 + 6600y^4 + 12000y^5)x^3 \\ & + (14y^2 + 220y^3 + 1150y^4 + 2000y^5)x^4 \\ & + (y^2 + 15y^3 + 75y^4 + 125y^5)x^5\end{aligned}$$

and D' is given in (5.5).

Similarly, if we multiply (4.1) by $q^{-1}\delta$ and apply the operator U, we see that $w_i = U(q^{-1}\delta\zeta^{i-1})$ satisfy (5.1) (with w for u).

Also,

$$w_1 = U(q^{-1}\delta) = \gamma(5 + 660\zeta + 14400\zeta^2 + 120000\zeta^3 + 448000\zeta^4 + 640000\zeta^5), \tag{5.12}$$

$$\begin{aligned}w_2 = U(q^{-1}\delta\zeta) = \gamma(&1 + 1705\zeta + 171700\zeta^2 + 6083200\zeta^3 + 110016000\zeta^4 \\ & + 178080000\zeta^5 + 797120000\zeta^6 + 34688000000\zeta^7 + 94720000000\zeta^8 \\ & + 148480000000\zeta^9 + 1024000000000\zeta^{10}),\end{aligned} \tag{5.13}$$

$$\begin{aligned}w_3 = U(q^{-1}\delta\zeta^2) = \gamma(&1545\zeta + 523885\zeta^2 + 48836000\zeta^3 + 2157580000\zeta^4 \\ & + 55972480000\zeta^5 + 950485600000\zeta^6 + 11233328000000\zeta^7 \\ & + 95713408000000\zeta^8 + 598718720000000\zeta^9 + 2762265600000000\zeta^{10} \\ & + 9317888000000000\zeta^{11} + 22405120000000000\zeta^{12} \\ & + 36454400000000000\zeta^{13} + 36044800000000000\zeta^{14} \\ & + 16384000000000000\zeta^{15}),\end{aligned} \tag{5.14}$$

$$\begin{aligned}w_4 = U(q^{-1}\delta\zeta^3) = \gamma(&686\zeta + 753625\zeta^2 + 161075075\zeta^3 + 14497246500\zeta^4 \\ & + 727863490000\zeta^5 + 23458401400000\zeta^6 + 526452595200000\zeta^7 \\ & + 8658501792000000\zeta^8 + 107918950400000000\zeta^9 \\ & + 1042082905600000000\zeta^{10} + 7904596864000000000\zeta^{11} \\ & + 47450048000000000000\zeta^{12} + 225774243840000000000\zeta^{13} \\ & + 847926476800000000000\zeta^{14} + 2486042624000000000000\zeta^{15} \\ & + 5577277440000000000000\zeta^{16} + 9255321600000000000000\zeta^{17} \\ & + 10715136000000000000000\zeta^{18} + 7733248000000000000000\zeta^{19} \\ & + 2621440000000000000000\zeta^{20})\end{aligned} \tag{5.15}$$

and

$$\begin{aligned}
w_5 = U(q^{-1}\delta\zeta^4) = \gamma(&163\zeta + 630970\zeta^2 + 295013300\zeta^3 \\
&+ 50030923625\zeta^4 + 4413689785000\zeta^5 + 240963519250000\zeta^6 \\
&+ 8992052284600000\zeta^7 + 244243690752000000\zeta^8 + 5037514186320000000\zeta^9 \\
&+ 81262009334400000000\zeta^{10} + 1047144506208000000000\zeta^{11} \\
&+ 10942698476160000000000\zeta^{12} + 93715045227520000000000\zeta^{13} \\
&+ 662259232256000000000000\zeta^{14} + 3875774510080000000000000\zeta^{15} \\
&+ 18796453150720000000000000\zeta^{16} + 75357109452800000000000000\zeta^{17} \\
&+ 248290942976000000000000000\zeta^{18} + 665623035904000000000000000\zeta^{19} \\
&+ 1429384069120000000000000000\zeta^{20} + 2401107968000000000000000000\zeta^{21} \\
&+ 3040870400000000000000000000\zeta^{22} + 2731540480000000000000000000\zeta^{23} \\
&+ 1551892480000000000000000000\zeta^{24} + 419430400000000000000000000\zeta^{25}).
\end{aligned} \tag{5.16}$$

It follows that for $i \geq 1$,

$$U(q^{-1}\delta\zeta^{i-1}) = \gamma \sum_{j=1}^{5i+1} \beta_{i,j}\zeta^{j-1}, \tag{5.17}$$

where

$$\sum_{i=1}^{\infty} \sum_{j=1}^{5i+1} \beta_{i,j} x^i y^j = \frac{N_\beta}{D'},$$

and where

$$\begin{aligned}
N_\beta = &(5y + 660y^2 + 14400y^3 + 120000y^4 + 448000y^5 + 640000y^6)x \\
&+ (y + 680y^2 + 14900y^3 + 123200y^4 + 456000y^5 + 640000y^6)x^2 \\
&+ (265y^2 + 5785y^3 + 47500y^4 + 174000y^5 + 240000y^6)x^3 \\
&+ (46y^2 + 1000y^3 + 8150y^4 + 29500y^5 + 40000y^6)x^4 \\
&+ (3y^2 + 65y^3 + 525y^4 + 1875y^5 + 2500y^6)x^5
\end{aligned}$$

and D' is given in (5.5).

6. Proof of the First Part of Theorem 1.1

The first part of Theorem 1.1 follows by a simple induction from (1.1), (5.11) and (5.17), as we now demonstrate.

We know that (1.3) is true for $\alpha = 1$. Suppose (1.3) is true for some $\alpha \geq 1$. Then

$$\sum_{n \geq 0} p_D\left(5^{2\alpha-1}n + \frac{5^{2\alpha}-1}{24}\right)q^n = \gamma \sum_{i=1}^{(5^{2\alpha}-1)/24} x_{2\alpha-1,i}\zeta^{i-1}. \tag{6.1}$$

If we apply the operator U to (6.1) and use (5.11), we find

$$\begin{aligned}\sum_{n\geq 0} p_D\left(5^{2\alpha-1}(5n)+\frac{5^{2\alpha}-1}{24}\right)q^n &= \sum_{i=1}^{(5^{2\alpha}-1)/24} x_{2\alpha-1,i}U(\gamma\zeta^{i-1}) \\ &= \sum_{i=1}^{(5^{2\alpha}-1)/24} x_{2\alpha-1,i}\delta\sum_{j=1}^{5i}\alpha_{i,j}\zeta^{j-1} \\ &= \delta\sum_{j=1}^{(5^{2\alpha+1}-5)/24}\left(\sum_{i=1}^{(5^{2\alpha}-1)/24} x_{2\alpha-1,i}\alpha_{i,j}\right)\zeta^{j-1} \\ &= \delta\sum_{j=1}^{(5^{2\alpha+1}-5)/24} x_{2\alpha,j}\zeta^{j-1},\end{aligned}$$

or,

$$\sum_{n\geq 0} p_D\left(5^{2\alpha}n+\frac{5^{2\alpha}-1}{24}\right)q^n = \delta\sum_{j=1}^{(5^{2\alpha+1}-5)/24} x_{2\alpha,j}\zeta^{j-1},$$

which is (1.4).

Now suppose (1.4) is true for some $\alpha\geq 1$. Then

$$\sum_{n\geq 0} p_D\left(5^{2\alpha}n+\frac{5^{2\alpha}-1}{24}\right)q^{n-1} = q^{-1}\delta\sum_{i=1}^{(5^{2\alpha+1}-5)/24} x_{2\alpha,i}\zeta^{i-1}. \tag{6.2}$$

If we apply the operator U to (6.2) and use (5.17), we find

$$\begin{aligned}\sum_{n\geq 0} p_D\left(5^{2\alpha}(5n+1)+\frac{5^{2\alpha}-1}{24}\right)q^n &= \sum_{i=1}^{(5^{2\alpha+1}-5)/24} x_{2\alpha,i}U(q^{-1}\delta\zeta^{i-1}) \\ &= \sum_{i=1}^{(5^{2\alpha+1}-5)/24} x_{2\alpha,i}\gamma\sum_{j=1}^{5i+1}\beta_{i,j}\zeta^{j-1} \\ &= \gamma\sum_{j=1}^{(5^{2\alpha+2}-1)/24}\left(\sum_{i=1}^{(5^{2\alpha+1}-5)/24} x_{2\alpha,i}\beta_{i,j}\right)\zeta^{j-1} \\ &= \gamma\sum_{j=1}^{(5^{2\alpha+2}-1)/24} x_{2\alpha+1,j}\zeta^{j-1},\end{aligned}$$

or,

$$\sum_{n\geq 0} p_D\left(5^{2\alpha+1}n+\frac{5^{2\alpha+2}-1}{24}\right)q^n=\gamma\sum_{j=1}^{(5^{2\alpha+2}-1)/24} x_{2\alpha+1,j}\zeta^{j-1},$$

which is (1.3) with $\alpha+1$ for α. □

7. Proof of the Second Part of Theorem 1.1

Let $\nu(n)$ denote the (highest) power of 5 that divides n.

We prove the following theorem.

Theorem 7.1.

$$\nu(\alpha_{i,j})\geq\left\lfloor\frac{5j-i-1}{6}\right\rfloor, \tag{7.1}$$

$$\nu(\beta_{i,j})\geq\left\lfloor\frac{5j-i-1}{6}\right\rfloor. \tag{7.2}$$

Proof. Let $\lambda_{i,j}=\nu(\alpha_{i,j})$, $\rho_{i,j}=\left\lfloor\frac{5j-i-1}{6}\right\rfloor$.

Observe that from the recurrence (5.1), for $i,j\geq 6$,

$$\begin{aligned}\lambda_{i,j}\geq\min(&\lambda_{i-1,j-1}+1,\lambda_{i-1,j-2}+2,\lambda_{i-1,j-3}+3,\lambda_{i-1,j-4}+4,\\ &\lambda_{i-1,j-5}+4,\lambda_{i-2,j-1}+1,\lambda_{i-2,j-2}+2,\lambda_{i-2,j-3}+4,\\ &\lambda_{i-2,j-4}+3,\lambda_{i-2,j-5}+4,\lambda_{i-3,j-1}+1,\lambda_{i-3,j-2}+3,\\ &\lambda_{i-3,j-3}+2,\lambda_{i-3,j-4}+3,\lambda_{i-3,j-5}+4,\lambda_{i-4,j-1}+1,\\ &\lambda_{i-4,j-2}+1,\lambda_{i-4,j-3}+2,\lambda_{i-4,j-4}+3,\lambda_{i-4,j-5}+4,\\ &\lambda_{i-5,j-1}+0,\lambda_{i-5,j-2}+1,\lambda_{i-5,j-3}+2,\lambda_{i-5,j-4}+3,\\ &\lambda_{i-5,j-5}+4).\end{aligned} \tag{7.3}$$

On the other hand,

$$\begin{aligned}\rho_{i,j}=\min(&\rho_{i-1,j-1}+1,\rho_{i-1,j-2}+2,\rho_{i-1,j-3}+3,\rho_{i-1,j-4}+4,\\ &\rho_{i-1,j-5}+4,\rho_{i-2,j-1}+1,\rho_{i-2,j-2}+2,\rho_{i-2,j-3}+4,\\ &\rho_{i-2,j-4}+3,\rho_{i-2,j-5}+4,\rho_{i-3,j-1}+1,\rho_{i-3,j-2}+3,\\ &\rho_{i-3,j-3}+2,\rho_{i-3,j-4}+3,\rho_{i-3,j-5}+4,\rho_{i-4,j-1}+1,\\ &\rho_{i-4,j-2}+1,\rho_{i-4,j-3}+2,\rho_{i-4,j-4}+3,\rho_{i-4,j-5}+4,\\ &\rho_{i-5,j-1}+0,\rho_{i-5,j-2}+1,\rho_{i-5,j-3}+2,\rho_{i-5,j-4}+3,\\ &\rho_{i-5,j-5}+4).\end{aligned} \tag{7.4}$$

The right side of (7.4)

$$
\begin{aligned}
&= \min\Bigg(\left\lfloor \frac{5j-i+1}{6} \right\rfloor, \left\lfloor \frac{5j-i+2}{6} \right\rfloor, \left\lfloor \frac{5u-i+3}{6} \right\rfloor, \left\lfloor \frac{5j-i+4}{6} \right\rfloor, \left\lfloor \frac{5j-i-1}{6} \right\rfloor, \\
&\quad \left\lfloor \frac{5j-i+2}{6} \right\rfloor, \left\lfloor \frac{5j-i+3}{6} \right\rfloor, \left\lfloor \frac{5j-i+10}{6} \right\rfloor, \left\lfloor \frac{5j-i-1}{6} \right\rfloor, \left\lfloor \frac{5j-i}{6} \right\rfloor, \\
&\quad \left\lfloor \frac{5j-i+3}{6} \right\rfloor, \left\lfloor \frac{5j-i+10}{6} \right\rfloor, \left\lfloor \frac{5j-i-1}{6} \right\rfloor, \left\lfloor \frac{5j-i}{6} \right\rfloor, \left\lfloor \frac{5j-i+1}{6} \right\rfloor, \\
&\quad \left\lfloor \frac{5j-i+4}{6} \right\rfloor, \left\lfloor \frac{5j-i-1}{6} \right\rfloor, \left\lfloor \frac{5j-i}{6} \right\rfloor, \left\lfloor \frac{5j-i+1}{6} \right\rfloor, \left\lfloor \frac{5j-i+3}{6} \right\rfloor, \\
&\quad \left\lfloor \frac{5j-i-1}{6} \right\rfloor, \left\lfloor \frac{5j-i}{6} \right\rfloor, \left\lfloor \frac{5j-i+1}{6} \right\rfloor, \left\lfloor \frac{5j-i+2}{6} \right\rfloor, \left\lfloor \frac{5j-i+3}{6} \right\rfloor \Bigg) \\
&= \left\lfloor \frac{5j-i-1}{6} \right\rfloor = \rho_{i,j}.
\end{aligned}
$$

The values of $\lambda_{i,j} - \rho_{i,j}$ for $1 \le i \le 5$ and for $1 \le j \le 5$ are given in the following tables. Note that they are all non-negative. (We use $\bullet$ for ∞.)

i \ j	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16	17	18	19	20	21	22	23	24	25	26
1	0	0	0	0	0	$\bullet$ $\cdots$																				
2	$\bullet$	0	0	0	0	0	0	0	1	0	$\bullet$ $\cdots$															
3	$\bullet$	0	0	0	0	0	0	0	0	0	0	0	0	0	0	$\bullet$ $\cdots$										
4	$\bullet$	0	0	1	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	$\bullet$ $\cdots$					
5	$\bullet$	0	0	0	0	0	1	0	0	0	0	0	1	0	0	0	0	0	1	0	1	1	1	0	0	$\bullet$ $\cdots$

(7.5)

j \ i	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16	17	18	19	20	21	22
1	0	$\bullet$ $\cdots$																				
2	0	0	0	0	0	0	$\bullet$ $\cdots$															
3	0	0	0	0	0	0	0	1	0	0	0	$\bullet$ $\cdots$										
4	0	0	0	1	0	0	0	0	0	0	0	0	0	0	0	0	$\bullet$ $\cdots$					
5	0	0	0	0	0	0	0	0	0	0	2	0	0	0	0	0	0	0	0	0	0	$\bullet$ $\cdots$

(7.6)

We see that (7.1) follows from (7.3)–(7.6) by induction.

The proof of (7.2) is essentially the same as that of (7.1). The boundary values are given by the following tables.

i \ j	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16	17	18	19	20	21	22	23	24	25	26	27
1	1	0	0	1	0	0	$\bullet$ $\cdots$																				
2	0	0	0	0	0	0	0	0	0	0	0	$\bullet$ $\cdots$															
3	$\bullet$	0	0	1	1	0	0	0	0	0	0	0	0	0	0	0	$\bullet$ $\cdots$										
4	$\bullet$	0	2	0	0	0	0	0	0	1	0	0	2	0	0	0	0	0	0	0	0	$\bullet$ $\cdots$					
5	$\bullet$	0	0	0	0	0	2	0	0	0	0	0	1	0	1	1	0	0	1	0	0	1	2	0	0	0	$\bullet$ $\cdots$

	i																							
	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16	17	18	19	20	21	22	23	
j 1	1	0	$\bullet$	$\cdots$																				
2	0	0	0	0	0	1	0	$\bullet$	$\cdots$															
3	0	0	0	2	0	0	0	0	0	0	1	0	$\bullet$	$\cdots$										
4	1	0	1	0	0	0	0	0	0	0	0	0	0	0	0	2	0	$\bullet$	$\cdots$					
5	0	0	1	0	0	0	0	0	1	0	1	0	0	0	1	0	0	0	0	0	1	0	$\bullet$	$\cdots$

$\square$

Theorem 7.2. *For $\alpha \geq 0$,*

$$\nu(x_{2\alpha+1,1}) \geq \alpha, \quad \nu(x_{2\alpha+1,i}) \geq \alpha + \left\lfloor \frac{5i-8}{6} \right\rfloor \text{ for } i \geq 2,$$

$$\nu(x_{2\alpha+2,i}) \geq \alpha + \left\lfloor \frac{5i-2}{6} \right\rfloor.$$

Proof. If we replace $\nu(A)$ by

$$\left(\left\lfloor \frac{5j-i-1}{6} \right\rfloor \right)_{i,j\geq 1}$$

and $\nu(B)$ by

$$\left(\left\lfloor \frac{5j-i-1}{6} \right\rfloor \right)_{i,j\geq 1}$$

with the exception $\nu(b_{1,1}) = 1$, and we start with $\nu(\mathbf{x}_1) = (0, \infty, \ \ldots)$, the results follow by induction. $\square$

This completes the proof of Theorem 1.1. $\square$

8. Calculations

We find that

$\mathbf{x}_1 = (1, 0, \ldots\ ),$

$\mathbf{x}_2 = (1, 160, 2800, 16000, 32000, 0, \ \ldots\ ),$

$$\begin{aligned}\mathbf{x}_3 = (&5*33,\ 2^2*5*1039573,\ 2^4*5^2*84358511,\ 2^6*5^3*1519417629,\\ &2^8*5^3*57468885219,\ 2^{10}*5^4*239126250621,\ 2^{20}*5^6*493702983,\\ &2^{16}*5^7*57851635449,\ 2^{17}*5^8*155363323153,\ 2^{22}*5^8*99443868167,\\ &2^{20}*5^9*1277863945093,\ 2^{23}*5^{11}*82117001559,\ 2^{24}*5^{12}*85675198911,\\ &2^{29}*5^{14}*916288433,\ 2^{29}*5^{13}*32357578059,\ 2^{33}*5^{14}*2366343709,\\ &2^{36}*5^{16}*57370733,\ 2^{37}*5^{17}*22998577,\ 2^{36}*5^{18}*30309607,\\ &2^{38}*5^{18}*20313321,\ 2^{40}*5^{19}*2181069,\ 2^{43}*5^{21}*18319,\\ &2^{48}*5^{23}*29,\ 2^{46}*5^{22}*521,\ 2^{49}*5^{22}*37,\ 2^{50}*5^{23},\ 0,\ \ \ldots\ ),\end{aligned}$$

in agreement with Baruah and Begum and

$$\begin{aligned}
\nu(\mathbf{x}_1) &= (0,\ \infty,\ \dots),\\
\nu(\mathbf{x}_2) &= (0,\ 1,\ 2,\ 3,\ 3,\ \infty,\ \dots),\\
\nu(\mathbf{x}_3) &= (1,\ 1,\ 2,\ 3,\ 3,\ 4,\ 6,\ 7,\ 8,\ 8,\ 9,\ 11,\ 12,\ 14,\ 13,\ 14,\ 16,\ 17,\ 18,\ 18,\ 19,\ 21,\\
&\qquad 23,\ 22,\ 22,\ 23,\ \infty,\ \dots).
\end{aligned}$$

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9. Appendix

We provide a proof of (4.2). The proofs of (4.3)–(4.6), (5.6)–(5.10) and (5.12)–(5.16) are similar but lengthier.

We require the following results.

Lemma 9.1. *Let*

$$K = \frac{\chi(-q^5)^5}{\chi(-q)}.$$

Then

$$\frac{R(q^2)}{R(q)^2} - \frac{R(q)^2}{R(q^2)} = \frac{4q}{K}, \tag{9.1}$$

$$\frac{1}{R(q)R(q^2)^2} - q^2R(q)R(q^2)^2 = K, \tag{9.2}$$

$$\frac{R(q)}{R(q^2)^3} + q^2\frac{R(q^2)^3}{R(q)} = K - 2q + \frac{4q^2}{K}, \tag{9.3}$$

$$\frac{1}{R(q)^3R(q^2)} + q^2R(q)^3R(q^2) = K + 2q + \frac{4q^2}{K}, \tag{9.4}$$

$$\frac{1}{R(q)^5} - q^2R(q)^5 = K + 4q + \frac{8q^2}{K} + \frac{16q^3}{K^2}, \tag{9.5}$$

$$\frac{R(q^2)}{R(q)^7} + q^2\frac{R(q)^7}{R(q^2)} = K + 6q + \frac{20q^2}{K} + \frac{32q^3}{K^2} + \frac{64q^4}{K^3}, \tag{9.6}$$

$$\begin{aligned}\frac{1}{R(q)^{10}} + q^4R(q)^{10} = K^2 + 8qK + 34q^2 + \frac{96q^3}{K}\\
+\frac{192q^4}{K^2} + \frac{2546q^5}{K^3} + \frac{256q^6}{K^4},\end{aligned} \tag{9.7}$$

$$\frac{1}{R(q)^8R(q^2)} - q^4R(q)^8R(q^2) = K^2 + 6qK + 20q^2 + \frac{44q^3}{K} + \frac{64q^4}{K^2} + \frac{64q^5}{K^3}, \tag{9.8}$$

$$\begin{aligned}\frac{R(q^2)}{R(q)^{12}} - q^4\frac{R(q)^{12}}{R(q^2)} = K^2 + 10qK + 52q^2 + \frac{180q^3}{K} + \frac{448q^4}{K^2}\\
+\frac{832q^5}{K^3} + \frac{1024q^6}{K^4} + \frac{1024q^7}{K^5},\end{aligned} \tag{9.9}$$

$$K+q=\frac{E(q^2)^4E(q^5)^2}{E(q)^2E(q^{10})^4}, \tag{9.10}$$

$$1-\frac{4q}{K}=\frac{E(q)^4E(q^{10})^2}{E(q^2)^2E(q^5)^4} \tag{9.11}$$

and

$$\frac{q}{K-4q}=\zeta. \tag{9.12}$$

Proofs of (9.1)–(9.4). We see that (9.1) is (2.5), (9.2) is (2.7), (9.3) is (2.9) and (9.4) is (2.10). □

Proof of (9.5).

$$\begin{aligned}\frac{1}{R(q)^5}-q^2R(q^5)&=\left(\frac{R(q^2)}{R(q)^2}-\frac{R(q)^2}{R(q^2)}\right)\left(\frac{1}{R(q)^3R(q^2)}+q^2R(q)^3R(q^2)\right)\\&\quad+\left(\frac{1}{R(q)R(q^2)^2}-q^2R(q)R(q^2)^2\right)\\&=\frac{4q}{K}\left(K+2q+\frac{4q^2}{K}\right)+K\\&=K+4q+\frac{8q^2}{K}+\frac{16q^3}{K^2}.\end{aligned}$$

□

Proof of (9.6).

$$\begin{aligned}\frac{R(q^2)}{R(q)^7}+q^2\frac{R(q)^7}{R(q^2)}&=\left(\frac{R(q^2)}{R(q)^2}-\frac{R(q)^2}{R(q^2)}\right)\left(\frac{1}{R(q)^5}-q^2R(q)^5\right)\\&\quad+\left(\frac{1}{R(q)^3R(q^2)}+q^2R(q)^3R(q^2)\right)\\&=\frac{4q}{K}\left(K+4q+\frac{8q^2}{K}+\frac{16q^3}{K^2}\right)+\left(K+2q+\frac{4q^2}{K}\right)\\&=K+6q+\frac{20q^2}{K}+\frac{32q^3}{K^2}+\frac{64q^4}{K^3}.\end{aligned}$$

□

Proof of (9.7).

$$\begin{aligned}\frac{1}{R(q)^{10}}+q^4R(q)^{10}&=\left(\frac{1}{R(q)^5}-q^2R(q)^5\right)^2+2q^2\\&=\left(K+4q+\frac{8q^2}{K}+\frac{16q^3}{K^2}\right)^2+2q^2\\&=K^2+8qK+34q^2+\frac{96q^3}{K}+\frac{192q^4}{K^2}+\frac{256q^5}{K^3}+\frac{256q^7}{K^4}.\end{aligned}$$

□

Proof of (9.8).

$$
\begin{aligned}
&\frac{1}{R(q)^8R(q^2)} - q^4R(q)^8R(q^2)\\
&\quad = \left(\frac{1}{R(q)^5} - q^2R(q)^5\right)\left(\frac{1}{R(q)^3R(q^2)} + q^2R(q)^3R(q^2)\right)\\
&\qquad - q^2\left(\frac{R(q^2)}{R(q)^2} - \frac{R(q)^2}{R(q^2)}\right)\\
&\quad = \left(K + 4q + \frac{8q^2}{K} + \frac{16q^3}{K^2}\right)\left(K + 2q + \frac{4q^2}{K}\right) - q^2\left(\frac{4q}{K}\right)\\
&\quad = K^2 + 6qK + 20q^2 + \frac{44q^3}{K} + \frac{64q^4}{K^2} + \frac{64q^5}{K^3}.
\end{aligned}
$$

□

Proof of (9.9).

$$
\begin{aligned}
&\frac{R(q^2)}{R(q)^{12}} - q^4\frac{R(q)^{12}}{R(q^2)}\\
&\quad = \left(\frac{R(q^2)}{R(q)^2} - \frac{R(q)^2}{R(q^2)}\right)\left(\frac{1}{R(q)^{10}} + q^4R(q)^{10}\right)\\
&\qquad + \left(\frac{1}{R(q)^8R(q^2)} - q^4R(q)^8R(q^2)\right)\\
&\quad = \frac{4q}{K}\left(K^2 + 8qK + 34q^2 + \frac{96q^3}{K} + \frac{192q^4}{K^2} + \frac{256q^5}{K^3} + \frac{256q^5}{K^3}\right)\\
&\qquad + \left(K^2 + 6qK + 20q^2 + \frac{44q^3}{K} + \frac{64q^4}{K^2} + \frac{64q^5}{K^3}\right)\\
&\quad = K^2 + 10qK + 52q^2 + \frac{180q^3}{K} + \frac{448q^4}{K^2} + \frac{832q^5}{K^3} + \frac{1024q^6}{K^4} + \frac{1024q^7}{K^5}.
\end{aligned}
$$

□

Proofs of (9.10) and (9.11). We see that (9.10) is (2.11) and (9.11) is (2.12). □

Proof of (9.12).

$$
K - 4q = K\left(1 - \frac{4q}{K}\right) = \frac{E(q^2)E(q^5)^5}{E(q)E(q^{10})^5} \cdot \frac{E(q)^4E(q^{10})^2}{E(q^2)^2E(q^5)^4} = \frac{E(q)^3E(q^5)}{E(q^2)E(q^{10})^3} = \frac{q}{\zeta},
$$

from which the result follows. □

Proof of (4.2). We start by noting that (4.2) is equivalent to

$$
U(\zeta) = 41\zeta + 860\zeta^2 + 6800\zeta^3 + 24000\zeta^4 + 32000\zeta^5.
$$

We have

$$
\begin{aligned}
U(\zeta) &= U\left(q\frac{E(q^2)E(q^{10})^3}{E(q)^3E(q^5)}\right)\\
&= \frac{E(q^2)^3}{E(q)}\,U\left(q\frac{E(q^2)}{E(q)^3}\right)\\
&= \frac{E(q^2)^3}{E(q)}\,U\left(qE(q^{50})\left(\frac{1}{R(q^{10})}-q^2-q^4R(q^{10})\right)\right.\\
&\quad\times\left(\frac{E(q^{25})^5}{E(q^5)^6}\right)^3\left(\frac{1}{R(q^5)^4}+\frac{q}{R(q^5)^3}+\frac{2q^2}{R(q^5)^2}+\frac{3q^3}{R(q^5)}+5q^4\right.\\
&\quad\left.\left.-3q^5R(q^5)+2q^6R(q^5)^2-q^7R(q^5)^3+q^8R(q^5)^4\right)^3\right)\\
&= \frac{E(q^2)^3E(q^5)^{15}E(q^{10})}{E(q)^{19}}\left(51q\left(\frac{1}{R(q)^8R(q^2)}-q^4R(q)^8R(q^2)\right)\right.\\
&\quad-9q\left(\frac{1}{R(q)^{10}}+q^4R(q)^{10}\right)-q\left(\frac{R(q^2)}{R(q)^{12}}-q^4\frac{R(q)^{12}}{R(q^2)}\right)\\
&\quad+153q^2\left(\frac{1}{R(q)^3R(q^2)}+q^2R(q)^3R(q^2)\right)-177q^2\left(\frac{1}{R(q)^5}-q^2R(q)^5\right)\\
&\quad\left.-78q^2\left(\frac{R(q^2)}{R(q)^7}+q^2\frac{R(q)^7}{R(q^2)}\right)-219q^3\left(\frac{R(q^2)}{R(q)^2}-\frac{R(q)^2}{R(q^2)}\right)-71q^3\right)\\
&= \frac{E(q^2)^3E(q^5)^{15}E(q^{10})}{E(q)^{19}}\\
&\quad\times\left(51q\left(K^2+6qK+20q^2+\frac{44q^3}{K}+\frac{64q^4}{K^2}+\frac{64q^5}{K^3}\right)\right.\\
&\quad-9q\left(K^2+8qK+34q^2+\frac{96q^3}{K}+\frac{192q^4}{K^2}+\frac{256q^5}{K^3}+\frac{256q^6}{K^4}\right)\\
&\quad-q\left(K^2+10qK+52q^2+\frac{180q^3}{K}+\frac{448q^4}{K^2}+\frac{832q^5}{K^3}+\frac{1024q^6}{K^4}+\frac{1024q^7}{K^5}\right)\\
&\quad+153q^2\left(K+2q+\frac{4q^2}{K}\right)-177q^2\left(K+4q+\frac{8q^2}{K}+\frac{16q^3}{K^2}\right)\\
&\quad\left.-78q^2\left(K+6q+\frac{20q^2}{K}+\frac{32q^3}{K^2}+\frac{64q^4}{K^3}\right)-219q^3\left(\frac{4q}{K}\right)-71q^3\right)\\
&= \frac{E(q^2)^3E(q^5)^{15}E(q^{10})}{E(q)^{19}}\\
&\quad\times\frac{(K+q)^2(K-4q)}{K^5}(41qK^4+204q^2K^3+416q^3K^2+384q^4K+256q^5)\\
&= \frac{E(q^2)^3E(q^5)^{15}E(q^{10})}{E(q)^{19}}\cdot\frac{(K+q)^2(K-4q)}{K^5}\\
&\quad\times(41q(K-4q)^4+860q^2(K-4q)^3+6800q^3(K-4q)^2\\
&\quad+24000q^4(K-4q)+32000q^5)
\end{aligned}
$$

$$
\begin{aligned}
&= \frac{E(q^2)^3E(q^5)^{15}E(q^{10})}{E(q)^{19}} \cdot \frac{(K+q)^2(K-4q)^6}{K^5} \\
&\quad \times \left(\frac{41q}{K-4q} + \frac{860q^2}{(K-4q)^2} + \frac{6800q^3}{(K-4q)^3} + \frac{24000q^4}{(K-4q)^4} + \frac{32000q^5}{(K-4q)^5} \right) \\
&= \left(\frac{E(q^2)^3E(q^5)^{15}E(q^{10})}{E(q)^{19}} \right) \left(\frac{E(q^2)^4E(q^5)^2}{E(q)^2E(q^{10})^4} \right)^2 \left(\frac{E(q)^3E(q^5)}{E(q^2)E(q^{10})^3} \right)^6 \\
&\quad \times \left(\frac{E(q)E(q^{10})^5}{E(q^2)E(q^5)^5} \right)^5 (41\zeta + 860\zeta^2 + 6800\zeta^3 + 24000\zeta^4 + 32000\zeta^5) \\
&= 41\zeta + 860\zeta^2 + 6800\zeta^3 + 24000\zeta^4 + 32000\zeta^5.
\end{aligned}
$$

□

In proceeding in the same manner with proofs of (4.3)–(4.6), (5.6)–(5.10) and (5.12)–(5.16), we encounter terms of the form

$$P(\alpha,\beta) = \frac{1}{R(q)^{\alpha+2\beta}R(q^2)^{2\alpha-\beta}} + (-1)^{\alpha+\beta}q^{2\alpha}R(q)^{\alpha+2\beta}R(q^2)^{2\alpha-\beta}$$

with $\alpha \geq 0$.

These terms can be expressed in terms of q, K and K^{-1} by making use of the recurrence relations

$$P(\alpha,\beta+1) = \frac{4q}{K}P(\alpha,\beta) + P(\alpha,\beta-1), \tag{9.13}$$

$$P(\alpha+2,0) = KP(\alpha+1,0) + q^2P(\alpha,0) \tag{9.14}$$

and

$$P(\alpha+2,-1) = \left(K - 2q + \frac{4q^2}{K}\right) P(\alpha+1,0) - q^2P(\alpha,1), \tag{9.15}$$

together with the initial values

$$P(0,0) = 2, \tag{9.16}$$

$$P(0,1) = \frac{R(q^2)}{R(q)^2} - \frac{R(q)^2}{R(q^2)} = \frac{4q}{K}, \tag{9.17}$$

$$P(1,0) = \frac{1}{R(q)R(q^2)^2} - q^2R(q)R(q^2)^2 = K \tag{9.18}$$

and

$$P(1,-1) = \frac{R(q)}{R(q^2)^3} + q^2\frac{R(q^2)^3}{R(q)} = K - 2q + \frac{4q^2}{K}. \tag{9.19}$$

We see that (9.17) is (9.1), (9.18) is (9.2) and (9.19) is (9.3).

Proofs of (9.13)–(9.15) were given by Chern and Tang [2].

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Shane Chern
Department of Mathematics
The Pennsylvania State University
University Park
State College
PA 16802
USA
e-mail: shanechern@psu.edu

Michael D. Hirschhorn
School of Mathematics and Statistics
University of New South Wales
Sydney 2052
Australia
e-mail: m.hirschhorn@unsw.edu.au

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The A_2 Rogers–Ramanujan Identities Revisited

To George Andrews on his 80th birthday

Sylvie Corteel and Trevor Welsh

Abstract. In this note, we show how to use cylindric partitions to rederive the four A_2 Rogers–Ramanujan identities originally proven by Andrews, Schilling and Warnaar, and provide a proof of a similar fifth identity.

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Keywords. Rogers–Ramanujan identities, Cylindric partitions, q-Series identities.

1. Introduction

The Rogers–Ramanujan identities were first proved in 1894 by Rogers and rediscovered in the 1910s by Ramanujan [14]. They are

$$\sum_{n\geq 0} \frac{q^{n(n+i)}}{(q;q)_n} = \frac{1}{(q^{1+i};q^5)_\infty (q^{4-i};q^5)_\infty} \tag{1.1}$$

with $i = 0, 1$, where $(a, q)_\infty = \prod_{i\geq 0}(1 - aq^i)$ and $(a; q)_n = (a; q)_\infty/(aq^n; q)_\infty$.

There have been many attempts to give combinatorial proofs of these identities and the first one is due to Garsia and Milne [11]. Unfortunately, it is not simple, and no simple combinatorial proof is known. Recently in [6], the first author presented a new bijective approach to the proofs of the Rogers–Ramanujan identities via the Robinson–Schensted–Knuth correspondence as presented in [13]. The bijection does not give the Rogers–Ramanujan identities but the Rogers–Ramanujan identities divided by $(q; q)_\infty$, namely

$$\frac{1}{(q;q)_\infty} \sum_{n\geq 0} \frac{q^{n(n+1)}}{(q;q)_n} = \frac{1}{(q,q^2,q^2,q^3,q^3,q^4,q^5;q^5)_\infty} \tag{1.2}$$

where $(a_1,\dots,a_k;q)_\infty = \prod_{i=1}^k (a_i;q)_\infty$. This proof uses the combinatorics of cylindric partitions [10]. We interpret both sides as the generating function of cylindric partitions of profile (3,0) and the bijection is a polynomial algorithm in the size of the cylindric partition. The idea to use cylindric partitions is due to Foda and the second author [12] in the more general setting of the Andrews–Gordon identities [2]. For $k > 0$ and $0 \le i \le k$, these identities divided by $(q;q)_\infty$ are

$$\frac{1}{(q;q)_\infty}\sum_{n_1,\dots,n_k}\frac{q^{\sum_{j=1}^k n_j^2+\sum_{j=i}^k n_j}}{(q)_{n_1-n_2}\cdots(q)_{n_{k-1}-n_k}(q)_{n_k}} = \frac{(q^i,q^{2k+3-i},q^{2k+3};q^{2k+3})_\infty}{(q;q)_\infty^2}.$$

In [12], the sum side is interpreted as a generating function for (what the authors call) decorated Bressoud paths, and the product side is interpreted as the generating function of cylindric partitions of profile $(2k+1-i,i)$, and a bijection between these two objects is provided. See [12] for more details.

In this note, we take the idea of applying cylindric partitions to Rogers–Ramanujan type identities a step further, using them to give an alternative proof of the A_2 Rogers–Ramanujan identities due to Andrews et al. [3].

Theorem 1.1. *We have*

$$\sum_{n_1=0}^{\infty}\sum_{n_2=0}^{2n_1}\frac{q^{n_1^2+n_2^2-n_1n_2+n_1+n_2}}{(q;q)_{n_1}}\begin{bmatrix}2n_1\\ n_2\end{bmatrix} = \frac{1}{(q^2,q^3,q^3,q^4,q^4,q^5;q^7)_\infty},$$

$$\sum_{n_1=0}^{\infty}\sum_{n_2=0}^{2n_1}\frac{q^{n_1^2+n_2^2-n_1n_2+n_2}}{(q;q)_{n_1}}\begin{bmatrix}2n_1\\ n_2\end{bmatrix} = \frac{1}{(q,q^2,q^3,q^4,q^5,q^6;q^7)_\infty},$$

$$\sum_{n_1=0}^{\infty}\sum_{n_2=0}^{2n_1+1}\frac{q^{n_1^2+n_2^2-n_1n_2+n_1}}{(q;q)_{n_1}}\begin{bmatrix}2n_1+1\\ n_2\end{bmatrix} = \frac{1}{(q,q^2,q^3,q^4,q^5,q^6;q^7)_\infty},$$

$$\sum_{n_1=0}^{\infty}\sum_{n_2=0}^{2n_1+1}\frac{q^{n_1^2+n_2^2-n_1n_2+n_2}}{(q;q)_{n_1}}\begin{bmatrix}2n_1+1\\ n_2\end{bmatrix} = \frac{1}{(q,q^2,q^2,q^5,q^5,q^6;q^7)_\infty},$$

$$\sum_{n_1=0}^{\infty}\sum_{n_2=0}^{2n_1}\frac{q^{n_1^2+n_2^2-n_1n_2}}{(q;q)_{n_1}}\begin{bmatrix}2n_1\\ n_2\end{bmatrix} = \frac{1}{(q,q,q^3,q^4,q^6,q^6;q^7)_\infty},$$

where the Gaussian polynomial $\begin{bmatrix}n\\ k\end{bmatrix}$ *is defined by*

$$\begin{bmatrix}n\\ k\end{bmatrix} = \frac{(q;q)_n}{(q;q)_k(q;q)_{n-k}}.$$

Note that the second and third expressions are equal. All but the fourth of these identities were obtained in Theorem 5.2 of [3], while the fourth was conjectured in Section 2.4 of [8] (and proved here for the first time). In [16], Warnaar gave another approach to proving these identities, making use of Hall–Littlewood functions.

In this note, we prove the following theorem, giving the generating functions $F_{c,n}(q)$ of cylindric partitions indexed by compositions c of 4 into 3 parts, with largest entry at most n:

Theorem 1.2.

$$F_{(4,0,0),n}(q) = \sum_{n_1=0}^{n} \sum_{n_2=0}^{2n_1} \frac{q^{n_1^2+n_2^2-n_1n_2+n_1+n_2}}{(q;q)_{n-n_1}(q;q)_{n_1}} \begin{bmatrix} 2n_1 \\ n_2 \end{bmatrix},$$

$$F_{(3,1,0),n}(q) = \sum_{n_1=0}^{n} \sum_{n_2=0}^{2n_1} \frac{q^{n_1^2+n_2^2-n_1n_2+n_2}}{(q;q)_{n-n_1}(q;q)_{n_1}} \begin{bmatrix} 2n_1 \\ n_2 \end{bmatrix},$$

$$F_{(3,0,1),n}(q) = \sum_{n_1=0}^{n} \sum_{n_2=0}^{2n_1} \frac{q^{n_1^2+n_2^2-n_1n_2}}{(q;q)_{n-n_1}} \frac{q^{n_1}}{(q;q)_{n_1}} \begin{bmatrix} 2n_1 \\ n_2 \end{bmatrix} + \sum_{n_1=1}^{n} \sum_{n_2=0}^{2n_1-2} \frac{q^{n_1^2+n_2^2-n_1n_2}}{(q;q)_{n-n_1}} \frac{q^{2n_2}}{(q;q)_{n_1-1}} \begin{bmatrix} 2n_1-2 \\ n_2 \end{bmatrix},$$

$$F_{(2,2,0),n}(q) = \sum_{n_1=0}^{n} \sum_{n_2=0}^{2n_1} \frac{q^{n_1^2+n_2^2-n_1n_2}}{(q;q)_{n-n_1}} \frac{q^{n_1}}{(q;q)_{n_1}} \begin{bmatrix} 2n_1 \\ n_2 \end{bmatrix} + \sum_{n_1=1}^{n} \sum_{n_2=0}^{2n_1-2} \frac{q^{n_1^2+n_2^2-n_1n_2}}{(q;q)_{n-n_1}} \frac{q^{n_2}(1+q^{n1+n2})}{(q;q)_{n_1-1}} \begin{bmatrix} 2n_1-2 \\ n_2 \end{bmatrix},$$

$$F_{(2,1,1),n}(q) = \sum_{n_1=0}^{n} \sum_{n_2=0}^{2n_1} \frac{q^{n_1^2+n_2^2-n_1n_2}}{(q;q)_{n-n_1}(q;q)_{n_1}} \begin{bmatrix} 2n_1 \\ n_2 \end{bmatrix}.$$

This result gives finite versions of the sum sides of the A_2 Rogers–Ramanujan identities.

In the $n \to \infty$ limit, we recover the sum sides of the identities of Theorem 1.1 divided by $(q;q)_\infty$. On the other hand, the product sides are obtained using a result of Borodin [4] on the generating functions of cylindric partitions. In Sect. 2, we start by defining cylindric partitions and then obtain the product sides of particular cylindric partitions. These yield the right-hand sides of the expressions in Theorem 1.1. The sum expressions on the left-hand sides are computed in Sect. 3.

2. Cylindric Partitions and the Product Side

Cylindric partitions were introduced by Gessel and Krattenthaler [10] and appeared naturally in different contexts [4,5,7,9,12,15]. Let ℓ and k be two positive integers. In this note, we choose to index cylindric partitions by compositions of ℓ into k non-negative parts.

Definition 2.1. Given a composition $c = (c_1, \dots, c_k)$, a cylindric partition of profile c is a sequence of k partitions $\Lambda = (\lambda^{(1)}, \dots \lambda^{(k)})$ such that

- $\lambda_j^{(i)} \geq \lambda_{j+c_{i+1}}^{(i+1)}$,
- $\lambda_j^{(k)} \geq \lambda_{j+c_1}^{(1)}$.

for all i and j.

For example, the sequence $\Lambda = ((3,2,1,1),(4,3,3,1),(4,1,1))$ is a cylindric partition of profile $(2,2,0)$. One can check that for all j, $\lambda_j^{(1)} \geq \lambda_{j+2}^{(2)}$, $\lambda_j^{(2)} \geq \lambda_j^{(3)}$ and $\lambda_j^{(3)} \geq \lambda_{j+2}^{(1)}$ for all j. Note that this definition implies that cylindric partitions of profile $(c_1,\dots,c_k)$ are in bijection with cylindric partitions of profile $(c_k,c_1,\dots,c_{k-1})$.

Our goal is to compute generating functions of cylindric partitions of a given profile c according to two statistics. Given a $\Lambda = (\lambda^{(1)},\dots,\lambda^{(k)})$, let

- $|\Lambda| = \sum_{i=1}^{k} \sum_{j\geq 1} \lambda_j^{(i)}$, the sum of the entries of the cylindric plane partition, and
- $\max(\Lambda) = \max(\lambda_1^{(1)},\dots \lambda_1^{(k)})$, the largest entry of the cylindric plane partition.

Going back to our example, we have $|\Lambda| = 24$, and $\max(\Lambda) = 4$.

Let $\mathcal{C}_c$ be the set of cylindric partitions of profile c and let $\mathcal{C}_{c,n}$ be the set of cylindric partitions of profile c and such that the largest entry is at most n. We are interested in the following generating functions:

$$F_c(q) = \sum_{\Lambda \in \mathcal{C}_c} q^{|\Lambda|}, \tag{2.1}$$

$$F_c(y,q) = \sum_{\Lambda \in \mathcal{C}_c} q^{|\Lambda|} y^{\max(\Lambda)}, \tag{2.2}$$

$$F_{c,n}(q) = \sum_{\Lambda \in \mathcal{C}_{c,n}} q^{|\Lambda|}. \tag{2.3}$$

A surprising and beautiful result is that for any c, the generating function $F_c(q)$ can be written as a product. Namely, with $t = k + \ell$,

Theorem 2.2. [4] *The generating function $F_c(q)$ is equal to*

$$\frac{1}{(q^t;q^t)} \prod_{i=1}^{k} \prod_{j=i+1}^{k} \prod_{m=1}^{c_i} \frac{1}{(q^{m+d_{i+1,j}+j-i};q^t)_\infty} \prod_{i=2}^{k} \prod_{j=2}^{i-1} \prod_{m=1}^{c_i} \frac{1}{(q^{t-(m+d_{j,i-1}+i-j)};q^t)_\infty}$$

where $d_{i,j} = c_i + c_{i+1} + \cdots + c_j$.

The original result is written in a different but equivalent form.

For what follows, we restrict attention to the case $\ell = 4$ and $k = 3$. As cylindric partitions of profile $(c_1,\dots,c_k)$ are in bijection with partitions of profile $(c_k,c_1,\dots,c_{k-1})$, we need only compute the generating functions for the compositions $(4,0,0)$, $(3,1,0)$, $(3,0,1)$, $(2,2,0)$, and $(2,1,1)$. We now apply the previous theorem:

Corollary 2.3.

$$
\begin{aligned}
F_{(4,0,0)}(q) &= \frac{1}{(q;q)_\infty (q^2,q^3,q^3,q^4,q^4,q^5;q^7)_\infty},\\
F_{(3,1,0)}(q) &= \frac{1}{(q;q)_\infty (q,q^2,q^3,q^4,q^5,q^6;q^7)_\infty},\\
F_{(3,0,1)}(q) &= \frac{1}{(q;q)_\infty (q,q^2,q^3,q^4,q^5,q^6;q^7)_\infty},\\
F_{(2,2,0)}(q) &= \frac{1}{(q;q)_\infty (q,q^2,q^2,q^5,q^5,q^6;q^7)_\infty},\\
F_{(2,1,1)}(q) &= \frac{1}{(q;q)_\infty (q,q,q^3,q^4,q^6,q^6;q^7)_\infty}.
\end{aligned}
$$

Note that these five products are precisely those in Theorem 1.1 divided by $(q;q)_\infty$.

3. The Sum Side

We first prove a general functional equation for $F_c(y,q)$ for any profile c. Suppose that $k > 1$ and $c = (c_1, \ldots, c_k)$. Let I_c be the subset of $\{1, \ldots, k\}$ such that $i \in I_c$ if and only if $c_i > 0$. For example if $c = (2,2,0)$ then $I_c = \{1,2\}$. Given a subset J of I_c, we define the composition $c(J) = (c_1(J), \ldots, c_k(J))$ by

$$
c_i(J) = \begin{cases} c_i - 1 & \text{if } i \in J \text{ and } (i-1) \notin J,\\ c_i + 1 & \text{if } i \notin J \text{ and } (i-1) \in J,\\ c_i & \text{otherwise.} \end{cases}
$$

Here, we set $c_0 = c_k$.

Proposition 3.1. *For any composition $c = (c_1, \ldots, c_k)$,*

$$
F_c(y,q) = \sum_{\emptyset \subset J \subseteq I_c} (-1)^{|J|-1} \frac{F_{c(J)}(yq^{|J|}, q)}{1 - yq^{|J|}}. \tag{3.1}
$$

with the conditions $F_c(0,q) = 1$ and $F_c(y,0) = 1$.

Proof. The proof makes use of an inclusion–exclusion argument.

First, for fixed J such that $\emptyset \subset J \subseteq I_c$, we require the generating function of cylindric partitions Λ of profile c such that $\lambda_1^{(j)} = \max(\Lambda)$ for all $j \in J$.

Let $M = (\mu^{(1)}, \ldots, \mu^{(k)})$ be a cylindric partition of profile $c(J)$, and set $n = \max(M)$. Then, for a fixed integer $m \geq 0$, create a cylindric partition $\Lambda = (\lambda^{(1)}, \ldots, \lambda^{(k)})$ using the following recipe:

$$
\lambda^{(j)} = \begin{cases} (m+n, \mu_1^{(j)}, \mu_2^{(j)}, \ldots) & \text{if } j \in J,\\ \mu^{(j)} & \text{if } j \notin J. \end{cases}
$$

It is easily checked that Λ is a cylindric partition of profile c and that $\max(\Lambda) = m+n$. Moreover, $\lambda_1^{(j)} = \max(\Lambda)$ for all $j \in J$. The generating function for all cylindric partitions Λ obtained from M in this way is

$$\sum_{m=0}^{\infty} y^{m+n} q^{|J|(m+n)} q^{|M|} = y^n q^{|J|n+|M|} \sum_{m=0}^{\infty} (yq^{|J|})^m = \frac{y^n q^{|J|n+|M|}}{1-yq^{|J|}}. \quad (3.2)$$

Then, the generating function for all cylindric partitions Λ obtained in this way from any cylindric partition M of profile $c(J)$ is

$$\sum_{M \in \mathcal{C}_{c(J)}} \frac{y^{\max(M)} q^{|J| \max(M)+|M|}}{1-yq^{|J|}} = \frac{F_{c(J)}(yq^{|J|}, q)}{1-yq^{|J|}}, \quad (3.3)$$

making use of the definition (2.2).

Let $\Lambda = (\lambda^{(1)}, \ldots, \lambda^{(k)})$ be an arbitrary cylindric partition of profile c, and let $p = \max(\Lambda)$. Because $\lambda_1^{(i-1)} \geq \lambda_1^{(i)}$ whenever $i \notin I_c$, it must be the case that $p = \lambda_1^{(j)}$ for some $j \in I_c$. Then, if $J \neq \emptyset$ is such that $p = \lambda_1^{(j)}$ for each $j \in J$ (this J might not be unique), we see that Λ is one of the cylindric partitions enumerated by (3.3). However, because Λ can arise from various different J, the generating function for cylindric partitions of profile c is obtained via the inclusion–exclusion process. This immediately gives (3.1). $\square$

Now, for each composition c, define

$$G_c(y, q) = (yq; q)_\infty \, F_c(y, q). \quad (3.4)$$

In terms of this, the previous result translates to

$$G_c(y, q) = \sum_{\emptyset \subset J \subseteq I} (-1)^{|J|-1} (yq; q)_{|J|-1} G_{c(J)}(yq^{|J|}, q) \quad (3.5)$$

with $G_c(0, q) = G_c(y, 0) = 1$.

Theorem 3.2.

$$G_{(4,0,0)}(y,q)=\sum_{n_1=0}^{\infty}\sum_{n_2=0}^{2n_1} y^{n_1}\frac{q^{n_1^2+n_2^2-n_1n_2+n_1+n_2}}{(q;q)_{n_1}}\begin{bmatrix}2n_1\\ n_2\end{bmatrix},$$

$$G_{(3,1,0)}(y,q)=\sum_{n_1=0}^{\infty}\sum_{n_2=0}^{2n_1} y^{n_1}\frac{q^{n_1^2+n_2^2-n_1n_2+n_2}}{(q;q)_{n_1}}\begin{bmatrix}2n_1\\ n_2\end{bmatrix},$$

$$G_{(3,0,1)}(y,q)=\sum_{n_1=0}^{\infty}\sum_{n_2=0}^{2n_1} y^{n_1}q^{n_1^2+n_2^2-n_1n_2}\frac{q^{n_1}}{(q;q)_{n_1}}\begin{bmatrix}2n_1\\ n_2\end{bmatrix}$$
$$+\sum_{n_1=1}^{\infty}\sum_{n_2=0}^{2n_1-2} y^{n_1}q^{n_1^2+n_2^2-n_1n_2}\frac{q^{2n_2}}{(q;q)_{n_1-1}}\begin{bmatrix}2n_1-2\\ n_2\end{bmatrix},$$

$$G_{(2,2,0)}(y,q)=\sum_{n_1=0}^{\infty}\sum_{n_2=0}^{2n_1} y^{n_1}q^{n_1^2+n_2^2-n_1n_2}\frac{q^{n_1}}{(q;q)_{n_1}}\begin{bmatrix}2n_1\\ n_2\end{bmatrix}$$
$$+\sum_{n_1=1}^{\infty}\sum_{n_2=0}^{2n_1-2} y^{n_1}q^{n_1^2+n_2^2-n_1n_2}\frac{q^{n_2}(1+q^{n1+n2})}{(q;q)_{n_1-1}}\begin{bmatrix}2n_1-2\\ n_2\end{bmatrix},$$

$$G_{(2,1,1)}(y,q)=\sum_{n_1=0}^{\infty}\sum_{n_2=0}^{2n_1} y^{n_1}\frac{q^{n_1^2+n_2^2-n_1n_2}}{(q;q)_{n_1}}\begin{bmatrix}2n_1\\ n_2\end{bmatrix}.$$

Proof. In this proof, we abbreviate $G_c(y,q)$ to $G_c(y)$ for convenience. Applying the form (3.5) of Proposition 3.1 to the case $\ell=4$ and $k=3$ yields

$$\begin{aligned}
G_{(4,0,0)}(y)&=G_{(3,1,0)}(yq),\\
G_{(3,1,0)}(y)&=G_{(3,0,1)}(yq)+G_{(2,2,0)}(yq)-(1-yq)G_{(2,1,1)}(yq^2),\\
G_{(3,0,1)}(y)&=G_{(4,0,0)}(yq)+G_{(2,1,1)}(yq)-(1-yq)G_{(3,1,0)}(yq^2),\\
G_{(2,2,0)}(y)&=G_{(3,0,1)}(yq)+G_{(2,1,1)}(yq)-(1-yq)G_{(2,1,1)}(yq^2),\\
G_{(2,1,1)}(y)&=G_{(2,1,1)}(yq)+G_{(2,2,0)}(yq)+G_{(3,1,0)}(yq)\\
&\quad-(1-yq)(G_{(2,2,0)}(yq^2)+G_{(2,1,1)}(yq^2)+G_{(3,0,1)}(yq^2))\\
&\quad+(1-yq)(1-yq^2)G_{(2,1,1)}(yq^3).
\end{aligned}$$

By manipulating these equations, we obtain

$$\begin{aligned}
G_{(4,0,0)}(y)&=G_{(3,1,0)}(yq),\\
G_{(3,1,0)}(y)&=G_{(2,2,0)}(yq)+yq^2G_{(3,1,0)}(yq^3)+yqG_{(2,1,1)}(yq^2),\\
G_{(3,0,1)}(y)&=G_{(2,1,1)}(yq)+yqG_{(3,1,0)}(yq^2),\\
G_{(2,2,0)}(y)&=G_{(2,1,1)}(yq)+yqG_{(2,1,1)}(yq^2)+yq^2G_{(3,1,0)}(yq^3),\\
G_{(2,1,1)}(y)&=G_{(2,1,1)}(yq)+yqG_{(2,2,0)}(yq)+yqG_{(2,2,0)}(yq^2)\\
&\quad+yq^3G_{(3,1,0)}(yq^4)+yq^2G_{(2,1,1)}(yq^3).
\end{aligned}\tag{3.6}$$

We claim that this system of Eq. (3.6) together with the boundary conditions $G_c(0,q)=G_c(y,0)=1$ for each composition c is uniquely solved by the expressions stated in the theorem. This is proved using an induction argument involving all five expressions.

We use induction on the exponents of y. For each composition c, let $g_c(n)$ denote the coefficient of y^n in the solution $G_c(y)$ of (3.6). The boundary conditions $G_c(0,q)=1$ imply that each $g_c(0)=1$. This holds for the expressions of the theorem. So now, for $n>0$, assume that $g_c(n_1)$ agrees with the coefficient of y^{n_1} in the statement of the theorem for each $n_1<n$ and each composition c. We must check that $g_c(n)$, as determined by the expressions (3.6), is equal to the coefficient of y^n in the statement of the theorem for each c.

The fifth expression in (3.6) implies that

$$\begin{aligned}(1-q^n)g_{(2,1,1)}(n) &= (q^n+q^{2n-1})g_{(2,2,0)}(n-1)\\ &\quad + q^{4n-1}g_{(3,1,0)}(n-1)+q^{3n-1}g_{(2,1,1)}(n-1).\end{aligned}$$

Using the expressions for $g_c(n)$ implied by the induction hypothesis then yields

$$g_{(2,1,1)}(n)=\sum_{n_2=0}^{2n}\frac{q^{n^2+n_2^2-nn_2}}{(q;q)_n}\begin{bmatrix}2n\\ n_2\end{bmatrix}.$$

This expression, along with the other expressions for $g_c(n)$ implied by (3.6), enables us to compute, in turn, $g_{(2,2,0)}(n)$, $g_{(3,0,1)}(n)$, $g_{(3,1,0)}(n)$ and finally $g_{(4,0,0)}(n)$. Because the expressions that result agree with the corresponding coefficients of y^n in the statement of the theorem, the induction argument is complete. □

Proof of Theorem 1.2. Comparing (2.2) and (2.3) leads to

$$\begin{aligned}F_{c,n}(q) &= [y^0]F_c(y,q)+[y^1]F_c(y,q)+\cdots+[y^n]F_c(y,q)\\ &= [y^n]\frac{F_c(y,q)}{1-y}=[y^n]\frac{G_c(y,q)}{(y;q)_n}\end{aligned}$$

using (3.4). By the q-binomial theorem [1], we have

$$\frac{1}{(y,q)_n}=\sum_{i=0}^{n}\frac{y^i}{(q;q)_i},$$

and, therefore, it follows that

$$F_{c,n}(q)=\sum_{n_1=0}^{n}\frac{1}{(q;q)_{n-n_1}}[y^{n_1}]G_c(y,q).$$

Applying this to the expressions of Theorem 3.2 then yields those of Theorem 1.2. □

Proof of Theorem 1.1. The product sides of the five identities result from multiplying each of the expressions in Corollary 2.3 by $(q;q)_\infty$. Because $F_c(q)=\lim_{n\to\infty}F_{c,n}(q)$, the sum sides of the identities arise by, for each expression in Theorem 1.2, taking the $n\to\infty$ limit and then multiplying by $(q;q)_\infty$. We get the sum side of the first, second and fifth identities directly in this way.

The other two identities require a bit more effort. They will require use of the Gaussian polynomial recurrence relations [1]:

$$\begin{bmatrix} n \\ k \end{bmatrix} = \begin{bmatrix} n-1 \\ k \end{bmatrix} + q^{n-k} \begin{bmatrix} n-1 \\ k-1 \end{bmatrix} = q^k \begin{bmatrix} n-1 \\ k \end{bmatrix} + \begin{bmatrix} n-1 \\ k-1 \end{bmatrix}. \tag{3.7}$$

For convenience, for $j, n \geq 0$ define

$$U^{(j)}(n) = \sum_{m \geq 0} q^{n^2+m^2-nm+jm} \begin{bmatrix} 2n \\ m \end{bmatrix}$$

and

$$V^{(j)}(n) = \sum_{m \geq 0} q^{n^2+m^2-nm+jm} \begin{bmatrix} 2n+1 \\ m \end{bmatrix}.$$

Using the first identity in (3.7) gives

$$\begin{aligned} V^{(j)}(n) &= \sum_{m \geq 0} q^{n^2+m^2-nm+jm} \begin{bmatrix} 2n \\ m \end{bmatrix} + \sum_{m \geq 0} q^{n^2+m^2-nm+2n+1+jm-m} \begin{bmatrix} 2n \\ m-1 \end{bmatrix} \\ &= U^{(j)}(n) + \sum_{m \geq 0} q^{n^2+(m-1)^2-n(m-1)+n+(j+1)(m-1)+j+1} \begin{bmatrix} 2n \\ m-1 \end{bmatrix}. \end{aligned}$$

After replacing m by $m+1$ in the second term (and noting that the original $m = 0$ summand is zero), we thus obtain

$$V^{(j)}(n) = U^{(j)}(n) + q^{n+j+1} U^{(j+1)}(n). \tag{3.8}$$

The sum side of the third identity in Theorem 1.1 is, via Theorem 1.2, given by

$$\begin{aligned} &(q;q)_\infty \lim_{n \to \infty} F_{(3,0,1),n}(q) \\ &\quad = \sum_{n_1,n_2} \frac{q^{n_1^2+n_2^2-n_1n_2+n_1}}{(q;q)_{n_1}} \begin{bmatrix} 2n_1 \\ n_2 \end{bmatrix} + \sum_{n_1,n_2} \frac{q^{n_1^2+n_2^2-n_1n_2+2n_2}}{(q;q)_{n_1-1}} \begin{bmatrix} 2n_1-2 \\ n_2 \end{bmatrix} \\ &\quad = \sum_{n_1} \frac{q^{n_1}}{(q;q)_{n_1}} U^{(0)}(n_1) + \sum_{n_1,n_2} \frac{q^{(n_1-1)^2+n_2^2-(n_1-1)n_2+2n_1+n_2-1}}{(q;q)_{n_1-1}} \begin{bmatrix} 2n_1-2 \\ n_2 \end{bmatrix} \\ &\quad = \sum_{n_1} \frac{q^{n_1}}{(q;q)_{n_1}} U^{(0)}(n_1) + \sum_{n_1} \frac{q^{2n_1+1}}{(q;q)_{n_1}} U^{(1)}(n_1) \end{aligned}$$

after replacing n_1 by n_1+1 in the second term. Via the $j = 0$ case of (3.8), this gives the sum side of the third identity in Theorem 1.1, as required.

Before proving the fourth identity, we apply the final expression in (3.7) to $V^{(j)}(n)$ to give

$$\begin{aligned} V^{(j)}(n) &= \sum_{m \geq 0} q^{n^2+m^2-nm+(j+1)m} \begin{bmatrix} 2n \\ m \end{bmatrix} + \sum_{m \geq 0} q^{n^2+m^2-nm+jm} \begin{bmatrix} 2n \\ m-1 \end{bmatrix} \\ &= U^{(j+1)}(n) + \sum_{m \geq 0} q^{n^2+(m-1)^2-n(m-1)-n+(j+2)(m-1)+j+1} \begin{bmatrix} 2n \\ m-1 \end{bmatrix}. \end{aligned}$$

Then, replacing m by $m+1$ in the second term gives

$$V^{(j)}(n) = U^{(j+1)}(n) + q^{-n+j+1}U^{(j+2)}(n). \tag{3.9}$$

The sum side of the fourth identity in Theorem 1.1 is, via Theorem 1.2, given by

$$\begin{aligned}
&(q;q)_\infty \lim_{n\to\infty} F_{(2,2,0),n}(q)\\
&\quad= \sum_{n_1,n_2} \frac{q^{n_1^2+n_2^2-n_1n_2+n_1}}{(q;q)_{n_1}} \begin{bmatrix} 2n_1 \\ n_2 \end{bmatrix}\\
&\qquad+ \sum_{n_1,n_2} \frac{q^{n_1^2+n_2^2-n_1n_2+n_2}(1+q^{n_1+n_2})}{(q;q)_{n_1-1}} \begin{bmatrix} 2n_1-2 \\ n_2 \end{bmatrix}\\
&\quad= \sum_{n_1} \frac{q^{n_1}}{(q;q)_{n_1}} U^{(0)}(n_1)\\
&\qquad+ \sum_{n_1,n_2} \frac{q^{(n_1-1)^2+n_2^2-(n_1-1)n_2+2n_1-1}(1+q^{n_1+n_2})}{(q;q)_{n_1-1}} \begin{bmatrix} 2n_1-2 \\ n_2 \end{bmatrix}\\
&\quad= \sum_{n_1} \frac{q^{n_1}}{(q;q)_{n_1}} U^{(0)}(n_1) + \sum_{n_1} \frac{q^{2n_1+1}}{(q;q)_{n_1}} U^{(0)}(n_1) + \sum_{n_1} \frac{q^{3n_1+2}}{(q;q)_{n_1}} U^{(1)}(n_1)\\
&\quad= \sum_{n_1} \frac{1}{(q;q)_{n_1}} \left(q^{n_1}(1+q^{n_1+1})U^{(0)}(n_1) + q^{3n_1+2}U^{(1)}(n_1) \right)
\end{aligned}$$

where the third equality results from replacing n_1 by n_1+1 in the second term. Using first the $j=0$ case of (3.8), then the $j=0$ case of (3.9), then the $j=1$ case of (3.8) shows that

$$\begin{aligned}
&q^{n_1}(1+q^{n_1+1})U^{(0)}(n_1) + q^{3n_1+2}U^{(1)}(n_1)\\
&\quad= q^{n_1}(1+q^{n_1+1}) \left(V^{(0)}(n_1) - q^{n_1+1}U^{(1)}(n_1) \right) + q^{3n_1+2}U^{(1)}(n_1)\\
&\quad= q^{n_1}(1+q^{n_1+1})V^{(0)}(n_1) - q^{2n_1+1}U^{(1)}(n_1)\\
&\quad= q^{n_1}(1+q^{n_1+1})V^{(0)}(n_1) - q^{2n_1+1}V^{(0)}(n_1) + q^{n_1+2}U^{(2)}(n_1)\\
&\quad= q^{n_1}V^{(0)}(n_1) + V^{(1)}(n_1) - U^{(1)}(n_1).
\end{aligned}$$

Therefore

$$\begin{aligned}
&(q;q)_\infty \lim_{n\to\infty} F_{(2,2,0),n}(q)\\
&\quad= \sum_{n_1} \frac{q^{n_1}}{(q;q)_{n_1}} V^{(0)}(n_1) + \sum_{n_1} \frac{1}{(q;q)_{n_1}} V^{(1)}(n_1) - \sum_{n_1} \frac{1}{(q;q)_{n_1}} U^{(1)}(n_1).
\end{aligned}$$

The first and third terms here cancel by virtue of the equality of the second and third identities in Theorem 1.1. This leaves the sum side of the fourth expression and, thus, the Proof of Theorem 1.1 is complete. ∎

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Sylvie Corteel
Department of Mathematics
University of California Berkeley
Berkeley
USA
e-mail: corteel@berkeley.edu

Trevor Welsh
Department of Mathematics and Statistics
University of Melbourne
Melbourne
Australia
e-mail: twelsh1@unimelb.edu.au

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Properties of Multivariate b-Ary Stern Polynomials

Dedicated to Professor George Andrews on the occasion of his eightieth birthday

Karl Dilcher and Larry Ericksen

Abstract. Given an integer base $b \geq 2$, we investigate a multivariate b-ary polynomial analogue of Stern's diatomic sequence which arose in the study of hyper b-ary representations of integers. We derive various properties of these polynomials, including a generating function and identities that lead to factorizations of the polynomials. We use some of these results to extend an identity of Courtright and Sellers on the b-ary Stern numbers $s_b(n)$. We also extend a result of Defant and a result of Coons and Spiegelhofer on the maximal values of $s_b(n)$ within certain intervals.

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1. Introduction

One of the most interesting integer sequences is the Stern (diatomic) sequence which can be defined by $s(0) = 0$, $s(1) = 1$, and for $n \geq 1$:

$$s(2n) = s(n), \qquad s(2n+1) = s(n) + s(n+1); \tag{1.1}$$

see entry A002487 in [9] for numerous properties and references.

It has been known for some time that this sequence is closely related to hyperbinary representations; see [10, Theorem 5.2], where it is proved that the number of hyperbinary representations of an integer $n \geq 1$ is given by the Stern number $s(n+1)$. More recently, this connection was refined by several authors through the introduction of various concepts of Stern polynomials, all extending the sequence (1.1); see [5] for references. These different Stern polynomials

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were subsequently unified by the present authors [5,6] through the introduction of the following sequence of bivariate Stern polynomials, where s and t are positive integer parameters and y and z are variables: set $\omega_{s,t}(0; y, z) = 0$, $\omega_{s,t}(1; y, z) = 1$, and for $n \geq 1$, let

$$\omega_{s,t}(2n; y, z) = y\,\omega_{s,t}(n; y^s, z^t), \tag{1.2}$$

$$\omega_{s,t}(2n+1; y, z) = z\,\omega_{s,t}(n; y^s, z^t) + \omega_{s,t}(n+1; y^s, z^t). \tag{1.3}$$

Various properties, including an explicit formula, a generating function, and some special cases, can be found in [5, Section 4]. By comparing (1.2), (1.3) with (1.1), we immediately see that for all $n \geq 0$ we have $\omega_{s,t}(n; 1, 1) = s(n)$, where s and t are arbitrary.

The purpose of this paper is to study the following sequence of polynomials, which was used in the recent paper [7] to characterize all hyper b-ary representations of a positive integer, just as the polynomials $\omega_{s,t}(n; y, z)$ served to characterize all hyperbinary expansions of a given positive integer.

Definition 1.1. Let $T = (t_1, \ldots, t_b)$ be a b-tuple of fixed positive integer parameters. We define the polynomial sequence $\omega_T(n; z_1, \ldots, z_b)$ in the b variables $z_1, \ldots, z_b$ by $\omega_T(0; z_1, \ldots, z_b) = 0$, $\omega_T(1; z_1, \ldots, z_b) = 1$, and for $n \geq 1$ by

$$\omega_T(b(n-1)+j+1; z_1, \ldots, z_b) = z_j\,\omega_T(n; z_1^{t_1}, \ldots, z_b^{t_b}) \quad (1 \leq j \leq b-1), \tag{1.4}$$

$$\omega_T(bn+1; z_1, \ldots, z_b) = z_b\,\omega_T(n; z_1^{t_1}, \ldots, z_b^{t_b}) + \omega_T(n+1; z_1^{t_1}, \ldots, z_b^{t_b}). \tag{1.5}$$

We immediately see that for $b = 2$ the identities (1.4), (1.5) reduce to (1.2) and (1.3), respectively. This sequence can also be seen as a polynomial analogue of a b-ary generalized Stern sequence that was earlier introduced and studied [2,3]. We will consider this integer sequence also here, in Sects. 2, 5 and 6.

As a special case that will provide us with examples later in this paper, we consider the ternary case, i.e., $b = 3$ in Definition 1.1.

Definition 1.2. Let $T = (r, s, t)$ be a triple of fixed positive integer parameters. We define the polynomial sequence $\omega_T(n; x, y, z)$ by $\omega_T(0; x, y, z) = 0$, $\omega_T(1; x, y, z) = 1$, and for $n \geq 1$ by

$$\omega_T(3n-1; x, y, z) = x\,\omega_T(n; x^r, y^s, z^t), \tag{1.6}$$

$$\omega_T(3n; x, y, z) = y\,\omega_T(n; x^r, y^s, z^t), \tag{1.7}$$

$$\omega_T(3n+1; x, y, z) = z\,\omega_T(n; x^r, y^s, z^t) + \omega_T(n+1; x^r, y^s, z^t). \tag{1.8}$$

The first 27 of the polynomials $\omega_T(n; x, y, z)$ are listed in Table 1.

This paper is structured as follows. In Sect. 2, we derive some basic properties of the polynomials $\omega_T(n; z_1, \ldots, z_b)$ that will be used later. In Sect. 3, we obtain a generating function for these polynomials, and in Sect. 4, we develop polynomial identities and describe factorizations of these b-variate polynomials. Finally, in Sects. 5 and 6, we derive some properties of the integer sequence $\omega_T(n; 1, \ldots, 1)$.

TABLE 1. $\omega_T(n;x,y,z)$ for $1 \le n \le 27$, $T=(r,s,t)$

n	$\omega_T(n;x,y,z)$	n	$\omega_T(n;x,y,z)$	n	$\omega_T(n;x,y,z)$
1	1	10	$x^{r^2}+y^sz+z^t$	19	$x^{r^2}y^sz+x^{r^2}z^t+y^{s^2}$
2	x	11	$x^{1+r^2}+xz^t$	20	$x^{1+r^2}z^t+xy^{s^2}$
3	y	12	$x^{r^2}y+yz^t$	21	$x^{r^2}yz^t+y^{1+s^2}$
4	x^r+z	13	$x^{r+r^2}+x^{r^2}z+z^{1+t}$	22	$x^{r^2}z^{1+t}+x^ry^{s^2}+y^{s^2}z$
5	x^{1+r}	14	x^{1+r+r^2}	23	$x^{1+r}y^{s^2}$
6	x^ry	15	$x^{r+r^2}y$	24	$x^ry^{1+s^2}$
7	x^rz+y^s	16	$x^{r+r^2}z+x^{r^2}y^s$	25	$x^ry^{s^2}z+y^{s+s^2}$
8	xy^s	17	$x^{1+r^2}y^s$	26	xy^{s+s^2}
9	y^{1+s}	18	$x^{r^2}y^{1+s}$	27	y^{1+s+s^2}

2. Some Basic Properties

From Definition 1.1, we obtain the following easy properties, instances of which can be observed in Table 1.

Lemma 2.1. *With b and T as in Definition 1.1, we have*

$$\omega_T(j; z_1, \ldots, z_b) = z_{j-1} \qquad (2 \le j \le b), \tag{2.1}$$

$$\omega_T(b+1; z_1, \ldots, z_b) = z_b + z_1^{t_1}, \tag{2.2}$$

$$\omega_T(b^\ell; z_1, \ldots, z_b) = z_{b-1}^{1+t_{b-1}+\cdots+t_{b-1}^{\ell-1}} \qquad (\ell \ge 1). \tag{2.3}$$

Proof. The identity (2.1) follows immediately from (1.4) with $n = 1$. We obtain (2.2) from (1.5) with $n = 1$, followed by (2.1) with $j = 2$. Finally, (2.3) is obtained by an easy induction, where (2.1) with $j = b$ is the induction beginning, and (1.4) with $j = b - 1$ provides the induction step. □

For the next result, and also for Sects. 5 and 6, we require an integer sequence analogue of Definition 1.1. The following definition and notation are specified in [3].

Definition 2.2. For a fixed integer $b \ge 2$, we define the generalized Stern sequence by $s_b(0) = 0$, $s_b(1) = 1$, and for $n \ge 1$ by

$$s_b(bn - j) = s_b(n) \qquad (j = 0, 1, \ldots, b-2), \tag{2.4}$$

$$s_b(bn + 1) = s_b(n) + s_b(n+1). \tag{2.5}$$

It is clear that the case $b = 2$ is the original Stern sequence (1.1). The sequence for $b = 3$ is listed as A054390 in [9], where various properties are given, including a close connection with hyperternary representations. Furthermore, by comparing Definition 1.1 with Definition 2.2, we see that for any $b \ge 2$ and $n \ge 0$, we have

$$\omega_T(n; 1, \ldots, 1) = s_b(n), \tag{2.6}$$

where the b-tuple T is arbitrary.

For the proof of the generating function in the next section, we need an estimate for the size of the polynomial $\omega_T(n; z_1, \ldots, z_b)$ when the variables are kept reasonably small.

Lemma 2.3. *Let $b \ge 2$ and suppose that $|z_j| \le 1$ for all $j = 1, 2, \ldots, b$. Then, there is a positive constant c_b, such that for all $n \ge 1$, we have*

$$|\omega_T(n; z_1, \ldots, z_b)| < c_b \cdot n^{\log_b \phi}, \tag{2.7}$$

where $\phi = \frac{1}{2}(1 + \sqrt{5})$ is the golden ratio.

Proof. By (1.4) and (1.5), it is clear that the monomials in each of the polynomials $\omega_T(n; z_1, \ldots, z_b)$ have positive coefficients. Therefore, whenever $|z_j| \le 1$ for $j = 1, \ldots, b$, by (2.6), we have

$$|\omega_T(n; z_1, \ldots, z_b)| \le \omega_T(n; 1, \ldots, 1) = s_b(n). \tag{2.8}$$

In [3], Defant proved that

$$\limsup_{n\to\infty} \frac{s_b(n)}{n^{\log_b \phi}} = \frac{(b^2-1)^{\log_b \phi}}{\sqrt{5}}; \tag{2.9}$$

see also [1] for a different proof. The identity (2.9) now implies that for some constant $c_b > 0$, we have

$$s_b(n) < c_b \cdot n^{\log_b \phi},$$

which, together with (2.8), proves (2.7). □

Table 1 indicates that the polynomials $\omega_T(n; x, y, z)$ have a special structure. This is in fact true for all bases $b \geq 2$, as shown in the following result.

Theorem 2.4. *Let $b \geq 2$, $n \geq 1$, and $T = (t_1, \dots, t_b)$ be such that $t_j \geq 2$ for $1 \leq j \leq b$. Then*

(i) The coefficients of all monomials of $\omega_T(n; z_1, \dots, z_b)$ are either 0 or 1.
(ii) In each monomial of $\omega_T(n; z_1, \dots, z_b)$, and for each j with $1 \leq j \leq b$, the exponent of z_j is a polynomial in t_j with only 0 or 1 as coefficients.

Proof. We set $n = mb + \ell$, $1 \leq \ell \leq b$, and proceed by induction on m. When $m = 0$, both statements of the theorem are true for $1 \leq \ell \leq b$, by (2.1) and by $\omega_T(1; z_1, \dots, z_b) = 1$.

Now, suppose that the statements (i) and (ii) are true up to some $m - 1$ ($m \geq 1$) and for all ℓ, $1 \leq \ell \leq b$. Then, by (1.4), we see that for $2 \leq \ell \leq b$, we have

$$\omega_T(bm + \ell; z_1, \dots, z_b) = z_{\ell-1}\, \omega_T(m; z_1^{t_1}, \dots, z_b^{t_b}).$$

Since $b \geq 2$, we have $m < bm$, so the induction hypothesis applies to the polynomial $\omega_T(m; z_1^{t_1}, \dots, z_b^{t_b})$, which means that all monomials in this last polynomial are of the required form. This does not change if we multiply them all by $z_{\ell-1}$.

Next, for $\ell = 1$ we use (1.5), namely

$$\omega_T(bm + 1; z_1, \dots, z_b) = z_b\, \omega_T(m; z_1^{t_1}, \dots, z_b^{t_b}) + \omega_T(m + 1; z_1^{t_1}, \dots, z_b^{t_b}). \tag{2.10}$$

As before, we have $m < bm$, and $m + 1 < bm$ also holds for all $m \geq 1$ when $b \geq 3$, and for $m \geq 2$ when $b = 2$. The remaining case $m = 1$, $b = 2$ is easily verified by (1.2) and (1.3). Therefore, the induction hypothesis applies to the right of (2.10), which means that all monomials in both summands are of the required form. Furthermore, since each monomial in the first summand is multiplied by z_b and $t_j \geq 2$ for $1 \leq j \leq b$, all monomials on the right of (2.10) are distinct. Hence, the coefficients of the monomials remain 1, and the proof is complete. □

3. Generating Functions

A generating function for the numerical sequence $(s_b(n))$ is given in [2]. It is rewritten here in our notation as

$$\sum_{n=1}^{\infty} s_b(n)q^n = q\prod_{j=0}^{\infty}\left(1+q^{b^j}+q^{2\cdot b^j}+\cdots+q^{b\cdot b^j}\right). \tag{3.1}$$

In this section we are going to extend the generating function (3.1) to the sequence of polynomials $\omega_T(n;z_1,\ldots,z_b)$. The case $b=2$ was earlier obtained in [5], where we showed that

$$\sum_{n=1}^{\infty}\omega_{s,t}(n;y,z)q^n = q\prod_{j=0}^{\infty}\left(1+y^{s^j}q^{2^j}+z^{t^j}q^{2\cdot 2^j}\right). \tag{3.2}$$

The following result is now easily seen as a natural extension of both (3.1) and (3.2).

Theorem 3.1. *Let $b\ge 2$, and let $T=(t_1,\ldots,t_b)$ be a b-tuple of positive integer parameters. Then, we have*

$$\sum_{n=1}^{\infty}\omega_T(n;z_1,\ldots,z_b)q^n = q\prod_{j=0}^{\infty}\left(1+z_1^{t_1^j}q^{b^j}+z_2^{t_2^j}q^{2\cdot b^j}+\cdots+z_b^{t_b^j}q^{b\cdot b^j}\right). \tag{3.3}$$

Proof. To simplify notation, we set

$$Z=(z_1,\ldots,z_b);\qquad Z^{T^j}=(z_1^{t_1^j},\ldots,z_b^{t_b^j}),\quad\text{where}\quad T=(t_1,\ldots,t_b). \tag{3.4}$$

We follow the general idea of the proof of a special case in Proposition 5.1 of [4] and further denote

$$F(Z,q)=\sum_{n=1}^{\infty}\omega_T(n;Z)q^{n-1}. \tag{3.5}$$

By Lemma 2.3, this power series has a positive radius of convergence, so the following operations are allowable. If we rewrite the series in (3.5) and use (1.4) and (1.5), then $F(Z,q)$ becomes

$$\begin{aligned}
1&+\sum_{n=1}^{\infty}\left(\sum_{j=1}^{b-1}\omega_T(b(n-1)+j+1;Z)q^{b(n-1)+j}+\omega_T(bn+1;Z)q^{bn}\right)\\
&=1+\sum_{n=1}^{\infty}\left(\sum_{j=1}^{b-1}z_j\omega_T(n;Z^T)q^j(q^b)^{n-1}+z_b\omega_T(n;Z^T)q^b(q^b)^{n-1}\right)\\
&\quad+\sum_{n=1}^{\infty}\omega_T(n+1;Z^T)(q^b)^n\\
&=1+\left(\sum_{j=1}^{b}z_jq^j\right)\sum_{n=1}^{\infty}\omega_T(n;Z^T)(q^b)^{n-1}+\sum_{n=1}^{\infty}\omega_T(n+1;Z^T)(q^b)^n.
\end{aligned}$$

If we add the initial 1 to the second summation and shift the summation index, we see that

$$F(Z,q)=(1+z_1q+z_2q^2+\cdots+z_bq^b)F(Z^T,q^b). \tag{3.6}$$

Upon iterating the functional equation (3.6), we get

$$\begin{aligned} F(Z,q) &= (1+z_1q+\cdots+z_bq^b)(1+z_1^{t_1}q^b+\cdots+z_b^{t_b}q^{b\cdot b})F(Z^{T^2},q^{b^2}) \\ &\ \vdots \\ &= \prod_{j=0}^{N}\left(1+z_1^{t_1^j}q^{b^j}+z_2^{t_2^j}q^{2\cdot b^j}+\cdots+z_b^{t_b^j}q^{b\cdot b^j}\right)F(Z^{T^{N+1}},q^{b^{N+1}}). \end{aligned} \tag{3.7}$$

We are done if we can show, for sufficiently small $z_1,\dots,z_b$ and q, that we have

$$F(Z^{T^{N+1}},q^{b^{N+1}})\to 1 \quad \text{as} \quad N\to\infty; \tag{3.8}$$

then (3.7) becomes the desired identity (3.3) as $N\to\infty$. To prove (3.8), we rewrite $F(Z,q)$ in (3.5) as

$$F(Z,q)=1+q\sum_{n=2}^{\infty}\omega_T(n;Z)q^{n-2}. \tag{3.9}$$

Then, we see that for any Z and q satisfying $|z_j|\le 1$, $j=1,\dots,b$ and $|q|\le 1-\varepsilon$ for some $\varepsilon>0$, the infinite series on the right of (3.9) remains bounded because of Lemma 2.3. But then, since $q^{b^{N+1}}\to 0$ as $N\to\infty$, we immediately get (3.8), which completes the proof of Theorem 3.1. □

4. Some General Polynomial Identities

The purpose of this section is to obtain a class of identities for the polynomial sequences $\omega_T(n;z_1,\dots,z_b)$. We begin with the following observation from Table 1: The entries for $n=14,\dots,18$ are exactly the entries for $n=5,\dots,9$ multiplied by x^{r^2}. Similarly, the entries for $n=23,\dots,27$ are again those for $n=5,\dots,9$, but this time multiplied by y^{s^2}. These are special instances of the following lemma, which will be needed as an auxiliary result later in this section.

Lemma 4.1. *Let $b\ge 2$ be an integer and let $T=(t_1,\dots,t_b)$ be a b-tuple of positive integers. Then, for any $\ell\ge 1$ and $1\le k\le b-1$, we have*

$$\omega_T(k\cdot b^\ell+j;z_1,\dots,z_b)=z_k^{t_k^\ell}\omega_T(j;z_1,\dots,z_b), \qquad \frac{b^\ell-1}{b-1}<j\le b^\ell. \tag{4.1}$$

Example 1. Let $b=3$, $\ell\ge 1$ and $T=(r,s,t)$. Then, for all j with $\frac{1}{2}(3^\ell-1)<j\le 3^\ell$, the identity (4.1) reduces to

$$\omega_T(3^\ell+j;x,y,z)=x^{r^\ell}\omega_T(j;x,y,z),$$
$$\omega_T(2\cdot 3^\ell+j;x,y,z)=y^{s^\ell}\omega_T(j;x,y,z).$$

For instance, if we take $\ell = 2$ and $5 \le j \le 9$, we obtain the first example mentioned at the beginning of this section.

For the remainder of this section, we adopt the notation (3.4), which was already used in the proof of Theorem 3.1.

Proof of Lemma 4.1. We proceed by induction on ℓ. For $\ell = 1$, (4.1) reduces to

$$\omega_T(k\cdot b + j; Z) = z_k^{t_k}\cdot \omega_T(j;Z) = z_{j-1}\omega_T(k+1;Z^T), \qquad 2\le j\le b,$$

where in the second equation, we have used (2.1) twice. But this is just (1.4) with $n = k+1$ and j replaced by $j-1$.

Now assume that (4.1) is true up to some $\ell \ge 1$. We wish to show that it also holds for ℓ replaced by $\ell+1$, for all j with $(b^{\ell+1}-1)/(b-1) < j \le b^{\ell+1}$. But this interval for j can be rewritten as

$$\frac{b^\ell - 1}{b-1}\cdot b + 2 \le j \le b^\ell \cdot b,$$

or divided into subintervals as

$$\frac{b^\ell - 1}{b-1}\cdot b + 2 \le j \le \left(\frac{b^\ell - 1}{b-1} + 1\right)\cdot b, \tag{4.2}$$

$$r\cdot b + 1 \le j \le (r+1)\cdot b, \quad \text{with} \quad \frac{b^\ell - 1}{b-1} + 1 \le r \le b^\ell - 1. \tag{4.3}$$

We now distinguish between two cases.

(a) We consider (4.2), combined with the interval (4.3) for j, without the left endpoints. In other words, we write

$$j = rb + i, \qquad \frac{b^\ell - 1}{b-1} \le r \le b^\ell - 1, \qquad 2 \le i \le b.$$

Now, we compute

$$\begin{aligned}
\omega_T(k\cdot b^{\ell+1} + j; Z) &= \omega_T((k\cdot b^\ell + r)b + i; Z)\\
&= z_{i-1}\omega_T(k\cdot b^\ell + r + 1; Z^T) \qquad \text{by (1.4)}\\
&= \omega_T(i;Z)\cdot z_k^{t_k^{\ell+1}}\cdot \omega_T(r+1;Z^T) \quad \text{by (2.1), (4.1)}\\
&= z_k^{t_k^{\ell+1}}\cdot \omega_T(rb+i;Z) \qquad \text{by (2.1), (1.4).}
\end{aligned}$$

Since $rb + i = j$, we have, therefore, proved (4.1) for $\ell+1$ in place of ℓ, which proves the first case by induction.

(b) We consider the left endpoints in the intervals (4.3) for j; that is, we write

$$j = rb + 1, \qquad \frac{b^\ell - 1}{b-1} + 1 \le r \le b^\ell - 1.$$

We compute

$$\begin{aligned}
\omega_T(k\cdot b^{\ell+1} + j; Z) &= \omega_T((k\cdot b^\ell + r)b + 1; Z)\\
&= z_b\omega_T(k\cdot b^\ell + r; Z^T) + \omega_T(k\cdot b^\ell + r + 1; Z^T) \quad \text{by (1.5)}
\end{aligned}$$

$$
\begin{aligned}
&= \left(z_b\omega_T(r;Z^T) + \omega_T(r+1;Z^T)\right) z_k^{t_k^{\ell+1}} \quad \text{by (4.1)}\\
&= \omega_T(rb+1;Z) z_k^{t_k^{\ell+1}} \qquad \text{by (1.5)}.
\end{aligned}
$$

Since $rb + 1 = j$, this proves (4.1) for $\ell + 1$ in place of ℓ. The proof of the lemma is now complete. $\square$

Independently of this work, and with different notations, Ulas [11] proved the following identity, where we set $\mathbb{1} = (1, 1, \ldots, 1)$. For integers $b \geq 2$, $\ell \geq 1$, $k \geq 1$ and $0 \leq j \leq b^\ell$, we have

$$
\begin{aligned}
\omega_{\mathbb{1}}(k \cdot b^\ell + j; Z) &= \omega_{\mathbb{1}}(k+1; Z) \cdot \omega_{\mathbb{1}}(j; Z)\\
&\quad + \omega_{\mathbb{1}}(k; Z) \cdot \left(\omega_{\mathbb{1}}(b^\ell + j; Z) - z_1\omega_{\mathbb{1}}(j; Z)\right). \qquad (4.4)
\end{aligned}
$$

The main result of this section is the following extension of (4.4) to arbitrary b-tuples T.

Theorem 4.2. *Let $b \geq 2$, $\ell \geq 1$, and $k \geq 1$ be integers. Then, for all $0 \leq j \leq b^\ell$, we have*

$$
\begin{aligned}
\omega_T(k \cdot b^\ell + j; Z) &= \omega_T(k+1; Z^{T^\ell}) \cdot \omega_T(j; Z)\\
&\quad + \omega_T(k; Z^{T^\ell}) \cdot \Big(\omega_T(b^\ell + j; Z) - z_1^{t_1^\ell}\omega_T(j; Z)\Big). \qquad (4.5)
\end{aligned}
$$

Before proving this result, we derive an interesting consequence and give a few examples.

Corollary 4.3. *Let $b \geq 2$, $\ell \geq 1$ and $k \geq 1$ be integers. Then*

$$
\omega_T(k \cdot b^\ell + j; Z) = \omega_T(k+1; Z^{T^\ell}) \cdot \omega_T(j; Z), \qquad \frac{b^\ell - 1}{b-1} < j \leq b^\ell. \quad (4.6)
$$

This follows immediately from Lemma 4.1. Indeed, if j is restricted as in (4.6), then by (4.1), the expression in large parentheses in (4.5) vanishes and we obtain the identity (4.6). We next note that by (2.1), we have

$$
\omega_T(k+1; Z^{T^\ell}) = z_k^{t_k^\ell}, \qquad 1 \leq k \leq b-1,
$$

and so (4.6) is an extension of (4.1). We also note that for $b = 2$, we get from (4.6) a single identity with $j = 2^\ell$, namely

$$
\omega_{s,t}((k+1) \cdot 2^\ell; y, z) = \omega_{s,t}(k+1; y^{s^\ell}, z^{t^\ell}) \cdot \omega_{s,t}(2^\ell; y, z).
$$

However, this identity is an immediate consequence of (1.2) and provides nothing new. Corollary 4.3 is, therefore, meaningful only for bases $b \geq 3$.

It is also clear that (4.6) gives large numbers of polynomials $\omega_T(n; Z)$ that are obviously reducible.

Example 2. (a) Let $b = 3$, $\ell = 2$, $k = 3$, and $j = 7$. Then

$$
\begin{aligned}
\omega_T(34; Z) &= \omega_T(4; Z^{T^2}) \cdot \omega_T(7; Z),\\
x^{r^3}y^s + x^{r+r^3}z + y^s z^{t^2} + x^r z^{1+t^2} &= (x^{r^3} + z^{t^2})(x^r z + y^s),
\end{aligned}
$$

see also Table 1. For $b = 3$, this is the example with smallest index, where both factors have more than one term.

(b) Similarly, by iterating (4.6), we get

$$\begin{aligned}\omega_T(304;Z) &= \omega_T(3\cdot 3^4+61;Z) = \omega_T(4;Z^{T^4})\cdot\omega_T(61;Z)\\ &= \omega_T(4;Z^{T^4})\cdot\omega_T(6\cdot 3^2+7;Z)\\ &= \omega_T(4;Z^{T^4})\cdot\omega_T(7;Z^{T^2})\cdot\omega_T(7;Z)\\ &= (x^{r^5}+z^{t^4})(x^{r^3}z^{t^2}+y^{s^3})(x^r z+y^s).\end{aligned}$$

For $b=3$, this is the smallest index polynomial with three factors, each of which has more than one term.

(c) The representation $n=k\cdot b^\ell+j$, with j in the allowable range of Corollary 4.3, may not be unique if it exists. For instance, the representations $26=8\cdot 3^1+2=2\cdot 3^2+8$ both satisfy the hypothesis of Corollary 4.3, and accordingly, we have

$$\begin{aligned}\omega_T(26;Z) &= \omega_T(9;Z^T)\cdot\omega_T(2;Z) = \omega_T(3;Z^{T^2})\cdot\omega_T(8;Z)\\ &= y^{s+s^2}\cdot x = y^{s^2}\cdot xy^s.\end{aligned}$$

The two factorizations are in fact the same, as they ought to be.

(d) The indices $b<n<100$ that do not lead to factorizations according to Corollary 4.3 are as follows:

$$\begin{aligned}&b=3\colon n=4,7,10,13,19,22,28,31,37,40,55,58,64,67,82,85,91,94;\\ &b=4\colon n=5,9,13,17,21,33,37,49,53,65,69,81,85;\\ &b=5\colon n=6,11,16,21,26,31,51,56,76,81.\end{aligned}$$

In particular, no n of the form $n=b^\ell+1$ satisfies the condition in (4.6).

Proof of Theorem 4.2. We first note that for $j=0$ the identity (4.5) reduces to

$$\omega_T(k\cdot b^\ell;Z) = \omega_T(k;Z^{T^\ell})\cdot\omega_T(b^\ell;Z),$$

which follows from (2.3) and by iterating (1.4). Now, we assume that $j\ge 1$, and we proceed by induction on ℓ. To simplify notation, we drop the subscript T. For $\ell=1$ and $0\le j\le b$, the identity (4.5) becomes

$$\omega(k\cdot b+j;Z) = \omega(k+1;Z^T)\,\omega(j;Z)+\omega(k;Z^T)\left[\omega(b+j;Z)-z_1^{t_1}\omega(j;Z)\right]. \tag{4.7}$$

When $j=1$, the identity (2.2) shows that the term in large brackets on the right of (4.7) reduces to z_b, and therefore, (4.7) becomes the same as (1.5), with k in place of n. Finally, when $2\le j\le b$, the bracket expression on the right of (4.7) vanishes by (4.1), and so (4.7) reduces to (1.4). This establishes the induction beginning.

We now assume that (4.5) is true for some $\ell\ge 1$; we wish to show that it also holds for $\ell+1$ and for all $1\le j\le b^{\ell+1}$. In analogy to the proof of Lemma 4.1, we set

$$j=rb+i,\qquad 0\le r\le b^\ell-1,\qquad 0<i\le b,$$

and distinguish between two cases.

(a) If $2 \le i \le b$, then

$$\begin{aligned}
&\omega(k\cdot b^{\ell+1}+j;Z)\\
&\quad = \omega(b(k\cdot b^{\ell}+r)+i;Z) = z_{i-1}\omega(k\cdot b^{\ell}+r+1;Z^{T})\\
&\quad = \omega(k+1;Z^{T^{\ell+1}})z_{i-1}\omega(r+1;Z^{T})\\
&\qquad + \omega(k;Z^{T^{\ell+1}})\left(z_{i-1}\omega(b^{\ell}+r+1;Z^{T}) - z_1^{t_1^{\ell+1}}z_{i-1}\omega(r+1;Z^{T})\right)\\
&\quad = \omega(k+1;Z^{T^{\ell+1}})\,\omega(br+i;Z)\\
&\qquad + \omega(k;Z^{T^{\ell+1}})\left(\omega(b(b^{\ell}+r)+i;Z) - z_1^{t_1^{\ell+1}}\omega(br+i;Z)\right),
\end{aligned}$$

where we have used (1.4), with $i-1$ in place of j, in the second equation, and then again three times in the last equation. Since $br+i=j$, this completes the proof by induction in the case $2 \le i \le b$.

(b) If $i=1$, then by (1.5), we have

$$\begin{aligned}
\omega(k\cdot b^{\ell+1}+1;Z) &= \omega(b(k\cdot b^{\ell}+r)+1;Z)\\
&= z_b\omega(k\cdot b^{\ell}+r;Z^{T}) + \omega(k\cdot b^{\ell}+r+1;Z^{T}).
\end{aligned}$$

Now, we proceed as in part (a), this time applying the induction hypothesis to both terms on the right-hand side of the last identity, then collecting appropriate terms, and finally applying (1.5) again (three times). This will once again give (4.5) with $\ell+1$ in place of ℓ, and the proof by induction is complete. □

To conclude this section, we give an example of Theorem 4.2 that is not covered by Corollary 4.3.

Example 3. Let $b=3, \ell=2, k=4$, and $j=1$ in Theorem 4.2. Then, by (4.5), we have

$$\begin{aligned}
\omega_T(37;x,y,z) &= \omega_T(5;x^{r^2},y^{s^2},z^{t^2})\cdot\omega_T(1;x,y,z)\\
&\quad + \omega_T(4;x^{r^2},y^{s^2},z^{t^2})\cdot\left(\omega_T(10;x,y,z) - x^{r^2}\omega_T(1;x,y,z)\right),
\end{aligned}$$

and using the entries in Table 1, we get

$$\omega_T(37;x,y,z) = x^{r^2+r^3} + \left(x^{r^3}+z^{t^2}\right)\left(y^{s}z+z^{t}\right).$$

On the other hand, the linear recurrence (1.8) gives

$$\begin{aligned}
\omega_T(37;x,y,z) &= z\,\omega_T(12;x^{r},y^{s},z^{t}) + \omega_T(13;x^{r},y^{s},z^{t})\\
&= zy^{s}\left(x^{r^3}+z^{t^2}\right) + x^{r^2+r^3} + x^{r^3}z^{t} + z^{t+t^2},
\end{aligned}$$

where we have used Table 1 again. It is easy to verify that the two forms of $\omega_T(37;x,y,z)$ are identical.

5. Identities for the Integer Sequence $s_b(n)$

In their study of arithmetic properties of the number of hyper b-ary representations, Courtright and Sellers [2] proved the following identity which we rewrite in our notation: For integers $b \geq 3$, $k \geq 0$, $\ell \geq 1$, and $2 \leq \nu \leq b-1$, we have

$$s_b(k \cdot b^\ell + \nu \cdot b^{\ell-1} + 1) = \ell \cdot s_b(k+1). \tag{5.1}$$

In this section, we show that (5.1) follows from Lemma 4.1 and that other similar identities can also be obtained, see also Corollary 6.5 for another analogue of (5.1).

Considering (4.6) and (2.6), it is clear that we need to evaluate $s_b(j)$ for the desired values of j. This is done in the following lemma.

Lemma 5.1. *For integers $b \geq 2$, $1 \leq k \leq b-1$, and $\ell \geq 2$ we have*

$$s_b(k \cdot b^\ell + \lambda \cdot b + \mu) = \begin{cases} \ell+1, & \text{if } \lambda = 0, \mu = 1, \\ \ell, & \text{if } \lambda = 0, 2 \leq \mu \leq b, \\ 2\ell-1, & \text{if } \lambda = 1, \mu = 1, \\ 2\ell-2, & \text{if } 2 \leq \lambda \leq b-1, \mu = 1, \\ \ell-1, & \text{if } 1 \leq \lambda \leq b-1, 2 \leq \mu \leq b, \end{cases} \tag{5.2}$$

where the first part also holds for $\ell \in \{0,1\}$ and the second part also holds for $\ell = 1$.

Proof. Throughout this proof, we will use the facts that for $\ell \geq 1$, we have

$$s_b(k) = 1 \qquad \text{and} \qquad s_b(k \cdot b^\ell) = 1, \qquad 1 \leq k \leq b. \tag{5.3}$$

This follows from (2.1) and (2.6), and from iterating (2.4) with $j = 1$.

We first prove the case $\lambda = 0, \mu = 1$ by induction on ℓ. For $\ell = 0$ we have $s_b(k \cdot 1 + 1) = 1$ by (5.3), and for $\ell = 1$ we see with (2.5) and (5.3) that

$$s_b(k \cdot b + 1) = s_b(k) + s_b(k+1) = 1 + 1 = 2.$$

Now, we assume that the first part of (5.2) holds for some $\ell \geq 1$. Using (2.5) again, we have

$$s_b(k \cdot b^{\ell+1} + 1) = s_b(k \cdot b^\ell) + s_b(k \cdot b^\ell + 1) = 1 + (\ell + 1),$$

where we have used (5.3) and the induction hypothesis. This proves the first part of (5.2).

Suppose now that $2 \leq \mu \leq b$, while still $\lambda = 0$. Then, with (2.4) and the first part of (5.2), we have

$$s_b(k \cdot b^\ell + \mu) = s_b((k \cdot b^{\ell-1} + 1)b - (b - \mu)) = s_b(k \cdot b^{\ell-1} + 1) = \ell,$$

which proves the second part of (5.2) for $\ell \geq 1$.

Next, we let $1 \leq \lambda \leq b-1$ and $\mu = 1$. Then, again, with (2.5) and the previous results, we find that for $\ell \geq 2$,

$$s_b(k \cdot b^\ell + \lambda \cdot b + 1) = s_b(k \cdot b^{\ell-1} + \lambda) + s_b(k \cdot b^{\ell-1} + \lambda + 1)$$
$$= \begin{cases} \ell + (\ell - 1) = 2\ell - 1, & \text{when } \lambda = 1, \\ (\ell - 1) + (\ell - 1) = 2\ell - 2, & \text{when } 2 \le \lambda \le b - 1, \end{cases}$$

which proves the third and fourth parts of (5.2).

Finally, we consider $1 \le \lambda \le b - 1$ and $2 \le \mu \le b$. Then, with (2.4) and previous parts, we find

$$s_b(k \cdot b^\ell + \lambda b + \mu) = s_b((k \cdot b^{\ell-1} + \lambda + 1)b - (b - \mu)) = s_b(k \cdot b^{\ell-1} + \lambda + 1) = \ell - 1.$$

This proves the fifth part of (5.2), and the proof of the lemma is complete. $\square$

The method of proof of this lemma could be used to obtain other identities of the type (5.2), beyond the ranges $0 \le \lambda \le b - 1$ and $1 \le \mu \le b$.

We now use Corollary 4.3 with $z_1 = \cdots = z_b = 1$ and apply Lemma 4.4 with $k = \nu$. With ν satisfying $2 \le \nu \le b - 1$, it is easy to verify that $\nu b^\ell + \lambda b + \mu$ lies in the interval required by (4.6), with ℓ shifted by 1. Therefore, Corollary 4.3 does indeed apply, and we immediately get the following result.

Corollary 5.2. *Let $b \ge 2$, $k \ge 0$, $\ell \ge 2$, and $2 \le \nu \le b - 1$ be integers. Then, for integers $0 \le \lambda \le b - 1$ and $1 \le \mu \le b$, we have*

$$s_b(k \cdot b^{\ell+1} + \nu \cdot b^\ell + \lambda b + \mu) = f_{\lambda,\mu}(\ell) \cdot s_b(k + 1), \tag{5.4}$$

where $f_{\lambda,\mu}(\ell)$ is given by the right-hand side of (5.2).

With $\lambda = 0$ and $\mu = 1$, the identity (5.4) reduces to (5.1), with ℓ shifted by 1.

6. Maximum Values

One of the well-known properties of Stern's diatomic sequence (1.1) is the fact that the maximum of $s(m)$, with $2^{n-2} \le m < 2^{n-1}$, is always the Fibonacci number F_n. Furthermore, when $n \ge 4$, this maximum is attained exactly twice in the given interval, at specific known values of m. This was apparently first proved by Lehmer [8]; see also [10].

More recently, Defant [3] and Coons and Spiegelhofer [1] proved the analogous result for an arbitrary base $b \ge 2$, namely

$$\max_{b^{n-2} \le m < b^{n-1}} s_b(m) = F_n, \tag{6.1}$$

and they showed that the smallest m at which the maximum occurs is

$$\alpha_n^b = \frac{b^n - 1}{b^2 - 1} + \left(\frac{1 - (-1)^n}{2}\right) \frac{b}{b + 1}. \tag{6.2}$$

Coons and Spiegelhofer also expressed this α_n^b value by the base-b expansion:

$$\alpha_n^b = \begin{cases} ((10)^{\ell-1}1)_b, & \text{when } n = 2\ell, \\ ((10)^{\ell-1}11)_b, & \text{when } n = 2\ell + 1. \end{cases} \tag{6.3}$$

Here, the notation $(10)^{\ell-1}$ indicates that the pair "10" of b-ary digits is repeated $\ell-1$ times.

It is the purpose of this section to extend the results (6.2), (6.3) and show that for $n \geq 4$, the maximum (6.1) is attained at $2(b-1)$ distinct values of m in each relevant interval. To establish this fact, we introduce two classes of integer sequences.

Definition 6.1. For a fixed integer j with $1 \leq j \leq b-1$, we define the following two recursive sequences. (1) Let $\alpha_{2,j}^b = j$, and for $n \geq 2$ set

$$\alpha_{n+1,j}^b = \begin{cases} b\alpha_{n,j}^b + 1, & \text{if } n \text{ is even,} \\ b\alpha_{n,j}^b + 1 - b, & \text{if } n \text{ is odd.} \end{cases} \tag{6.4}$$

(2) Let $\beta_{2,j}^b = j+1$, and for $n \geq 2$ set

$$\beta_{n+1,j}^b = \begin{cases} b\beta_{n,j}^b + 1 - b, & \text{if } n \text{ is even,} \\ b\beta_{n,j}^b + 1, & \text{if } n \text{ is odd.} \end{cases} \tag{6.5}$$

For $n \geq 4$, and in analogy to (6.3), the terms $\alpha_{n,j}^b$ and $\beta_{n,j}^b$ can be written in terms of base-b expansions.

Lemma 6.2. *Let $b \geq 2$, $n \geq 4$, and $1 \leq j \leq b-1$. Then*

$$\alpha_{n,j}^b = \begin{cases} (j0(10)^{\ell-2}1)_b, & \text{when} \quad n = 2\ell, \\ (j0(10)^{\ell-2}11)_b, & \text{when} \quad n = 2\ell+1, \end{cases} \tag{6.6}$$

and

$$\beta_{n,j}^b = \begin{cases} (j(10)^{\ell-2}11)_b, & \text{when} \quad n = 2\ell, \\ (j(10)^{\ell-1}1)_b, & \text{when} \quad n = 2\ell+1. \end{cases} \tag{6.7}$$

Proof. By (6.6), we have $\alpha_{4,j}^b = (j01)_b = jb^2+1$, which is consistent with (6.4). Similarly, we have $\beta_{4,j}^b = (j11)_b = jb^2+b+1$, consistent with (6.5). Finally, it is easy to verify that the base-b expansions (6.6) and (6.7) satisfy the recurrence relations (6.4) and (6.5), respectively. This completes the proof of the lemma. $\square$

Explicit formulas for $\alpha_{n,j}^b$ and $\beta_{n,j}^b$, analogous to and extending (6.2) can easily be obtained from Lemma 6.2 for $n \geq 4$, and from Definition 6.1 for $n = 2, 3$. We, therefore, state the following identities without proofs.

Corollary 6.3. *Let $b \geq 2$ and $n \geq 2$ be integers. Then, for $1 \leq j \leq b-1$, we have*

$$\alpha_{n,j}^b = \begin{cases} (j-1)b^{n-2} + \dfrac{b^n-1}{b^2-1}, & \text{if } n \text{ is even,} \\ (j-1)b^{n-2} + \dfrac{b^n+b^2-b-1}{b^2-1}, & \text{if } n \text{ is odd;} \end{cases}$$

$$\beta_{n,j}^b = \begin{cases} (j-1)b^{n-2} + \dfrac{(b^2+b-1)\cdot b^{n-2}+b^2-b-1}{b^2-1}, & \text{if } n \text{ is even,} \\ (j-1)b^{n-2} + \dfrac{(b^2+b-1)\cdot b^{n-2}-1}{b^2-1}, & \text{if } n \text{ is odd.} \end{cases}$$

We are now ready to state and prove the main result of this section.

Theorem 6.4. *Let $b \geq 2$ and $n \geq 4$. Then, the $2(b-1)$ distinct values*

$$m \in \{\alpha_{n,1}^b, \beta_{n,1}^b, \ldots, \alpha_{n,b-1}^b, \beta_{n,b-1}^b\} \tag{6.8}$$

give the maximum value in (6.1), that is, they satisfy

$$b^{n-2} \leq m < b^{n-1} \qquad \text{and} \qquad s_b(m) = F_n. \tag{6.9}$$

Proof. First, we note that all four base-b representations in (6.6) and (6.7) have exactly $n-1$ b-ary digits, which means that they lie in the required interval $b^{n-2} \leq m < b^{n-1}$.

Now, we fix a j, $1 \leq j < b-1$, and use induction on n to show that

$$s_b(\alpha_{n,j}^b) = F_n, \quad n \geq 2. \tag{6.10}$$

By (2.4), we have

$$s_b(\alpha_{2,j}^b) = s_b(j) = 1 = F_2,$$

and with (6.4), (2.5), and again (2.4), we get

$$s_b(\alpha_{3,j}^b) = s_b(jb+1) = s_b(j) + s_b(j+1) = 1 + 1 = F_3.$$

Now, suppose that (6.10) is true up to some $n \geq 3$. First, if n is even, then by (6.4) and (2.5), we have

$$s_b(\alpha_{n+1,j}^b) = s_b(b\alpha_{n,j}^b + 1) = s_b(\alpha_{n,j}^b) + s_b(\alpha_{n,j}^b + 1). \tag{6.11}$$

Now, by the second part of (6.4) and by (2.4), we have

$$s_b(\alpha_{n,j}^b + 1) = s_b(b\alpha_{n-1,j}^b - (b-2)) = s_b(\alpha_{n-1,j}^b),$$

and with (6.11), we get the Fibonacci recursion

$$s_b(\alpha_{n+1,j}^b) = s_b(\alpha_{n,j}^b) + s_b(\alpha_{n-1,j}^b). \tag{6.12}$$

Second, if n is odd, then again by (6.4) and (2.5), we have

$$s_b(\alpha_{n+1,j}^b) = s_b(b(\alpha_{n,j}^b - 1) + 1) = s_b(\alpha_{n,j}^b - 1) + s_b(\alpha_{n,j}^b). \tag{6.13}$$

Using now the first part of (6.4), followed by (2.4), we get

$$s_b(\alpha_{n,j}^b - 1) = s_b(b\alpha_{n-1,j}^b) = s_b(\alpha_{n-1,j}^b),$$

and with (6.13), we get again the Fibonacci recursion (6.12). This completes the proof of (6.10) by induction. The analogue of (6.10), with α replaced by β, can be proved in essentially the same way. ☐

Remark 6.5.

(1) When $n = 3$, we have $\alpha_{3,j}^b = \beta_{3,j}^b = jb + 1$ for $1 \leq j \leq b-1$; hence in the interval $b \leq m < b^2$ we have only $b-1$ maximum values $F_3 = 2$. Furthermore, when $n = 2$, we have $\alpha_{2,j}^b = j$ and $\beta_{2,j}^b = j+1$, while $s_b(j) = 1 = F_2$ for $1 \leq j \leq b$.

(2) The base-b expansions (6.6) and (6.7) show that for all $n \geq 4$ and $1 \leq j \leq b-2$, we have the order relations:

$$\alpha_{n,j}^b < \beta_{n,j}^b < \alpha_{n,j+1}^b < \beta_{n,j+1}^b.$$

This is illustrated by Example 4.

Example 4. Let $b = 3$. Then, we have

$$\alpha_{4,1}^3 = (101)_3 = 10, \quad \alpha_{4,2}^3 = (201)_3 = 19,$$
$$\beta_{4,1}^3 = (111)_3 = 13, \quad \beta_{4,2}^3 = (211)_3 = 22.$$

By Theorem 6.4, the maximum value $F_4 = 3$ is attained at these four values. This is consistent with Table 1 if we note that $s_3(m)$ is the number of monomials in the mth entry.

Furthermore, we have $\alpha_{3,j}^3 = \beta_{3,j}^3 = 3j+1$, so the maximum value $F_3 = 2$ between 3 and 9 is attained at $m = 4$ and $m = 7$, which is again consistent with Table 1.

We conclude this paper with an analogue of the identity (5.1) of Courtright and Sellers.

Corollary 6.6. *Let $b \geq 3$, $k \geq 0$, and $\ell \geq 2$ be integers. Then, for all $2 \leq j \leq b-1$, we have*

$$s_b(k \cdot b^\ell + \alpha_{\ell+1,j}^b) = F_{\ell+1} \cdot s_b(k+1),$$
$$s_b(k \cdot b^\ell + \beta_{\ell+1,j}^b) = F_{\ell+1} \cdot s_b(k+1).$$

Proof. By (2.6), Corollary 4.3 with $Z = (1, \ldots, 1)$ gives

$$s_b(k \cdot b^\ell + m) = s_b(m) \cdot s_b(k+1), \quad \frac{b^\ell - 1}{b-1} < m \leq b^\ell. \tag{6.14}$$

We now use Theorem 6.4 with $n = \ell + 1$, and note that by (6.6) and (6.7), the integers $\alpha_{\ell+1,j}^b$ and $\beta_{\ell+1,j}^b$, for $2 \leq j \leq b-1$, satisfy the inequality on the right of (6.14). Finally, the case $\ell = 2$ follows from Remark (1) preceding Example 4. $\square$

If we leave Z as a b-tuple of variables, then Corollary 4.3 will provide us with a polynomial analogue of Corollary 6.5.

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Karl Dilcher
Department of Mathematics and Statistics
Dalhousie University
Halifax
Nova Scotia B3H 4R2
Canada
e-mail: dilcher@mathstat.dal.ca

Larry Ericksen
P.O. Box 172
Millville
NJ 08332-0172
USA
e-mail: LE22@cornell.edu

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Annals of Combinatorics

A Simple Proof of a Congruence for a Series Involving the Little q-Jacobi Polynomials

Dedicated to Professor George E. Andrews on the occasion of his 80th birthday

Atul Dixit

Abstract. We give a simple and a more explicit proof of a mod 4 congruence for a series involving the little q-Jacobi polynomials which arose in a recent study of a certain restricted overpartition function.

Mathematics Subject Classification. Primary 11P81, Secondary 05A17.

Keywords. Overpartitions, Congruence, Little q-Jacobi Polynomials.

1. Introduction

In [3], Andrews, Schultz, Yee and the author studied the overpartition function $\overline{p}_\omega(n)$, namely, the number of overpartitions of n such that all odd parts are less than twice the smallest part, and in which the smallest part is always overlined. In the same paper, they obtained a representation for the generating function of $\overline{p}_\omega(n)$ in terms of a ${}_3\phi_2$ basic hypergeometric series and an infinite series involving the little q-Jacobi polynomials. The latter are given by [2, Equation (3.1)]

$$p_n(x;\alpha,\beta:q) := {}_2\phi_1\left(\begin{matrix} q^{-n},\, \alpha\beta q^{n+1} \\ \alpha q \end{matrix}\,;qx\right), \tag{1.1}$$

where the basic hypergeometric series ${}_{r+1}\phi_r$ is defined by

$${}_{r+1}\phi_r\left(\begin{matrix} a_1, a_2, \ldots, a_{r+1} \\ b_1, b_2, \ldots, b_r \end{matrix}\,;q,z\right) := \sum_{n=0}^{\infty} \frac{(a_1;q)_n(a_2;q)_n\cdots(a_{r+1};q)_n}{(q;q)_n(b_1;q)_n\cdots(b_r;q)_n} z^n,$$

and where we use the notation

$$(A;q)_0 = 1; \quad (A;q)_n = (1-A)(1-Aq)\cdots(1-Aq^{n-1}), \quad n \geq 1,$$
$$(A;q)_1 = \lim_{n\to 1}(A;q)_n \quad (|q|<1).$$

The precise representation for the generating function of $\overline{p}_\omega(n)$ obtained in [3] is as follows.

Theorem 1.1. *The following identity holds for* $|q|<1$*:*

$$\begin{aligned}\overline{P}_\omega(q) &:= \sum_{n=1}^{\infty} \overline{p}_\omega(n) q^n \\ &= -\frac{1}{2}\frac{(q;q)_\infty (q;q^2)_\infty}{(-q;q)_\infty(-q;q^2)_\infty}\, {}_3\phi_2\left(\begin{matrix}-1, & iq^{1/2}, & -iq^{1/2} \\ q^{1/2}, & -q^{1/2} & \end{matrix};q,q\right) \\ &\quad + \frac{(-q;q)_\infty}{(q;q)_\infty}\sum_{n=0}^{\infty}\frac{(q;q^2)_n(-q)^n}{(-q;q^2)_n(1+q^{2n})}p_{2n}(-1;q^{-2n-1},-1:q). \qquad (1.2)\end{aligned}$$

Later, Bringmann, Jennings-Shaffer and Mahlburg [4, Theorem 1.1] showed that

$$\overline{P}_\omega(q) + \frac{1}{4} - \frac{\eta(4\tau)}{2\eta(2\tau)^2},$$

where $q = e^{2\pi i\tau}$ and $\eta(\tau)$ is the Dedekind eta function, can be completed to a function $\hat{P}_\omega(\tau)$, which transforms like a weight 1 modular form. They called the function

$$\overline{P}_\omega(q) + \frac{1}{4} - \frac{\eta(4\tau)}{2\eta(2\tau)^2}$$

a *higher depth mock modular form.*

While the series involving the little q-Jacobi polynomials in Theorem 1.1 itself looks formidable, it was shown in [3, Theorem 1.3] that it is a simple q-product modulo 4. The mod 4 congruence proved in there is given below.

Theorem 1.2. *The following congruence holds:*

$$\sum_{n=0}^{\infty}\frac{(q;q^2)_n(-q)^n}{(-q;q^2)_n(1+q^{2n})}p_{2n}(-1;q^{-2n-1},-1:q) \equiv \frac{1}{2}\frac{(q;q^2)_\infty}{(-q;q^2)_\infty} \pmod 4. \qquad (1.3)$$

The proof of this congruence in [3] is beautiful but somewhat involved. The objective of this short note is to give a very simple proof of it. In fact, we derive it as a corollary of the following result.

Theorem 1.3. *For* $|q|<1$*, we have*

$$\begin{aligned}&\sum_{n=0}^{\infty}\frac{(q;q^2)_n(-q)^n}{(-q;q^2)_n(1+q^{2n})}p_{2n}(-1;q^{-2n-1},-1:q) \\ &\quad = \frac{1}{2}\frac{(q;q^2)_\infty}{(-q;q^2)_\infty} + \frac{4q^2}{(1+q)}\sum_{n=0}^{\infty}\frac{(q^3;q^2)_n(-q)^n}{(-q^3;q^2)_n(1+q^{2n+2})}\sum_{j=0}^{n}\frac{(-q;q)_{2j}q^{2j}}{(q^2;q)_{2j}}. \qquad (1.4)\end{aligned}$$

The presence of 4 in front of the series on the right-hand side in the above equation immediately implies that Theorem 1.2 holds.

2. Proof of Theorem 1.3

Observe that from (1.1),

$$\sum_{n=0}^{\infty} \frac{(q;q^2)_n(-q)^n}{(-q;q^2)_n(1+q^{2n})} p_{2n}(-1;q^{-2n-1},-1:q)$$
$$= \sum_{n=0}^{\infty} \frac{(q;q^2)_n(-q)^n}{(-q;q^2)_n(1+q^{2n})} \sum_{j=0}^{2n} \frac{(-1;q)_j}{(q;q)_j}(-q)^j. \tag{2.1}$$

However, let us first consider

$$A(q) := \sum_{n=0}^{\infty} \frac{(q;q^2)_n(-q)^n}{(-q;q^2)_n(1+q^{2n})} \sum_{j=0}^{2n} \frac{(-1;q)_j}{(q;q)_j} q^j. \tag{2.2}$$

The only difference in the series on the right-hand side of (2.1) and the series in (2.2) is the presence of $(-1)^j$ inside the finite sum in the former.

To simplify $A(q)$, we start with a result of Alladi [1, p. 215, Equation (2.6)]:

$$\frac{(abq;q)_n}{(bq;q)_n} = 1 + b(1-a)\sum_{j=1}^{n} \frac{(abq;q)_{j-1}q^j}{(bq;q)_j}. \tag{2.3}$$

Let $a=-1, b=1$ and replace n by $2n$ so that

$$\sum_{j=0}^{2n} \frac{(-1;q)_j q^j}{(q;q)_j} = \frac{(-q;q)_{2n}}{(q;q)_{2n}}. \tag{2.4}$$

Substitute (2.4) in (2.2) to see that

$$A(q) = \sum_{n=0}^{\infty} \frac{(q;q^2)_n(-q)^n}{(-q;q^2)_n(1+q^{2n})} \frac{(-q;q)_{2n}}{(q;q)_{2n}}$$
$$= \frac{1}{2} + \sum_{n=1}^{\infty} \frac{(-q^2;q^2)_{n-1}}{(q^2;q^2)_n}(-q)^n$$
$$= \frac{1}{2}\frac{(q;q^2)_\infty}{(-q;q^2)_\infty}, \tag{2.5}$$

where in the last step we used the q-binomial theorem

$$\sum_{n=0}^{\infty} \frac{(a;q)_n}{(q;q)_n} z^n = \frac{(az;q)_\infty}{(z;q)_\infty},$$

valid for $|z|<1$ and $|q|<1$.

From (2.1) and (2.2),

$$\sum_{n=0}^{\infty} \frac{(q;q^2)_n(-q)^n}{(-q;q^2)_n(1+q^{2n})} p_{2n}(-1;q^{-2n-1},-1:q) - A(q)$$
$$= \sum_{n=0}^{\infty} \frac{(q;q^2)_n(-q)^n}{(-q;q^2)_n(1+q^{2n})} \sum_{j=0}^{2n}((-1)^j-1)\frac{(-1;q)_j q^j}{(q;q)_j}$$

$$= -2\sum_{n=0}^{\infty}\frac{(q;q^2)_n(-q)^n}{(-q;q^2)_n(1+q^{2n})}\sum_{j=1}^{n}\frac{(-1;q)_{2j-1}q^{2j-1}}{(q;q)_{2j-1}}$$
$$= \frac{4q^2}{(1+q)}\sum_{n=0}^{\infty}\frac{(q^3;q^2)_n(-q)^n}{(-q^3;q^2)_n(1+q^{2n+2})}\sum_{j=0}^{n}\frac{(-q;q)_{2j}q^{2j}}{(q^2;q)_{2j}}. \tag{2.6}$$

Invoking (2.5), we see that the proof of Theorem 1.3 is complete.

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Atul Dixit
Department of Mathematics
Indian Institute of Technology Gandhinagar
Palaj, Gandhinagar
Gujarat 382355
India
e-mail: adixit@iitgn.ac.in

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Check for updates

D.H. Lehmer's Tridiagonal Determinant: An Étude in (Andrews-Inspired) Experimental Mathematics

Dedicated to George Andrews on his 80th birthday.

Shalosh B. Ekhad and Doron Zeilberger

Abstract. We use George Andrews' "reverse-engineering" method to re-prove, using experimental mathematics, a conjecture of D.H. Lehmer.

Mathematics Subject Classification. 05A17.

Keywords. Partitions, Experimental mathematics, George Andrews, Ansatz.

1. Lehmer's Theorem and Its Finite Form

Define, with Lehmer [2, p. 54], $M(n) = M(n)(X, q)$, to be the following tridiagonal $n \times n$ matrix (we changed a to $\sqrt{X}$ and r to q):

$$M(n)_{i,j} = \begin{cases} 1, & \text{if} \quad i-j=0; \\ \sqrt{X}q^{(i-1)/2}, & \text{if} \quad i-j=-1; \\ \sqrt{X}q^{(i-2)/2}, & \text{if} \quad i-j=1; \\ 0, & \text{otherwise.} \end{cases}$$

Theorem 1.1. (Lehmer [2])

$$\lim_{n\to\infty} \det M(n)(X, q) = \sum_{a=0}^{\infty} \frac{(-1)^a X^a q^{a(a-1)}}{(1-q)(1-q^2)\cdots(1-q^a)}.$$

(As noted by Lehmer, when $X = -q$ and $X = -1$, one gets the sum sides of the famous Rogers–Ramanujan identities.)

Our new result is an explicit expression for the **finite form**, that immediately implies Lehmer's theorem, by taking the limit $n \to \infty$, and gives it a new (and shorter!) proof.

Theorem 1.2.

$$\det\ M(n)(X,q) = \sum_{a=0}^{\lfloor n/2 \rfloor} \frac{(-1)^a X^a q^{a(a-1)}(1-q^{n-a})(1-q^{n-a-1})\cdots(1-q^{n-2a+1})}{(1-q)(1-q^2)\cdots(1-q^a)}.$$

Proof. As noted by Lehmer [2, Eq. (3)], by expanding with respect to the last row, we have:

$$\det\ M(n)(X,q) = \det\ M(n-1)(X,q) - Xq^{n-2} \det\ M(n-2)(X,q). \qquad (LehmerRecurrence)$$

Using the q-Zeilberger algorithm[1]([4,7], see also [3] for a nice Mathematica version), we see that the right side of Theorem 1.2 also satisfies the very same recurrence. Since it holds for the **initial conditions** $n = 1$ and $n = 2$ (check!), the theorem follows by induction. □

2. Secrets from the Kitchen

Our paper could have ended here. We have increased human knowledge by extending a result of a famous number theorist, and proved it rigorously. *However*, at least as interesting as the **statement** of the theorem (and far more interesting than the **proof**) is the *way* it was *discovered*, and the rest of this paper will consist in describing **two** ways of doing it. The first way is a direct adaptation of George Andrews' "*reverse-engineering*" approach beautifully illustrated in the last chapter of his delightful booklet [1] (based on ten amazing lectures, given at Arizona State University, May 1985 that we were fortunate to attend). In that masterpiece (Section 10.2), he described how he used the computer algebra system SCRATCHPAD to prove a deep conjecture by three notable mathematicians: George Lusztig, Ian Macdonald, and C.T.C. Wall. In Andrews's approach, it is assumed that the discoverer knows about Gaussian polynomials, and knows how to spot them. In other words, the 'atoms' are Gaussian polynomials. In the second, more basic, approach, the only pre-requisite is the notion of *polynomials*, and Gaussian polynomials pop-up naturally in the act of discovery.

We will start completely *from scratch*, pretending that we did not read Lehmer's paper. In fact, we did not have to 'pretend'. We had no clue that Lehmer's paper existed until way after we discovered (and proved) Theorem 1.2, (and hence reproved Lehmer's Theorem 1.1). This is the time for a short "commercial break", since this paper (like so many other ones!) owes it existence to the OEIS.

[Start of commercial break.]

[1]Typing qzeil((-1)**a*X**a*q**(a*(a-1))*qbin(n-a,a),S,a,n,[]); in `qEKHAD` gives $Xq^n - S + S^2, Xq^n$ that is the recurrence operator annihilating the sum (S is the forward shift operator in n) followed by the 'certificate' (i.e., the proof, see [4])).

3. Serendipity and the OEIS

We learned about Lehmer's Theorem 1 via *serendipity*, thanks to that amazing tool that we are so lucky to have, the *On-Line Encyclopedia of Integer Sequences* [5] (OEIS).

Recall that a *composition* of n is an array of positive integers $(p_1, \ldots, p_k)$, such that $p_1 + \cdots + p_k = n$, and they are very easy to count (there are 2^{n-1} of them). A *partition* of n is a composition with the additional property that it is *weakly decreasing*, that is

$$p_i - p_{i+1} \geq 0 \quad (1 \leq i < k),$$

(and they are much harder to count).

My current Ph.D. student, Mingjia Yang [6], is investigating *relaxed partitions*, that she calls r-partitions, that are compositions of n with the condition:

$$p_i - p_{i+1} \geq r.$$

When $r = 1$, we get the familiar partitions into distinct parts, and when $r = 2$, we get one of the actors in the Rogers Ramanujan identities. However, what about negative r? In particular what about (-1)-partitions? After generating the first 20 terms:

$$1, 2, 4, 7, 13, 23, 41, 72, 127, 222, 388, 677, 1179, \\ 2052, 3569, 6203, 10778, 18722, 32513, 56455,$$

we copied-and-pasted it to the OEIS, and sure enough, we were *scooped*! It is sequence **A003116**, whose (former) description was 'reciprocal of an expansion of a determinant', that pointed to sequence **A039924**, mentioning Lehmer's Theorem 1.1 (in fact, the special case $X = q$). As a reference, it cited 'personal communication' by Herman P. Robinson, a friend and disciple of Lehmer. The OEIS entry for **A039924** also referenced Lehmer's "lecture notes on number theory", but we could not find it either on-line or off-line.

Since Lehmer's proof seemed to have been lost, we tried to prove it ourselves and succeeded. Our approach, inspired by Andrews' [1], was to first find an explicit expression for the *finite form*, and then take the limit as n goes to infinity (like Andrews did for the L–M–W conjecture). Only *after* we had the proof, we searched **MathSciNet** for

"Lehmer AND determinant AND tridiagonal",

and discovered [2]. To our relief, Lehmer's proof was longer than ours, and did not go via the finite form, Theorem 1.2. As far as we know, Theorem 1.2 is new. Once we discovered the reference [2], we notified Neil Sloane, and he added that reference to the relevant sequences **A003116** and **A039924**. Therefore, the present paper is yet another paper that owes its existence to the OEIS!

[End of commercial break.]

4. How the Statement of Theorem 1.2 Would Have Been (Easily!) Discovered by George Andrews

In Andrews's proof of the L–M–W conjecture, he used the *Gaussian polynomials* (aka *q-binomial coefficients*) as **building blocks**. With his approach, Theorem 1.2 could have been found by him fairly quickly. Let $Q_n(X,q) := \det M(n)(X,q)$.

Recall that the Gaussian polynomials $GP(m,n)(q)$ are defined by:

$$GP(m,n)(q) := \frac{(1-q^{m+1})(1-q^{m+2})\cdots(1-q^{m+n})}{(1-q)\cdots(1-q^n)}$$

(in spite of their appearance, they are *polynomials!*).

The way George Andrews would have discovered Theorem 1.2 is as follows.

Initially, crank out the first, say, 20 terms of the sequence of polynomials $Q_n(X,q)$, either by evaluating the determinants, or, more efficiently, via (*LehmerRecurrence*).

You do not need a computer to realize that the coefficient of X^0, i.e., the constant term, is always 1.

The coefficients of $X = X^1$ in $Q_n(X,q)$ for n from 1 to 8 are:

$$[0,-1,-1-q,-1-q-q^2,-1-q-q^2-q^3,-1-q-q^2-q^3-q^4,$$
$$-1-q-q^2-q^3-q^4-q^5,-1-q-q^2-q^3-q^4-q^5-q^6].$$

A quick glance by George Andrews would have made him conjecture that it is:

$$-GP(n-2,1)(q).$$

Moving right along, here are the coefficients of X^2 for $1 \le n :\le 10$:

$$[0,0,0,q^2,q^2+q^3+q^4,q^2+q^3+2q^4+q^5+q^6,$$
$$q^2+q^3+2q^4+2q^5+2q^6+q^7+q^8,$$
$$q^2+q^3+2q^4+2q^5+3q^6+2q^7+2q^8+q^{10}+q^9,$$
$$q^2+q^3+2q^4+2q^5+3q^6+3q^7+3q^8+q^{11}+2q^{10}+2q^9+q^{12},$$
$$q^2+q^3+2q^4+2q^5+3q^6+3q^7+4q^8+2q^{11}+3q^{10}+3q^9+2q^{12}+q^{13}+q^{14}].$$

Dividing by q^2 and checking against the Gaussian polynomials 'data base' suggests that the coefficient of X^2 is always:

$$q^2GP(n-4,2)(q).$$

Similarly, the coefficient of X^3 would have emerged as:

$$-q^6GP(n-6,3)(q).$$

The coefficient of X^4 would have emerged as:

$$q^{12}GP(n-8,4)(q).$$

The coefficient of X^5 would have emerged as:

$$-q^{20}GP(n-10,5)(q).$$

And *bingo*, it requires no great leap of an Andrews's imagination to conjecture that:

$$Q_n(X,q) = \sum_{a=0}^{\lfloor n/2 \rfloor} (-1)^a X^a q^{a(a-1)} GP(n-2a,a)(q),$$

that is identical to the statement of Theorem 1.2.

5. How the Statement of Theorem 1.2 Could Have Been Discovered by Someone Who is Not George Andrews?

Suppose that you have never heard of the Gaussian polynomials. You still could have conjectured the statement of Theorem 1.2. Even if you have never heard of *Gaussian* polynomials, you probably *did* hear of *polynomials*. Therefore, assuming the *ansatz* that, for each a, the coefficient of X^a is a certain polynomial in q^n, try and fit it with a 'generic' polynomial with *undetermined coefficients*.[2]

Setting $N = q^n$, your computer would have guessed the following polynomial expressions (in $N = q^n$) for the first five coefficients of X in $Q_n(X)$.

- The coefficient of X in $Q_n(X,q)$ is:
$$\frac{N-q}{q\,(1-q)}.$$
- The coefficient of X^2 in $Q_n(X,q)$ is:
$$\frac{\left(N-q^2\right)\left(N-q^3\right)}{q^3\,(1+q)\,(1-q)^2}.$$
- The coefficient of X^3 in $Q_n(X,q)$ is:
$$-\frac{\left(N-q^3\right)\left(N-q^4\right)\left(N-q^5\right)}{q^6\,(1+q)\,(q^2+q+1)\,(q-1)^3}.$$
- The coefficient of X^4 in $Q_n(X,q)$ is:
$$\frac{\left(N-q^4\right)\left(N-q^5\right)\left(N-q^6\right)\left(N-q^7\right)}{q^{10}\,(q^2+1)\,(q-1)^4\,(1+q)^2\,(q^2+q+1)}.$$
- The coefficient of X^5 in $Q_n(X,q)$ is:
$$-\frac{\left(N-q^5\right)\left(N-q^6\right)\left(N-q^7\right)\left(N-q^8\right)\left(N-q^9\right)}{q^{15}\,(q-1)^5\,(q^4+q^3+q^2+q+1)\,(1+q)^2\,(q^2+q+1)\,(q^2+1)}.$$

This immediately leads one to guess that the **numerator** is always:

$$(-1)^a\,(N-q^a)(N-q^{a+1})\cdots(N-q^{2a-1}).$$

[2] You start out with a generic polynomial of degree 0, and keep raising the degree until success (or failure).

On the other hand, the sequence of denominators, let us then call them $d(a)$, for $1 \le a \le 5$, happens to be:

$$[-q(q-1), q^3(1+q)(q-1)^2, -q^6(1+q)\left(q^2+q+1\right)(q-1)^3,$$
$$q^{10}\left(q^2+1\right)(q-1)^4(1+q)^2\left(q^2+q+1\right),$$
$$-q^{15}(q-1)^5\left(q^4+q^3+q^2+q+1\right)(1+q)^2\left(q^2+q+1\right)\left(q^2+1\right)].$$

This looks a bit complicated, but let us form the sequence of *ratios* $d(a)/d(a-1)$ for $a = 2, 3, 4, 5$ and expand, getting

$$[q^2 - q^4, q^3 - q^6, -q^8 + q^4, q^5 - q^{10}],$$

that is clearly $q^a(1-q^a)$. Hence, the coefficient of X^a in $Q_n(X, q)$ is guessed to be:

$$\frac{(-1)^a(N-q^a)(N-q^{a+1})\cdots(N-q^{2a-1})}{q^{a(a+1)/2}(1-q)\cdots(1-q^a)}.$$

By putting $N = q^n$, we get the statement of Theorem 1.2.

Therefore, with this second approach, we discovered the Gaussian polynomials *ab initio*, our only gamble was that the coefficients of X in $Q_n(X, q)$ are always *polynomials* in q^n.

6. Concluding Words

Let us quote the last sentence of Section 10.2 of [1], where Andrews described his pioneering (*experimental mathematics!*) approach illustrated by his discovery process of the proof of the L–M–W conjecture.

"From here the battle with the L–M–W conjecture is 90 % won. Standard techniques allow one to establish the [finite form] of the conjecture, and a simple argument leads to the original conjecture."

Today, the 90 % may be replaced by 99.999 %, since the final *verification* can be done automatically using the so-called q-Zeilberger algorithm [3,4,7]. In the much more difficult L–M–N case, this would have saved George Andrews a few hours, and would have made it accessible to anyone else. In the present case, you can still use the q-Zeilberger algorithm, if you are feeling lazy, but it is not too hard to do it purely humanly. Can **you** do it?

Added March 12, 2019: While the point of this article is *methodological* and *pedagogical*, we have to mention that our main result appears, in an equivalent form, in

K. Garrett, M. E. H. Ismail, and D. Stanton, *Variants of the Rogers–Ramanujan identities*, Adv. in Appl. Math. **23** (1999), 274–299.

See also

Mourad E. H. Ismail, Helmut Prodinger, Dennis Stanton, *Schur's Determinants and Partition Theorems* , Séminaire Lotharingien de Combinatoire, **B44a** (2000), 10 pp. https://www.mat.univie.ac.at/~slc/wpapers/s44ismail.html.

We thank Christian Krattenthaler for pointing it out.

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Shalosh B. Ekhad and Doron Zeilberger
Department of Mathematics
Rutgers University (New Brunswick)
Hill Center-Busch Campus
110 Frelinghuysen Rd.
Piscataway NJ08854-8019
USA

Doron Zeilberger
e-mail: doronzeil@gmail.com

Shalosh B. Ekhad
e-mail: ShaloshBEkhad@gmail.com

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Gaussian Binomial Coefficients with Negative Arguments

Dedicated to Professor George Andrews on the occasion of his eightieth birthday

Sam Formichella and Armin Straub

Abstract. Loeb showed that a natural extension of the usual binomial coefficient to negative (integer) entries continues to satisfy many of the fundamental properties. In particular, he gave a uniform binomial theorem as well as a combinatorial interpretation in terms of choosing subsets of sets with a negative number of elements. We show that all of this can be extended to the case of Gaussian binomial coefficients. Moreover, we demonstrate that several of the well-known arithmetic properties of binomial coefficients also hold in the case of negative entries. In particular, we show that Lucas' theorem on binomial coefficients modulo p not only extends naturally to the case of negative entries, but even to the Gaussian case.

Mathematics Subject Classification. 05A10, 05A30, 11B65, 11A07.

Keywords. q-Binomial coefficients, q-Binomial theorem, Lucas congruences.

1. Introduction

Occasionally, the binomial coefficient $\binom{n}{k}$, with integer entries n and k, is considered to be zero when $k < 0$ (see Remark 1.9, where it is further indicated that the common extension, via the gamma function, of binomial coefficients to complex n and k does not immediately lend itself to the case of negative integers k). However, as demonstrated by Loeb [14], an alternative extension of the binomial coefficients to negative arguments is arguably more natural for many combinatorial or number theoretic applications. The q-binomial coefficients $\binom{n}{k}_q$ (often also referred to as Gaussian polynomials) are a polynomial generalization of the binomial coefficients that occur naturally in varied contexts, including combinatorics, number theory, representation theory and mathematical physics. For instance, if q is a prime power, then they count the number of k-dimensional subspaces of an n-dimensional vector space over the finite field $\mathbb{F}_q$. We refer to the book [11] for a very pleasant introduction to the q-calculus. Yet, surprisingly, q-binomial coefficients with general integer entries have, to the best of our knowledge, not been studied in the literature (Gasper and Rahman define q-binomial coefficients with complex entries in [9, Ex. 1.2 and (I.40)], see Remark 1.8, but do not pursue the case of integer entries; Loeb [14] does briefly discuss such q-binomial coefficients but only in the case $k \geq 0$). The goal of this paper is to fill this gap, and to demonstrate that these generalized q-binomial coefficients are natural, by showing that they satisfy many of the fundamental combinatorial and arithmetic properties of the usual binomial coefficients. In particular, we extend Loeb's interesting combinatorial interpretation [14] in terms of sets with negative numbers of elements. On the arithmetic side, we prove that Lucas' theorem can be uniformly generalized to both binomial coefficients and q-binomial coefficients with negative entries.

In the context of q-series, it is common to introduce the q-binomial coefficient, for $n, k \geq 0$, as the quotient

$$\binom{n}{k}_q = \frac{(q;q)_n}{(q;q)_k (q;q)_{n-k}}, \tag{1.1}$$

where $(a;q)_n$ denotes the q-Pochhammer symbol

$$(a;q)_n = \prod_{j=0}^{n-1} (1 - aq^j), \quad n \geq 0. \tag{1.2}$$

In particular, $(a;q)_0 = 1$. It is not difficult to see that (1.1) reduces to the usual binomial coefficient in the limit $q \to 1$. In order to extend (1.1) to the case of negative integers n and k, we employ the natural convention that, for all integers r and s,

$$\prod_{j=r}^{s-1} a_j = \prod_{j=s}^{r-1} a_j^{-1}.$$

Applied to (1.2), we, therefore, define, as is common, that

$$(a;q)_{-n} = \prod_{j=1}^{n} \frac{1}{1 - aq^{-j}}, \quad n \geq 0. \tag{1.3}$$

With the above convention in place, both product formulas in (1.2) and (1.3) for the q-Pochhammer symbol are equivalent and hold for all integers n.

Note that $(q;q)_n = \infty$ whenever $n < 0$, so that (1.1) does not immediately extend to the case when n or k is negative. We, therefore, make the following definition, which clearly reduces to (1.1) when $n, k \geq 0$:

Definition 1.1. For all integers n and k,

$$\binom{n}{k}_q = \lim_{a \to q} \frac{(a;q)_n}{(a;q)_k (a;q)_{n-k}}. \tag{1.4}$$

Though not immediately obvious from (1.4) when n or k is negative, these generalized q-binomial coefficients are Laurent polynomials in q with integer coefficients. In particular, upon setting $q = 1$, we always obtain integers.

Example 1.2.

$$\binom{-3}{-5}_q = \lim_{a\to q} \frac{(a;q)_{-3}}{(a;q)_{-5}(a;q)_2} = \lim_{a\to q} \frac{\left(1-\frac{a}{q^4}\right)\left(1-\frac{a}{q^5}\right)}{(1-a)(1-aq)} = \frac{(1+q^2)(1+q+q^2)}{q^7}.$$

In Sect. 2, we observe that, for integers n and k, the q-binomial coefficients are also characterized by the Pascal relation:

$$\binom{n}{k}_q = \binom{n-1}{k-1}_q + q^k\binom{n-1}{k}_q, \tag{1.5}$$

provided that $(n,k) \neq (0,0)$ (this exceptional case excludes itself naturally in the proof of Lemma 2.1), together with the initial conditions

$$\binom{n}{0}_q = \binom{n}{n}_q = 1.$$

In the case $q = 1$, this extension of Pascal's rule to negative parameters was observed by Loeb [14, Proposition 4.4].

Among the other basic properties of the generalized q-binomial coefficients are the following: All of these are well known in the classical case $k \geq 0$ (see, for instance, [9, Appendix I]). That they extend uniformly to all integers n and k (though, as illustrated by (1.5) and item (c), some care has to be applied when generalizing certain properties) serves as a first indication that the generalized q-binomial coefficients are natural objects. For (c), the sign function $\operatorname{sgn}(k)$ is defined to be 1 if $k \geq 0$, and -1 if $k < 0$.

Lemma 1.3. *For all integers n and k,*

(a) $\binom{n}{k}_q = q^{k(n-k)}\binom{n}{k}_{q^{-1}}$,

(b) $\binom{n}{k}_q = \binom{n}{n-k}_q$,

(c) $\binom{n}{k}_q = (-1)^k \operatorname{sgn}(k) q^{\frac{1}{2}k(2n-k+1)}\binom{k-n-1}{k}_q$,

(d) $\binom{n}{k}_q = \frac{1-q^n}{1-q^k}\binom{n-1}{k-1}_q$, *if $k \neq 0$.*

Properties (b) and (d) follow directly from the definition (1.4), while property (a) is readily deduced from (1.5) combined with (b). In the classical case $n, k \geq 0$, property (a) reflects the fact that the q-binomial coefficient is a self-reciprocal polynomial in q of degree $k(n-k)$. In contrast to that and as illustrated in Example 1.2, the q-binomial coefficients with negative entries are Laurent polynomials, whose degrees are recorded in Corollary 3.3.

The reflection rule (c) is the subject of Sect. 3 and is proved in Theorem 3.1. Rule (c) reduced to the case $q = 1$ is the main object in [19], where Sprugnoli observed the necessity of including the sign function when extending the binomial coefficient to negative entries. Sprugnoli further realized that the basic symmetry (b) and the negation rule (c) act on binomial coefficients as a group of transformations isomorphic to the symmetric group on three letters. In Sect. 3, we observe that the same is true for q-binomial coefficients.

Note that property (d), when combined with (b), implies that, for $n \neq k$,

$$\binom{n}{k}_q = \frac{1-q^n}{1-q^{n-k}} \binom{n-1}{k}_q.$$

In particular, the q-binomial coefficient is a q-hypergeometric term.

Example 1.4. It follows from Lemma 1.3(c) that, for all integers k,

$$\binom{-1}{k}_q = (-1)^k \operatorname{sgn}(k) \frac{1}{q^{k(k+1)/2}}.$$

In Sect. 4, we review the remarkable and beautiful observation of Loeb [14] that the combinatorial interpretation of binomial coefficients as counting subsets can be naturally extended to the case of negative entries. We then prove that this interpretation can be generalized to q-binomial coefficients. Theorem 4.5, our main result of that section, is a precise version of the following:

Theorem 1.5. *For all integers n and k,*

$$\binom{n}{k}_q = \pm \sum_Y q^{\sigma(Y) - k(k-1)/2},$$

where the sum is over all k-element subsets Y of the n-element set X_n.

The notion of sets (and subsets) with a negative number of elements, as well as the definitions of σ and X_n, are deferred to Sect. 4. In the previously known classical case $n, k \geq 0$, the sign in that formula is positive, $X_n = \{0, 1, 2, \ldots, n-1\}$, and $\sigma(Y)$ is the sum of the elements of Y. As an application of Theorem 1.5, we demonstrate at the end of Sect. 4 how to deduce from it generalized versions of the Chu–Vandermonde identity as well as the (commutative) q-binomial theorem.

In Sect. 5, we discuss the binomial theorem, which interprets the binomial coefficients as coefficients in the expansion of $(x + y)^n$. Loeb showed that, by also considering expansions in inverse powers of x, one can extend this interpretation to the case of binomial coefficients with negative entries. Once more, we are able to show that the generalized q-binomial coefficients share this property in a uniform fashion.

Theorem 1.6. *Suppose that $yx = qxy$. Then, for all integers n, k,*

$$\binom{n}{k}_q = \{x^k y^{n-k}\}(x+y)^n.$$

Here, the operator $\{x^k y^{n-k}\}$, which is defined in Sect. 5, extracts the coefficient of $x^k y^{n-k}$ in the appropriate expansion of what follows.

A famous theorem of Lucas [15] states that, if p is a prime, then

$$\binom{n}{k} \equiv \binom{n_0}{k_0}\binom{n_1}{k_1}\cdots\binom{n_d}{k_d} \pmod{p},$$

where n_i and k_i are the p-adic digits of the nonnegative integers n and k, respectively. In Sect. 6, we show that this congruence in fact holds for all integers n and k. In Sect. 7, we prove that generalized Lucas congruences uniformly hold for q-binomial coefficients.

Theorem 1.7. *Let $m \geq 2$ be an integer. Then, for all integers n and k,*

$$\binom{n}{k}_q \equiv \binom{n_0}{k_0}_q \binom{n'}{k'} \pmod{\Phi_m(q)},$$

where $n = n_0 + n'm$ and $k = k_0 + k'm$ with $n_0, k_0 \in \{0, 1, \ldots, m-1\}$.

Here, $\Phi_m(q)$ is the m-th cyclotomic polynomial. The classical special case $n, k \geq 0$ of this result has been obtained by Olive [16] and Désarménien [7].

We conclude this introduction with some comments on alternative approaches to and conventions for binomial coefficients with negative entries. In particular, we remark on the current state of computer algebra systems and on how it could benefit from the generalized q-binomial coefficients introduced in this paper.

Remark 1.8. Using the gamma function, binomial coefficients are commonly introduced as

$$\binom{n}{k} = \frac{\Gamma(n+1)}{\Gamma(k+1)\Gamma(n-k+1)} \tag{1.6}$$

for all complex n and k such that $n, k \notin \{-1, -2, \ldots\}$. This definition, however, does not immediately lend itself to the case of negative integers; though the structure of poles (and lack of zeros) of the underlying gamma function is well understood, the binomial function (1.6) has a subtle structure when viewed as a function of two variables. For a study of this function, as well as a historical account on binomials, we refer to [8]. For instance, let us note that, employing (1.6) as the definition of the binomial coefficients, we have

$$\lim_{\varepsilon \to 0} \binom{-3+\varepsilon}{-5+r\varepsilon} = \frac{1}{2} \lim_{\varepsilon \to 0} \frac{\Gamma(-2+\varepsilon)}{\Gamma(-4+r\varepsilon)} = 6r,$$

where the final equality follows because, for integers $n \geq 0$,

$$\Gamma(-n+\varepsilon) = \frac{(-1)^n}{n!}\frac{1}{\varepsilon} + O(1)$$

as $\varepsilon \to 0$. This illustrates that the values of the binomial coefficients at negative integers cannot be defined by simply appealing to (1.6) and continuity. A natural way to extend (1.6) to negative integers is to set

$$\binom{n}{k} = \lim_{\varepsilon \to 0} \frac{\Gamma(n+1+\varepsilon)}{\Gamma(k+1+\varepsilon)\Gamma(n-k+1+\varepsilon)}, \tag{1.7}$$

where n and k are now allowed to take any complex values. This is in fact the definition that Loeb [14] and Sprugnoli [19] adopt. (That the q-binomial coefficients we introduce in (1.4) reduce to (1.7) when $q = 1$ can be seen, for instance, from observing that the q-Pascal relation (1.5) reduces to the Pascal relation established by Loeb for (1.7).)

Similarly, Gasper and Rahman [9, Appendix I] define the q-binomial coefficient for complex arguments n and k (and $|q| < 1$) using the q-gamma function as

$$\binom{n}{k}_q = \frac{\Gamma_q(n+1)}{\Gamma_q(k+1)\Gamma_q(n-k+1)} = \frac{(q^{k+1};q)_\infty (q^{n-k+1};q)_\infty}{(q;q)_\infty (q^{n+1};q)_\infty}, \tag{1.8}$$

where $(a;q)_\infty = \lim_{n\to\infty}(a;q)_n$. We note that definition (1.8) is equivalent to extending (1.1) by defining

$$(a;q)_n = \frac{(a;q)_\infty}{(aq^n;q)_\infty} \tag{1.9}$$

for complex values of n (for negative integers n, formula (1.9) is compatible with (1.3)). When n or k is a negative integer, however, the right-hand side of (1.8) must be interpreted appropriately by cancelling matching zeros in the infinite products. Interpreting (1.8) in this way, it follows from (1.9) that definition (1.8) is necessarily equivalent to (1.4).

Remark 1.9. Other conventions for binomial coefficients with negative integer entries exist and have their merit. Most prominently, if, for instance, one insists that Pascal's relation (1.5) should hold for all integers n and k, then the resulting version of the binomial coefficients is zero when $k < 0$ (see, for instance, [12, Section 1.2.6 (3)]). On the other hand, as illustrated by the results in [14] and this paper, it is reasonable and preferable for many purposes to extend the classical binomial coefficients (as well as its polynomial counterpart) to negative arguments as done here.

As an unfortunate consequence, both conventions are implemented in current computer algebra systems, which can be a source of confusion. For instance, SageMath currently (as of Version 8.0) uses the convention that all binomial coefficients with $k < 0$ are evaluated as zero. On the other hand, recent versions of Mathematica (at least Version 9 and higher) and Maple (at least Version 18 and higher) evaluate binomial coefficients with negative entries in the way advertised in [14] and here.

In Version 7, Mathematica introduced the `QBinomial[n,k,q]` function; however, as of Version 11, this function evaluates the q-binomial coefficient as zero whenever $k < 0$. Similarly, Maple implements these coefficients as `QBinomial(n,k,q)`, but, as of Version 18, choosing $k < 0$ results in a division-by-zero error. We hope that this paper helps to adjust these inconsistencies with the classical case $q = 1$ by offering a natural extension of the q-binomial coefficient for negative entries.

2. Characterization via a q-Pascal Relation

The generalization of the binomial coefficients to negative entries by Loeb satisfies Pascal's rule

$$\binom{n}{k} = \binom{n-1}{k-1} + \binom{n-1}{k} \tag{2.1}$$

for all integers n and k that are not both zero [14, Proposition 4.4]. In this brief section, we show that the q-binomial coefficients (with arbitrary integer entries), defined in (1.4), are also characterized by a q-analog of the Pascal rule. It is well known that this is true for the familiar q-binomial coefficients when $n, k \geq 0$ (see, for instance, [11, Proposition 6.1]).

Lemma 2.1. *For integers n and k, the q-binomial coefficients are characterized by*

$$\binom{n}{k}_q = \binom{n-1}{k-1}_q + q^k \binom{n-1}{k}_q, \tag{2.2}$$

provided that $(n, k) \neq (0, 0)$, together with the initial conditions

$$\binom{n}{0}_q = \binom{n}{n}_q = 1.$$

Observe that $\binom{0}{0}_q = 1$, while the corresponding right-hand side of (2.2) is $\binom{-1}{-1}_q + q^0 \binom{-1}{0}_q = 2 \neq 1$, illustrating the need to exclude the case $(n, k) = (0, 0)$. It should also be noted that the initial conditions are natural but not minimal: the case $\binom{n}{0}_q$ with $n \leq -2$ is redundant (but consistent).

Proof of Lemma 2.1. We note that the relation (2.2) and the initial conditions indeed suffice to deduce values for each q-binomial coefficient. It, therefore, only remains to show that (2.2) holds for the q-binomial coefficient as defined in (1.4). For the purpose of this proof, let us write

$$\binom{n}{k}_{a,q} = \frac{(a;q)_n}{(a;q)_k (a;q)_{n-k}},$$

and observe that, for all integers n and k,

$$\binom{n-1}{k}_{a,q} = \frac{1 - aq^{n-k-1}}{1 - aq^{n-1}} \binom{n}{k}_{a,q}$$

as well as

$$\binom{n-1}{k-1}_{a,q} = \frac{1 - aq^{k-1}}{1 - aq^{n-1}} \binom{n}{k}_{a,q}.$$

It then follows that

$$\binom{n}{k}_{a,q} = \binom{n-1}{k-1}_{a,q} + aq^{k-1} \frac{1 - q^{n-k}}{1 - aq^{n-k-1}} \binom{n-1}{k}_{a,q} \tag{2.3}$$

for all integers n and k. If $n \neq k$, then

$$\lim_{a\to q}\left[aq^{k-1}\frac{1-q^{n-k}}{1-aq^{n-k-1}}\right]=q^k,$$

so that (2.2) follows for these cases. On the other hand, if $n=k$, then $\binom{n-1}{k}_q=0$, provided that $(n,k)\neq(0,0)$, so that (2.2) also holds in the remaining cases. □

Remark 2.2. Applying Pascal's relation (2.2) to the right-hand side of Lemma 1.3(b), followed by applying the symmetry Lemma 1.3(b) to each q-binomial coefficient, we find that Pascal's relation (2.2) is equivalent to the alternative form

$$\binom{n}{k}_q=q^{n-k}\binom{n-1}{k-1}_q+\binom{n-1}{k}_q. \tag{2.4}$$

3. Reflection Formulas

In [19], Sprugnoli, likely unaware of the earlier work of Loeb [14], introduces binomial coefficients with negative entries via the gamma function (see Remark 1.8). Sprugnoli then observes that the familiar negation rule

$$\binom{n}{k}=(-1)^k\binom{k-n-1}{k},$$

as stated, for instance, in [12, Section 1.2.6], does not continue to hold when k is allowed to be negative. Instead, he shows that, for all integers n and k,

$$\binom{n}{k}=(-1)^k\operatorname{sgn}(k)\binom{k-n-1}{k}, \tag{3.1}$$

where $\operatorname{sgn}(k)=1$ for $k\geq 0$ and $\operatorname{sgn}(k)=-1$ for $k<0$. We generalize this result to the q-binomial coefficients. Observe that the result of Sprugnoli [19] is immediately obtained as the special case $q=1$.

Theorem 3.1. *For all integers n and k,*

$$\binom{n}{k}_q=(-1)^k\operatorname{sgn}(k)q^{\frac{1}{2}k(2n-k+1)}\binom{k-n-1}{k}_q. \tag{3.2}$$

Proof. Let us begin by observing that, for all integers n and k,

$$(a;q)_n(aq^n;q)_k=(a;q)_{n+k}. \tag{3.3}$$

Further, for all integers n,

$$(a;q)_n=(-a)^nq^{n(n-1)/2}(q^{-n+1}/a;q)_n. \tag{3.4}$$

Applying (3.3) and then (3.4), we find that

$$\frac{(a;q)_n}{(a;q)_{n-k}}=\frac{1}{(aq^n;q)_{-k}}=\frac{(-a)^kq^{\frac{1}{2}k(2n-k-1)}}{(q^{k-n+1}/a;q)_{-k}}. \tag{3.5}$$

By another application of (3.3),

$$\frac{1}{(q^{k-n+1}/a;q)_{-k}} = \frac{(1/a;q)_{k-n+1}}{(1/a;q)_{-n+1}} = \frac{(q^2/a;q)_{k-n-1}}{(q^2/a;q)_{-n-1}}, \tag{3.6}$$

where, for the second equality, we used the basic relation $(a;q)_n = (1-a)(aq;q)_{n-1}$ twice for each Pochhammer symbol. Combining (3.5) and (3.6), we thus have

$$\frac{(a;q)_n}{(a;q)_{n-k}} = (-a)^k q^{\frac{1}{2}k(2n-k-1)} \frac{(q^2/a;q)_{k-n-1}}{(q^2/a;q)_{-n-1}}$$

for all integers n and k. Suppose we have already shown that, for any integer n,

$$\lim_{a\to q} \frac{(q^2/a;q)_n}{(a;q)_n} = \operatorname{sgn}(n). \tag{3.7}$$

Then,

$$\begin{aligned}
\binom{n}{k}_q &= \lim_{a\to q} \frac{(a;q)_n}{(a;q)_k (a;q)_{n-k}} \\
&= \lim_{a\to q} (-a)^k q^{\frac{1}{2}k(2n-k-1)} \frac{(q^2/a;q)_{k-n-1}}{(a;q)_k (q^2/a;q)_{-n-1}} \\
&= \operatorname{sgn}(k-n-1)\operatorname{sgn}(-n-1) \\
&\quad \times \lim_{a\to q} (-a)^k q^{\frac{1}{2}k(2n-k-1)} \frac{(a;q)_{k-n-1}}{(a;q)_k (a;q)_{-n-1}} \\
&= (-1)^k \operatorname{sgn}(k) q^{\frac{1}{2}k(2n-k+1)} \binom{k-n-1}{k}_q .
\end{aligned}$$

For the final equality, we used that

$$\operatorname{sgn}(k-n-1)\operatorname{sgn}(-n-1) = \operatorname{sgn}(k),$$

whenever the involved q-binomial coefficients are different from zero (for more details on this argument, see [19, Theorem 2.2]).

It remains to prove (3.7). The limit clearly is 1 if $n \geq 0$. On the other hand, if $n < 0$, then

$$\begin{aligned}
\lim_{a\to q} \frac{(q^2/a;q)_n}{(a;q)_n} &= \lim_{a\to q} \frac{\left(1-\frac{a}{q}\right)\left(1-\frac{a}{q^2}\right)\cdots\left(1-\frac{a}{q^n}\right)}{\left(1-\frac{q}{a}\right)\left(1-\frac{1}{a}\right)\cdots\left(1-\frac{1}{aq^{n-2}}\right)} \\
&= \lim_{a\to q} \frac{\left(1-\frac{a}{q}\right)}{\left(1-\frac{q}{a}\right)} = -1,
\end{aligned}$$

as claimed. □

It was observed in [19, Theorem 3.2] that the basic symmetry (Lemma 1.3(b)) and the negation rule (3.2) act on (formal) binomial coefficients as a group of transformations isomorphic to the symmetric group on three letters.

The same is true for q-binomial coefficients. Since the argument is identical, we only record the resulting six forms for the q-binomial coefficients.

Corollary 3.2. *For all integers n and k,*

$$\begin{aligned}\binom{n}{k}_q &= \binom{n}{n-k}_q \\ &= (-1)^{n-k}\operatorname{sgn}(n-k)q^{\frac{1}{2}(n(n+1)-k(k+1))}\binom{-k-1}{n-k}_q \\ &= (-1)^{n-k}\operatorname{sgn}(n-k)q^{\frac{1}{2}(n(n+1)-k(k+1))}\binom{-k-1}{-n-1}_q \\ &= (-1)^{k}\operatorname{sgn}(k)q^{\frac{1}{2}k(2n-k+1)}\binom{k-n-1}{-n-1}_q \\ &= (-1)^{k}\operatorname{sgn}(k)q^{\frac{1}{2}k(2n-k+1)}\binom{k-n-1}{k}_q.\end{aligned}$$

Proof. These equalities follow from alternately applying the basic symmetry from Lemma 1.3(b) and the negation rule (3.2). Moreover, for the fourth equality, we use that

$$-\operatorname{sgn}(n-k)\operatorname{sgn}(-n-1) = \operatorname{sgn}(k)$$

whenever the involved q-binomial coefficients are different from zero (again, see [19, Theorem 2.2] for more details on this argument). □

It follows directly from the definition (1.4) that the q-binomial coefficient $\binom{n}{k}_q$ is zero if $k > n \geq 0$ or if $n \geq 0 > k$. The third equality in Corollary 3.2 then makes it plainly visible that the q-binomial coefficient also vanishes if $0 > k > n$. Moreover, we can read off from Corollary 3.2 that the q-binomial coefficient is nonzero otherwise, that is, it is nonzero precisely in the three regions $0 \leq k \leq n$ (the classical case), $n < 0 \leq k$ and $k \leq n < 0$. More precisely, we have the following, of which the first statement is, of course, well known (see, for instance, [11, Corollary 6.1]).

Corollary 3.3. (a) *If $0 \leq k \leq n$, then $\binom{n}{k}_q$ is a polynomial of degree $k(n-k)$.*

(b) *If $n < 0 \leq k$, then $\binom{n}{k}_q$ is $q^{\frac{1}{2}k(2n-k+1)}$ times a polynomial of degree $k(-n-1)$.*

(c) *If $k \leq n < 0$, then $\binom{n}{k}_q$ is $q^{\frac{1}{2}(n(n+1)-k(k+1))}$ times a polynomial of degree $(-n-1)(n-k)$.*

In each case, the polynomials are self-reciprocal and have integer coefficients.

Observe that Corollary 3.2 together with the defining product (1.1), spelled out as

$$\binom{n}{k}_q = \frac{(1-q^{k+1})(1-q^{k+2})\cdots(1-q^n)}{(1-q)(1-q^2)\cdots(1-q^{n-k})},$$

and valid when $0 \leq k \leq n$, provides explicit product formulas for all choices of n and k. Indeed, the three regions in which the binomial coefficients are nonzero are $0 \leq k \leq n$, $n < 0 \leq k$ and $k \leq n < 0$, and these three are permuted by the transformations in Corollary 3.2.

4. Combinatorial Interpretation

For integers $n, k \geq 0$, the binomial coefficient $\binom{n}{k}$ counts the number of k-element subsets of a set with n elements. It is a remarkable and beautiful observation of Loeb [14] that this interpretation (up to an overall sign) can be extended to all integers n and k by a natural notion of sets with a negative number of elements. In this section, after briefly reviewing Loeb's result, we generalize this combinatorial interpretation to the case of q-binomial coefficients.

Let U be a collection of elements (the "universe"). A set X with elements from U can be thought of as a map $M_X : U \to \{0, 1\}$ with the understanding that $u \in X$ if and only if $M_X(u) = 1$. Similarly, a multiset X can be thought of as a map

$$M_X : U \to \{0, 1, 2, \ldots\},$$

in which case $M_X(u)$ is the multiplicity of an element u. In this spirit, Loeb introduces a *hybrid set* X as a map $M_X : U \to \mathbb{Z}$. We will denote hybrid sets in the form $\{\ldots | \ldots\}$, where elements with a positive multiplicity are listed before the bar, and elements with a negative multiplicity after the bar.

Example 4.1. The hybrid set $\{1, 1, 4|2, 3, 3\}$ contains the elements $1, 2, 3, 4$ with multiplicities $2, -1, -2, 1$, respectively.

A hybrid set Y is a subset of a hybrid set X, if one can repeatedly remove elements from X (here, removing means decreasing by one the multiplicity of an element with nonzero multiplicity) and thus obtain Y or have removed Y. We refer to [14] for a more formal definition and further discussion, including a proof that this notion of being a subset is a well-defined partial order (but not a lattice). The interested reader will find there also connections to symmetric functions and, in particular, the involutive relation between elementary and complete symmetric functions.

Example 4.2. From the hybrid set $\{1, 1, 4|2, 3, 3\}$ we can remove the element 4 to obtain $\{1, 1|2, 3, 3\}$ (at which point, we cannot remove 4 again). We can further remove 2 twice to obtain $\{1, 1|2, 2, 2, 3, 3\}$. Consequently, $\{4|\}$ and $\{1, 1|2, 3, 3\}$ as well as $\{2, 2, 4|\}$ and $\{1, 1|2, 2, 2, 3, 3\}$ are subsets of $\{1, 1, 4|2, 3, 3\}$.

Following [14], a *new set* is a hybrid set such that either all multiplicities are 0 or 1 (a "positive set") or all multiplicities are 0 or -1 (a "negative set").

Theorem 4.3 [14]. *For all integers n and k, the number of k-element subsets of an n-element new set is $\left|\binom{n}{k}\right|$.*

Example 4.4. Consider the new set $\{|-1,-2,-3\}$ with -3 elements (the reason for choosing the elements to be negative numbers will become apparent when we revisit this example in Example 4.7). Its 2-element subsets are

$$\{-1,-1|\},\quad \{-1,-2|\},\quad \{-1,-3|\},\quad \{-2,-2|\},\quad \{-2,-3|\},\quad \{-3,-3|\},$$

so that $\left|\binom{-3}{2}\right| = 6$. On the other hand, its -4-element subsets are

$$\{|-1,-1,-2,-3\},\quad \{|-1,-2,-2,-3\},\quad \{|-1,-2,-3,-3\},$$

so that $\left|\binom{-3}{-4}\right| = 3$.

Let X_n denote the standard new set with n elements, by which we mean $X_n = \{0,1,\ldots,n-1|\}$, if $n \geq 0$, and $X_n = \{|-1,-2,\ldots,n\}$, if $n < 0$. For a hybrid set $Y \subseteq X_n$ with multiplicity function M_Y, we write

$$\sigma(Y) = \sum_{y\in Y} M_Y(y)y.$$

Note that, if Y is a classic set, then $\sigma(Y)$ is just the sum of its elements. With this setup, we prove the following uniform generalization of [14, Theorem 5.2], which is well known in the case that $n,k \geq 0$ (see, for instance, [11, Theorem 6.1]):

Theorem 4.5. *For all integers n and k,*

$$\binom{n}{k}_q = \varepsilon \sum_{Y} q^{\sigma(Y)-k(k-1)/2}, \quad \varepsilon = \pm 1, \tag{4.1}$$

where the sum is over all k-element subsets Y of the n-element set X_n. If $0 \leq k \leq n$, then $\varepsilon = 1$. If $n < 0 \leq k$, then $\varepsilon = (-1)^k$. If $k \leq n < 0$, then $\varepsilon = (-1)^{n-k}$.

Proof. The case $n,k \geq 0$ is well known. A proof can be found, for instance, in [11, Theorem 6.1]. On the other hand, if $n \geq 0 > k$, then both sides vanish.

Let us consider the case $n < 0 \leq k$. It follows from the reflection formula (3.2) that (4.1) is equivalent to the (arguably cleaner, but less uniform because restricted to $n < 0 \leq k$) identity

$$\binom{k-n-1}{k}_q = \sum_{Y\in C(n,k)} q^{\sigma(Y)}, \tag{4.2}$$

where $C(n,k)$ is the collection of k-element subsets of the n-element set $X_n^+ = \{|0,1,2,\ldots,|n|-1\}$ (note that a natural bijection $X_n \to X_n^+$ is given by $x \mapsto |n| + x$).

Fix n, k and suppose that (4.2) holds whenever n and k are replaced with n' and k' such that $n < n' < 0$ or $n = n' < 0 \le k' < k$. Then,

$$\begin{aligned}\sum_{Y\in C(n,k)} q^{\sigma(Y)} &= \sum_{\substack{Y\in C(n,k)\\ -n-1\notin Y}} q^{\sigma(Y)} + \sum_{\substack{Y\in C(n,k)\\ -n-1\in Y}} q^{\sigma(Y)}\\ &= \sum_{Y\in C(n+1,k)} q^{\sigma(Y)} + \sum_{Y\in C(n,k-1)} q^{\sigma(Y)-n-1}\\ &= \binom{k-n-2}{k}_q + q^{-n-1}\binom{k-n-2}{k-1}_q\\ &= \binom{k-n-1}{k}_q,\end{aligned}$$

where the last equality follows from Pascal's relation in the form (2.4). Since (4.2) holds trivially if $n = -1$ or if $k = 0$, it, therefore, follows by induction that (4.2) is true whenever $n < 0 \le k$.

Finally, consider the case $n, k < 0$. It is clear that both sides vanish unless $k \le n < 0$. By the third equality in Corollary 3.2,

$$\binom{n}{k}_q = (-1)^{n-k} q^{\frac{1}{2}(n(n+1)-k(k+1))}\binom{-k-1}{-n-1}_q,$$

so that (4.1) becomes equivalent to

$$\binom{-k-1}{-n-1}_q = \sum_{Y\in D(n,k)} q^{\sigma(Y)+k-n(n+1)/2}, \tag{4.3}$$

where $D(n,k)$ is the collection of k-element subsets Y of the n-element set $X_n = \{|-1,-2,\ldots,n\}$. If $n = -1$, then (4.3) holds because the only contribution comes from $Y = \{|-1,-1,\ldots,-1\}$, with $M_Y(-1) = |k|$ and $\sigma(Y) = -k$. If, on the other hand, $k = -1$, then (4.3) holds as well because a contributing Y only exists if $n = -1$. Fix $n, k < -1$ and suppose that (4.3) holds whenever n and k are replaced with n' and k' such that $k < k' < 0$ and $n \le n' < 0$. Then the right-hand side of (4.3) equals

$$\sum_{\substack{Y\in D(n,k)\\ M_Y(n)=-1}} q^{\sigma(Y)+k-n(n+1)/2} + \sum_{\substack{Y\in D(n,k)\\ M_Y(n)<-1}} q^{\sigma(Y)+k-n(n+1)/2}.$$

We now remove the element n from Y (once) and, to make up for that, replace $\sigma(Y)$ with $\sigma(Y) - n$. Proceeding this way, we see that the right-hand side of (4.3) equals

$$\begin{aligned}&\sum_{Y\in D(n+1,k+1)} q^{\sigma(Y)+k+1-(n+1)(n+2)/2} + q^{-n-1}\sum_{Y\in D(n,k+1)} q^{\sigma(Y)+k+1-n(n+1)/2}\\ &\quad= \binom{-k-2}{-n-2}_q + q^{-n-1}\binom{-k-2}{-n-1}_q\\ &\quad= \binom{-k-1}{-n-1}_q,\end{aligned}$$

with the final equality following from Pascal's relation (2.2). We conclude, by induction, that (4.3) is true for all $n, k < 0$. $\square$

Remark 4.6. The number of possibilities to choose k elements from a set of n elements with replacement is

$$\binom{k+n-1}{k} = \binom{k+n-1}{n-1}.$$

The usual "trick" to arrive at this count is to encode each choice of k elements by lining them up in some order with elements of the same kind separated by dividers (since there are n kinds of elements, we need $n-1$ dividers). The $n-1$ positions of the dividers among all $k+n-1$ positions then encode a choice of k elements. Formula (4.2) is a q-analog of this combinatorial fact.

Example 4.7. Let us revisit and refine Example 4.4, which concerns subsets of $X_{-3} = \{|-1,-2,-3\}$. Letting $k = 2$, the 2-element subsets have element-sums

$$\begin{aligned} \sigma(\{-1,-1|\}) &= -2, \quad & \sigma(\{-1,-2|\}) &= -3, \\ \sigma(\{-1,-3|\}) &= -4, \quad & \sigma(\{-2,-2|\}) &= -4, \\ \sigma(\{-2,-3|\}) &= -5, \quad & \sigma(\{-3,-3|\}) &= -6. \end{aligned}$$

Subtracting $k(k-1)/2 = 1$ from these sums to obtain the weight, we find

$$\binom{-3}{2}_q = q^{-3} + q^{-4} + 2q^{-5} + q^{-6} + q^{-7}.$$

Next, let us consider the case $k = -4$. The -4-element subsets have element-sums

$$\begin{aligned} \sigma(\{|-1,-1,-2,-3\}) &= 7, \quad & \sigma(\{|-1,-2,-2,-3\}) &= 8, \\ & & \sigma(\{|-1,-2,-3,-3\}) &= 9. \end{aligned}$$

Subtracting $k(k-1)/2 = 10$ from these sums and noting that $(-1)^{n-k} = -1$, we conclude that

$$\binom{-3}{-4}_q = -(q^{-3} + q^{-2} + q^{-1}).$$

In the remainder of this section, we consider two applications of Theorem 4.5. The first of these is the following extension of the classical Chu–Vandermonde identity:

Lemma 4.8. *For all integers n, m and k, with $k \geq 0$,*

$$\sum_{j=0}^{k} q^{(k-j)(n-j)} \binom{n}{j}_q \binom{m}{k-j}_q = \binom{n+m}{k}_q. \tag{4.4}$$

Proof. Throughout this proof, if Y is a k-element set, write $\tau(Y) = \sigma(Y) - k(k-1)/2$.

Suppose $n, m \geq 0$. Let Y_1 be a j-element subset of X_n, and Y_2 a $(k-j)$-element subset of X_m. Let $Y_2' = \{y + n : y \in Y_2\}$, so that $Y = Y_1 \cup Y_2'$ is a k-element subset of X_{n+m}. Then, since

$$\sigma(Y) = \sigma(Y_1) + \sigma(Y_2') = \sigma(Y_1) + \sigma(Y_2) + (k-j)n,$$

we have

$$\tau(Y) = \tau(Y_1) + \tau(Y_2) + (k-j)(n-j).$$

Then this follows from Theorem 4.5 because

$$\binom{j}{2} + \binom{k-j}{2} - \binom{k}{2} + (k-j)n = (k-j)(n-j).$$

□

Similarly, one can deduce from Theorem 4.5 the following version for the case when k is a negative integer. It also holds if $n, m \geq 0$, but the identity does not generally hold in the case when n and m have mixed signs.

Lemma 4.9. *For all negative integers n, m and k,*

$$\sum_{j \in \{-1,-2,\ldots,k+1\}} q^{(k-j)(n-j)} \binom{n}{j}_q \binom{m}{k-j}_q = \binom{n+m}{k}_q.$$

As another application of the combinatorial characterization in Theorem 4.5, we readily obtain the following identity. In the case $n \geq 0$, this identity is well known and often referred to as the (commutative version of the) q-binomial theorem (in which case the sum only extends over $k = 0, 1, \ldots, n$). We will discuss the noncommutative q-binomial theorem in the next section.

Theorem 4.10. *For all integers n,*

$$(-x; q)_n = \sum_{k \geq 0} q^{k(k-1)/2} \binom{n}{k}_q x^k.$$

Proof. Suppose that $n \geq 0$, so that

$$(-x; q)_n = (1+x)(1+xq)\cdots(1+xq^{n-1}). \tag{4.5}$$

Let, as before, $X_n = \{0, 1, \ldots, n-1|\}$. To each subset $Y \subseteq X_n$ we associate the product of the terms xq^y with $y \in Y$ in the expansion of (4.5). This results in

$$(-x; q)_n = \sum_{Y \subseteq X_n} q^{\sigma(Y)} x^{|Y|},$$

which, by Theorem 4.5, reduces to the claimed sum.

Next, let us consider the case $n < 0$. Then, $X_n = \{|-1, -2, \ldots, n\}$ and

$$(x; q)_n = \prod_{j=1}^{|n|} \frac{1}{1 - xq^{-j}} = \prod_{j=1}^{|n|} \sum_{m \geq 0} x^m q^{-jm}.$$

Similar to the previous case, terms of the expansion of this product are in natural correspondence with (hybrid) subsets $Y \subseteq X_n$ with $|Y| \geq 0$. Namely, to Y we associate the product of the terms $x^m q^{ym}$ where $y \in Y$ and $m = M_Y(y)$ is the multiplicity of y. Therefore,

$$(-x; q)_n = \sum_{\substack{Y \subseteq X_n \\ |Y| \geq 0}} (-1)^{|Y|} q^{\sigma(Y)} x^{|Y|},$$

and the claim again follows directly from Theorem 4.5 (note that $\varepsilon = (-1)^k$ in the present case). $\square$

5. The Binomial Theorem

When introducing binomial coefficients with negative entries, Loeb [14] also provided an extension of the binomial theorem

$$(x+y)^n = \sum_{k=0}^{n} \binom{n}{k} x^k y^{n-k},$$

the namesake of the binomial coefficients, to the case when n and k may be negative integers. In this section, we show that this extension can also be generalized to the case of q-binomial coefficients.

Suppose that $f(x)$ is a function with Laurent expansions

$$f(x) = \sum_{k \geq -N} a_k x^k, \quad f(x) = \sum_{k \geq -N} b_{-k} x^{-k}, \tag{5.1}$$

around $x = 0$ and $x = \infty$, respectively. Let us extract coefficients of these expansions by writing

$$\{x^k\} f(x) = \begin{cases} a_k, \text{ if } k \geq 0, \\ b_k, \text{ if } k < 0. \end{cases}$$

Loosely speaking, $\{x^k\} f(x)$ is the coefficient of x^k in the appropriate Laurent expansion of $f(x)$.

Theorem 5.1 [14]. *For all integers n and k,*

$$\binom{n}{k} = \{x^k\}(1+x)^n.$$

Example 5.2. As $x \to \infty$,

$$(1+x)^{-3} = x^{-3} - 3x^{-4} + 6x^{-5} + O\left(x^{-6}\right),$$

so that, for instance,

$$\binom{-3}{-5} = 6.$$

It is well known (see, for instance, [11, Theorem 5.1]) that, if x and y are noncommuting variables such that $yx = qxy$, then the q-binomial coefficients arise from the expansion of $(x+y)^n$.

Theorem 5.3. *Let $n \geq 0$. If $yx = qxy$, then*

$$(x+y)^n = \sum_{k=0}^{n} \binom{n}{k}_q x^k y^{n-k}. \tag{5.2}$$

Our next result shows that the restriction to $n \geq 0$ is not necessary. In fact, we prove the following result, which extends both the noncommutative q-binomial Theorem 5.3 and Loeb's Theorem 5.1. In analogy with the classical case, we consider expansions of $f_n(x, y) = (x + y)^n$ in the two q-commuting variables x, y. As before, we can expand $f_n(x, y)$ in two different ways, that is,

$$f_n(x,y) = \sum_{k\geq 0} a_k x^k y^{n-k}, \qquad f_n(x,y) = \sum_{k\geq n} b_{-k} x^{-k} y^{n+k}.$$

Again, we extract coefficients of these expansions by writing

$$\{x^k y^{n-k}\} f_n(x,y) = \begin{cases} a_k, & \text{if } k \geq 0, \\ b_k, & \text{if } k < 0. \end{cases}$$

Theorem 5.4. *Suppose that $yx = qxy$. Then, for all integers n and k,*

$$\binom{n}{k}_q = \{x^k y^{n-k}\}(x+y)^n.$$

Proof. Using the geometric series,

$$(x+y)^{-1} = y^{-1}(xy^{-1}+1)^{-1} = y^{-1} \sum_{k\geq 0} (-1)^k (xy^{-1})^k,$$

and, applying the q-commutativity,

$$(x+y)^{-1} = \sum_{k\geq 0} (-1)^k q^{-k(k+1)/2} x^k y^{-k-1} = \sum_{k\geq 0} \binom{-1}{k}_q x^k y^{-1-k}.$$

(Consequently, the claim holds when $n = -1$ and $k \geq 0$.) More generally, we wish to show that, for all $n \geq 1$,

$$(x+y)^{-n} = \sum_{k\geq 0} \binom{-n}{k}_q x^k y^{-n-k}. \tag{5.3}$$

We just found that (5.3) holds for $n = 1$. On the other hand, assume that (5.3) holds for some n. Then,

$$\begin{aligned} (x+y)^{-n-1} &= (x+y)^{-n}(x+y)^{-1} \\ &= \left(\sum_{k\geq 0} \binom{-n}{k}_q x^k y^{-n-k} \right) \left(\sum_{k\geq 0} \binom{-1}{k}_q x^k y^{-1-k} \right) \\ &= \sum_{k\geq 0} \sum_{j=0}^{k} \binom{-n}{j}_q \binom{-1}{k-j}_q q^{(k-j)(-n-j)} x^k y^{-n-1-k} \\ &= \sum_{k\geq 0} \binom{-n-1}{k}_q x^k y^{-n-1-k}, \end{aligned}$$

where the last step is an application of the generalized Chu–Vandermonde identity (4.4) with $m = -1$. By induction, (5.3), therefore, is true for all $n \geq 1$.

We have, therefore, shown that (5.2) holds for all integers n. This implies the present claim in the case $k \geq 0$. The case when $k < 0$ can also be deduced from (5.3). Indeed, observe that $xy = q^{-1}yx$, so that, for any integer n, by (5.2) and (5.3),

$$\begin{aligned}(x+y)^n &= \sum_{k \geq 0} \binom{n}{k}_{q^{-1}} y^k x^{n-k} \\ &= \sum_{k \leq n} q^{k(n-k)} \binom{n}{k}_{q^{-1}} x^k y^{n-k} \\ &= \sum_{k \leq n} \binom{n}{k}_q x^k y^{n-k}.\end{aligned}$$

When $n \geq 0$, this is just a version of (5.2). However, when $k < 0$, we deduce that

$$\{x^k y^{n-k}\}(x+y)^n = \binom{n}{k}_q,$$

as claimed. □

6. Lucas' Theorem

Lucas' famous theorem [15] states that, if p is a prime, then

$$\binom{n}{k} \equiv \binom{n_0}{k_0}\binom{n_1}{k_1} \cdots \binom{n_d}{k_d} \pmod{p},$$

where n_i and k_i are the p-adic digits of the nonnegative integers n and k, respectively. Our first goal is to prove that this congruence in fact holds for all integers n and k. The next section is then concerned with further extending these congruences to the polynomial setting.

Example 6.1. The base p expansion of a negative integer is infinite. However, only finitely many digits are different from $p - 1$. For instance, in base 7,

$$-11 = 3 + 5 \cdot 7 + 6 \cdot 7^2 + 6 \cdot 7^3 + \cdots,$$

which we will abbreviate as

$$-11 = (3, 5, 6, 6, \ldots)_7.$$

Similarly,

$$-19 = (2, 4, 6, 6, \ldots)_7.$$

The extension of the Lucas congruences that is proved below shows that

$$\binom{-11}{-19} \equiv \binom{3}{2}\binom{5}{4}\binom{6}{6}\binom{6}{6} \cdots = 3 \cdot 5 \equiv 1 \pmod{7},$$

without computing that the left-hand side is 43,758.

The main result of this section, Theorem 6.2, can also be deduced from the polynomial generalization in the next section. However, we give a direct and uniform proof here to make the ingredients more transparent. A crucial ingredient in the usual proofs of Lucas' classical theorem is the simple congruence

$$(1+x)^p \equiv 1 + x^p \pmod{p}, \tag{6.1}$$

sometimes jokingly called a freshman's dream, which encodes the observation that $\binom{p}{k}$ is divisible by the prime p, except in the boundary cases $k = 0$ and $k = p$.

Theorem 6.2. *Let p be a prime. Then, for any integers n and k,*

$$\binom{n}{k} \equiv \binom{n_0}{k_0}\binom{n'}{k'} \pmod{p},$$

where $n = n_0 + n'p$ and $k = k_0 + k'p$ with $n_0, k_0 \in \{0, 1, \ldots, p-1\}$.

Proof. It is a consequence of (6.1) (and the algebra of Laurent series) that, for any prime p,

$$(1+x)^{-p} \equiv (1+x^p)^{-1} \pmod{p}, \tag{6.2}$$

where it is understood that both sides are expanded, as in (5.1), either around 0 or ∞. Hence, in the same sense,

$$(1+x)^{np} \equiv (1+x^p)^n \pmod{p}, \tag{6.3}$$

for any integer n.

With the notation from the previous section, we observe that

$$\begin{aligned}\{x^k\}(1+x)^n &= \{x^k\}(1+x)^{n_0}(1+x)^{n'p}\\ &\equiv \{x^k\}(1+x)^{n_0}(1+x^p)^{n'} \pmod{p},\end{aligned}$$

where the congruence is a consequence of (6.3). Since

$$n_0 \in \{0, 1, \ldots, p-1\},$$

we conclude that

$$\{x^k\}(1+x)^n \equiv (\{x^{k_0}\}(1+x)^{n_0})(\{x^{k'p}\}(1+x^p)^{n'}) \pmod{p}.$$

This is obvious if $k \geq 0$, but remains true for negative k as well (because $(1+x)^{n_0}$ is a polynomial, in which case the expansions (5.1) around 0 and ∞ agree). Thus,

$$\{x^k\}(1+x)^n \equiv \left(\{x^{k_0}\}(1+x)^{n_0}\right)\left(\{x^{k'}\}(1+x)^{n'}\right) \pmod{p}.$$

Applying Theorem 5.1 to each term, it follows that

$$\binom{n}{k} \equiv \binom{n_0}{k_0}\binom{n'}{k'} \pmod{p},$$

as claimed. ∎

7. A q-Analog of Lucas' Theorem

Let $\Phi_m(q)$ be the m-th cyclotomic polynomial. In this section, we prove congruences of the type $A(q) \equiv B(q)$ modulo $\Phi_m(q)$, where $A(q)$ and $B(q)$ are Laurent polynomials. The congruence is to be interpreted in the natural sense that the difference $A(q) - B(q)$ is divisible by $\Phi_m(q)$.

Example 7.1. Following the notation in Theorem 6.2, in the case $(n, k) = (-4, -8)$, we have $(n_0, k_0) = (2, 1)$ and $(n', k') = (-2, -3)$. We reduce modulo $\Phi_3(q) = 1 + q + q^2$. The result we prove below shows that

$$\binom{-4}{-8}_q \equiv \binom{2}{1}_q \binom{-2}{-3} \pmod{\Phi_3(q)}.$$

Here,

$$\begin{aligned}\binom{-4}{-8}_q &= \frac{1}{q^{22}} \Phi_5(q)\Phi_6(q)\Phi_7(q) \\ &= \frac{1}{q^{22}} (1 - q + q^2)(1 + q + q^2 + q^3 + q^4)(1 + q + q^2 + \cdots + q^6)\end{aligned}$$

as well as

$$\binom{2}{1}_q \binom{-2}{-3} = -2(1 + q),$$

and the meaning of the congruence is that

$$\binom{-4}{-8}_q - \binom{2}{1}_q \binom{-2}{-3} = \Phi_3(q) \cdot \frac{p_{21}(q)}{q^{22}},$$

where

$$p_{21}(q) = 1 + q^2 + 2q^3 + q^4 + \cdots - 2q^{19} + 2q^{21}$$

is a polynomial of degree 21. Observe how, upon setting $q = 1$, we obtain the Lucas congruence

$$\binom{-4}{-8} \equiv \binom{2}{1}\binom{-2}{-3} \pmod{3},$$

provided by Theorem 6.2 (the two sides of the congruence are equal to 35 and -4, respectively).

In the case $n, k \geq 0$, the following q-analog of Lucas' classical binomial congruence has been obtained by Olive [16] and Désarménien [7]. A nice proof based on a group action is given by Sagan [17], who attributes the combinatorial idea to Strehl. We show that these congruences extend uniformly to all integers n and k. A minor difference to keep in mind is that the q-binomial coefficients in this extended setting are Laurent polynomials (see Example 7.1).

Theorem 7.2. *Let $m \geq 2$ be an integer. For any integers n and k,*

$$\binom{n}{k}_q \equiv \binom{n_0}{k_0}_q \binom{n'}{k'} \pmod{\Phi_m(q)},$$

where $n = n_0 + n'm$ and $k = k_0 + k'm$ with $n_0, k_0 \in \{0, 1, \ldots, m-1\}$.

Proof. Suppose throughout that x and y satisfy $yx = qxy$. It follows from the (noncommutative) q-binomial Theorem 5.3 that, for nonnegative integers m,

$$(x+y)^m \equiv x^m + y^m \pmod{\Phi_m(q)}.$$

As in the proof of Theorem 6.2 (and in the analogous sense), we conclude that

$$(x+y)^{nm} \equiv (x^m + y^m)^n \pmod{\Phi_m(q)}, \tag{7.1}$$

for any integer n.

With the notation from Sect. 5, we observe that, by (7.1),

$$\left\{x^k y^{n-k}\right\}(x+y)^n \equiv \left\{x^k y^{n-k}\right\}(x+y)^{n_0}(x^m+y^m)^{n'} \pmod{\Phi_m(q)}.$$

Since $n_0 \in \{0, 1, \ldots, p-1\}$, the right-hand side equals

$$q^{(n_0-k_0)k'm}\left(\{x^{k_0}y^{n_0-k_0}\}(x+y)^{n_0}\right)\left(\{x^{k'm}y^{(n'-k')m}\}(x^m+y^m)^{n'}\right).$$

As $q^m \equiv 1$ modulo $\Phi_m(q)$, we conclude that $\{x^k y^{n-k}\}(x+y)^n$ is congruent to

$$\left(\{x^{k_0}y^{n_0-k_0}\}(x+y)^{n_0}\right)\left(\{x^{k'm}y^{(n'-k')m}\}(x^m+y^m)^{n'}\right)$$

modulo $\Phi_m(q)$. Observe that the variables $X = x^m$ and $Y = y^m$ satisfy the commutation relation $YX = q^{m^2}XY$. Hence, applying Theorem 5.4 to each term, we conclude that

$$\binom{n}{k}_q \equiv \binom{n_0}{k_0}_q \binom{n'}{k'}_{q^{m^2}} \pmod{\Phi_m(q)}.$$

Since $q^{m^2} \equiv 1$ modulo $\Phi_m(q)$, the claim follows. □

In [2], Adamczewski, Bell, Delaygue and Jouhet consider congruences modulo cyclotomic polynomials for multidimensional q-factorial ratios and are thus able to generalize many Lucas-type congruences. In particular, specializing [2, Proposition 1.4] (the case $q = 1$ of which had previously been proved in [1]) to $d = 2$, $u = 1$, $v = 2$, $e_1 = (1; 0)$, $f_1 = (1; -1)$ and $f_2 = (0; 1)$, we obtain the classical case $n, k \geq 0$ of Theorem 7.2. As pointed out by Adamczewski, Bell, Delaygue and Jouhet in private communication, an alternative, a little more tricky, proof of the general case of Theorem 7.2 can be obtained by reducing it, via Corollary 3.2, to the nonnegative case.

8. Conclusion

We believe (and hope that the results of this paper provide some evidence to that effect) that the binomial and q-binomial coefficients with negative entries are natural and beautiful objects. On the other hand, let us indicate an application, taken from [21], of binomial coefficients with negative entries.

Example 8.1. A crucial ingredient in Apéry's proof [4] of the irrationality of $\zeta(3)$ is played by the *Apéry numbers*

$$A(n) = \sum_{k=0}^{n} \binom{n}{k}^2 \binom{n+k}{k}^2. \tag{8.1}$$

These numbers have many interesting properties. For instance, they satisfy remarkably strong congruences, including

$$A(p^r m - 1) \equiv A(p^{r-1} m - 1) \pmod{p^{3r}}, \tag{8.2}$$

established by Beukers [5], and

$$A(p^r m) \equiv A(p^{r-1} m) \pmod{p^{3r}}, \tag{8.3}$$

proved by Coster [6]. Both congruences hold for all primes $p \geq 5$ and positive integers m, r. The definition of the Apéry numbers $A(n)$ can be extended to all integers n by setting

$$A(n) = \sum_{k \in \mathbb{Z}} \binom{n}{k}^2 \binom{n+k}{k}^2, \tag{8.4}$$

where the binomial coefficients are now allowed to have negative entries. Applying the reflection rule (3.1)–(8.4), we obtain

$$A(-n) = A(n-1). \tag{8.5}$$

In particular, we find that the congruence (8.2) is equivalent to (8.3) with m replaced with $-m$. By working with binomial coefficients with negative entries, the second author gave a uniform proof of both sets of congruences in [21]. In addition, the symmetry (8.5), which becomes visible when allowing negative indices, explains why other Apéry-like numbers satisfy (8.3) but not (8.2).

We illustrated that the Gaussian binomial coefficients can be usefully extended to the case of negative arguments. More general binomial coefficients, formed from an arbitrary sequence of integers, are considered, for instance, in [13] and it is shown by Hu and Sun [10] that Lucas' theorem can be generalized to these. It would be interesting to investigate the extent to which these coefficients and their properties can be extended to the case of negative arguments. Similarly, an elliptic analog of the binomial coefficients has recently been introduced by Schlosser [18], who further obtains a general noncommutative binomial theorem of which Theorem 5.3 is a special case. It is natural to wonder whether these binomial coefficients have a natural extension to negative arguments as well.

In the last section, we showed that the generalized q-binomial coefficients satisfy Lucas congruences in a uniform fashion. It would be of interest to determine whether other well-known congruences for the q-binomial coefficients, such as those considered in [3] or [20], have similarly uniform extensions.

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Sam Formichella and Armin Straub
Department of Mathematics and Statistics
University of South Alabama
Mobile
AL 36688
USA

Armin Straub
e-mail: straub@southalabama.edu

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A Lecture Hall Theorem for m-Falling Partitions

To our mentor and friend, George E. Andrews, on his 80th birthday

Shishuo Fu, Dazhao Tang and Ae Ja Yee

Abstract. For an integer $m \geq 2$, a partition $\lambda = (\lambda_1, \lambda_2, \ldots)$ is called m-falling, a notion introduced by Keith, if the least non-negative residues mod m of λ_i's form a nonincreasing sequence. We extend a bijection originally due to the third author to deduce a lecture hall theorem for such m-falling partitions. A special case of this result gives rise to a finite version of Pak–Postnikov's (m, c)-generalization of Euler's theorem. Our work is partially motivated by a recent extension of Euler's theorem for all moduli, due to Xiong and Keith. We note that their result actually can be refined with one more parameter.

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Keywords. Partitions, Stockhofe–Keith map, Lecture hall partitions, m-Falling partitions.

1. Introduction

A *partition* λ of a positive integer n is a nonincreasing sequence of positive integers $(\lambda_1, \lambda_2, \ldots, \lambda_r)$, such that $\sum_{i=1}^{r} \lambda_i = n$. The λ_i's are called the parts of λ, and n is called the weight of λ, usually denoted as $|\lambda|$. For convenience, we often allow parts of size zero and append as many zeros as needed.

Being widely perceived as the genesis of the theory of partitions, Euler's theorem asserts that the set of partitions of n into odd parts and the set of partitions of n into distinct parts are equinumerous. Equivalently

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$$\prod_{i=1}^{\infty}(1+q^i)=\prod_{i=1}^{\infty}\frac{1}{1-q^{2i-1}}.$$

Among numerous generalizations and refinements of Euler's theorem [1, 2, 5–10, 12, 14–19, 21, 23], the one that arguably attracted the most attention is the following finite version, called the Lecture Hall Theorem, discovered by Bousquet-Mélou and Eriksson [5]. Andrews [4] gave a proof using the method of partition analysis.

If $\lambda=(\lambda_1,\ldots,\lambda_n)$ is a partition of length n with some parts possibly zero, such that

$$\frac{\lambda_1}{n}\geq\frac{\lambda_2}{n-1}\geq\cdots\geq\frac{\lambda_n}{1}\geq 0, \tag{1.1}$$

then λ is called a *lecture hall partition of length* n. Let $\mathcal{L}_n$ be the set of lecture hall partitions of length n.

Theorem 1.1. ([5, Theorem 1.1]) *For* $n\geq 1$

$$\sum_{\lambda\in\mathcal{L}_n}q^{|\lambda|}=\prod_{i=1}^{n}\frac{1}{1-q^{2i-1}}. \tag{1.2}$$

It can be easily checked that any partition λ into distinct parts less than or equal to n satisfies the inequality condition in (1.1). That is, $\lambda\in\mathcal{L}_n$ as long as $n\geq\lambda_1$, which shows that (1.2) indeed yields Euler's theorem when $n\to\infty$.

Glaisher [11] found a purely bijective proof of Euler's theorem and was able to extend it to the equinumerous relationship between partitions with parts repeated less than m times and partitions into non-multiples of m for any $m\geq 2$. That is

$$\prod_{i=1}^{\infty}\left(1+q^i+\cdots+q^{(m-1)i}\right)=\prod_{\substack{i=1\\ i\not\equiv 0 \pmod m}}^{\infty}\frac{1}{(1-q^i)}.$$

Recently, Xiong and Keith [24] obtained a substantial refinement of Glaisher's result with respect to certain partition statistics, which we define next.

Throughout this paper, we will assume that $m\geq 2$. For any partition $\lambda=(\lambda_1,\lambda_2,\ldots)$, let

$$s_i(\lambda)=\lambda_i-\lambda_{i+1}+\lambda_{m+i}-\lambda_{m+i+1}+\lambda_{2m+i}-\lambda_{2m+i+1}+\cdots,\quad 1\leq i\leq m.$$

We define its m-alternating sum type to be the $(m-1)$-tuple $\mathbf{s}(\lambda):=(s_1(\lambda),\ldots,s_{m-1}(\lambda))$ and its m-alternating sum

$$s(\lambda):=\sum_{i=1}^{m-1}s_i(\lambda).$$

We note that the m-alternating sum type of λ does not put any restriction on s_m.

Similarly, let

$$\ell_i(\lambda)=\#\{j:\lambda_j\equiv i \pmod m\},\quad 1\leq i\leq m.$$

We define its m-length type to be the $(m-1)$-tuple $\mathbf{l}(\lambda) := (\ell_1(\lambda), \ell_2(\lambda), \ldots, \ell_{m-1}(\lambda))$ and its m-length $\ell(\lambda) = \sum_{i=1}^{m-1} \ell_i(\lambda)$. Note that the m-length type of λ is independent of the parts in λ that are multiples of m.

Let us define the following two subsets of partitions:

- $\mathcal{D}_m$: the set of partitions in which each non-zero part can be repeated at most $m-1$ times;
- $\mathcal{O}_m$: the set of partitions in which each non-zero part is not divisible by m, called m-regular partitions.

Theorem 1.2. ([24, Theorem 2.1]) *For $m \geq 2$,*

$$\sum_{\mu \in \mathcal{D}_m} z_1^{s_1(\mu)} \cdots z_{m-1}^{s_{m-1}(\mu)} q^{|\mu|} = \sum_{\lambda \in \mathcal{O}_m} z_1^{\ell_1(\lambda)} \cdots z_{m-1}^{\ell_{m-1}(\lambda)} q^{|\lambda|}.$$

The natural desire to find certain "lecture hall version" for the result of Xiong and Keith motivated us to take on this investigation. While the version with full generality matching their result is yet to be found, we do obtain a lecture hall theorem for m-falling partitions.

A partition $\lambda = (\lambda_1, \lambda_2, \ldots)$ is called m-falling, which was introduced by Keith [13], if the least non-negative residues mod m of λ_i's form a nonincreasing sequence. We denote the set of m-falling and m-regular partitions (m-falling regular partitions for short) as $\mathcal{O}_{m\searrow}$. For $n \geq 1$, let

$$\mathcal{O}_{m\searrow}^n := \{\lambda \in \mathcal{O}_{m\searrow} : \lambda_1 < nm\}$$

and $\mathcal{L}_{m\searrow}^n$ be a subset of $\mathcal{D}_m$ with certain ratio conditions between parts. Due to the complexity of the conditions, the definition of $\mathcal{L}_{m\searrow}^n$ is postponed to Sect. 3. A partition in $\mathcal{L}_{m\searrow}^n$ is called an m-falling lecture hall partition of order n.

We now state the main result of this paper.

Theorem 1.3. (m-falling lecture hall theorem) *For $m \geq 2$ and $n \geq 1$,*

$$\sum_{\mu \in \mathcal{L}_{m\searrow}^n} z_1^{s_1(\mu)} \cdots z_{m-1}^{s_{m-1}(\mu)} q^{|\mu|} = \sum_{\lambda \in \mathcal{O}_{m\searrow}^n} z_1^{\ell_1(\lambda)} \cdots z_{m-1}^{\ell_{m-1}(\lambda)} q^{|\lambda|}. \tag{1.3}$$

Moreover, we obtain the following generating function of m-falling regular partitions with the largest part less than nm. We shall adopt the common notation $(q;q)_n = (1-q)(1-q^2)\cdots(1-q^n)$ for $n \geq 1$ with $(q;q)_0 = 1$. As usual, the Gaussian coefficients are given by

$$\begin{bmatrix} m+n \\ m \end{bmatrix}_q = \frac{(q;q)_{m+n}}{(q;q)_m (q;q)_n}.$$

Theorem 1.4. *We have*

$$\sum_{\lambda \in \mathcal{O}_{m\searrow}^n} z^{\ell(\lambda)} q^{|\lambda|} = \sum_{i=0}^{\infty} \begin{bmatrix} m-2+i \\ i \end{bmatrix}_q \begin{bmatrix} n-1+i \\ i \end{bmatrix}_{q^m} z^i q^i.$$

The final result of this paper is a refinement of Theorem 1.2. Let us consider the residue sequence of a partition. Namely, for $\lambda = (\lambda_1, \lambda_2, \ldots)$, we take for each part the least non-negative residue mod m and denote the resulting sequence as $v(\lambda) = v_1 v_2 \cdots$. Recall the permutation statistic ascent:

$$\mathrm{asc}(w) = \#\{i : 1 \le i < n,\ w_i < w_{i+1}\},$$

for any word $w = w_1 \cdots w_n$, which consists of totally ordered letters. We extend this statistic to partitions via their residue sequences and let $\mathrm{asc}(\lambda) = \mathrm{asc}(v(\lambda))$.

We have the following refinement of Theorem 1.2.

Theorem 1.5. *For $m \ge 2$,*

$$\sum_{\mu \in \mathcal{D}_m} z_1^{s_1(\mu)} \cdots z_{m-1}^{s_{m-1}(\mu)} z_m^{s_m(\mu)} q^{|\mu|} = \sum_{\lambda \in \mathcal{O}_m} z_1^{\ell_1(\lambda)} \cdots z_{m-1}^{\ell_{m-1}(\lambda)} z_m^{\left\lfloor \frac{\lambda_1}{m} \right\rfloor - \mathrm{asc}(\lambda)} q^{|\lambda|}. \tag{1.4}$$

To make this paper self-contained, in the next section we first recall the Stockhofe–Keith map and then prove Theorem 1.5. In Sect. 3, we define m-falling lecture hall partitions and prove Theorem 1.3, one special case of which gives rise to a lecture hall theorem (see Theorem 3.1) for Pak–Postnikov's (m, c)-generalization [17] of Euler's theorem. In the end, we sketch a proof of Theorem 1.4.

2. Preliminaries and a Proof of Theorem 1.5

In this section, we first recall further definitions and notions involving partitions for later use. After that, we will review the Stockhofe–Keith map and prove Theorem 1.5.

Given two (infinite) sequences $\lambda = (\lambda_1, \lambda_2, \ldots)$ and $\mu = (\mu_1, \mu_2, \ldots)$, we define the usual linear combination $k\lambda + l\mu$ as

$$k\lambda + l\mu = (k\lambda_1 + l\mu_1, k\lambda_2 + l\mu_2, \ldots)$$

for any two non-negative integers k and l.

For a partition $\lambda = (\lambda_1, \ldots, \lambda_r)$, its conjugate partition $\lambda' = (\lambda'_1, \ldots, \lambda'_s)$ is a partition resulting from choosing λ'_i as the number of parts of λ that are not less than i [3, Definition 1.8].

The following lemma (see for instance [24, Lemma 1]) follows via the conjugation of partitions.

Lemma 2.1. *The conjugation map $\lambda \mapsto \lambda'$ is a weight-preserving bijection, such that*

1. $\boldsymbol{s}(\lambda) = \boldsymbol{\ell}(\lambda')$,
2. $\lambda_1 - s(\lambda) = \ell_m(\lambda')$.

Proof. 1. This immediately follows via the conjugation of partitions, so we omit the details.

2. Again, by conjugation, we see that $s_m(\lambda) = \ell_m(\lambda')$. In addition, by the definition
$$\lambda_1 = s_1(\lambda) + \cdots + s_m(\lambda) = s(\lambda) + s_m(\lambda).$$
Thus $\lambda_1 - s(\lambda) = \ell_m(\lambda')$.

$\square$

Using conjugation, we can derive an interesting set of partitions that are equinumerous to $\mathcal{D}_m$, namely, m-flat partitions:

- $\mathcal{F}_m$: the set of partitions in which the differences between consecutive parts are at most $m-1$ and the smallest positive part must also be less than m, called *m-flat* partitions.

Remark 2.2. The two sets $\mathcal{D}_m$ and $\mathcal{F}_m$ are clearly in one-to-one correspondence via conjugation.

2.1. Stockhofe–Keith Map ϕ: $\mathcal{O}_m \to \mathcal{D}_m$

Given any partition λ, we define its base m-flat partition, denoted as $\beta(\lambda)$, as follows. Whenever there are two consecutive parts λ_i and λ_{i+1} with $\lambda_i - \lambda_{i+1} \geq m$, we subtract m from each of the parts $\lambda_1, \lambda_2, \ldots, \lambda_i$. We repeat this until we reach a partition in $\mathcal{F}_m$, which is taken to be $\beta(\lambda)$.

Suppose we are given a partition $\lambda \in \mathcal{O}_m$. We now describe step-by-step how to get a partition $\phi(\lambda) = \mu \in \mathcal{D}_m$ via the aforementioned Stockhofe–Keith map ϕ.

Step 1: Decompose $\lambda = m\sigma + \beta(\lambda)$.
Step 2: Insert each part in $m\sigma'$, from the largest one to the smallest one, into $\beta(\lambda)$ according to the following insertion method. Note that after each insertion, we always arrive at a new m-flat partition. In particular, the final partition we get, say τ, is in $\mathcal{F}_m$ as well.
Step 3: Conjugate τ to get $\mu = \tau' \in \mathcal{D}_m$.

Insertion method to get $\tau \in \mathcal{F}_m$

Initiate $\tau = (\tau_1, \tau_2, \ldots) = \beta(\lambda)$. Note that parts in $m\sigma'$ are necessarily multiples of m. Suppose we currently want to insert a part km into τ.

If $km - \tau_1 \geq m$, then find the unique integer i, $1 \leq i \leq k$, such that
$$(\tau_1 + m, \tau_2 + m, \ldots, \tau_i + m, (k-i)m, \tau_{i+1}, \ldots)$$
is still a partition in $\mathcal{F}_m$. Replace τ with this new partition.

Otherwise, we simply insert km into $\beta(\lambda)$ as a new part and replace τ with this new partition.

For example, let us take $m = 3$ and $\lambda = (19, 17, 14, 13, 13, 8, 1) \in \mathcal{O}_3$. We use 3-modular Ferrers graphs [3] to illustrate the process of deriving μ, see Fig. 1, where the inserted entries are displayed in boldface in Step 2.

We should remark that the original description of the Stockhofe–Keith map [13, 22] consists of only Steps 1 and 2. Thus, the map [13, 22] accounts for the following theorem.

Theorem 2.3. ([24, Theorem 3.1]) *m-Regular partitions of any given m-length type are in bijection with m-flat partitions of the same m-length type.*

Theorem 2.3 with Remark 2.2 proves Theorem 1.2, so we will adopt the modified definition of the Stockhofe–Keith map which consists of the above three steps.

2.2. Proof of Theorem 1.5

In view of the proof of Theorem 1.2 in [24], all it remains is to examine the map ϕ with respect to the extra parameter z_m. Suppose $\lambda \in \mathcal{O}_m$, and the largest part of $\beta(\lambda)$ is b_1, then we have $\left\lfloor \frac{b_1}{m} \right\rfloor = \mathrm{asc}(\lambda)$, according to the definition of m-flat partitions. Next, during Step 2, we insert columns of $m\sigma$ into τ, and each insertion will give rise to a new part in τ that is divisible by m; therefore, we see that $\frac{\lambda_1 - b_1}{m} = s_m(\mu)$. The above discussion gives

$$s_m(\mu) = \frac{\lambda_1 - b_1}{m} = \left\lfloor \frac{\lambda_1}{m} \right\rfloor - \left\lfloor \frac{b_1}{m} \right\rfloor = \left\lfloor \frac{\lambda_1}{m} \right\rfloor - \mathrm{asc}(\lambda),$$

as desired. $\square$

Remark 2.4. For $\lambda \in \mathcal{O}_m$, let $\phi(\lambda) = \mu$. Then, the extra parameter tracked by z_m gives

$$\mu_1 = s(\mu) + s_m(\mu) = s(\mu) + \left\lfloor \frac{\lambda_1}{m} \right\rfloor - \mathrm{asc}(\lambda) = \ell(\lambda) + \left\lfloor \frac{\lambda_1}{m} \right\rfloor - \mathrm{asc}(\lambda),$$

which has previously been derived by Keith [13, Theorem 6] as well (he used f_λ instead of $\mathrm{asc}(\lambda)$). Moreover, this refinement is reminiscent of Sylvester's bijection for proving Euler's theorem, in which case $m = 2$ and we always have $\mathrm{asc}(\lambda) = 0$, see, for example, Theorem 1 (item 4) in [26].

3. A Lecture Hall Theorem for m-Falling Partitions

We will first handle the case with a single residue class. Let us fix $c, 1 \le c \le m-1$. For $n \ge 1$, let

$$\mathcal{O}_{c,m} := \{\lambda \in \mathcal{O}_m : \lambda_i \equiv c \pmod{m}, \text{ for all } i\},$$

$$\mathcal{O}_{c,m}^n := \{\lambda \in \mathcal{O}_{c,m} : \lambda_1 < nm\},$$

$$\mathcal{D}_{c,m} := \left\{ \lambda \in \mathcal{D}_m : \mathbf{s}(\lambda) = (\overbrace{0,\ldots,0}^{c-1}, a, 0\ldots,0) \text{ for some } a > 0, \text{ or } |\lambda| = 0 \right\},$$

$$\mathcal{L}_{c,m}^n := \Bigg\{ \lambda \in \mathcal{D}_{c,m} : l(\lambda) \le \left\lfloor \frac{n+1}{2} \right\rfloor (m-2) + n \text{ and}$$
$$\frac{\lambda_{km+c}}{n-2k} \ge \frac{\lambda_{km+m}}{n-2k-1} \ge \frac{\lambda_{(k+1)m+c}}{n-2k-2} \text{ for } 0 \le k < \left\lceil \frac{n}{2} \right\rceil \Bigg\},$$

where $l(\lambda)$ is the number of non-zero parts in λ, and we make the convention that for a fraction $\frac{p}{q}$, if $q = 0$ we replace the relevant fraction with 0.

Step 1: $\lambda = \begin{array}{ccccccc} 3&3&3&3&3&3&1\\ 3&3&3&3&3&2&\\ 3&3&3&3&2&&\\ 3&3&3&3&1&&\\ 3&3&3&3&1&&\\ 3&3&2&&&&\\ 1&&&&&& \end{array} = \begin{array}{cccc} 3&3&3&3\\ 3&3&3&3\\ &3&3&3\\ &3&3&3\\ &3&3&3\\ &&3&3 \end{array} + \begin{array}{ccc} 3&3&1\\ 3&2&\\ 3&2&\\ 3&1&\\ 3&1&\\ 2&&\\ 1&& \end{array}$

Step 2: $\begin{array}{c} 3\\3\\3\\3\\3\\3 \end{array} \xrightarrow{\text{insert}} \begin{array}{ccc} 3&3&1\\ 3&2&\\ 3&2&\\ 3&1&\\ 3&1&\\ 2&&\\ 1&& \end{array} = \begin{array}{cccc} \mathbf{3}&3&3&1\\ \mathbf{3}&3&2&\\ \mathbf{3}&3&2&\\ \mathbf{3}&3&1&\\ \mathbf{3}&\mathbf{3}&&\\ 3&1&&\\ 2&&&\\ 1&&& \end{array}$

$\begin{array}{c} 3\\3\\3\\3\\3\\3 \end{array} \xrightarrow{\text{insert}} \begin{array}{cccc} 3&3&3&1\\ 3&3&2&\\ 3&3&2&\\ 3&3&1&\\ 3&3&&\\ 3&1&&\\ 2&&&\\ 1&&& \end{array} = \begin{array}{ccccc} \mathbf{3}&3&3&3&1\\ \mathbf{3}&3&3&2&\\ \mathbf{3}&3&3&2&\\ \mathbf{3}&\mathbf{3}&\mathbf{3}&&\\ 3&3&1&&\\ 3&3&&&\\ 3&1&&&\\ 2&&&&\\ 1&&&& \end{array} \quad \cdots$

Step 3: $\begin{array}{ccccc} 3&3&3&3&3\\ 3&3&3&3&1\\ 3&3&3&2&\\ 3&3&3&2&\\ 3&3&3&&\\ 3&3&1&&\\ 3&3&&&\\ 3&3&&&\\ 3&1&&&\\ 2&&&&\\ 1&&&& \end{array} \xrightarrow{\text{conjugate}} \mu = (11, 10, 9, 9, 8, 8, 6, 5, 5, 4, 4, 2, 2, 1, 1)$

FIGURE 1. Three steps to get $\phi(\lambda) = \mu$

Theorem 3.1. *For any $n \geq 1$*

$$\sum_{\lambda \in \mathcal{O}_{c,m}^n} z^{\ell(\lambda)} q^{|\lambda|} = \sum_{\mu \in \mathcal{L}_{c,m}^n} z^{s(\mu)} q^{|\mu|}$$

$$= \frac{1}{(1-zq^c)(1-zq^{m+c})\cdots(1-zq^{(n-1)m+c})}. \tag{3.1}$$

The above result can be viewed as a finite (or "lecture hall") version of the following Pak–Postnikov's (m,c)-generalization [17, Theorem 1] of Euler's partition theorem since $\lim_{n\to\infty} \mathcal{O}_{c,m}^n \to \mathcal{O}_{c,m}$ and $\lim_{n\to\infty} \mathcal{L}_{c,m}^n \to \mathcal{D}_{c,m}$.

Theorem 3.2. *For $n \geq 1, m \geq 2$ and $1 \leq c \leq m-1$, the number of partitions of n into parts congruent to c mod m equals the number of partitions of n with exactly c parts of maximal size, $m-c$ (if any) second by size parts, c (if any) third by size parts, etc.*

Proof of Theorem 3.1. Since a partition $\lambda \in \mathcal{O}_{c,m}^n$ has all its parts congruent to c mod m, with the largest part $\lambda_1 < nm$, we see that $\ell(\lambda) = \ell_c(\lambda) = l(\lambda)$ and we have

$$\sum_{\lambda \in \mathcal{O}_{c,m}^n} z^{\ell(\lambda)} q^{|\lambda|} = \frac{1}{(1-zq^c)(1-zq^{m+c})\cdots(1-zq^{(n-1)m+c})}.$$

It remains to prove the first equality. We achieve this by constructing a weight-preserving bijection φ_n from $\mathcal{O}_{c,m}^n$ to $\mathcal{L}_{c,m}^n$, such that $\ell(\lambda) = s(\varphi_n(\lambda))$. We extend the bijection from $\mathcal{O}_n$ to $\mathcal{L}_n$ constructed in [25] to deal with the m-falling partitions considered here.

Define the maps $\varphi_n : \mathcal{O}_{c,m}^n \longrightarrow \mathcal{L}_{c,m}^n$ as follows. For $\lambda \in \mathcal{O}_{c,m}^n$, let μ be the sequence obtained from the empty sequence $(0,0,\ldots)$ by recursively inserting the parts of λ in nonincreasing order according to the following insertion procedure. We define $\varphi_n(\lambda) = \mu$.

Insertion procedure

Let $(\mu_1, \mu_2, \ldots) \in \mathcal{L}_{c,m}^n$. To insert $km + c$ into $(\mu_1, \mu_2, \ldots)$, set $j = 0$.

If $j < k$ and $\dfrac{\mu_{mj+c}+1}{n-2j} \geq \dfrac{\mu_{mj+m}+1}{n-2j-1}$, (Test I)

then increase j by 1 and go to (Test I);

otherwise stop testing and return

$$(\mu_1, \mu_2, \ldots) + (\overbrace{1, \ldots, 1}^{jm}, \overbrace{k-j+1, \ldots, k-j+1}^{c}, \overbrace{k-j, \ldots, k-j}^{m-c}, 0, 0, \ldots).$$

The effect of this insertion is that we use up a complete part $km + c$, so the weight of the sequence $(\mu_1, \mu_2, \ldots)$ and its m-alternating sum are increased by $km + c$ and 1, respectively. In addition, it can be checked easily that the returned sequence satisfies the condition for $\mathcal{L}_{c,m}^n$. We omit the details.

The map φ_n is indeed invertible, since the parts of λ were inserted in nonincreasing order, i.e., from the largest to the smallest. If the parts are not inserted in this order, φ_n is not necessarily invertible. The inverse of φ_n, namely, ψ_n, can be described similarly in this algorithmic fashion. For a given partition $\mu \in \mathcal{L}^n_{c,m}$ with $s(\mu) = k$, define $\psi_n(\mu)$ to be the sequence $\lambda = (\lambda_1, \ldots, \lambda_k, 0, 0, \ldots)$ obtained from the empty sequence $(0, 0, \ldots)$ by adding nondecreasing parts one at a time that are derived from peeling off partially or entirely certain parts of μ according to the following deletion procedure.

Deletion procedure

Let $(\mu_1, \mu_2, \ldots) \neq (0, 0, \ldots)$ be in $\mathcal{L}^n_{c,m}$. Set $k = 0$ and $j = 0$.

If $(\mu_1, \mu_2, \ldots) - (\overbrace{1, \ldots, 1}^{jm}, \overbrace{k-j+1, \ldots, k-j+1}^{c}, \overbrace{k-j, \ldots, k-j}^{m-c}, 0, 0, \ldots) \in \mathcal{L}^n_{c,m}$, (Test D)

then stop testing and return $km + c$ with

$$\mu = (\mu_1, \mu_2, \ldots) - (\overbrace{1, \ldots, 1}^{jm}, \overbrace{k-j+1, \ldots, k-j+1}^{c}, \overbrace{k-j, \ldots, k-j}^{m-c}, 0, 0, \ldots);$$

otherwise, if $j < k$, then increase j by 1 and go to (Test D);

otherwise increase k by 1, set $j = 0$ and go to (Test D).

The effect of this deletion is that the weight of the sequence $(\mu_1, \mu_2, \ldots)$ and its m-alternating sum are decreased by $km + c$ and 1, respectively. In addition, it should be noted that this deletion process must stop after a finite number of steps. Since $(\mu_1, \mu_2, \ldots) \neq (0, 0, \ldots)$ belongs to $\mathcal{L}^n_{c,m}$, there must be i, such that

$$\mu_{im+c} > \mu_{im+m}.$$

Let j be the largest such i. Then, $\mu_l = 0$ for any $l > jm + m$ and

$$(\mu_1, \mu_2, \ldots) - (\overbrace{1, \ldots, 1}^{jm}, \overbrace{k-j+1, \ldots, k-j+1}^{c}, \overbrace{k-j, \ldots, k-j}^{m-c}, 0, 0, \ldots) \in \mathcal{L}^n_{c,m},$$

which shows such j must pass (Test D).

To finish the proof, we make the following claims about φ_n and ψ_n without giving the proofs, since all of them are essentially the same as those found in [25], which is the case when $m = 2$ and $c = 1$.

- Each insertion outputs a new $\mu \in \mathcal{L}^n_{c,m}$, and in particular, φ_n is well-defined.
- Each deletion outputs a new $\lambda \in \mathcal{O}^n_{c,m}$, and in particular, ψ_n is well-defined.

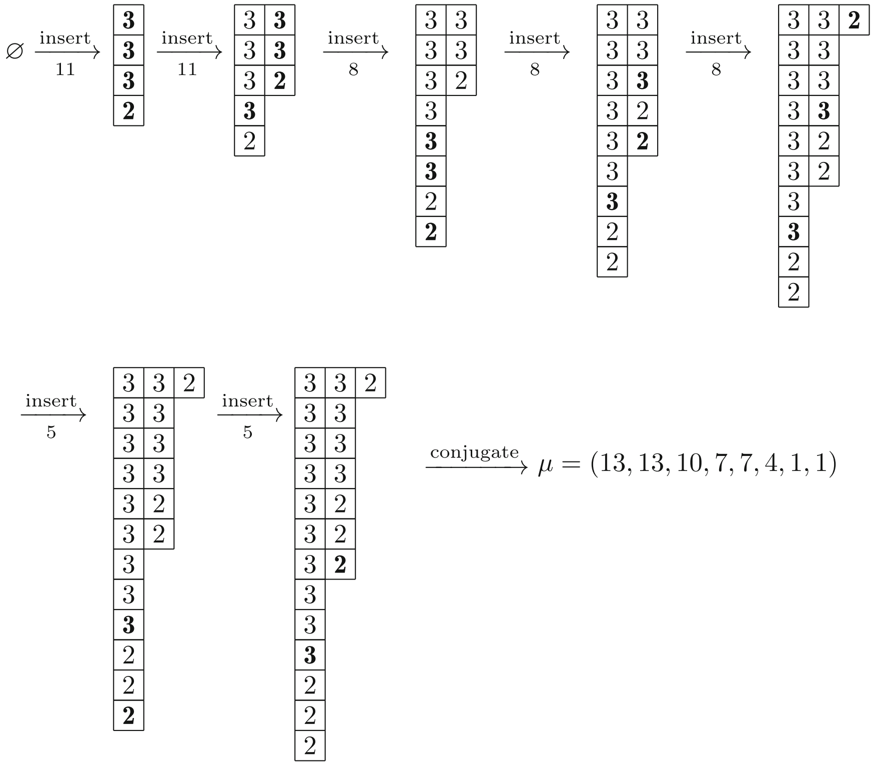

FIGURE 2. $\varphi_5((11,11,8,8,8,5,5)) = (13,13,10,7,7,4,1,1)$

- The deletion procedure reverses the insertion procedure, consequently ψ_n is the inverse of φ_n.

$\square$

Before we move on, we provide an example for the insertion procedure.

Example 3.3. Let $m = 3$, $c = 2$, $n = 5$, and $\mu = (3,3,2,0,0,\ldots) \in \mathcal{L}^5_{2,3}$. We insert 8 into μ as follows. Note that

$$\frac{\mu_2+1}{5} = \frac{4}{5} \geq \frac{\mu_3+1}{4} = \frac{3}{4} \quad \text{but} \quad \frac{\mu_5+1}{3} = \frac{1}{3} \not\geq \frac{\mu_6+1}{2} = \frac{1}{2}.$$

Therefore, we get

$$(3,3,2,0,0,\ldots) + (1,1,1,2,2,1) = (4,4,3,2,2,1).$$

In Fig. 2, we illustrate the process of getting μ by applying φ_5 to $\lambda = (11,11,8,8,8,5,5) \in \mathcal{O}^5_{2,3}$ using 3-modular Ferrers graphs. Newly inserted entries after each step are displayed in boldface.

Remark 3.4. In general, our bijection φ_n works on m-modular Ferrers graphs. For the special case when $m = 2$ and $c = 1$, φ_n actually reduces to the original bijection constructed in [25]. Another notable generalization can be found in [20].

As we will see, the bijection φ_n plays a crucial role in our proof of Theorem 1.3. We need a few more definitions.

Definition 3.5. For a partition $\lambda \in \mathcal{D}_m$, we define two local statistics "first bigger" and "last bigger". For each i, $0 \le i \le \left\lfloor \dfrac{l(\lambda)-1}{m} \right\rfloor$, where $l(\lambda)$ is the number of non-zero parts in λ, suppose

$$\lambda_{im+1} = \cdots = \lambda_{im+j} > \lambda_{im+j+1} \ge \cdots \ge \lambda_{im+k} > \lambda_{im+k+1} = \cdots = \lambda_{im+m}.$$

Then, we let $\mathrm{fb}_i = j$, $\mathrm{lb}_i = k$.

Note that since $\lambda \in \mathcal{D}_m$, such j and k must exist and $j \le k$, so fb_i and lb_i are well-defined.

Definition 3.6. Fix a positive integer n. For a partition $\lambda \in \mathcal{D}_m$, we call it an m-falling lecture hall partition of order n, if $l(\lambda) \le \left\lfloor \dfrac{n+1}{2} \right\rfloor (m-2) + n$, and the following two conditions hold.

1. For i, $0 \le i < \left\lfloor \dfrac{l(\lambda)-1}{m} \right\rfloor$, $\mathrm{lb}_i \le \mathrm{fb}_{i+1}$.
2. $$\frac{\lambda_1}{n} \ge \frac{\lambda_m}{n-1} \ge \frac{\lambda_{m+1}}{n-2} \ge \frac{\lambda_{2m}}{n-3} \ge \cdots \ge \frac{\lambda_{(k-1)m+1}}{2} \ge \frac{\lambda_{km}}{1}, \quad \text{if } n = 2k,$$
$$\frac{\lambda_1}{n} \ge \frac{\lambda_m}{n-1} \ge \frac{\lambda_{m+1}}{n-2} \ge \frac{\lambda_{2m}}{n-3} \ge \cdots \ge \frac{\lambda_{km+1}}{1} \ge \frac{\lambda_{km+m}}{0}, \quad \text{if } n = 2k+1.$$

We denote the set of all m-falling lecture hall partitions of order n as $\mathcal{L}^n_{m\searrow}$.

Remark 3.7. Partitions in $\mathcal{D}_m$ satisfying the above condition (1) are said to be of m-alternating type in [13].

Recall the definition of m-falling regular partitions. A partition is m-falling regular if the parts are not multiples of m and their positive residues are nonincreasing.

For a chosen vector $\mathbf{v} = (v_1, v_2, \ldots, v_{m-1})$, let

$$\mathcal{O}^{\mathbf{v}}_{m\searrow} := \left\{\lambda \in \mathcal{O}^n_{m\searrow} : \mathbf{l}(\lambda) = \mathbf{v}\right\},$$
$$\mathcal{L}^{\mathbf{v}}_{m\searrow} := \left\{\mu \in \mathcal{L}^n_{m\searrow} : \mathbf{s}(\mu) = \mathbf{v}\right\}.$$

Proof of Theorem 1.3. For any $c, 1 \le c \le m-1$ and a given vector $\mathbf{v} = (v_1, v_2, \ldots, v_{m-1})$, we consider two embeddings:

$$f_{\mathbf{v}} : \mathcal{O}^{\mathbf{v}}_{m\searrow} \hookrightarrow \mathcal{O}^n_{c,m} \quad \text{and} \quad g_{\mathbf{v}} : \mathcal{L}^{\mathbf{v}}_{m\searrow} \hookrightarrow \mathcal{L}^n_{c,m},$$

such that $\ell(\lambda) = \ell(f_{\mathbf{v}}(\lambda))$ and $s(\mu) = s(g_{\mathbf{v}}(\mu))$. To be precise, for a given partition λ, both $f_{\mathbf{v}}$ and $g_{\mathbf{v}}$ change the residue of each part of λ mod m to be uniformly the predetermined value c. In terms of the corresponding m-modular Ferrers graph, the two maps keep all the cells labelled m, but relabel all the remaining cells by c. Therefore, in general, neither of these two maps preserves the weight of the partition, but they do keep the number of cells in their m-modular Ferrers graphs unchange.

Table 1. $\phi_3 \colon \mathcal{O}^{(3,2)}_{3\searrow} \longrightarrow \mathcal{L}^{(3,2)}_{3\searrow}$

$\mathcal{O}^{(3,2)}_{3\searrow}$	$\mathcal{L}^{(3,2)}_{3\searrow}$
$8,8,7,7,7$	$15,12,10$
$8,8,7,7,4$	$14,11,9$
$8,8,7,7,1$	$13,10,8$
$8,8,7,4,4$	$12,9,8,1,1$
$8,8,7,4,1$	$12,9,7$
$8,8,7,1,1$	$11,8,6$
$8,8,4,4,4$	$11,8,7,1,1$
$8,8,4,4,1$	$10,7,6,1,1$
$8,8,4,1,1$	$10,7,5$
$8,8,1,1,1$	$9,6,4$
$8,5,4,4,4$	$9,6,6,2,2$
$8,5,4,4,1$	$9,6,5,1,1$
$8,5,4,1,1$	$8,5,4,1,1$
$8,5,1,1,1$	$8,5,3$
$8,2,1,1,1$	$7,4,2$
$5,5,4,4,4$	$8,5,5,2,2$
$5,5,4,4,1$	$7,4,4,2,2$
$5,5,4,1,1$	$7,4,3,1,1$
$5,5,1,1,1$	$6,3,2,1,1$
$5,2,1,1,1$	$6,3,1$
$2,2,1,1,1$	$5,2$

Moreover, the given vector $\mathbf{v}$ and the m-falling condition uniquely determine the preimage of any partition in $f_{\mathbf{v}}(\mathcal{O}^{\mathbf{v}}_{m\searrow})$. Similarly, the condition (1) in Definition 3.6 together with $\mathbf{v}$ dictates the preimage of any partition in $g_{\mathbf{v}}(\mathcal{L}^{\mathbf{v}}_{m\searrow})$. This enables us to define a bijection

$$\phi_n = g_{\mathbf{v}}^{-1} \circ \varphi_n \circ f_{\mathbf{v}} \colon \mathcal{O}^n_{m\searrow} \longrightarrow \mathcal{L}^n_{m\searrow},$$

where $\mathbf{v}$ is the m-length type of the partition it acts on.

It has been proved in Theorem 3.1 that φ_n is a bijection satisfying $\ell(\lambda) = s(\varphi_n(\lambda))$, and the discussion above shows that both $f_{\mathbf{v}}$ and $g_{\mathbf{v}}$ are invertible. Consequently, we see that ϕ_n is indeed a bijection such that $\mathbf{l}(\lambda) = \mathbf{s}(\phi_n(\lambda))$ for any $\lambda \in \mathcal{O}^n_{m\searrow}$, and we complete the proof. $\square$

Example 3.8. As an illustrative example, we take $m = 3, n = 3$ and fix the vector $\mathbf{v} = (3,2)$. In Table 1, we list out all the 21 partitions λ in $\mathcal{O}^{(3,2)}_{3\searrow}$ with $\lambda_1 < nm = 9$, as well as all the 21 partitions μ in $\mathcal{L}^{(3,2)}_{3\searrow}$ with $l(\mu) \le 5$. They are matched up via our map ϕ_3. The derivation of one particular partition $(8,5,5,2,2)$ from $(5,5,4,4,4)$ using 3-modular Ferrers graphs is detailed in Fig. 3.

Now, we turn to the proof of Theorem 1.4.

$$\text{Step 1: } \lambda = \begin{array}{|c|c|}\hline 3 & 2 \\ \hline 3 & 2 \\ \hline 3 & 1 \\ \hline 3 & 1 \\ \hline 3 & 1 \\ \hline \end{array} \xrightarrow{f_{(3,2)}} \begin{array}{|c|c|}\hline 3 & c \\ \hline 3 & c \\ \hline 3 & c \\ \hline 3 & c \\ \hline 3 & c \\ \hline \end{array}$$

$$\text{Step 2: } \varnothing \xrightarrow[\boxed{3 \mid c}]{\text{insert}} \begin{array}{|c|}\hline \mathbf{3} \\ \hline \mathbf{c} \\ \hline \end{array} \xrightarrow[\boxed{3 \mid c}]{\text{insert}} \begin{array}{|c|c|}\hline 3 & \mathbf{c} \\ \hline \mathbf{3} \\ \cline{1-1} c \\ \cline{1-1} \end{array} \xrightarrow[\boxed{3 \mid c}]{\text{insert}} \begin{array}{|c|c|}\hline 3 & c \\ \hline 3 \\ \cline{1-1} \mathbf{3} \\ \cline{1-1} c \\ \cline{1-1} \mathbf{c} \\ \cline{1-1} \end{array} \xrightarrow[\boxed{3 \mid c}]{\text{insert}}$$

$$\begin{array}{|c|c|}\hline 3 & c \\ \hline 3 & \mathbf{c} \\ \hline 3 \\ \cline{1-1} \mathbf{3} \\ \cline{1-1} c \\ \cline{1-1} c \\ \cline{1-1} \end{array} \xrightarrow[\boxed{3 \mid c}]{\text{insert}} \begin{array}{|c|c|}\hline 3 & c \\ \hline 3 & c \\ \hline 3 \\ \cline{1-1} 3 \\ \cline{1-1} \mathbf{3} \\ \cline{1-1} c \\ \cline{1-1} c \\ \cline{1-1} \mathbf{c} \\ \cline{1-1} \end{array}$$

$$\text{Step 3: } \begin{array}{|c|c|}\hline 3 & c \\ \hline 3 & c \\ \hline 3 \\ \cline{1-1} 3 \\ \cline{1-1} 3 \\ \cline{1-1} c \\ \cline{1-1} c \\ \cline{1-1} c \\ \cline{1-1} \end{array} \xrightarrow{g^{-1}_{(3,2)}} \begin{array}{|c|c|}\hline 3 & 2 \\ \hline 3 & 2 \\ \hline 3 \\ \cline{1-1} 3 \\ \cline{1-1} 3 \\ \cline{1-1} 1 \\ \cline{1-1} 1 \\ \cline{1-1} 1 \\ \cline{1-1} \end{array} \xrightarrow{\text{conjugate}} \mu = (8,5,5,2,2)$$

FIGURE 3. $\phi_3((5,5,4,4,4)) = (8,5,5,2,2)$

Proof of Theorem 1.4. Based on the m-modular Ferrers graphs of m-regular falling partitions, we obtain

$$\sum_{\mu \in \mathcal{O}^n_{m\searrow}} z_1^{\ell_1(\mu)} \cdots z_{m-1}^{\ell_{m-1}(\mu)} q^{|\mu|} = \sum_{i=0}^{\infty} h_i(z_1 q, z_2\, q^2, \ldots, z_{m-1} q^{m-1}) \begin{bmatrix} n-1+i \\ i \end{bmatrix}_{q^m}, \tag{3.2}$$

where the ith homogeneous symmetric function $h_i(x_1, \ldots, x_k)$ is defined by

$$h_i(x_1, \ldots, x_k) = \sum_{1 \le j_1 \le j_2 \le \cdots \le j_i \le k} x_{j_1} x_{j_2} \cdots x_{j_i}.$$

Setting $z_1 = \cdots = z_{m-1} = z$ in (3.2), we get

$$\sum_{\mu \in \mathcal{O}^n_{m\searrow}} z^{\ell(\mu)} q^{|\mu|} = \sum_{i=0}^{\infty} h_i\left(q, q^2, \ldots, q^{m-1}\right) \begin{bmatrix} n-1+i \\ i \end{bmatrix}_{q^m} z^i. \tag{3.3}$$

Since

$$h_i(q, q^2, \ldots, q^{m-1}) = q^i \begin{bmatrix} m-2+i \\ i \end{bmatrix}_q,$$

it follows from (3.3) that

$$\sum_{\mu \in \mathcal{O}^n_{m\searrow}} z^{\ell(\mu)} q^{|\mu|} = \sum_{i=0}^{\infty} \begin{bmatrix} m-2+i \\ i \end{bmatrix}_q \begin{bmatrix} n-1+i \\ i \end{bmatrix}_{q^m} z^i q^i,$$

as claimed. □

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Shishuo Fu and Dazhao Tang
College of Mathematics and Statistics
Chongqing University
Huxi Campus
Chongqing 401331
People's Republic of China

Dazhao Tang
e-mail: dazhaotang@sina.com ;
dazhaotang@cqu.edu.cn

Shishuo Fu
e-mail: fsshuo@cqu.edu.cn

Ae Ja Yee
Department of Mathematics
The Pennsylvania State University
University Park
PA 16802
USA
e-mail: yee@psu.edu

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Annals of Combinatorics

New Fifth and Seventh Order Mock Theta Function Identities

Dedicated to George Andrews on the occasion of his eightieth birthday

Frank G. Garvan

Abstract. We give simple proofs of Hecke–Rogers indefinite binary theta series identities for the two Ramanujan's fifth order mock theta functions $\chi_0(q)$ and $\chi_1(q)$ and all three of Ramanujan's seventh order mock theta functions. We find that the coefficients of the three mock theta functions of order 7 are surprisingly related.

Mathematics Subject Classification. Primary 33D15; Secondary 11B65, 11F27.

Keywords. Mock theta functions, Hecke–Rogers double sums, Bailey pairs, Conjugate Bailey pairs.

1. Introduction

In his last letter to Hardy, Ramanujan described new functions that he called mock theta functions and listed mock theta functions of order 3, 5, and 7. Watson studied the behaviour of the third order functions under the modular group, but was unable to find similar transformation properties for the fifth and seventh order functions. The first substantial progress towards finding such transformation properties was made by Andrews [1], who found double sum representations for the fifth and seventh order functions. These double sum representations were reminiscent of certain identities for modular forms found by Hecke and Rogers. Andrews' results for the fifth and seventh order mock theta functions were crucial to Zwegers [14], who later showed how to complete these functions to harmonic Maass forms. For more details on this aspect, see Zagier's survey [12].

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Throughout this paper, we use the following standard notation:

$$
\begin{aligned}
(a;q)_\infty &= (a)_\infty = \prod_{n=0}^{\infty}(1-aq^n),\\
(a;q)_n &= (a)_n = (a;q)_\infty/(aq^n;q)_\infty\\
&\qquad \big(= (1-a)(1-aq)\cdots(1-aq^{n-1}) \text{ for } n \text{ a nonnegative integer}\big)\,.
\end{aligned}
$$

Andrews [1] found Hecke–Rogers indefinite binary theta series identities for all the fifth order mock theta functions except for the following two:

$$
\begin{aligned}
\chi_0(q) &= \sum_{n=0}^{\infty}\frac{q^n}{(q^{n+1};q)_n} = \sum_{n=0}^{\infty}\frac{q^n\,(q)_n}{(q)_{2n}}\\
&= 1+q+q^2+2\,q^3+q^4+3\,q^5+2\,q^6+3\,q^7+\cdots\,,
\end{aligned}
$$

and

$$
\begin{aligned}
\chi_1(q) &= \sum_{n=0}^{\infty}\frac{q^n}{(q^{n+1};q)_{n+1}} = \sum_{n=0}^{\infty}\frac{q^n\,(q)_n}{(q)_{2n+1}}\\
&= 1+2\,q+2\,q^2+3\,q^3+3\,q^4+4\,q^5+4\,q^6+6\,q^7+\cdots\,.
\end{aligned}
$$

Zwegers [13] found triple sum identities for $\chi_0(q)$ and $\chi_1(q)$. Zagier [12] stated indefinite binary theta series identities for these two functions but gave few details. We find new Hecke–Rogers indefinite binary theta series identities for these two functions. In Sect. 5, we compare our results with Zagier's:

Theorem 1.1.

$$
\begin{aligned}
&(q)_\infty(\chi_0(q)-2)\\
&\quad= \sum_{j=0}^{\infty}\sum_{-j\le 3m\le j}\operatorname{sgn}(m)(-1)^{m+j+1}q^{j(3j+1)/2-m(15m+1)/2}(1-q^{2j+1})\\
&\qquad+\sum_{j=1}^{\infty}\sum_{-j-1\le 3m\le j-1}\operatorname{sgn}(m)(-1)^{m+j+1}\\
&\qquad\qquad\times q^{j(3j+1)/2-m(15m+11)/2-1}(1-q^{2j+1}),
\end{aligned}
\tag{1.1}
$$

and

$$
\begin{aligned}
&(q)_\infty\chi_1(q)\\
&\quad= \sum_{j=1}^{\infty}\sum_{-j\le 3m\le j-1}\operatorname{sgn}(m)(-1)^{m+j+1}q^{j(3j-1)/2-m(15m+7)/2-1}(1+q^j)\\
&\qquad+\sum_{j=1}^{\infty}\sum_{-j-1\le 3m\le j-2}\operatorname{sgn}(m)(-1)^{m+j+1}q^{j(3j-1)/2-m(15m+13)/2-2}(1+q^j),
\end{aligned}
\tag{1.2}
$$

where

$$
\operatorname{sgn}(m) = \begin{cases} 1, & \text{if } m\ge 0,\\ -1, & \text{if } m<0.\end{cases}
$$

Idea of Proof. We need the following conjugate Bailey pair (with $a = q$):

$$\delta_n = \frac{q^n (q)_n (q)_\infty}{(1-q)},$$
$$\gamma_n = \sum_{j=n+1}^{\infty} (-1)^{j+n+1} q^{j(3j-1)/2-3n(n+1)/2-1}(1+q^j).$$

The proof of this only uses Heine's transformation [5, Eq. (III.I)] and an exercise from Andrews' book [2, Ex. 10, p. 29]. The rest of the proof of Theorem 1.1 uses this conjugate Bailey pair, the Bailey Transform, and Slater's Bailey pairs A(4) and A(2) (with $a = q$) [8, p. 463]. The necessary background on conjugate Bailey pairs, Bailey pairs, and the Bailey Transform is given in Sect. 2. In Sect. 3, the proof of Theorem 1.1 is completed.

Using the same conjugate Bailey pair and Slater's A(7*), A(8), and A(6) (with $a = q$) leads to new Hecke–Rogers indefinite binary theta series identities for Ramanujan's three seventh order mock theta functions. A(7*) is actually a variant of A(7) adjusted to work with $a = q$ instead of $a = 1$. The three identities given below in Theorem 1.2 appear to be new. The following are Ramanujan's three seventh order mock theta functions:

$$\begin{aligned}
\mathcal{F}_0(q) &= \sum_{n=0}^{\infty} \frac{q^{n^2}}{(q^{n+1};q)_n} \\
&= 1 + q + q^3 + q^4 + q^5 + 2\,q^7 + q^8 + 2\,q^9 + \cdots, \\
\mathcal{F}_1(q) &= \sum_{n=1}^{\infty} \frac{q^{n^2}}{(q^{n};q)_n} \\
&= q + q^2 + q^3 + 2\,q^4 + q^5 + 2\,q^6 + 2\,q^7 + 2\,q^8 + \cdots, \\
\mathcal{F}_2(q) &= \sum_{n=0}^{\infty} \frac{q^{n^2+n}}{(q^{n+1};q)_{n+1}} \\
&= 1 + q + 2\,q^2 + q^3 + 2\,q^4 + 2\,q^5 + 3\,q^6 + 2\,q^7 + \cdots.
\end{aligned}$$

We have the following theorem:

Theorem 1.2.

$$\begin{aligned}
&(q)_\infty \mathcal{F}_0(q) \\
&= \sum_{j=1}^{\infty} \sum_{-j \le 3m \le j-1} \operatorname{sgn}(m)(-1)^{m+j+1} q^{j(3j-1)/2-m(21m+13)/2-1} \\
&\qquad \times (1+q^j)(1-q^{6m+1}),
\end{aligned} \tag{1.3}$$

$$\begin{aligned}
&(q)_\infty \mathcal{F}_1(q) \\
&= \sum_{j=1}^{\infty} \sum_{-j \le 3m \le j-1} \operatorname{sgn}(m)(-1)^{m+j+1} q^{j(3j-1)/2-m(21m+5)/2}(1+q^j) \\
&\quad + \sum_{j=2}^{\infty} \sum_{-j-1 \le 3m \le j-2} \operatorname{sgn}(m)(-1)^{m+j+1} \\
&\qquad \times q^{j(3j-1)/2-m(21m+19)/2-2}(1+q^j),
\end{aligned} \tag{1.4}$$

$$\begin{aligned}
&(q)_\infty \mathcal{F}_2(q)\\
&\quad = \sum_{j=1}^{\infty} \sum_{-j \le 3m \le j-1} \operatorname{sgn}(m)(-1)^{m+j+1} q^{j(3j-1)/2-m(21m+11)/2-1}(1+q^j)\\
&\qquad + \sum_{j=2}^{\infty} \sum_{-j-1 \le 3m \le j-2} \operatorname{sgn}(m)(-1)^{m+j+1}\\
&\qquad\qquad \times q^{j(3j-1)/2-m(21m+17)/2-2}(1+q^j).
\end{aligned} \tag{1.5}$$

We prove this theorem in Sect. 4. In his last letter to Hardy, all that Ramanujan said about the seventh order functions was that they were not related to each other. Surprisingly, we show that the coefficients of the three seventh order functions are indeed related, although this is probably not the kind of relationship that Ramanujan had in mind. For example, we find for $n \ge 0$ that

$$f_0(25n+8) = f_2(n), \tag{1.6}$$

$$f_1(25n+1) = f_0(n), \tag{1.7}$$

$$f_2(25n-3) = -f_1(n), \tag{1.8}$$

where we define $f_j(n)$ by

$$\sum_{n=0}^{\infty} f_j(n) q^n = (q)_\infty \, \mathcal{F}_j(q),$$

for $j = 0,\ 1,\ 2$. This and more general results including analogous results for the fifth order functions are proved in Sect. 5.

2. The Bailey Transform and Conjugate Bailey Pairs

Theorem 2.1. (The Bailey Transform). *Subject to suitable convergence conditions, if*

$$\beta_n = \sum_{r=0}^{n} \alpha_r u_{n-r} v_{n+r}, \quad \textit{and} \quad \gamma_n = \sum_{r=n}^{\infty} \delta_r u_{r-n} v_{r+n}, \tag{2.1}$$

then

$$\sum_{n=0}^{\infty} \alpha_n \gamma_n = \sum_{n=0}^{\infty} \beta_n \delta_n. \tag{2.2}$$

When applying his transform, Bailey [4] chose $u_n = 1/(q)_n$ and $v_n = 1/(aq;q)_n$. This motivates the following definitions:

Definition 2.2. A pair of sequences (α_n, β_n) is a **Bailey pair** relative to (a, q) if

$$\beta_n = \sum_{r=0}^{n} \frac{\alpha_r}{(q)_{n-r}(aq)_{n+r}}, \tag{2.3}$$

for $n \ge 0$.

Definition 2.3. A pair of sequences (γ_n, δ_n) is a **conjugate Bailey pair** relative to (a, q) if

$$\gamma_n = \sum_{r=n}^{\infty} \frac{\delta_r}{(q)_{r-n}(aq)_{r+n}}, \tag{2.4}$$

for $n \geq 0$.

The basic idea is to find a suitable conjugate Bailey pair and apply the Bailey Transform using known Bailey pairs.

Theorem 2.4. *The sequences*

$$\delta_n = \frac{q^n (q)_n (q)_\infty}{(1-q)}, \tag{2.5}$$

$$\gamma_n = \sum_{j=n+1}^{\infty} (-1)^{j+n+1} q^{j(3j-1)/2-3n(n+1)/2-1}(1+q^j), \tag{2.6}$$

form a conjugate Bailey pair relative to (q, q), i.e., $a = q$.

Remark 2.5. We note that this result can be deduced from a special case of a result of Lovejoy [7, Thm 1.1(4), p. 53]. We give a simple proof that uses only Heine's transformation and a combinatorial result of Andrews [2, Ex. 10, p. 29].

Proof of Theorem 2.4. We let

$$\delta_n = (q)_n (q)_\infty \frac{q^n}{(1-q)},$$

and

$$\gamma_n = \sum_{r=n}^{\infty} \frac{\delta_r}{(q)_{r-n}(q^2; q)_{r+n}} = (q)_\infty \sum_{r=n}^{\infty} \frac{(q)_r q^r}{(q)_{r-n}(q; q)_{r+n+1}}.$$

We must show that γ_n is given by (2.6). Note that

$$\begin{aligned}
&\sum_{r=n}^{\infty} \frac{(q)_r q^r}{(q)_{r-n}(q; q)_{r+n+1}} \\
&\quad= \sum_{r=0}^{\infty} \frac{(q)_{r+n} q^{r+n}}{(q)_r (q)_{r+2n+1}} \\
&\quad= q^n \frac{(q)_n}{(q)_{2n+1}} \sum_{r=0}^{\infty} \frac{(q^{n+1}; q)_r q^r}{(q)_r (q^{2n+2}; q)_r} \\
&\quad= q^n \frac{(q)_n}{(q)_{2n+1}} {}_2\phi_1 \left[\begin{matrix} 0, q^{n+1} \\ q^{2n+2} \end{matrix} ; q, q \right] \\
&\quad= q^n \frac{(q)_n}{(q)_{2n+1}} \frac{(q^{n+1}; q)_\infty}{(q^{2n+2}; q)_\infty (q)_\infty} {}_2\phi_1 \left[\begin{matrix} q^{n+1}, q \\ 0 \end{matrix} ; q, q^{n+1} \right] \\
&\quad= q^n \frac{1}{(q)_\infty} \sum_{j=0}^{\infty} (q^{n+1}; q)_j q^{(n+1)j},
\end{aligned}$$

by Heine's transformation [5, Eq. (III.I)], so that

$$\gamma_n = q^n \sum_{j=0}^{\infty} (q^{n+1};q)_j q^{(n+1)j}. \tag{2.7}$$

From Andrews [2, Ex. 10, p. 29], we have:

$$\sum_{j=0}^{\infty} (xq)_j x^{j+1} q^{j+1} = \sum_{m=1}^{\infty} (-1)^{m-1} q^{m(3m-1)/2} x^{3m-2} (1 + xq^m). \tag{2.8}$$

Using (2.7) and (2.8) with $x = q^n$, we have:

$$\begin{aligned}
\gamma_n &= q^n \sum_{j=0}^{\infty} (q^{n+1};q)_j q^{(n+1)j} \\
&= \sum_{m=1}^{\infty} (-1)^{m-1} q^{m(3m-1)/2 + n(3m-2) - 1} (1 + q^{m+n}) \\
&= \sum_{m=n+1}^{\infty} (-1)^{m+n+1} q^{m(3m-1)/2 - 3n(n+1)/2 - 1} (1 + q^m),
\end{aligned}$$

as required. We note that Subbarao [9] gave a combinatorial proof of (2.8) using a variant of Franklin's involution [2, pp. 10–11]. □

3. Proof of Theorem 1.1

To prove Theorem 1.1, we will apply the Bailey Transform, with $u_n = 1/(q)_n$, $v_n = 1/(q^2;q)_n$, using the conjugate Bailey pair in Theorem 2.4, and Slater's Bailey pairs $A(4)$ and $A(2)$. By [8, p. 463], the following gives Slater's $A(4)$ Bailey pair relative to (q,q):

$$\beta_n = \frac{q^n}{(q^2;q)_{2n}}, \quad \alpha_n = \begin{cases} q^{6m^2-4m}, & \text{if } n = 3m-1, \\ q^{6m^2+4m}, & \text{if } n = 3m, \\ -q^{6m^2+8m+2} - q^{6m^2+4m}, & \text{if } n = 3m+1. \end{cases} \tag{3.1}$$

By [11, Eq. (A_0), p. 278], we have:

$$\begin{aligned}
\chi_0(q) &= \sum_{n=0}^{\infty} \frac{q^n}{(q^{n+1};q)_n} \\
&= 1 + \sum_{n=0}^{\infty} \frac{q^{2n+1}}{(q^{n+1};q)_{n+1}} \\
&= 1 + q \sum_{n=0}^{\infty} \frac{q^n}{(q^2;q)_{2n}} \cdot \frac{q^n (q)_n}{(1-q)} \\
&= 1 + \frac{q}{(q)_\infty} \sum_{n=0}^{\infty} \beta_n \delta_n,
\end{aligned}$$

where δ_n is given in (2.5). Thus, by the Bailey Transform and (3.1), we have:

$$\begin{aligned}
&(q)_\infty\,(\chi_0(q)-1)\\
&\quad= q\sum_{n=0}^{\infty}\beta_n\delta_n\\
&\quad= q\sum_{n=0}^{\infty}\alpha_n\gamma_n\\
&\quad= \sum_{m=1}^{\infty}\sum_{j=3m}^{\infty}(-1)^{m+j}q^{j(3j-1)/2-m(15m-1)/2}(1+q^j)\\
&\qquad+\sum_{m=0}^{\infty}\sum_{j=3m+1}^{\infty}(-1)^{m+j+1}q^{j(3j-1)/2-m(15m+1)/2}(1+q^j)\\
&\qquad+\sum_{m=0}^{\infty}\sum_{j=3m+2}^{\infty}(-1)^{m+j+1}\Big\{q^{j(3j-1)/2-m(15m+11)/2-1}\\
&\qquad\qquad+q^{j(3j-1)/2-m(15m+19)/2-3}\Big\}(1+q^j)\\
&\quad= \sum_{j=1}^{\infty}\sum_{-j\le 3m\le j-1}\operatorname{sgn}(m)(-1)^{m+j+1}q^{j(3j-1)/2-m(15m+1)/2}(1+q^j)\\
&\qquad+\sum_{j=2}^{\infty}\sum_{-j-1\le 3m\le j-2}\operatorname{sgn}(m)(-1)^{m+j+1}\\
&\qquad\qquad\times q^{j(3j-1)/2-m(15m+11)/2-1}(1+q^j),
\end{aligned}\tag{3.2}$$

by noting that

$$-(-m-1)(15(-m-1)+19)/2-3=-m(15m+11)/2-1.$$

Now, from Euler's Pentagonal Number Theorem [2, p. 11], we have:

$$(q)_\infty=\sum_{n=-\infty}^{\infty}(-1)^nq^{n(3n-1)/2}=\sum_{m=-\infty}^{\infty}q^{6m^2+m}-\sum_{m=-\infty}^{\infty}q^{6m^2+5m+1}.\tag{3.3}$$

By (3.2) and (3.3), we have:

$$\begin{aligned}
&(q)_\infty\,(\chi_0(q)-2)\\
&\quad=(q)_\infty\,(\chi_0(q)-1)-(q)_\infty\\
&\quad=\sum_{j=1}^{\infty}\sum_{-j\le 3m\le j-1}\operatorname{sgn}(m)(-1)^{m+j+1}q^{j(3j-1)/2-m(15m+1)/2}(1+q^j)\\
&\qquad+\sum_{j=1}^{\infty}\sum_{-j-1\le 3m\le j-2}\operatorname{sgn}(m)(-1)^{m+j+1}q^{j(3j-1)/2-m(15m+11)/2-1}(1+q^j)\\
&\qquad-\sum_{m=-\infty}^{\infty}q^{6m^2+m}+\sum_{m=-\infty}^{\infty}q^{6m^2+5m+1}
\end{aligned}$$

$$
\begin{aligned}
&= \sum_{j=1}^{\infty} \sum_{-j+1\le 3m\le j-1} \operatorname{sgn}(m)(-1)^{m+j+1} q^{j(3j-1)/2-m(15m+1)/2}\\
&\quad + \sum_{j=0}^{\infty} \sum_{-j\le 3m\le j} \operatorname{sgn}(m)(-1)^{m+j+1} q^{j(3j+1)/2-m(15m+1)/2}\\
&\quad + \sum_{j=2}^{\infty} \sum_{-j\le 3m\le j-2} \operatorname{sgn}(m)(-1)^{m+j+1} q^{j(3j-1)/2-m(15m+11)/2-1}\\
&\quad + \sum_{j=1}^{\infty} \sum_{-j-1\le 3m\le j-1} \operatorname{sgn}(m)(-1)^{m+j+1} q^{j(3j+1)/2-m(15m+11)/2-1}.
\end{aligned}
$$

On the right side of the last equation above, replace j by $j+1$ in the first and third double sums to obtain:

$$
\begin{aligned}
&(q)_\infty(\chi_0(q)-2)\\
&\quad= \sum_{j=0}^{\infty} \sum_{-j\le 3m\le j} \operatorname{sgn}(m)(-1)^{m+j+1} q^{j(3j+1)/2-m(15m+1)/2}(1-q^{2j+1})\\
&\qquad + \sum_{j=1}^{\infty} \sum_{-j-1\le 3m\le j-1} \operatorname{sgn}(m)(-1)^{m+j+1}\\
&\qquad\qquad \times q^{j(3j+1)/2-m(15m+11)/2-1}(1-q^{2j+1}),
\end{aligned}
$$

which is (1.1).

To prove (1.2), we need Slater's [8, p. 463] A(2) Bailey pair relative to (q, q):

$$
\beta_n = \frac{1}{(q^2;q)_{2n}}, \quad \alpha_n = \begin{cases} q^{6m^2-m}, & \text{if } n = 3m-1,\\ q^{6m^2+m}, & \text{if } n = 3m,\\ -q^{6m^2+5m+1} - q^{6m^2+7m+2}, & \text{if } n = 3m+1. \end{cases} \tag{3.4}
$$

We have:

$$
\begin{aligned}
\chi_1(q) &= \sum_{n=0}^{\infty} \frac{q^n}{(q^{n+1};q)_{n+1}}\\
&= \sum_{n=0}^{\infty} \frac{q^n(q)_n}{(q)_{2n+1}}\\
&= \sum_{n=0}^{\infty} \frac{1}{(q^2;q)_{2n}} \cdot \frac{q^n(q)_n}{(1-q)}\\
&= \frac{1}{(q)_\infty} \sum_{n=0}^{\infty} \beta_n \delta_n.
\end{aligned}
$$

By the Bailey Transform and (3.4), we have:

$$
\begin{aligned}
&(q)_\infty \chi_1(q)\\
&\quad = \sum_{n=0}^{\infty} \beta_n \delta_n = \sum_{n=0}^{\infty} \alpha_n \gamma_n\\
&\quad = \sum_{m=1}^{\infty} \sum_{j=3m}^{\infty} (-1)^{m+j} q^{j(3j-1)/2-m(15m-7)/2-1}(1+q^j)\\
&\qquad + \sum_{m=0}^{\infty} \sum_{j=3m+1}^{\infty} (-1)^{m+j+1} q^{j(3j-1)/2-m(15m+7)/2-1}(1+q^j)\\
&\qquad + \sum_{m=0}^{\infty} \sum_{j=3m+2}^{\infty} (-1)^{m+j+1} \Big\{ q^{j(3j-1)/2-m(15m+17)/2-3}\\
&\qquad\qquad + q^{j(3j-1)/2-m(15m+13)/2-2} \Big\} (1+q^j)\\
&\quad = \sum_{j=1}^{\infty} \sum_{-j \le 3m \le j-1} \operatorname{sgn}(m)(-1)^{m+j+1} q^{j(3j-1)/2-m(15m+7)/2-1}(1+q^j)\\
&\qquad + \sum_{j=1}^{\infty} \sum_{-j-1 \le 3m \le j-2} \operatorname{sgn}(m)(-1)^{m+j+1} q^{j(3j-1)/2-m(15m+13)/2-2}(1+q^j),
\end{aligned}
\tag{3.5}
$$

by noting that

$$-(-m-1)(15(-m-1)+17)/2-3 = -m(15m+13)/2-2.$$

This completes the proof of Theorem 1.1.

4. Proof of Theorem 1.2

To prove Theorem 1.2, we proceed as in Sect. 3. This time we need Slater's Bailey pairs $A(6)$ and $A(8)$, and a variant of her Bailey pair $A(7)$.

From [8, Eq. (3.4), p. 464], we have:

$$
\begin{aligned}
\frac{q^{n^2-n}}{(q)_{2n}} &= \sum_{r=-[(n+1)/3]}^{[n/3]} \frac{(1-q^{6r+1})q^{3r^2-2r}}{(q)_{n+3r+1}(q)_{n-3r}}\\
&= \sum_{r=0}^{[n/3]} \frac{(1-q^{6r+1})q^{3r^2-2r}}{(q)_{n-3r}(q)_{n+3r+1}} + \sum_{r=1}^{[(n+1)/3]} \frac{(1-q^{-6r+1})q^{3r^2+2r}}{(q)_{n+3r}(q)_{n+1-3r}}\\
&= \sum_{r=1}^{[(n+1)/3]} \frac{q^{3r^2+2r}-q^{3r^2-4r+1}}{(q)_{n-(3r-1)}(q)_{n+(3r-1)+1}} + \sum_{r=0}^{[n/3]} \frac{q^{3r^2-2r}-q^{3r^2+4r+1}}{(q)_{n-3r}(q)_{n+3r+1}},
\end{aligned}
$$

so that

$$
\frac{(1-q)q^{n^2-n}}{(q)_{2n}} = \sum_{r=1}^{[(n+1)/3]} \frac{q^{3r^2+2r}-q^{3r^2-4r+1}}{(q)_{n-(3r-1)}\,(q^2;q)_{n+(3r-1)}} + \sum_{r=0}^{[n/3]} \frac{q^{3r^2-2r}-q^{3r^2+4r+1}}{(q)_{n-3r}\,(q^2;q)_{n+3r}}.
$$

This implies the following Bailey pair relative to (q,q):

$$\beta_n = \frac{(1-q)q^{n^2-n}}{(q)_{2n}}, \quad \alpha_n = \begin{cases} q^{3m^2+2m} - q^{3m^2-4m+1}, & \text{if } n = 3m-1, \\ q^{3m^2-2m} - q^{3m^2+4m+1}, & \text{if } n = 3m, \\ 0, & \text{if } n = 3m+1. \end{cases} \tag{4.1}$$

We note that this Bailey pair was found by Warnaar [10, p. 375] using a different method. We have:

$$\begin{aligned} \mathcal{F}_0(q) &= \sum_{n=0}^{\infty} \frac{q^{n^2}}{(q^{n+1};q)_n} \\ &= \sum_{n=0}^{\infty} \frac{(1-q)q^{n^2-n}}{(q)_{2n}} \cdot \frac{q^n(q)_n}{(1-q)} \\ &= \frac{1}{(q)_\infty} \sum_{n=0}^{\infty} \beta_n \delta_n. \end{aligned}$$

Thus, by the Bailey Transform and (4.1), we have:

$$\begin{aligned} &(q)_\infty \mathcal{F}_0(q) \\ &\quad = \sum_{n=0}^{\infty} \beta_n \delta_n = \sum_{n=0}^{\infty} \alpha_n \gamma_n \\ &\quad = \sum_{m=1}^{\infty} \sum_{j=3m}^{\infty} (-1)^{m+j} q^{j(3j-1)/2 - m(21m-13)/2-1}(1+q^j) \\ &\qquad + \sum_{m=1}^{\infty} \sum_{j=3m}^{\infty} (-1)^{m+j+1} q^{j(3j-1)/2 - m(21m-1)/2}(1+q^j) \\ &\qquad + \sum_{m=0}^{\infty} \sum_{j=3m+1}^{\infty} (-1)^{m+j+1} q^{j(3j-1)/2 - m(21m+13)/2-1}(1+q^j) \\ &\qquad + \sum_{m=0}^{\infty} \sum_{j=3m+1}^{\infty} (-1)^{m+j} q^{j(3j-1)/2 - m(21m+1)/2}(1+q^j) \\ &\quad = \sum_{j=1}^{\infty} \sum_{-j \le 3m \le j-1} \operatorname{sgn}(m)(-1)^{m+j+1} q^{j(3j-1)/2 - m(21m+13)/2-1}(1+q^j) \\ &\qquad + \sum_{j=1}^{\infty} \sum_{-j \le 3m \le j-1} \operatorname{sgn}(m)(-1)^{m+j} q^{j(3j-1)/2 - m(21m+1)/2}(1+q^j) \\ &\quad = \sum_{j=1}^{\infty} \sum_{-j \le 3m \le j-1} \operatorname{sgn}(m)(-1)^{m+j+1} q^{j(3j-1)/2 - m(21m+13)/2-1} \\ &\qquad\qquad \times (1+q^j)(1-q^{6m+1}), \end{aligned} \tag{4.2}$$

which is (1.3).

To prove (1.4), we need Slater's [8, p. 463] A(8) Bailey pair relative to (q,q):

$$\beta_n = \frac{q^{n^2+n}}{(q^2;q)_{2n}}, \quad \alpha_n = \begin{cases} q^{3m^2-2m}, & \text{if } n = 3m-1, \\ q^{3m^2+2m}, & \text{if } n = 3m, \\ -q^{3m^2+4m+1} - q^{3m^2+2m}, & \text{if } n = 3m+1. \end{cases} \tag{4.3}$$

We have

$$\begin{aligned} \mathcal{F}_1(q) &= \sum_{n=1}^{\infty} \frac{q^{n^2}}{(q^n;q)_n} \\ &= q\sum_{n=0}^{\infty} \frac{q^{n^2+2n}}{(q^{n+1};q)_{n+1}} \\ &= q\sum_{n=0}^{\infty} \frac{q^{n^2+n}}{(q^2;q)_{2n}} \cdot \frac{q^n(q)_n}{(1-q)} \\ &= \frac{q}{(q)_\infty} \sum_{n=0}^{\infty} \beta_n \delta_n. \end{aligned}$$

Thus, by the Bailey Transform and (4.3), we have:

$$\begin{aligned} &(q)_\infty \mathcal{F}_1(q) \\ &\quad= \sum_{n=0}^{\infty} \beta_n \delta_n = \sum_{n=0}^{\infty} \alpha_n \gamma_n \\ &\quad= \sum_{m=1}^{\infty} \sum_{j=3m}^{\infty} (-1)^{m+j} q^{j(3j-1)/2-m(21m-5)/2}(1+q^j) \\ &\qquad+ \sum_{m=0}^{\infty} \sum_{j=3m+1}^{\infty} (-1)^{m+j+1} q^{j(3j-1)/2-m(21m+5)/2}(1+q^j) \\ &\qquad+ \sum_{m=0}^{\infty} \sum_{j=3m+2}^{\infty} (-1)^{m+j+1} \Big\{ q^{j(3j-1)/2-m(21m+19)/2-2}(1+q^j) \\ &\qquad\qquad + q^{j(3j-1)/2-m(21m+23)/2-3}(1+q^j) \Big\} \\ &\quad= \sum_{j=1}^{\infty} \sum_{-j\le 3m\le j-1} \operatorname{sgn}(m)(-1)^{m+j+1} q^{j(3j-1)/2-m(21m+5)/2}(1+q^j) \\ &\qquad+ \sum_{j=2}^{\infty} \sum_{-j-1\le 3m\le j-2} \operatorname{sgn}(m)(-1)^{m+j+1} q^{j(3j-1)/2-m(21m+19)/2-2}(1+q^j), \end{aligned} \tag{4.4}$$

which is (1.4).

To prove (1.5), we need Slater's [8, p. 463] A(6) Bailey pair relative to (q,q):

$$\beta_n = \frac{q^{n^2}}{(q^2;q)_{2n}}, \quad \alpha_n = \begin{cases} q^{3m^2+m}, & \text{if } n = 3m-1, \\ q^{3m^2-m}, & \text{if } n = 3m, \\ -q^{3m^2+m} - q^{3m^2+5m+2}, & \text{if } n = 3m+1. \end{cases} \tag{4.5}$$

We have

$$\mathcal{F}_2(q) = \sum_{n=0}^{\infty} \frac{q^{n^2+n}}{(q^{n+1};q)_{n+1}} = \sum_{n=0}^{\infty} \frac{q^{n^2}}{(q^2;q)_{2n}} \cdot \frac{q^n(q)_n}{(1-q)} = \frac{1}{(q)_\infty} \sum_{n=0}^{\infty} \beta_n \delta_n.$$

Thus, by the Bailey Transform and (4.5), we have:

$$\begin{aligned}
&(q)_\infty \mathcal{F}_2(q) \\
&\quad = \sum_{n=0}^{\infty} \beta_n \delta_n = \sum_{n=0}^{\infty} \alpha_n \gamma_n \\
&\quad = \sum_{m=1}^{\infty} \sum_{j=3m}^{\infty} (-1)^{m+j} q^{j(3j-1)/2 - m(21m-11)/2 - 1}(1+q^j) \\
&\qquad + \sum_{m=0}^{\infty} \sum_{j=3m+1}^{\infty} (-1)^{m+j+1} q^{j(3j-1)/2 - m(21m+11)/2 - 1}(1+q^j) \\
&\qquad + \sum_{m=0}^{\infty} \sum_{j=3m+2}^{\infty} (-1)^{m+j+1} \Big\{ q^{j(3j-1)/2 - m(21m+25)/2 - 4}(1+q^j) \\
&\qquad\qquad + q^{j(3j-1)/2 - m(21m+17)/2 - 2}(1+q^j) \Big\} \\
&\quad = \sum_{j=1}^{\infty} \sum_{-j \le 3m \le j-1} \operatorname{sgn}(m)(-1)^{m+j+1} q^{j(3j-1)/2 - m(21m+11)/2 - 1}(1+q^j) \\
&\qquad + \sum_{j=2}^{\infty} \sum_{-j-1 \le 3m \le j-2} \operatorname{sgn}(m)(-1)^{m+j+1} q^{j(3j-1)/2 - m(21m+17)/2 - 2}(1+q^j),
\end{aligned} \tag{4.6}$$

which is (1.5). This completes the proof of Theorem 1.2.

5. Zagier's Mock Theta Function Identities and Related Results

In this section, we write our double-series identities for the two fifth order functions $\chi_0(q)$ and $\chi_1(q)$ and all three seventh order functions $\mathcal{F}_j(q)$ ($j = 0, 1, 2$) using Dirichlet characters. This leads naturally to relations between the coefficients of these series as in Theorems 5.5 and 5.6.

As mentioned before, Andrews [1] obtained indefinite theta series identities for all of Ramanujan's fifth order functions except $\chi_0(q)$ and $\chi_1(q)$. Using Andrews' results, Zwegers [14] showed how to complete all of Andrews' fifth order functions to weak harmonic Maass forms. As noted by Watson [11, pp.

277–279], Ramanujan gave identities for $\chi_0(q)$ and $\chi_1(q)$ in terms of the other fifth order functions. Zagier suggested that indefinite theta function identities for $\chi_0(q)$ and $\chi_1(q)$ could be obtained from Ramanujan's results and Zwegers' transformation formulas, although he gave no details. We state Zagier's results in a modified form in the following:

Theorem 5.1.

$$(q)_\infty(2-\chi_0(q)) = \sum_{\substack{5|b|<|a| \\ a+b\equiv 2 \pmod 4 \\ a\equiv 2 \pmod 5}} (-1)^a \operatorname{sgn}(a)\left(\frac{-3}{a^2-b^2}\right) q^{\frac{1}{120}(a^2-5b^2)-\frac{1}{30}},$$

and

$$(q)_\infty\chi_1(q) = \sum_{\substack{5|b|<|a| \\ a+b\equiv 2 \pmod 4 \\ a\equiv 4 \pmod 5}} (-1)^a \operatorname{sgn}(a)\left(\frac{-3}{a^2-b^2}\right) q^{\frac{1}{120}(a^2-5b^2)-\frac{19}{30}}.$$

Remark 5.2. Here, $\left(\frac{-3}{\cdot}\right)$ is the Kronecker symbol, and is a Dirichlet character mod 3.

Our Theorem 1.1 seems to differ from Zagier's theorem. In contrast to Zagier's theorem which involves a character mod 3, our version involves the Dirichlet character mod 60:

$$\chi_{60}(m) = \begin{cases} 1, & \text{if } m \equiv 1, 11, 19, 29 \pmod{60}, \\ i, & \text{if } m \equiv 7, 13, 17, 23 \pmod{60}, \\ -1, & \text{if } m \equiv 31, 41, 49, 59 \pmod{60}, \\ -i, & \text{if } m \equiv 37, 43, 47, 53 \pmod{60}: \end{cases}$$

Theorem 5.3.

$$(q)_\infty(2-\chi_0(q)) = \sum_{\substack{3|b|<5|a| \\ a\equiv 1 \pmod 6 \\ b\equiv 1,11 \pmod{30}}} \operatorname{sgn}(b)\left(\frac{12}{a}\right)\chi_{60}(b) q^{\frac{1}{120}(5a^2-b^2)-\frac{1}{30}}, \tag{5.1}$$

and

$$(q)_\infty\chi_1(q) = i \sum_{\substack{3|b|<5|a| \\ a\equiv b\equiv 1 \pmod 6 \\ b\equiv \pm 2 \pmod 5}} \operatorname{sgn}(b)\left(\frac{12}{a}\right)\chi_{60}(b) q^{\frac{1}{120}(5a^2-b^2)-\frac{19}{30}}. \tag{5.2}$$

We find analogous identities for the seventh order functions. Also Andrews [1] obtained indefinite theta series identities for these functions. Hickerson [6, Theorem 2.0, p. 666] found nice versions of Andrews' identities, which he used to prove his seventh order analogs of Ramanujan's mock theta conjectures [3] for the fifth order functions. Our identities differ from Andrews' and Hickerson's and appear to be new:

Theorem 5.4.

$$(q)_\infty \mathcal{F}_0(q) = \sum_{\substack{3|b|<7|a| \\ a\equiv 1 \pmod 6 \\ b\equiv 1,13 \pmod{42}}} \operatorname{sgn}(b)\left(\frac{12}{a}\right)\left(\frac{12}{b}\right)\left(\frac{b}{7}\right) q^{\frac{1}{168}(7a^2-b^2)-\frac{1}{28}}, \quad (5.3)$$

$$(q)_\infty \mathcal{F}_1(q) = -\sum_{\substack{3|b|<7|a| \\ a\equiv 1 \pmod 6 \\ b\equiv 5,19 \pmod{42}}} \operatorname{sgn}(b)\left(\frac{12}{a}\right)\left(\frac{12}{b}\right)\left(\frac{b}{7}\right) q^{\frac{1}{168}(7a^2-b^2)+\frac{3}{28}}, \quad (5.4)$$

and

$$(q)_\infty \mathcal{F}_2(q) = -\sum_{\substack{3|b|<7|a| \\ a\equiv 1 \pmod 6 \\ b\equiv 11,17 \pmod{42}}} \operatorname{sgn}(b)\left(\frac{12}{a}\right)\left(\frac{12}{b}\right)\left(\frac{b}{7}\right) q^{\frac{1}{168}(7a^2-b^2)-\frac{9}{28}}. \quad (5.5)$$

We sketch the proof of (5.2). First, we observe that:

$$\tfrac{1}{2}j(3j\pm 1) - \tfrac{1}{2}m(15m+7) - 1 = \tfrac{1}{120}\left(5(6j\pm 1)^2 - (30m+7)^2\right) - \tfrac{19}{30},$$
$$\tfrac{1}{2}j(3j\pm 1) - \tfrac{1}{2}m(15m+13) - 2 = \tfrac{1}{120}\left(5(6j\pm 1)^2 - (30m+13)^2\right) - \tfrac{19}{30}.$$

In the summations in Eq. (5.2), we let $a = 6(\pm j)+1$, and $b = 30m + r$, where $j \ge 1$, $m \in \mathbb{Z}$, and $r = 7$, 13. We have:

$$\left(\frac{12}{a}\right) = \left(\frac{12}{6(\pm j)+1}\right) = (-1)^j,$$
$$i\chi_{60}(b) = i\chi_{60}(30m+r) = (-1)^{m+1},$$
$$\operatorname{sgn}(b) = \operatorname{sgn}(30m+r) = \operatorname{sgn}(m).$$

Next, we consider the inequalities for the variables in the summations.

Case 1

$m \ge 0$ and $r = 7$. Then, we see that

$$3|b| < 5|a| \Leftrightarrow 3m < j + \left(\frac{\pm 5 - 21}{30}\right) \Leftrightarrow 3m \le j-1.$$

Case 2

$m < 0$ and $r = 7$. Then, we see that

$$3|b| < 5|a| \Leftrightarrow -j < 3m + \left(\frac{\pm 5 + 21}{30}\right) \Leftrightarrow -j \le 3m.$$

Case 3

$m \ge 0$ and $r = 13$. Then, we see that

$$3|b| < 5|a| \Leftrightarrow 3m < j + \left(\frac{\pm 5 - 39}{30}\right) \Leftrightarrow 3m \le j-2.$$

Case 4

$m < 0$ and $r = 13$. Then, we see that

$$3|b| < 5|a| \Leftrightarrow -j + \left(\frac{-39 \pm 5}{30}\right) < 3m \Leftrightarrow -j - 1 \leq 3m.$$

It follows that

$$\sum_{j=1}^{\infty} \sum_{-j \leq 3m \leq j-1} \operatorname{sgn}(m)(-1)^{m+j+1} q^{j(3j-1)/2 - m(15m+7)/2-1}(1+q^j)$$
$$= i \sum_{\substack{3|b|<5|a| \\ a \equiv 1 \pmod 6 \\ b \equiv 7 \pmod{30}}} \operatorname{sgn}(b) \left(\frac{12}{a}\right) \chi_{60}(b) q^{\frac{1}{120}(5a^2-b^2) - \frac{19}{30}},$$

and

$$\sum_{j=1}^{\infty} \sum_{-j-1 \leq 3m \leq j-2} \operatorname{sgn}(m)(-1)^{m+j+1} q^{j(3j-1)/2 - m(15m+13)/2-2}(1+q^j)$$
$$= i \sum_{\substack{3|b|<5|a| \\ a \equiv 1 \pmod 6 \\ b \equiv 13 \pmod{30}}} \operatorname{sgn}(b) \left(\frac{12}{a}\right) \chi_{60}(b) q^{\frac{1}{120}(5a^2-b^2) - \frac{19}{30}}.$$

Therefore, we see that Eq. (1.2) implies (5.2). The proofs of the remaining parts of Theorems 5.3 and 5.4 are analogous.

Theorems 5.3 and 5.4 imply simple relations between the coefficients. We define the coefficients $C_0(n)$ and $C_1(n)$ by:

$$\sum_{n=0}^{\infty} C_0(n) q^n = (q)_\infty \left(2 - \chi_0(q)\right),$$
$$\sum_{n=0}^{\infty} C_1(n) q^n = (q)_\infty \, \chi_1(q),$$

define

$$\varepsilon_p = \begin{cases} -1, & \text{if } p \equiv 3 \pmod{10}, \\ 1, & \text{if } p \equiv 7 \pmod{10}, \end{cases} \tag{5.6}$$

and, for an integer n and a prime p, define $\nu_p(n)$ to be the exact power of p dividing n.

Theorem 5.5. *If $p > 5$ is any prime congruent to 3 or 7 modulo 10, then*

$$C_0(n) = 0, \quad \text{if } \nu_p(30n+1) = 1, \tag{5.7}$$
$$C_0\left(p^2 n + \tfrac{1}{30}(19p^2 - 1)\right) = -\varepsilon_p \, C_1(n), \quad \text{for } n \geq 0, \tag{5.8}$$
$$C_1(n) = 0, \quad \text{if } \nu_p(30n+19) = 1, \tag{5.9}$$
$$C_1\left(p^2 n + \tfrac{1}{30}(p^2 - 19)\right) = \varepsilon_p \, C_0(n), \quad \text{for } n \geq 0. \tag{5.10}$$

Proof. Suppose $p > 5$ is any prime congruent to 3 or 7 mod 10. Then, 5 is a quadratic nonresidue mod p. Therefore $5a^2 - b^2 \equiv 0 \pmod{p}$ implies that $a \equiv b \equiv 0 \pmod{p}$ and (5.7) clearly follows from (5.1). Similarly, (5.9) follows from (5.2).

We suppose $a \equiv 1 \pmod{6}$, $b \equiv 1, 11 \pmod{30}$, $3|b| < 5|a|$, and $a \equiv b \equiv 0 \pmod{p}$. Letting $a = pa'$, $b = pb'$, we have the following table:

$p \pmod{30}$	$a' \pmod{6}$	$b' \pmod{30}$
7	1	$13, -7$
13	1	$7, -13$
17	-1	$-7, 13$
23	-1	$-13, 7$

By considering the table and noting that the summation term:

$$\operatorname{sgn}(b)\left(\frac{12}{a}\right)\chi_{60}(b)q^{\frac{1}{120}(5a^2-b^2)-\frac{1}{30}}$$

is invariant under both $a \mapsto -a$ and $b \mapsto -b$, we see that

$$\begin{aligned}
&\sum_{n=0}^{\infty} C_0(p^2n + \tfrac{1}{30}(19p^2-1))\, q^{p^2n+\frac{1}{30}(19p^2-1)} \\
&\quad = \sum_{\substack{3|b'|<5|a'| \\ a'\equiv 1 \pmod{6} \\ b'\equiv 7,13 \pmod{30}}} \operatorname{sgn}(pb')\left(\frac{12}{pa'}\right)\chi_{60}(pb')q^{\frac{1}{120}(p^2(5(a')^2-(b')^2))-\frac{1}{30}} \\
&\quad = \left(\frac{12}{p}\right)\chi_{60}(p) \sum_{\substack{3|b|<5|a| \\ a\equiv 1 \pmod{6} \\ b\equiv 7,13 \pmod{30}}} \operatorname{sgn}(b)\left(\frac{12}{a}\right)\chi_{60}(b)q^{\frac{1}{120}(p^2(5a^2-b^2))-\frac{1}{30}}
\end{aligned}$$

and

$$\begin{aligned}
&\sum_{n=0}^{\infty} C_0(p^2n + \tfrac{1}{30}(19p^2-1))\, q^{n} \\
&\quad = -i\,\varepsilon_p \sum_{\substack{3|b|<5|a| \\ a\equiv 1 \pmod{6} \\ b\equiv 7,13 \pmod{30}}} \operatorname{sgn}(b)\left(\frac{12}{a}\right)\chi_{60}(b)q^{\frac{1}{120}(5a^2-b^2)-\frac{19}{30}} \\
&\quad = -\varepsilon_p(q)_\infty\,\chi_1(q) = -\varepsilon_p \sum_{n=0}^{\infty} C_1(n)q^n,
\end{aligned}$$

and (5.8) follows. The proof of (5.10) is analogous. □

In a similar fashion, Theorem 5.4 implies relations between the coefficients of the seventh order mock theta functions. For $j = 0$, 1, and 2, we define $f_j(n)$ by:

$$\sum_{n=0}^{\infty} f_j(n)q^n = (q)_\infty\,\mathcal{F}_j(q).$$

Theorem 5.6. *Let p be any prime for which 7 is a quadratic nonresidue modulo p, i.e., $p \equiv \pm 5, \pm 11$ or $\pm 13 \pmod{28}$.*

1. *Then*

$$\begin{aligned} f_0(n) &= 0, \quad \textit{if } \nu_p(28n+1) = 1, \\ f_1(n) &= 0, \quad \textit{if } \nu_p(28n-3) = 1, \\ f_2(n) &= 0, \quad \textit{if } \nu_p(28n+9) = 1. \end{aligned}$$

2. *If $p \equiv \pm 5 \pmod{28}$, then*

$$\begin{aligned} f_0(p^2 n + \tfrac{1}{28}(9p^2-1)) &= \pm f_2(n), \\ f_1(p^2 n + \tfrac{1}{28}(p^2+3)) &= \pm f_0(n), \\ f_2(p^2 n + \tfrac{1}{28}(25p^2-9)) &= \mp f_1(n+1). \end{aligned}$$

3. *If $p \equiv \pm 11 \pmod{28}$, then*

$$\begin{aligned} f_0(p^2 n + \tfrac{1}{28}(25p^2-1)) &= \mp f_1(n+1), \\ f_1(p^2 n + \tfrac{1}{28}(9p^2+3)) &= \pm f_2(n), \\ f_2(p^2 n + \tfrac{1}{28}(p^2-9)) &= \mp f_0(n). \end{aligned}$$

4. *If $p \equiv \pm 13 \pmod{28}$, then*

$$\begin{aligned} f_0(p^2 n + \tfrac{1}{28}(p^2-1)) &= \mp f_0(n), \\ f_1(p^2 n + \tfrac{1}{28}(25p^2+3)) &= \mp f_1(n+1), \\ f_2(p^2 n + \tfrac{1}{28}(9p^2-9)) &= \mp f_2(n). \end{aligned}$$

We omit the proof of Theorem 5.6. The proof is analogous to that of Theorem 5.5.

6. Concluding Remarks

In Theorems 5.3 and 5.4, we found new identities for the fifth order mock theta functions $\chi_0(q)$, $\chi_1(q)$ and all three seventh order mock theta functions $\mathcal{F}_0(q)$, $\mathcal{F}_1(q)$, $\mathcal{F}_2(q)$, in terms of Hecke–Rogers indefinite binary theta series. This suggests the problem of relating these theorems directly to the results of Zagier (Theorem 5.1) for the fifth order functions, and to the results of Andrews [1, Theorem 13, p. 132–133] and Hickerson [6, Theorem 2.0, p. 666] for the seventh order functions.

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Frank G. Garvan
Department of Mathematics
University of Florida
Gainesville FL 32611-8105
USA
e-mail: fgarvan@ufl.edu

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On Pattern-Avoiding Fishburn Permutations

Juan B. Gil and Michael D. Weiner

Abstract. The class of permutations that avoid the bivincular pattern $(231, \{1\}, \{1\})$ is known to be enumerated by the Fishburn numbers. In this paper, we call them *Fishburn permutations* and study their pattern avoidance. For classical patterns of size 3, we give a complete enumerative picture for regular and indecomposable Fishburn permutations. For patterns of size 4, we focus on a Wilf equivalence class of Fishburn permutations that are enumerated by the Catalan numbers. In addition, we also discuss a class enumerated by the binomial transform of the Catalan numbers and give conjectures for other equivalence classes of pattern-avoiding Fishburn permutations.

Mathematics Subject Classification. Primary 05A05; Secondary 05A15, 05A19.

Keywords. Pattern avoiding permutation, Fishburn number, Bivincular pattern.

1. Introduction

Motivated by a recent paper by Andrews and Sellers [1], we became interested in the Fishburn numbers $\xi(n)$, defined by the formal power series

$$\sum_{n=0}^{\infty} \xi(n) q^n = 1 + \sum_{n=1}^{\infty} \prod_{j=1}^{n} (1 - (1-q)^j).$$

They are listed as Sequence A022493 in [7] and have several combinatorial interpretations. For example, $\xi(n)$ gives the:

- ▷ number of linearized chord diagrams of degree n,
- ▷ number of unlabeled $(2+2)$-free posets on n elements,
- ▷ number of ascent sequences of length n,

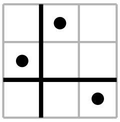
TABLE 1. σ-avoiding Fishburn permutations

Pattern σ	$\lvert\mathscr{F}_n(\sigma)\rvert$	OEIS
123, 132, 213, 312	1, 2, 4, 8, 16, 32, 64, 128, 256, 512, ...	A000079
231	1, 2, 5, 14, 42, 132, 429, 1430, 4862, ...	A000108
321	1, 2, 4, 9, 22, 57, 154, 429, 1223, 3550, ...	A105633

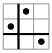
▹ number of permutations in S_n that avoid a certain bivincular pattern. [1]

For more information on these interpretations, we refer the reader to [2] and the references therein.

In this note, we are primarily concerned with the aforementioned class of permutations. That they are enumerated by the Fishburn numbers was proved in [2] by Bousquet-Mélou, Claesson, Dukes, and Kitaev, where the authors introduced bivincular patterns (permutations with restrictions on the adjacency of positions and values) and gave a bijection to ascent sequences. More specifically, a permutation $\pi \in S_n$ is said to contain the bivincular pattern $(231, \{1\}, \{1\})$ if there are positions $i < k$ with $\pi(i) > 1$, $\pi(k) = \pi(i) - 1$, such that the subsequence $\pi(i)\pi(i+1)\pi(k)$ forms a 231 pattern. Such a bivincular pattern may be visualized by the plot



where bold lines indicate adjacent entries and gray lines indicate an elastic distance between the entries.

We let $\mathscr{F}_n$ denote the class of permutations in S_n that avoid the pattern , and since $|\mathscr{F}_n| = \xi(n)$ (see [2]), we call the elements of $\mathscr{F} = \bigcup_n \mathscr{F}_n$ *Fishburn permutations*. Further, we let $\mathscr{F}_n(\sigma)$ denote the class of Fishburn permutations in $\mathscr{F}_n$ that avoid the pattern σ.

Our goal is to study $F_n(\sigma) = |\mathscr{F}_n(\sigma)|$ for classical patterns of size 3 or 4. In Sect. 2, we give a complete picture for regular and indecomposable Fishburn permutations that avoid a classical pattern of size 3. Table 1 and Table 2 provide a summary of our findings. In Sect. 3, we discuss patterns of size 4, focusing on a Wilf equivalence class of Fishburn permutations that are enumerated by the Catalan numbers C_n (see Table 4). We also prove the formula $F_n(1342) = \sum_{k=1}^{n} \binom{n-1}{k-1} C_{n-k}$, and conjecture two other equivalence classes (see Table 3). Finally, in Sect. 4, we briefly discuss indecomposable Fishburn permutations that avoid a pattern of size 4. In Table 5, we make some conjectures based on preliminary computations.

[1] S_n denotes the set of permutations on $[n] = \{1, \dots, n\}$.

TABLE 2. σ-avoiding indecomposable Fishburn permutations

Pattern σ	$\lvert\mathscr{F}_n^{\mathsf{ind}}(\sigma)\rvert$	OEIS
123	1, 1, 2, 5, 12, 27, 58, 121, 248, 503, ...	A000325
132, 213	1, 1, 2, 4, 8, 16, 32, 64, 128, 256, ...	A000079
231	1, 1, 2, 5, 14, 42, 132, 429, 1430, 4862, ...	A000108
312	1, 1, 1, 1, 1, 1, 1, 1, 1, 1, ...	A000012
321	1, 1, 1, 2, 5, 13, 35, 97, 275, 794, ...	A082582

TABLE 3. Equivalence classes with a single pattern

Pattern σ	$\lvert\mathscr{F}_n(\sigma)\rvert$	OEIS
1342	1, 2, 5, 15, 51, 188, 731, 2950, ...	A007317
1432	1, 2, 5, 14, 43, 142, 495, 1796, ...	
2314	1, 2, 5, 15, 52, 200, 827, 3601, ...	
2341	1, 2, 5, 15, 52, 202, 858, 3910, ...	
3412	1, 2, 5, 15, 52, 201, 843, 3764, ...	A202062(?)
3421	1, 2, 5, 15, 52, 203, 874, 4076, ...	
4123	1, 2, 5, 14, 42, 133, 442, 1535, ...	
4231	1, 2, 5, 15, 52, 201, 843, 3765, ...	
4312	1, 2, 5, 14, 43, 143, 508, 1905, ...	
4321	1, 2, 5, 14, 45, 162, 639, 2713, ...	

TABLE 4. Catalan equivalent class

Pattern σ	$\lvert\mathscr{F}_n(\sigma)\rvert$	OEIS
1234, 1243, 1324, 1423, 2134, 2143, 3124, 3142	1, 2, 5, 14, 42, 132, 429, 1430, 4862, ...	A000108

Basic notation. Permutations will be written in one-line notation. Given two permutations σ and τ of sizes k and ℓ, respectively, their direct sum $\sigma \oplus \tau$ is the permutation of size $k+\ell$ consisting of σ followed by a shifted copy of τ. Similarly, their skew sum $\sigma \ominus \tau$ is the permutation consisting of τ preceded by a shifted copy of σ. For example, $312 \oplus 21 = 31254$ and $312 \ominus 21 = 53421$.

A permutation is said to be indecomposable if it cannot be written as a direct sum of two nonempty permutations.

$\mathsf{Av}_n(\sigma)$ denotes the class of permutations in S_n that avoid the pattern σ. It is well known that if $\sigma \in S_3$ then $\lvert\mathsf{Av}_n(\sigma)\rvert = C_n$, where C_n is the Catalan number $\frac{1}{n+1}\binom{2n}{n}$, see e.g. [4].

TABLE 5. Some σ-avoiding indecomposable classes

Pattern σ	$\lvert\mathscr{F}_n^{\mathsf{ind}}(\sigma)\rvert$	OEIS
1234	1, 1, 2, 6, 22, 85, 324, 1204, ...	
1243, 2134	1, 1, 2, 6, 21, 75, 266, 938, ...	A289597(?)
1324	1, 1, 2, 6, 22, 84, 317, 1174, ...	
1342	1, 1, 2, 6, 22, 88, 367, 1568, ...	A165538
1423, 3124	1, 1, 2, 6, 20, 68, 233, 805, ...	A279557
1432	1, 1, 2, 6, 20, 71, 263, 1002, ...	
2143	1, 1, 2, 6, 19, 62, 207, 704, ...	A026012
2314	1, 1, 2, 6, 23, 99, 450, 2109, ...	
2341	1, 1, 2, 6, 22, 91, 409, 1955, ...	
2413, 2431, 3241	1, 1, 2, 6, 22, 90, 395, 1823, ...	A165546(?)
3142	1, 1, 2, 5, 14, 42, 132, 429, ...	A000108
3214	1, 1, 2, 6, 20, 72, 275, 1096, ...	
3412	1, 1, 2, 6, 22, 90, 396, 1840, ...	
3421	1, 1, 2, 6, 22, 92, 423, 2088, ...	
4123	1, 1, 2, 5, 14, 43, 143, 507, ...	
4132, 4213	1, 1, 2, 5, 15, 51, 188, 732, ...	
4231	1, 1, 2, 6, 22, 90, 396, 1841, ...	
4312	1, 1, 2, 5, 15, 51, 188, 733, ...	
4321	1, 1, 2, 5, 17, 66, 279, 1256, ...	

2. Avoiding Patterns of Size 3

Clearly, $\mathsf{Av}_n(231) \subset \mathscr{F}_n$. Now, since every Fishburn permutation that avoids the classical pattern 231 is contained in the set of 231-avoiding permutations, we get

$$\mathscr{F}_n(231) = \mathsf{Av}_n(231), \text{ and so } F_n(231) = C_n. \tag{2.1}$$

Enumeration of the Fishburn permutations that avoid the other five classical patterns of size 3 is less obvious.

Theorem 2.1. *For $\sigma \in \{123, 132, 213, 312\}$, we have $F_n(\sigma) = 2^{n-1}$.*

Proof. First of all, note that for every σ of size 3, we have $F_1(\sigma) = 1$ and $F_2(\sigma) = 2$.

CASE $\sigma = 132$: If $\pi \in \mathscr{F}_{n-1}(132)$, the permutations $1 \ominus \pi$ and $\pi \oplus 1$ are both in $\mathscr{F}_n(132)$. On the other hand, if τ is a permutation in $\mathsf{Av}_n(132)$ with $\tau(i) = n$ for some $1 < i < n$, then we must have $\tau(j) > \tau(k)$ for every $j \in \{1, \dots, i-1\}$ and $k \in \{i+1, \dots, n\}$. Thus, $n - i = \tau(k')$ for some $k' > i$ and $n - i + 1 = \tau(j')$ for some $j' < i$. But this violates the Fishburn condition since $n - i + 1$ is the smallest value to the left of n and must, therefore, be part of an ascent in $\tau(1) \cdots \tau(i-1)\, n$. In other words, $\mathscr{F}_n(132)$ is the disjoint union of the sets $\{1 \ominus \pi : \pi \in \mathscr{F}_{n-1}(132)\}$ and $\{\pi \oplus 1 : \pi \in \mathscr{F}_{n-1}(132)\}$. Thus

$$F_n(132) = 2F_{n-1}(132)$$

for $n > 1$,which implies $F_n(132) = 2^{n-1}$.

CASE $\sigma = 123$: For $n > 2$, the permutation $(n-1)(n-2)\cdots 21n$ is the only permutation in $\mathscr{F}_n(123)$ that ends with n, and if $\pi \in \mathscr{F}_{n-1}(123)$, then $1 \ominus \pi \in \mathscr{F}_n(123)$.

Assume $\tau \in \mathscr{F}_n(123)$ is such that $\tau(i) = n$ for some $1 < i < n$. Since τ avoids the pattern 123, we must have $\tau(1) > \tau(2) > \cdots > \tau(i-1)$. Moreover, the Fishburn condition forces $\tau(i-1) = 1$, which implies $\tau(i+1) > \tau(i+2) > \cdots > \tau(n)$. In other words, τ may be any permutation with $\tau(i-1) = 1$, $\tau(i) = n$ for which the entries to the left of 1 and to the right of n form two decreasing sequences. There are $\binom{n-2}{i-2}$ such permutations.

In conclusion, we have the recurrence

$$F_n(123) = F_{n-1}(123) + 1 + \sum_{i=2}^{n-1} \binom{n-2}{i-2} = F_{n-1}(123) + 2^{n-2},$$

which implies $F_n(123) = 2^{n-1}$.

CASE $\sigma = 213$: For $n > 2$, the permutation $12\cdots n$ is the only permutation in $\mathscr{F}_n(213)$ that ends with n, and if $\pi \in \mathscr{F}_{n-1}(213)$, then $1 \ominus \pi \in \mathscr{F}_n(213)$.

Assume $\tau \in \mathscr{F}_n(213)$ is such that $\tau(i) = n$ for some $1 < i < n$. Since τ avoids the pattern 213, we must have $\tau(1) < \tau(2) < \cdots < \tau(i-1)$ and the Fishburn condition forces $\tau(j) = j$ for every $j \in \{1, \ldots, i-1\}$. Thus τ must be of the form $\tau = 1 \cdots (i-1)n\pi_R$, where π_R may be any element of $\mathscr{F}_{n-i}(213)$. This implies

$$F_n(213) = 1 + \sum_{i=1}^{n-1} F_{n-i}(213),$$

and we conclude $F_n(213) = 2^{n-1}$.

CASE $\sigma = 312$: If $\pi \in \mathscr{F}_{n-1}(312)$, the permutation $1 \oplus \pi$ is in $\mathscr{F}_n(312)$. On the other hand, if τ is a permutation in $\mathsf{Av}_n(312)$ with $\tau(i) = 1$ for some $1 < i \le n$, then we must have $\tau(j) < \tau(k)$ for every $j \in \{1, \ldots, i-1\}$ and $k \in \{i+1, \ldots, n\}$. Moreover, the Fishburn condition forces $\tau(j) = i + 1 - j$ for every $j \in \{1, \ldots, i-1\}$. Thus, τ must be of the form $\tau = i \cdots 21\pi_R$, where $\pi_R = \emptyset$ if $i = n$, or $\pi_R \in \mathscr{F}_{n-i}(312)$ if $i < n$. This implies

$$F_n(312) = 1 + \sum_{i=1}^{n-1} F_{n-i}(312),$$

hence $F_n(312) = 2^{n-1}$. □

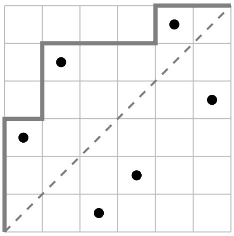
For our next result, we use a bijection between $\mathsf{Av}_n(321)$ and the set of Dyck paths of semilength n, via the left-to-right maxima.[2] Here, a Dyck path of semilength n is a simple lattice path from $(0,0)$ to (n,n) that stays weakly above the diagonal $y = x$ (with vertical unit steps U and horizontal unit steps D). On the other hand, a left-to-right maximum of a permutation π is an element π_i such that $\pi_j < \pi_i$ for every $j < i$.

[2]This is a slightly different version of a bijection given by Krattenthaler [5].

The bijective map between $\mathsf{Av}_n(321)$ and the set of Dyck paths of semilength n is defined as follows: Given $\pi \in \mathsf{Av}_n(321)$, write

$$\pi = m_1 w_1 m_2 w_2 \cdots m_s w_s,$$

where $m_1, \dots, m_s$ are the left-to-right maxima of π, and each w_i is a subword of π. Let $|w_i|$ denote the length of w_i. Reading the decomposition of π from left to right, we construct a path starting with m_1 U-steps, $|w_1| + 1$ D-steps, and for every other subword $m_i w_i$, we add $m_i - m_{i-1}$ U-steps followed by $|w_i| + 1$ D-steps. In short, identify the left-to-right maxima in the plot of π and draw your path over them. For example, for $\pi = 351264 \in \mathsf{Av}_6(321)$ we get:

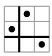


Note that $\pi = 351264 \notin \mathscr{F}_6$.

Theorem 2.2. *The set* $\mathscr{F}_n(321)$ *is in bijection with the set of Dyck paths of semilength* n *that avoid the subpath* $UUDU$*. By* [6, Proposition 5] *we then have*

$$F_n(321) = \sum_{j=0}^{\lfloor (n-1)/2 \rfloor} \frac{(-1)^j}{n-j} \binom{n-j}{j} \binom{2n-3j}{n-j+1}.$$

This is Sequence [7, A105633]*.*

Proof. Under the above bijection, an ascent $\pi_i < \pi_{i+1}$ in $\pi \in \mathsf{Av}_n(321)$ with $k = \pi_{i+1} - \pi_i$ generates the subpath UDU^k in the corresponding Dyck path P_π, and if $\pi_i - 1 = \pi_j$ for some $j > i+1$, then P_π must necessarily contain the subpath $UUDU^k$. Thus, we have that π avoids the pattern  if and only if P_π avoids $UUDU$. $\square$

2.1. Indecomposable Permutations

Let $\mathscr{F}_n^{\mathsf{ind}}(\sigma)$ be the set of indecomposable Fishburn permutations that avoid the pattern σ, and let $IF_n(\sigma)$ denote the number of elements in $\mathscr{F}_n^{\mathsf{ind}}(\sigma)$. Observe that for every σ of size ≥ 3, we have $IF_1(\sigma) = 1$ and $IF_2(\sigma) = 1$.

We start with a fundamental known lemma, see e.g. [3, Lemma 3.1].

Lemma 2.3. *If a pattern* σ *is indecomposable, then the sequence* $|\mathsf{Av}_n(\sigma)|$ *is the* INVERT *transform of the sequence* $|\mathsf{Av}_n^{ind}(\sigma)|$*. That is, if* $A^\sigma(x)$ *and* $A_I^\sigma(x)$ *are the corresponding generating functions, then*

$$1 + A^\sigma(x) = \frac{1}{1 - A_I^\sigma(x)}, \quad \textit{and so} \quad A_I^\sigma(x) = \frac{A^\sigma(x)}{1 + A^\sigma(x)}.$$

In particular, since 231 *is indecomposable, these identities are also valid for Fishburn permutations. The sequence* $(IF_n)_{n\in\mathbb{N}}$ *that enumerates indecomposable Fishburn permutations of size* n *starts with*

$$1, 1, 2, 6, 23, 104, 534, 3051, 19155, 130997, \dots.$$

Theorem 2.4. *For* $n > 1$*, we have* $IF_n(123) = 2^{n-1} - (n-1)$*.*

Proof. As discussed in the proof of Theorem 2.1, for $n > 2$ the set $\mathscr{F}_n(123)$ consists of elements of the form $1 \ominus \pi$ with $\pi \in \mathscr{F}_{n-1}(123)$, and elements of the form $\tau = (i-1)\cdots 1 n \pi_R$ for some $1 < i \le n$ and $\pi_R \in \mathscr{F}_{n-i}(123)$. This forces $\pi_R = \emptyset$ if $i = n$, and $\pi_R = (n-1)\cdots i$ if $i < n$. Thus the only decomposable elements of $\mathscr{F}_n(123)$ are the $n-1$ permutations

$$\begin{gathered} 1n(n-1)\cdots 32, \\ 21n(n-1)\cdots 3, \\ \vdots \\ (n-1)\cdots 321n. \end{gathered}$$

In conclusion, $IF_n(123) = F_n(123) - (n-1) = 2^{n-1} - (n-1)$. ☐

Theorem 2.5. *For* $n > 1$ *and* $\sigma \in \{132, 213\}$*, we have* $IF_n(\sigma) = 2^{n-2}$*.*

Proof. From the proof of Theorem 2.1, we know that for $n > 1$ every element of $\mathscr{F}_n(132)$ must be of the form $1 \ominus \pi$ or $\pi \oplus 1$ with $\pi \in \mathscr{F}_{n-1}(132)$. Since $\pi \oplus 1$ is decomposable and $1 \ominus \pi$ is indecomposable, we have

$$\mathscr{F}_n^{\mathsf{ind}}(132) = \{1 \ominus \pi \colon \pi \in \mathscr{F}_{n-1}(132)\}.$$

We also know that $\mathscr{F}_n(213)$ consists of elements of the form $1 \ominus \pi$ with $\pi \in \mathscr{F}_{n-1}(213)$, and elements of the form $\tau = (1\cdots(i-1)) \oplus (1 \ominus \pi_R)$ for some $1 < i \le n$ (with $\pi_R = \emptyset$ when $i = n$). Thus

$$\mathscr{F}_n^{\mathsf{ind}}(213) = \{1 \ominus \pi \colon \pi \in \mathscr{F}_{n-1}(213)\}.$$

In conclusion, if $\sigma \in \{132, 213\}$, we have $IF_n(\sigma) = F_{n-1}(\sigma) = 2^{n-2}$. ☐

Let $F^\sigma(x)$ and $IF^\sigma(x)$ be the generating functions associated with the sequences $(F_n(\sigma))_{n\in\mathbb{N}}$ and $(IF_n(\sigma))_{n\in\mathbb{N}}$, respectively.

Theorem 2.6. *For* $\sigma \in \{231, 312, 321\}$*, we have*

$$IF^\sigma(x) = \frac{F^\sigma(x)}{1 + F^\sigma(x)}.$$

In particular, $IF_n(231) = C_{n-1}$ *and* $IF_n(312) = 1$*.*

Proof. This follows from (2.1), Theorem 2.1, and Lemma 2.3. ☐

Theorem 2.7. *The sequence* $a_n = IF_n(321)$ *satisfies the recurrence relation*

$$a_n = a_{n-1} + \sum_{j=2}^{n-2} a_j a_{n-j}$$

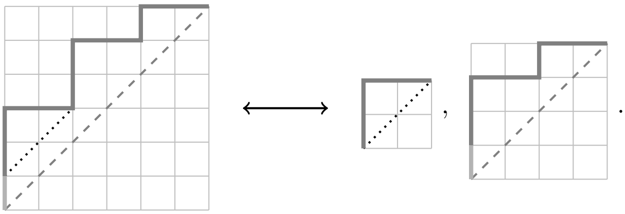
for $n \ge 4$*, with* $a_1 = a_2 = a_3 = 1$*. This is Sequence* [7, A082582]*.*

Proof. We use the same Dyck path approach as in the proof of Theorem 2.2. Under this bijection, indecomposable permutations correspond to Dyck paths that do not touch the line $y = x$ except at the end points.

Let $\mathcal{A}_n$ be the set of Dyck paths corresponding to $\mathscr{F}_n^{\mathsf{ind}}(321)$. We will prove that $a_n = |\mathcal{A}_n|$ satisfies the claimed recurrence relation. Clearly, for $n = 1, 2, 3$, the only indecomposable Fishburn permutations are 1, 21, and 312, which correspond to the Dyck paths UD, U^2D^2, and U^3D^3, respectively. Thus, $a_1 = a_2 = a_3 = 1$.

Note that indecomposable permutations may not start with 1 or end with n. Moreover, every element of $\pi \in \mathscr{F}_n^{\mathsf{ind}}(321)$ must be of the form $m1\pi(3)\cdots\pi(n)$ with $m \geq 3$. Therefore, the elements of $\mathcal{A}_n$ have no peaks at the points $(0, 1)$, $(0, 2)$, or $(n-1, n)$, and for $n > 3$ their first return to the line $y = x+1$ happens at a lattice point $(x, x + 1)$ with $x \in [2, n-1]$.

Dyck paths in $\mathcal{A}_n$ having $(n-1, n)$ as their first return to $y = x+1$, are in one-to-one correspondence with the elements of $\mathcal{A}_{n-1}$ (just remove the first U and the last D of the longer path). Now, for $j \in \{2, \ldots, n-2\}$, the set of paths $P \in \mathcal{A}_n$ having first return to $y = x + 1$ at the point $(j, j + 1)$ corresponds uniquely to the set of all pairs (P_j, P_{n-j}) with $P_j \in \mathcal{A}_j$ and $P_j \in \mathcal{A}_{n-j}$. For example,



This implies that there are $a_j a_{n-j}$ paths in $\mathcal{A}_n$ having the point $(j, j + 1)$ as their first return to the line $y = x + 1$. Finally, summing over j gives the claimed identity. □

Here is a summary of our enumeration results for patterns of size 3:

3. Avoiding Patterns of Size 4

In this section, we discuss the enumeration of Fishburn permutations that avoid a pattern of size 4. There are at least 13 Wilf equivalence classes that we break down into three categories: 10 classes with a single pattern, 2 classes with (conjecturally) three patterns each, and a larger class with eight patterns enumerated by the Catalan numbers.

We will provide a proof for the enumeration of the class $\mathscr{F}_n(1342)$, but our main focus in this paper will be on the enumeration of the equivalence class given in Table 4.

For the remaining patterns we have the following conjectures.

Conjecture 3.1. *$\mathscr{F}_n(2413) \sim \mathscr{F}_n(2431) \sim \mathscr{F}_n(3241)$.*

Conjecture 3.2. *$\mathscr{F}_n(3214) \sim \mathscr{F}_n(4132) \sim \mathscr{F}_n(4213)$.*

Our first result of this section involves the binomial transform of the Catalan numbers, namely the sequence [7, A007317].

Theorem 3.3.

$$F_n(1342) = \sum_{k=1}^{n} \binom{n-1}{k-1} C_{n-k}.$$

Proof. Let $\mathcal{A}_{n,k}$ be the set of all permutations $\pi \in S_n$ such that

- $\pi(k) = 1$ and $\pi(1) > \pi(2) > \cdots > \pi(k-1)$,
- $\pi(k+1)\cdots\pi(n) \in \mathsf{Av}_{n-k}(231)$,

and let $\mathscr{A}_n = \bigcup_{k=1}^{n} \mathcal{A}_{n,k}$. Clearly,

$$|\mathscr{A}_n| = \sum_{k=1}^{n} |\mathcal{A}_{n,k}| = \sum_{k=1}^{n} \binom{n-1}{k-1} C_{n-k}.$$

We will prove the theorem by showing that $\mathscr{A}_n = \mathscr{F}_n(1342)$.

First of all, since $\mathcal{A}_{n,k} \subset \mathsf{Av}_n(1342)$ and $\mathsf{Av}_{n-k}(231) = \mathscr{F}_{n-k}(231)$ for every n and k, we have $\mathscr{A}_n \subset \mathscr{F}_n(1342)$.

Going in the other direction, let $\pi \in \mathscr{F}_n(1342)$ and let k be such that $\pi(k) = 1$. Thus, π is of the form $\pi = \pi(1)\cdots\pi(k-1)\,1\,\pi(k+1)\cdots\pi(n)$, which implies $\pi(k+1)\cdots\pi(n) \in \mathsf{Av}_{n-k}(231)$. Now, if there is a $j \in \{1,\dots,k-2\}$ such that

$$\pi(1) > \cdots > \pi(j) < \pi(j+1),$$

then $\pi(j) - 1 = \pi(\ell)$ for some $\ell > j+1$, and the pattern $\pi(j)\pi(j+1)\pi(\ell)$ would violate the Fishburn condition. In other words, the entries to the left of $\pi(k) = 1$ must form a decreasing sequence, which implies $\pi \in \mathcal{A}_{n,k} \subset \mathscr{A}_n$. Thus $\mathscr{F}_n(1342) \subset \mathscr{A}_n$, and we conclude that $\mathscr{A}_n = \mathscr{F}_n(1342)$. $\square$

3.1. Catalan Equivalence Class

The remaining part of this section is devoted to prove that $|\mathscr{F}_n(\sigma)| = C_n$ for every $\sigma \in \{1234, 1243, 1324, 1423, 2134, 2143, 3124, 3142\}$.

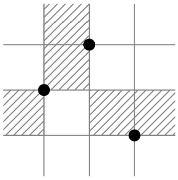
Theorem 3.4. *We have* $\mathscr{F}_n(3142) = \mathscr{F}_n(231)$*, hence* $F_n(3142) = C_n$.

Proof. Since 3142 contains the pattern 231, we have $\mathscr{F}_n(231) \subseteq \mathscr{F}_n(3142)$.

To prove the reverse inclusion, suppose that there exists $\pi \in \mathscr{F}_n(3142)$ such that π contains the pattern 231. Let $i < j < k$ be the positions of the 231 pattern such that

- $\pi(k) < \pi(i) < \pi(j)$,
- $\pi(i)$ is the left-most entry of π involved in a 231 pattern,
- $\pi(j)$ is the first entry with $j > i$ such that $\pi(i) < \pi(j)$,
- $\pi(k)$ is the largest entry with $k > j$ such that $\pi(k) < \pi(i)$.

In other words, assume the plot of π is of the form



where no elements of π may occur in the shaded regions. It follows that, if ℓ is the position of $\pi(k)+1$, then $i \le \ell < j$. But this is not possible since, $\pi(\ell) < \pi(\ell+1)$ violates the Fishburn condition, and $\pi(\ell) > \pi(\ell+1)$ implies $\pi(k) > \pi(\ell+1)$ which forces the existence of a 3142 pattern. In conclusion, no permutation $\pi \in \mathscr{F}_n(3142)$ is allowed to contain a 231 pattern. Therefore, $\mathscr{F}_n(3142) \subseteq \mathscr{F}_n(231)$ and we obtain the claimed equality. $\square$

Theorem 3.5. *$\mathscr{F}_n(1234) \sim \mathscr{F}_n(1243)$ and $\mathscr{F}_n(2134) \sim \mathscr{F}_n(2143)$.*

Proof. To prove both Wilf equivalence relations, we use a bijection

$$\phi : \mathsf{Av}_n(\tau \oplus 12) \to \mathsf{Av}_n(\tau \oplus 21)$$

given by West in [8], which we proceed to describe.

For $\pi \in S_n$ and $\sigma \in S_k$, $k < n$, let $B_\pi(\sigma)$ be the set of maximal values of all instances of the pattern σ in the permutation π. For example, $B_\pi(\sigma) = \emptyset$ if π avoids σ, and for $\pi = 531968274$ we have $B_\pi(123) = \{4, 7, 8\}$ (Fig. 1).

For $\pi \in \mathsf{Av}_n(\tau \oplus 12)$, let ℓ be the number of elements in $B_\pi(\tau \oplus 1)$. If $\ell = 0$, we define $\phi(\pi) = \pi$. If $\ell > 0$, we let $i_1 < \cdots < i_\ell$ be the positions in π of the elements of $B_\pi(\tau \oplus 1)$ and define

$$\tilde{\pi}(j) = \pi(j) \text{ if } j \notin \{i_1, \ldots, i_\ell\}.$$

Note that $\pi \in \mathsf{Av}_n(\tau \oplus 12)$ implies $\pi(i_1) > \cdots > \pi(i_\ell)$.

Let b_1 be the smallest element of $B_\pi(\tau \oplus 1)$ such that $\tilde{\pi}(1) \cdots \tilde{\pi}(i_1 - 1)b_1$ contains the pattern $\tau \oplus 1$. Define

$$\tilde{\pi}(i_1) = b_1.$$

Iteratively, for $k = 2, \ldots, \ell$, we let b_k be the smallest element of $B_\pi(\tau \oplus 1)\backslash\{b_1, \ldots, b_{k-1}\}$ such that $\tilde{\pi}(1) \cdots \tilde{\pi}(i_k - 1)b_k$ contains the pattern $\tau \oplus 1$. We

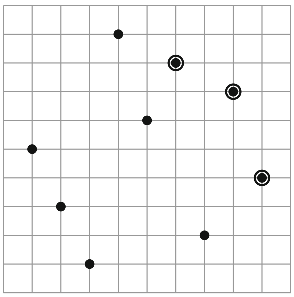


FIGURE 1. $\pi = 531968274$ and $B_\pi(123) = \{4, 7, 8\}$

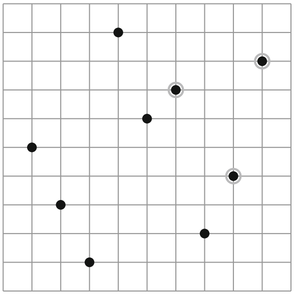


FIGURE 2. $\tilde{\pi} = \phi(531968274) = 531967248$

then define

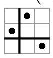
$$\tilde{\pi}(i_k) = b_k \ \text{ for } k = 2, \ldots, \ell,$$

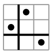
to complete the definition of $\tilde{\pi} = \phi(\pi)$.

For example, for $\pi = 53196\mathbf{8274}$ and $\tau = 12$, we have $\ell = 3$ $\tilde{\pi} = 53196\mathbf{7248}$ (Fig. 2).

It is easy to check that the map ϕ induces a bijection

$$\phi \colon \mathscr{F}_n(\tau \oplus 12) \to \mathscr{F}_n(\tau \oplus 21).$$

Indeed, if $\pi(i_k) \in B_\pi(\tau \oplus 1)$ is such that $\pi(i_k - 1) < \pi(i_k)$, then $\tilde{\pi}(i_k - 1) = \pi(i_k - 1)$ and the pair $\tilde{\pi}(i_k - 1), \tilde{\pi}(i_k)$ does not create a pattern .

On the other hand, if $\pi(i_1) > \cdots > \pi(i_k)$ is a maximal descent of elements from $B_\pi(\tau \oplus 1)$, and if $\pi(i_j) - 1 > 0$ (for $j \in \{1, \ldots, k\}$) is not part of that descent, then $\pi(i_j) - 1$ must be to the left of $\pi(i_1)$ and so any ascent in $\tilde{\pi}(i_1) \cdots \tilde{\pi}(i_k)$ cannot create the pattern .

Thus, if $\pi \in \mathsf{Av}_n(\tau \oplus 12)$ is Fishburn, so is $\tilde{\pi} = \phi(\pi) \in \mathsf{Av}_n(\tau \oplus 21)$. $\square$

Theorem 3.6. $\mathscr{F}_n(1423) \sim \mathscr{F}_n(1243) \sim \mathscr{F}_n(1234) \sim \mathscr{F}_n(1324)$.

Proof. Let $\alpha \colon \mathscr{F}_n(1423) \to \mathscr{F}_n(1243)$ be the map defined through the following algorithm.

Algorithm α: Let $\pi \in \mathscr{F}_n(1423)$ and set $\tilde{\pi} = \pi$.

Step 1: If $\tilde{\pi} \notin \mathsf{Av}_n(1243)$, let $i < j < k < \ell$ be the positions of the left-most 1243 pattern contained in $\tilde{\pi}$. Redefine $\tilde{\pi}$ by moving $\tilde{\pi}(k)$ to position j, shifting the entries at positions j through $k - 1$ one step to the right.

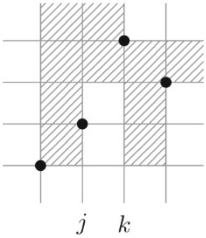
Step 2: If $\tilde{\pi} \in \mathsf{Av}_n(1243)$, then return $\alpha(\pi) = \tilde{\pi}$; otherwise go to Step 1.

For example, for $\pi = 2135476 \in \mathscr{F}(1423)$, the above algorithm yields

$$\begin{array}{ccc} \mathbf{2}1\mathbf{35}476 & \longrightarrow & 21\mathbf{5}3476 \notin \mathsf{Av}(1243) \\ & \swarrow & \\ \mathbf{2}153\mathbf{476} & \longrightarrow & 21\mathbf{7}5346 \in \mathsf{Av}(1243) \end{array}$$

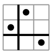
and so $\alpha(\pi) = 2175346 \in \mathscr{F}(1243)$.

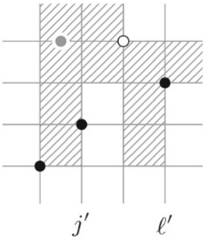
Observe that the map α changes every 1243 pattern into a 1423 pattern. To see that it preserves the Fishburn condition, let $\pi \in \mathscr{F}_n(1423)$ be such that $\pi(i)$, $\pi(j)$, $\pi(k)$, $\pi(\ell)$ form a left-most 1243 pattern. Thus, at first, π must be of the form

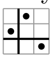


where no elements of π may occur in the shaded regions. In particular, we must have

$$\pi(j-1) < \pi(j) \quad \text{and} \quad \pi(k-1) < \pi(k). \tag{3.1}$$

Hence the step of moving $\pi(k)$ to position j does not create a new ascent and, therefore, it cannot create a pattern . After one iteration, $\tilde{\pi}$ takes the form



and if the left-most 1243 pattern $\tilde{\pi}(i)$, $\tilde{\pi}(j)$, $\tilde{\pi}(k)$, $\tilde{\pi}(\ell)$ contained in $\tilde{\pi}$ has its second entry at a position different from j', then $\tilde{\pi}$ must satisfy (3.1) and no pattern  will be created.

Otherwise, if $j = j'$, then either $k = \ell'$ or $\ell = \ell'$. In the first case, we have $\tilde{\pi}(k-1) < \tilde{\pi}(k)$ and $\tilde{\pi}(k) < \tilde{\pi}(j-1)$, so moving $\tilde{\pi}(k)$ to position j does not create a new ascent. On the other hand, if $\ell = \ell'$, then $\tilde{\pi}(k) > \tilde{\pi}(j-1)$ but $\tilde{\pi}(j-1)-1$ must be to the left of $\tilde{\pi}(i)$. Therefore, also in this case, applying an iteration of α will preserve the Fishburn condition.

We conclude that, if π is Fishburn, so is $\alpha(\pi)$.

The reverse map $\beta\colon \mathscr{F}_n(1243) \to \mathscr{F}_n(1423)$ is given by the following algorithm.

Algorithm β: Let $\tau \in \mathscr{F}_n(1243)$ and set $\tilde{\tau} = \tau$.

- Step 1: If $\tilde{\tau} \notin \mathsf{Av}_n(1423)$, let $i < j < k < \ell$ be the positions of the right-most 1423 pattern contained in $\tilde{\tau}$. Redefine $\tilde{\tau}$ by moving $\tilde{\tau}(j)$ to position k, shifting the entries at positions $j+1$ through k one step to the left.
- Step 2: If $\tilde{\tau} \in \mathsf{Av}_n(1423)$, then return $\beta(\tau) = \tilde{\tau}$; otherwise go to Step 1.

In conclusion, the map α gives a bijection $\mathscr{F}_n(1423) \to \mathscr{F}_n(1243)$.

With a similar argument, it can be verified that α also maps $\mathscr{F}_n(1324) \to \mathscr{F}_n(1234)$ bijectively. Finally, the equivalence $\mathscr{F}_n(1243) \sim \mathscr{F}_n(1234)$ was shown in Theorem 3.5. $\square$

Theorem 3.7. $\mathscr{F}_n(3142) \sim \mathscr{F}_n(3124) \sim \mathscr{F}_n(1324)$.

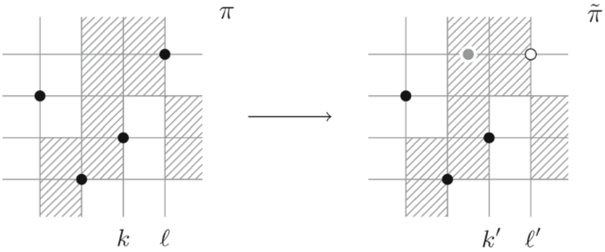
Proof. We will define two maps

$$\mathscr{F}_n(3142) \xrightarrow{\alpha_1} \mathscr{F}_n(3124) \quad \text{and} \quad \mathscr{F}_n(3124) \xrightarrow{\alpha_2} \mathscr{F}_n(1324)$$

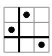
through algorithms similar to the one introduced in the proof of Theorem 3.6.
Algorithm α_1: Let $\pi \in \mathscr{F}_n(3142)$ and set $\tilde{\pi} = \pi$.

Step 1: If $\tilde{\pi} \notin \mathsf{Av}_n(3124)$, let $i < j < k < \ell$ be the positions of the left-most 3124 pattern contained in $\tilde{\pi}$. Redefine $\tilde{\pi}$ by moving $\tilde{\pi}(\ell)$ to position k, shifting the entries at positions k through $\ell - 1$ one step to the right.
Step 2: If $\tilde{\pi} \in \mathsf{Av}_n(3124)$, then return $\alpha_1(\pi) = \tilde{\pi}$; otherwise go to Step 1.

As α in Theorem 3.6, the map α_1 is reversible and preserves the Fishburn condition. For an illustration of the latter claim, here is a sketch of a permutation $\pi \in \mathscr{F}_n(3142)$ having a left-most 3124 pattern, together with the sketch of $\tilde{\pi}$ after one iteration of α_1:



where no elements of the permutation π may occur in the shaded regions.

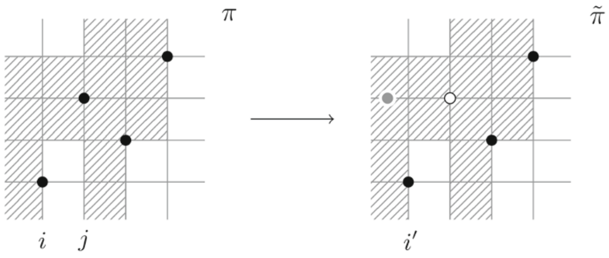
Since $\pi(k-1) < \pi(k)$ and $\pi(\ell - 1) < \pi(\ell)$, the Fishburn condition of π is preserved after the first iteration of α_1. Further, if $\tilde{\pi}$ has a left-most 3124 pattern with the third entry at position k', then we must have $\ell > \ell'$ and $\tilde{\pi}(\ell) > \tilde{\pi}(i)$. If $\tilde{\pi}(\ell) < \tilde{\pi}(k'-1)$, no new ascent can be created when moving $\tilde{\pi}(\ell)$ to position k'. Otherwise, if $\tilde{\pi}(\ell) > \tilde{\pi}(k'-1)$, then either π has ascents at the positions of these two entries or every entry between $\tilde{\pi}(\ell')$ and $\tilde{\pi}(\ell)$ must be smaller than $\tilde{\pi}(j)$. Since $\pi \in \mathsf{Av}_n(3142)$, the latter would imply that $\tilde{\pi}(k'-1) - 1$ is to the left of $\tilde{\pi}(j)$. In any case, no pattern  will be created in the next iteration of α_1.

Since any later iteration of α_1 may essentially be reduced to one of the above cases, we conclude that α_1 preserves the Fishburn condition.
Algorithm α_2: Let $\pi \in \mathscr{F}_n(3124)$ and set $\tilde{\pi} = \pi$.

Step 1: If $\tilde{\pi} \notin \mathsf{Av}_n(1324)$, let $i < j < k < \ell$ be the positions of the left-most 1324 pattern contained in $\tilde{\pi}$. Redefine $\tilde{\pi}$ by moving $\tilde{\pi}(j)$ to position i, shifting the entries at positions i through $j - 1$ one step to the right.
Step 2: If $\tilde{\pi} \in \mathsf{Av}_n(1324)$, then return $\alpha_2(\pi) = \tilde{\pi}$; otherwise go to Step 1.

This map is reversible and preserves the Fishburn condition. As before, we will illustrate the Fishburn property by sketching the plot of a permutation

$\pi \in \mathscr{F}_n(3124)$ that contains a left-most 1324 pattern $\pi(i)$, $\pi(j)$, $\pi(k)$, $\pi(\ell)$, together with the sketch of the permutation $\tilde{\pi}$ obtained after one iteration of α_2:



Since no elements of the permutation π may occur in the shaded regions, we must have either $i = 1$ or $\pi(i-1) > \pi(j)$. Consequently, moving $\pi(j)$ to position i will not create a new ascent and the Fishburn condition will be preserved.

Similarly, if $\tilde{\pi}$ has a left-most 1324 pattern with first entry at a position different from i', or if $\tilde{\pi}(i) = \tilde{\pi}(i')$ and $\tilde{\pi}(j) < \tilde{\pi}(i'-1)$, then no new ascent will be created and the next $\tilde{\pi}$ will be Fishburn. It is not possible to have $\tilde{\pi}(i) = \tilde{\pi}(i')$ and $\tilde{\pi}(j) > \tilde{\pi}(i'-1)$.

In summary, α_1 and α_2 are both bijective maps. $\square$

The following theorem completes the enumeration of the Catalan class (see Table 4).

Theorem 3.8. *$\mathscr{F}_n(3142) \sim \mathscr{F}_n(2143)$.*

Proof. Let $\gamma\colon \mathscr{F}_n(3142) \to \mathscr{F}_n(2143)$ be the map defined through the following algorithm.

Algorithm γ: Let $\pi \in \mathscr{F}_n(3142)$ and set $\tilde{\pi} = \pi$.

Step 1: If $\tilde{\pi} \notin \mathsf{Av}_n(2143)$, let $i < j < k$ be the positions of the left-most 213 pattern contained in $\tilde{\pi}$ such that $\tilde{\pi}(i)$, $\tilde{\pi}(j)$, $\tilde{\pi}(k)$, $\tilde{\pi}(\ell)$ form a 2143 pattern for some $\ell > k$. Let ℓ_m be the position of the smallest such $\tilde{\pi}(\ell)$, and let

$$Q = \{q \in [n] : \tilde{\pi}(i) \le \tilde{\pi}(q) < \tilde{\pi}(\ell_m)\}.$$

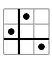
Redefine $\tilde{\pi}$ by replacing $\tilde{\pi}(\ell_m)$ with $\tilde{\pi}(i)$, adding 1 to $\tilde{\pi}(q)$ for every $q \in Q$.

Step 2: If $\tilde{\pi} \in \mathsf{Av}_n(2143)$, then return $\gamma(\pi) = \tilde{\pi}$; otherwise go to Step 1.

For example, if $\pi = 4312576$, then $\gamma(\pi) = 5412673$ (after 2 iterations, see Fig. 3).

The map γ is reversible. Moreover,

(a) since $\tilde{\pi}(\ell_m)$ is the smallest entry such that $\tilde{\pi}(i) < \tilde{\pi}(\ell_m) < \tilde{\pi}(k)$, replacing $\tilde{\pi}(\ell_m)$ with $\tilde{\pi}(i)$ (which is equivalent to moving the plot of $\tilde{\pi}(\ell_m)$ down to height $\tilde{\pi}(i)$) will not create any new ascent at position ℓ_m;

(b) since $\tilde{\pi}(i)$ is chosen to be the first entry of a left-most 2143 pattern, $\tilde{\pi}(i) - 1$ must be to the right of $\tilde{\pi}(i)$. Hence, replacing $\tilde{\pi}(i)$ by $\tilde{\pi}(i) + 1$ cannot create a new pattern .

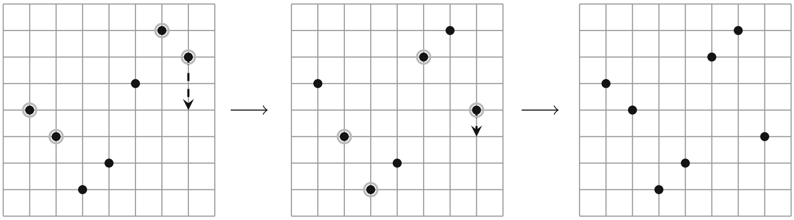


FIGURE 3. Algorithm γ: 4312576 → 5312674 → 5412673

In conclusion, γ preserves the Fishburn condition and gives the claimed bijection. □

4. Further Remarks

In this paper, we have discussed the enumeration of Fishburn permutations that avoid a pattern of size 3 or a pattern of size 4. In Sect. 2, we offer the complete picture for patterns of size 3, including the enumeration of indecomposable permutations.

Regarding patterns of size 4, we have proved the Wilf equivalence of eight permutation families counted by the Catalan numbers. We have also shown that $\mathscr{F}_n(1342)$ is enumerated by the binomial transform of the Catalan numbers. In general, there seems to be 13 Wilf equivalence classes of permutations that avoid a pattern of size 4, some of which appear to be in bijection with certain pattern avoiding ascent sequences ([7, A202061, A202062]). At this point in time, we do not know how the pattern avoidance of a Fishburn permutation is related to the pattern avoidance of an ascent sequence. It would be interesting to pursue this line of investigation.

Concerning indecomposable permutations, we leave the field open for future research. Note that Theorem 3.4 and Lemma 2.3 imply

$$|\mathscr{F}_n^{\mathsf{ind}}(3142)| = C_{n-1}.$$

The study of other patterns is unexplored territory, and our preliminary data suggests the existence of 19 Wilf equivalence classes listed in Table 5.

We are particularly curious about the class $\mathscr{F}_n^{\mathsf{ind}}(2413)$ as it appears (based on limited data) to be equinumerous with the set $\mathsf{Av}_{n-1}(2413, 3412)$, cf. [7, A165546].

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Juan B. Gil and Michael D. Weiner
Department of Mathematics
Penn State Altoona
3000 Ivyside Park
Altoona PA 16601
USA

Juan B. Gil
e-mail: jgil@psu.edu

Michael D. Weiner
e-mail: mdw8@psu.edu

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Annals of Combinatorics

A q-Analogue for Euler's $\zeta(6) = \pi^6/945$

In honour of Prof. George Andrews on his 80th birthday

Ankush Goswami

Abstract. Recently, Sun (Two q-analogues of Euler's formula $\zeta(2) = \pi^2/6$. arXiv:1802.01473, 2018) obtained q-analogues of Euler's formula for $\zeta(2)$ and $\zeta(4)$. Sun's formulas were based on identities satisfied by triangular numbers and properties of Euler's q-Gamma function. In this paper, we obtain a q-analogue of $\zeta(6) = \pi^6/945$. Our main results are stated in Theorems 2.1 and 2.2 below.

Mathematics Subject Classification. 11N25, 11N37, 11N60.

Keywords. q-Analogue, Triangular numbers.

1. Introduction

Recently, Sun [3] obtained a very nice q-analogue of Euler's formula $\zeta(2) = \pi^2/6$.

Theorem 1.1. (Sun [3]) *For a complex q with $|q| < 1$, we have:*

$$\sum_{k=0}^{\infty} \frac{q^k(1+q^{2k+1})}{(1-q^{2k+1})^2} = \prod_{n=1}^{\infty} \frac{(1-q^{2n})^4}{(1-q^{2n-1})^4}. \tag{1.1}$$

Motivated by Theorem 1.1, the present author obtained the q-analogue of $\zeta(4) = \pi^4/90$ and noted that it was simultaneously and independently obtained by Sun in his subsequent revised paper.

Theorem 1.2. (Sun [3]) *For a complex q with $|q| < 1$, we have:*

$$\sum_{k=0}^{\infty} \frac{q^{2k}(1+4q^{2k+1}+q^{4k+2})}{(1-q^{2k+1})^4} = \prod_{n=1}^{\infty} \frac{(1-q^{2n})^8}{(1-q^{2n-1})^8}. \tag{1.2}$$

Furthermore, Sun commented that one does not know how to find q-analogues of Euler's formula for $\zeta(6)$ and beyond, similar to Theorems 1.1 and 1.2. This further motivated the author to consider the problem, and indeed, we obtained the q-analogue of $\zeta(6)$. As we shall see shortly, the q-analogue formulation of $\zeta(6)$ is more difficult as compared to $\zeta(2)$ and $\zeta(4)$ due to an extra term that shows up in the identity; however, in the limit as $q \uparrow 1$ (where $q \uparrow 1$ means q is approaching 1 from inside the unit disk), this term vanishes. We also state the q-analogue of $\zeta(4) = \pi^4/90$, since we found it independently of Sun's result; however, we skip the proof of this, since it essentially uses the same idea as Sun.

We emphasize here that the q-analogue of $\zeta(6) = \pi^6/945$ is the first non-trivial case where we notice the occurrence of an interesting extra term which essentially is the twelfth power of a well-known function of Euler (see Theorem 2.2). After obtaining this result, we obtained q-analogues of Euler's general formula for $\zeta(2k), k = 4, 5, \dots$ (see [1]). Each of these q-analogues has an extra term that arises from the general theory of modular forms all of which approach zero in the limit $q \uparrow 1$. The case $k = 3$ or the q-analogue of $\zeta(6)$ is special, since the extra term that we obtain in this case has a beautiful product representation, and has connections to well-known identities of Euler (see below).

2. Main Theorems

Theorem 2.1. *For a complex q with $|q| < 1$, we have:*

$$\sum_{k=0}^{\infty} \frac{q^{2k}\, P_2(q^{2k+1})}{(1-q^{2k+1})^4} = \prod_{n=1}^{\infty} \frac{(1-q^{2n})^8}{(1-q^{2n-1})^8}, \tag{2.1}$$

where $P_2(x) = x^2 + 4x + 1$. In other words, (2.1) gives a q-analogue of $\zeta(4) = \pi^4/90$.

Theorem 2.2. *For a complex q with $|q| < 1$, we have:*

$$\sum_{k=0}^{\infty} \frac{q^k(1+q^{2k+1})\, P_4(q^{2k+1})}{(1-q^{2k+1})^6} - \phi^{12}(q) = 256q \prod_{n=1}^{\infty} \frac{(1-q^{2n})^{12}}{(1-q^{2n-1})^{12}}, \tag{2.2}$$

where $P_4(x) = x^4 + 236x^3 + 1446x^2 + 236x + 1$ and $\phi(q) = \prod_{n=1}^{\infty}(1-q^n)$ is Euler's function. In other words, (2.2) gives a q-analogue of $\zeta(6) = \pi^6/945$.

Remark 2.3. We note that $\phi^{12}(q)$ has a beautiful product representation and is uniquely determined by:

$$\phi^{12}(q) = \sum_{k=0}^{\infty} \frac{q^k(1+q^{2k+1})\, P_4(q^{2k+1})}{(1-q^{2k+1})^6} - 256q \prod_{n=1}^{\infty} \frac{(1-q^{2n})^{12}}{(1-q^{2n-1})^{12}}. \tag{2.3}$$

In the general q-analogue formulation (see [1]), we do not have very elegant representations of these functions, although we obtain expressions for them similar to (2.3).

Remark 2.4. Since the coefficients in the q-series expansion of $\phi^{12}(q)$ are related to the pentagonal numbers by Euler's pentagonal number theorem, and the coefficients of the product in the right-hand side of (2.2) are related to the triangular numbers, it will be worthwhile to understand the relationships of these coefficients via identity (2.2).

3. Some Useful Lemmas

Let $q = e^{2\pi i\tau}$, $\tau \in \mathcal{H}$ where $\mathcal{H} = \{\tau \in \mathbb{C} : \mathrm{Im}(\tau) > 0\}$. Then, the Dedekind η-function defined by:

$$\eta(\tau) = q^{1/24} \prod_{n=1}^{\infty} (1 - q^n), \tag{3.1}$$

is a modular form of weight 1/2. Also, let us denote by $\psi(q)$ the following sum:

$$\psi(q) = \sum_{n=0}^{\infty} q^{T_n}, \tag{3.2}$$

where $T_n = \dfrac{n(n+1)}{2}$ (for $n = 0, 1, 2, \ldots$) are triangular numbers. Then, we have the following well-known result due to Gauss:

Lemma 3.1.

$$\psi(q) = \prod_{n=1}^{\infty} \frac{(1 - q^{2n})}{(1 - q^{2n-1})}. \tag{3.3}$$

Thus, we have from Lemma 3.1 that:

$$\prod_{n=1}^{\infty} \frac{(1 - q^{2n})^{12}}{(1 - q^{2n-1})^{12}} = \psi^{12}(q) = \sum_{n=1}^{\infty} t_{12}(n) q^n, \tag{3.4}$$

where $t_{12}(n)$ is the number of ways of representing a positive integer n as a sum of 12 triangular numbers. Next, we have the following well-known result of Ono, Robins and Wahl [2].

Theorem 3.2. *Let $\eta^{12}(2\tau) = \sum_{k=0}^{\infty} a(2k+1) q^{2k+1}$. Then, for a positive integer n, we have:*

$$t_{12}(n) = \frac{\sigma_5(2n+3) - a(2n+3)}{256}, \tag{3.5}$$

where

$$\sigma_5(n) = \sum_{d|n} d^5. \tag{3.6}$$

4. Proof of Theorem 2.2

Since $\zeta(6) = \dfrac{\pi^6}{945}$ has the following equivalent form:

$$\sum_{k=0}^{\infty} \frac{1}{(2k+1)^6} = \frac{63}{64}\zeta(6) = \frac{\pi^6}{960}, \tag{4.1}$$

it will be sufficient to get the q-analogue of (4.1). Now, from q-analogue of Euler's Gamma function, we know that:

$$\lim_{q\uparrow 1} (1-q) \prod_{n=1}^{\infty} \frac{(1-q^{2n})^2}{(1-q^{2n-1})^2} = \frac{\pi}{2}, \tag{4.2}$$

so that from (4.2), we have:

$$\lim_{q\uparrow 1} (1-q)^6 \prod_{n=1}^{\infty} \frac{(1-q^{2n})^{12}}{(1-q^{2n-1})^{12}} = \frac{\pi^6}{64}. \tag{4.3}$$

Next, we consider the following infinite series

$$S_6(q) := \sum_{k=0}^{\infty} \frac{q^k(1+q^{2k+1})\, P_4(q^{2k+1})}{(1-q^{2k+1})^6}, \tag{4.4}$$

where $P_4(x) = x^4 + 236x^3 + 1446x^2 + 236x + 1$.

By partial fractions, we have:

$$\begin{aligned} S_6(q) = \sum_{k=0}^{\infty} q^k \Bigg\{ &\frac{3840}{(1-q^{2k+1})^6} - \frac{9600}{(1-q^{2k+1})^5} + \frac{8160}{(1-q^{2k+1})^4} \\ &- \frac{2640}{(1-q^{2k+1})^3} + \frac{242}{(1-q^{2k+1})^2} - \frac{1}{(1-q^{2k+1})} \Bigg\}. \end{aligned} \tag{4.5}$$

Lemma 4.1. *With $S_6(q)$ represented by (4.5), we have:*

$$S_6(q) = 256q \sum_{n=0}^{\infty} t_{12}(n) q^n + \phi^{12}(q). \tag{4.6}$$

Proof. From (4.5), we have:

$$\begin{aligned} S_6(q) &= \sum_{k=0}^{\infty}\sum_{j=0}^{\infty} q^k \left\{ 3840\binom{-6}{j} - 9600\binom{-5}{j} + 8160\binom{-4}{j} \right. \\ &\quad \left. -2640\binom{-3}{j} + 242\binom{-2}{j} - \binom{-1}{j} \right\} (-q)^{j(2k+1)} \\ &= \sum_{k=0}^{\infty}\sum_{j=0}^{\infty} \{32(j+1)(j+2)(j+3)(j+4)(j+5) \\ &\quad -400(j+1)(j+2)(j+3)(j+4) + 1360(j+1)(j+2)(j+3) \\ &\quad -1320(j+1)(j+2) + 242(j+1) - 1\}\, q^{k+j(2k+1)} \end{aligned}$$

$$
\begin{aligned}
&= \sum_{k=0}^{\infty}\sum_{j=0}^{\infty}(2j+1)^5 q^{\frac{(2j+1)(2k+1)-1}{2}} \\
&= \sum_{n=0}^{\infty} \sigma_5(2n+1)q^n \\
&= 1 + \sum_{n=1}^{\infty} \sigma_5(2n+1)q^n \\
&= 1 + q\sum_{n=0}^{\infty} \sigma_5(2n+3)q^n.
\end{aligned}
$$

Also from (3.1), we have:

$$
\begin{aligned}
\phi^{12}(q) &= \frac{\eta^{12}(\tau)}{q^{\frac{1}{2}}} \\
&= \sum_{n=0}^{\infty} a(2n+1)q^n \\
&= 1 + \sum_{n=1}^{\infty} a(2n+1)q^n \\
&= 1 + q\sum_{n=0}^{\infty} a(2n+3)q^n.
\end{aligned}
$$

Thus, from above, we have:

$$
\begin{aligned}
S_6(q) - \phi^{12}(q) &= q\sum_{n=0}^{\infty} \left\{\sigma_5(2n+3) - a(2n+3)\right\} q^n \\
&= 256\, q\sum_{n=0}^{\infty} t_{12}(n)q^n,
\end{aligned}
$$

where the last step follows from Theorem 3.2. This completes the proof of Theorem 2.2. □

We also note that

$$
\begin{aligned}
\lim_{q\uparrow 1}\,(1-q)^6(S_6(q) - \phi^{12}(q)) &= \lim_{q\uparrow 1}\,(1-q)^6 S_6(q) - \lim_{q\uparrow 1}\,(1-q)^6\phi^{12}(q) \\
&= \sum_{k=0}^{\infty} \frac{3840}{(2k+1)^6}, \qquad (4.7)
\end{aligned}
$$

where $\lim_{q\uparrow 1}\,(1-q)^6\phi^{12}(q) = 0$ and $q \uparrow 1$ indicates $q \to 1$ from within the unit disk. Hence, combining Eqs. (4.1), (4.3), (4.7), and Lemma 4.1, Theorem 2.2 follows.

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Ankush Goswami
Department of Mathematics
University of Florida
Gainesville
Fl 32603
USA
e-mail: ankush04@ufl.edu

Present Address
Ankush Goswami
Research Institute for Symbolic Computation (RISC)
Johannes Kepler University
Altenbergerstraße 69
4040 Linz
Austria
e-mail: agoswami@risc.jku.at

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A Variant of `IdentityFinder` and Some New Identities of Rogers–Ramanujan–MacMahon Type

To George E. Andrews, with great respect and gratitude

Shashank Kanade, Debajyoti Nandi and Matthew C. Russell

Abstract. We report on findings of a variant of `IdentityFinder`—a Maple program that was used by two of the authors to conjecture several new identities of Rogers–Ramanujan kind. In the present search, we modify the parametrization of the search space by taking into consideration several aspects of Lepowsky and Wilson's Z-algebraic mechanism and its variant by Meurman and Primc. We search for identities based on forbidding the appearance of "flat" partitions as sub-partitions. Several new identities of Rogers–Ramanujan–MacMahon type are found and proved.

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Keywords. MacMahon identity, Rogers–Ramanujan identities, Sylvester's bijection.

1. Introduction and Motivation

The aim of this paper is to report on several new integer partition identities of Rogers–Ramanujan–MacMahon type. This paper should be viewed largely as a continuation of [18] with a search space that is broader and differently parametrized than the one in [18]. For the general background on experimentally finding new partition identities and the importance thereof we refer the reader to a small, but by no means, exhaustive selection of articles: [3,4,18,23,28]. We ask the reader to recall the relevant terminology (sum-sides, product-sides, difference conditions, initial conditions etc.) from [18].

Several important considerations motivated the present search for identities.

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First, in [25], the second author was able to deduce conjectures of Rogers–Ramanujan-type by analyzing the standard modules for the affine Lie algebra $A_2^{(2)}$ at level 4. These were found using Meurman and Primc's variant [24] of Lepowsky and Wilson's Z-algebraic mechanism [20, 21] for finding and proving new identities using standard modules for affine Lie algebras. The remarkably striking feature of these identities is the appearance of an infinite list of conditions on the sum-sides. It remains a worthy (and perhaps a difficult) goal to author an automated search targeted towards such kinds of identities that vastly generalizes `IdentityFinder`. Therefore, understanding how the Z-algebraic mechanism (or the variant in [24]) works is an important first step.

In Lepowsky–Wilson's Z-algebraic interpretation [20, 21] of the sum-sides in Rogers–Ramanujan identities, sub-partitions are eliminated (equivalently, forbidden to appear) based on their lexicographical ordering (see also [24]). In other words, they treat the "difference-2-at-distance-1" condition in the Rogers–Ramanujan identities as forbidding the appearance of "flattest" length 2-partitions as sub-partitions. The flattest length 2 partition of $2n$ is $n + n$, while that of $2n+1$ is $(n+1)+n$. A large subset of conditions on the identities in [25] could be interpreted this way.

We are specifically motivated by the following "affine rank 2"-type, that is, similar to the affine Lie algebras $A_1^{(1)}$ (also known as $\widehat{\mathfrak{sl}_2}$) and $A_2^{(2)}$ situations. Roughly speaking, a relation with leading term being a "square" of a certain vertex operator (as in $\widehat{\mathfrak{sl}_2}$ level 3) corresponds to the flattest 2-partitions being forbidden (for example, in the case of Rogers–Ramanujan identities). If one has several such "generic" relations with leading term being a quadratic, then one can eliminate as many first flattest 2-partitions. Sometimes, one has "non-generic" quadratic relations, meaning that the relevant matrices formed by the leading terms are sometimes singular, which leads to conditional elimination of the flattest partitions (for example, in the case of Capparelli's identities). At higher levels, higher powers (as opposed to quadratics) of vertex operators are present as the leading terms in the relations, thereby engendering the "difference-at-a-distance"-type conditions. Our aim was to capture such phenomena. Genuinely higher rank situations give rise to various families of Z-operators and, therefore, to multi-color partition identities. Note that many seemingly higher rank algebras at low levels also yield "affine rank 2"-type conditions. Also observe that this is a very crude guideline to the inner mechanics of Z-algebras; particular situations often involve many quirks.

Second, it was a question of Drew Sills if `IdentityFinder` (possibly with some modifications) is able to capture identities like the following identity of MacMahon:

> Partitions of n with no appearance of consecutive integers as parts and all parts at least 2 are equinumerous with partitions of n in which each part is divisible by either 2 or 3.

and a generalization due to Andrews:

> Partitions of n with no appearance of consecutive integers as parts, no part being repeated thrice and all parts are greater than 1 are equinumerous with partitions of n in which each part is $\equiv 2, 3, 4 \pmod 6$.

Sum-sides in both of these identities could be easily recast into the “forbidding-flattest-partitions” language. For instance, in the case of MacMahon’s identity, forbidding the appearance of consecutive integers as parts is equivalent to forbidding the appearance, as sub-partitions, of the flattest 2-partitions of any odd integer. For other examples of such reinterpretations of various known identities, see Sect. 2. Sequence avoiding partitions as the ones appearing in MacMahon’s identity are also of an independent interest, see for instance, [5,6,8,16] etc.

It should now be clear to the reader that a framework built on forbidding flattest partitions can unlock a treasure of many new such identities (and it indeed does, as we report in this paper).

One further observation proved quite useful in searching for these identities. In many well-known identities, the initial conditions are implied by the difference conditions if one appends one or more fictitious 0 parts to the partitions (for example, MacMahon’s identity and its generalization due to Andrews recalled above, the second of the Rogers–Ramanujan identities and so on). This phenomenon is quite well-known (as was pointed out to us by Drew Sills); see the description of identities in [2] for instance.

We present several new families of identities. Many of the identities reported here have the “sequence avoiding” feature as in the MacMahon identity above, and many identities are direct generalizations of MacMahon’s identity. Hence, we loosely chose to call these identities as identities of Rogers–Ramanujan–MacMahon type. Quite contrary to our expectations, to the best of our knowledge, none of the identities presented here are principally specialized characters of standard modules for affine Lie algebras at positive integral levels. Some such identities may lie much deeper in the search space, or perhaps even more innovative searching parameters are required. As a testimony to the former, several ideas of this article along with [17] helped us identify conjectures related to certain level 2 modules for $A_9^{(2)}$ which we presented in our article [19]. As will be clear from our discussion below, these identities lie quite deep in our current search space and hence had to be found by completely different methods. Many of the conjectures reported in [19] now stand proved thanks to the efforts of Bringmann et al. [7]. Last, we mention that the present search has a rather broad search space; hence many times an ad-hoc zooming into the search space was required.

A majority of the identities presented below are proved bijectively, using the works of Xiong and Keith [32], Pak and Postnikov [27], Stockhofe [29] and Sylvester [31]. One family of identities is proved using Appell’s theorem.

Future Work and Work in Progress

Extending the proof technique in Family 1 is work in progress.

In our search, we worked with a specific ordering on the partitions (explained below). It would be very interesting to search with different orderings.

Several of the identities reported in this article quickly generalize to "multi-color" identities. We are in the process of significantly generalizing our current search to include these multi-color generalizations.

It will be very interesting to search for identities solely based on the recurrences for sum-side conditions. One advantage is that such a search is fast. This is an ongoing project.

2. Some Known Identities

Let us first standardize the conventions used in this paper. Let $\mu = m_1 + \cdots + m_r$ and $\pi = p_1 + \cdots + p_s$ be partitions of n. If π is a partition of n, we say that the *weight* of π is n. Partitions will always be in *non-increasing* order (however, we shall present the identities in a manner independent of order).

By a *k-partition* of n, we mean a partition of length k of n.

We say that $\mu < \pi$ (or that μ is *flatter* than π) if either of the following holds:

- $r > s$ (this will not really be needed; we will only compare partitions of same length.)
- $r = s$ and $m_1 = p_1, m_2 = p_2, \ldots, m_{i-1} = p_{i-1}$ but $m_i < p_i$ for some i with $1 \leq i \leq r$.

Example 2.1. Here are the 4-partitions of 10 arranged from flattest to steep, i.e., from lexicographically smallest to largest:

$$\begin{aligned}(3,3,2,2) &< (3,3,3,1) < (4,2,2,2) < (4,3,2,1) < (4,4,1,1)\\ &< (5,2,2,1) < (5,3,1,1) < (6,2,1,1) < (7,1,1,1).\end{aligned}$$

We say that a partition

$$\pi = p_1 + p_2 + \cdots + p_s$$

is *forbidden as a sub-partition* (or simply, *forbidden*) in another partition

$$\mu = m_1 + m_2 + \cdots + m_r$$

if for all indices i with $1 \leq i \leq r - s + 1$, we have

$$(p_1, p_2, \ldots, p_s) \neq (m_i, m_{i+1}, \ldots, m_{i+s-1}).$$

In other words, π does not appear as a contiguous sub-partition of μ. As an example, if μ is a partition in which consecutive parts differ by at least 2, then all partitions of the form $\pi = n + n$ and $\pi' = (n+1) + n$ with $n \geq 1$ are forbidden from appearing as sub-partitions of μ. Following is a (highly non-exhaustive) list of difference conditions in some well-known partition identities recast in terms of forbidden sub-partitions.

We ignore initial conditions, focusing only on the global difference conditions.

1. Rogers–Ramanujan: the flattest 2-partitions are forbidden.
2. Gordon–Andrews (modulo $2k+1$): the flattest k-partitions are forbidden.

3. Andrews–Bressoud (modulo $2k$): the flattest k-partitions are forbidden, the flattest $(k-1)$-partition of n' is forbidden if n' satisfies a specific parity condition.
4. Capparelli: the flattest 2-partitions are forbidden and for all $n' \not\equiv 0 \pmod 3$, the second flattest 2-partition of n' is forbidden.
5. Schur: the flattest 2-partitions are forbidden, the second flattest 2-partitions of even numbers are forbidden, the second flattest 2-partitions of numbers divisible by 3 are forbidden. The last two conditions can be combined to give: the second flattest 2 partitions of numbers $\not\equiv \pm 1 \pmod 6$ are forbidden.
6. Göllnitz-Gordon: the flattest 2-partitions are forbidden, the second flattest 2-partitions of numbers $\equiv 2 \pmod 4$ are forbidden.
7. MacMahon: the flattest 2-partitions of odd numbers are forbidden.
8. Andrews: recall that this identity states that the number of partitions of n into parts congruent to 2, 3 or 4, modulo 6 equals the number of partitions of n into parts greater than 1 where no two consecutive integers may appear as parts and a given part may be repeated, but not more than twice. Recast: the flattest 2-partitions of odd numbers are forbidden, the flattest 3-partitions of numbers divisible by 3 are forbidden.
9. Symmetric Mod$-$9s [18, I_1, I_2, I_3]: the flattest 2-partition of n' if $n' \not\equiv 0 \pmod 3$ is forbidden, the first two flattest 3-partitions for all n' are forbidden.
10. Identities 1, 2, 3 from [19]: the flattest 2-partitions of odd numbers are forbidden, the flattest 2-partition of n' with $n' \equiv 2 \pmod 4$ is forbidden, the second flattest 3-partitions of any n' with $n' \equiv \pm 2 \pmod 6$ are forbidden, the third and fourth flattest 3-partitions of n' with $n' \equiv 3 \pmod 6$ are forbidden. As one can see, these identities lie deep in our current search space.

3. The Method

For $n \geq 0$, let $\mathcal{C}(n)$ be a certain subset of partitions of n. We prescribe $\mathcal{C}$ by imposing flattest-partition conditions on the partitions. Let $\mathcal{C}_j(n)$ be those partitions in $\mathcal{C}(n)$ with the largest part at most j.

Let

$$P(q) = 1 + \sum_{m \geq 1} |\mathcal{C}(m)|\, q^m, \qquad P_j(q) = 1 + \sum_{m \geq 1} |\mathcal{C}_j(m)|\, q^m$$

be the corresponding generating functions. We calculate several coefficients of P (say, up to order q^{25}) then employ Euler's algorithm [4] to see if P has a chance to factor as an interesting (periodic) infinite product of the form $\prod_{m \geq 1}(1-q^m)^{a_m}$. If so, we have a potential candidate for an identity. We use Euler's algorithm as implemented in Garvan's q-series maple package [14].

To verify a given potential candidate to a high degree of certainty, we proceed as in [18]. We first find recursions satisfied by P_j. We utilize these recursions to calculate P_N up to the order q^N for a large value of N. Finally, we check if P_N also factorizes similarly. Note that

$$P - P_N \in q^{N+1}\mathbb{N}[[q]].$$

Seldom, these recursions will lead to easy proofs, for example, in the case of Identity 5.10.

4. Search Space

The natural search space here is a collection of conditions:

Parameters: $N, A_i, B_i, C_i, D_i, Bool_i$

For each $i = 1, \ldots, N$:

The A_ith flattest length B_i partition of any n' is forbidden to appear as a sub-partition if $n' \equiv C_i \pmod{D_i}$. The boolean bit $Bool_i$ toggles between $\equiv$ and $\not\equiv$.

Many well-known identities have the following property of initial conditions:

A partition π satisfies the difference conditions and the initial conditions if and only if $\pi + 0$, i.e., π adjoined with a "fictitious 0" part satisfies the difference conditions.

We utilize this criterion to impose natural initial conditions. Sometimes, adding more than one fictitious zeros could lead to interesting identities.

Remark 4.1. Many identities come in pairs or sets (like Rogers–Ramanujan), and in such cases, at least one identity in the set seems to satisfy this criterion. For the second Capparelli identity, the initial condition that 2 does not appear could be replaced by assuming a fictitious -1 as a part.

Remark 4.2. Six new conjectural identities were found in [18]. It can be checked that the initial conditions in the identities I_2–I_6 in [18] are all given by one or more fictitious zeros. I_1 does not have an initial condition.

In [28] three more identities, called I_{4a}, I_{5a}, I_{6a}, were found as companions to the corresponding identities in [18]. These identities involved initial conditions which at first sight seem very mysterious. However, again, it can be checked that the initial conditions in I_{4a}, I_{5a}, I_{6a} can be substituted with fictitious zero(s). There is a tiny bit of adjustment needed for I_{6a} which we leave to the reader.

One may find more examples of this phenomenon in [19], for example, initial conditions in Identity 3 could be replaced by two fictitious zeros.

5. Results

We will express the identities in the following way:
Product: Condition "P"
Sum: Condition "S"
Conjugate: Condition "C"

Flat form: Condition "F"

This corresponds to the statement that for any n, partitions satisfying Condition "P" are equinumerous with partitions satisfying Condition "S", and moreover, the generating function for the former class of partitions can be expressed as a periodic infinite product.

In Condition "C", we will describe the conditions obtained when the sum-side partitions are replaced by their conjugates (transposing the Ferrer's diagram). We will omit the proof of equivalence of Condition "S" and Condition "C".

In Condition "F", we will encode the difference conditions on the sum-sides in the "forbidding flattest partitions" format using the following convention:

> $[A, B; \equiv C\,(D)]$ corresponds to forbidding the appearance, as a sub-partition, of the Ath flattest length B partition of any number that is $\equiv C \pmod{D}$. We may also use $\not\equiv$ as necessary.

The first few families of identities are either direct generalizations of MacMahon's identity recalled in the Introduction or resemble it closely. We shall provide bijective proofs of these identities.

Generalizations of MacMahon's partition identity were provided by Andrews [1], and later by Subbarao [30]. Then, Andrews, Eriksson, Petrov, and Romik provided a bijective proof of MacMahon's partition identity [6]. A different bijective proof of MacMahon's partition identity was provided by Fu and Sellers [13], who also extended this new bijection to cover the generalizations of Andrews and of Subbarao, along with a new extension of their own.

► Family 1

This family is composed of three infinite sub-families:

Fix $k \geq 1$.

Family 1.1

Product: Parts are either multiples of 3 or congruent to $\pm 2 \pmod{3k+3}$.

Sum:

- Difference between adjacent parts is not 1.
- If the difference between adjacent parts is in $\{2, 5, \ldots, 3k-4\}$, then the smaller of these parts must be $\not\equiv 2 \pmod 3$.
- If the difference between adjacent parts is in $\{4, 7, \ldots, 3k-2\}$, then the smaller of these parts must be $\equiv 1 \pmod 3$.
- Initial conditions are given by a fictitious zero, i.e., no parts are equal to $1, 4, \ldots, 3k-2$.

Conjugate:

- No part appears exactly once.
- If the frequency of a part belongs to $\{2, 5, 8, \ldots, 3k-4\}$, then the number of parts that are strictly greater than it must be $\not\equiv 2 \pmod 3$.
- If the frequency of a part belongs to $\{4, 7, 10, \ldots, 3k-2\}$, then the number of parts that are strictly greater than it must be $\equiv 1 \pmod 3$.

Family 1.2
Product: Parts are either multiples of 3 or $\equiv -4, -2 \pmod{3k+3}$.
Sum:

- Difference between adjacent parts is not 1.
- If the difference between adjacent parts is in $\{2, 5, \ldots, 3k-4\}$, then the smaller of these parts must be $\not\equiv 0 \pmod 3$.
- If the difference between adjacent parts is in $\{4, 7, \ldots, 3k-2\}$, then the smaller of these parts must be $\equiv 2 \pmod 3$.
- Initial conditions are given by a fictitious zero, i.e., no parts are equal to 1, 4, ..., $3k-2$ or $2, 5, \ldots, 3k-4$.

Conjugate:

- No part appears exactly once.
- If the frequency of a part belongs to $\{2, 5, 8, \ldots, 3k-4\}$, then the number of parts that are strictly greater than it must be $\not\equiv 0 \pmod 3$.
- If the frequency of a part belongs to $\{4, 7, 10, \ldots, 3k-2\}$, then the number of parts that are strictly greater than it must be $\equiv 2 \pmod 3$.

Family 1.3
Product: Parts are either multiples of 3 or $\equiv 2, 4 \pmod{3k+3}$.
Sum:

- Difference between adjacent parts is not 1.
- If the difference between adjacent parts is in $\{2, 5, \ldots, 3k-4\}$, then the smaller of these parts must be $\not\equiv 1 \pmod 3$.
- If the difference between adjacent parts is in $\{4, 7, \ldots, 3k-2\}$, then the smaller of these parts must be $\equiv 0 \pmod 3$.
- Initial conditions are given by a fictitious zero, i.e., no part is equal to 1.

Conjugate:

- No part appears exactly once.
- If the frequency of a part belongs to $\{2, 5, 8, \ldots, 3k-4\}$, then the number of parts that are strictly greater than it must be $\not\equiv 1 \pmod 3$.
- If the frequency of a part belongs to $\{4, 7, 10, \ldots, 3k-2\}$, then the number of parts that are strictly greater than it must be $\equiv 0 \pmod 3$.

We now recall the necessary tools required to prove this family of identities.

Glaisher's Theorem (due to Glaisher [15]), a generalization of Euler's Identity, states that, for fixed modulus $m \geq 2$ and all nonnegative integers n, the number of partitions of n with no parts congruent to $0 \pmod m$ equals the number of partitions of n with no part occurring m or more times. A natural question to ask is whether or not there is a bijective proof of Glaisher's Theorem that "acts" similarly to Sylvester's bijection. As it turns out, a bijection originally due to Stockhofe [29] does the trick. Accordingly, Xiong and Keith [32] provided a refinement of Glaisher's Theorem using a small extension of Stockhofe's bijection. We will give this refinement immediately after defining a few new bits of terminology.

Let the length type of a partition with no parts congruent to m be the $(m-1)$-tuple $(\alpha_1, \alpha_2, \ldots, \alpha_{m-1})$, where there are α_i parts congruent to $i \pmod{m}$. Let the alternating sum type of a partition in which no part occurs m or more times be the $(m-1)$-tuple $(M_1 - M_2, M_2 - M_3, \ldots, M_{m-1} - M_m)$, where M_i is the sum of all parts in the partition whose index is congruent to $i \pmod{m}$.

Theorem 5.1 [32]. *Consider a modulus m and a nonnegative integer n. The number of partitions of n with no parts congruent to $0 \pmod{m}$ and with length type $(\alpha_1, \alpha_2, \ldots, \alpha_{m-1})$ equals the number of partitions of n with no part occurring m or more times with alternating sum type $(\alpha_1, \alpha_2, \ldots, \alpha_{m-1})$.*

We will not give the details of the bijection here, but direct the reader to the works of Xiong and Keith [32] and Stockhofe [29] for more information. The reader is invited to verify for herself that, in the case $m = 2$, this reduces to the properties of Sylvester's bijection to be used for the identities below. For our purposes, we will use the case $m = 3$ to provide a proof of Family 1 which gives a new generalization of MacMahon's identity.

Proof of Family 1. We shall only prove Family 1.1, the other two families being similar.

Consider a partition of n counted in product side. For the parts that are congruent to $0 \pmod{3}$, replace all parts $3j$ with three copies of the part j. Set these parts aside for the time being.

Now, consider the parts that are congruent to $\pm 2 \pmod{3k+3}$. Let the number of parts congruent to $2 \pmod{3k+3}$ be α_1, and the number of parts congruent to $-2 \pmod{3k+3}$ be α_2. These parts can be written as either $(3k+3)\,m_j - (3k-1)$ or $(3k+3)\,m_j - 2$, respectively. Map these parts to $3m_j - 2$ and $3m_j - 1$, respectively. This now provides a partition in which no part is a multiple of 3. At this stage, use Stockhofe's bijection to obtain a partition $\mu_1 + \mu_2 + \mu_3 + \cdots$ in which each part appears at most twice which has length type (α_1, α_2).

Let

$$
\begin{aligned}
M_1 &= \mu_1 + \mu_4 + \mu_7 + \cdots\,, \\
M_2 &= \mu_2 + \mu_5 + \mu_8 + \cdots\,, \\
M_3 &= \mu_3 + \mu_6 + \mu_9 + \cdots\,,
\end{aligned}
$$

so the alternating sum type is $(M_1 - M_2, M_2 - M_3)\,.$

Now:

- Replace each part $\mu_1, \mu_4, \mu_7, \ldots$ with two copies of that part.
- Replace each part $\mu_2, \mu_5, \mu_8, \ldots$ with $(3k-1)$ copies of that part.
- Replace each part $\mu_3, \mu_6, \mu_9, \ldots$ with two copies of that part.

We need to verify that all of these operations restore the partition to its original weight. We just added back in a sum of $M_1 + (3k-2)M_2 + M_3$. But, we know

$$\begin{aligned} M_1 - M_2 &= \alpha_1, \\ M_2 - M_3 &= \alpha_2, \\ M_1 + M_2 + M_3 &= 3\sum m_j - 2\alpha_1 - \alpha_2. \end{aligned}$$

Now,

$$\begin{aligned} &M_1 + (3k-2)M_2 + M_3 \\ &\quad = k\left(M_1 + M_2 + M_3\right) - (k-1)\left(M_1 - M_2\right) + (k-1)\left(M_2 - M_3\right) \\ &\quad = K\left(3\sum m_j - 2\alpha_1 - \alpha_2\right) - (k-1)\,\alpha_1 + (k-1)\,\alpha_2 \\ &\quad = \sum 3km_j - (3k-1)\,\alpha_1 - \alpha_2, \end{aligned}$$

so we are adding $\sum 3km_j - (3k-1)\,\alpha_1 - \alpha_2$ back into our partition.

Restoring the "set aside" parts that come in triples from the very start of the proof, we have obtained a partition that satisfies the conditions as in the Conjugate formulation.

Now we produce a candidate for the inverse map. Consider a partition π satisfying the conditions of the conjugate formulation. Break π into five pieces:

1. In π_1 collect those parts of π whose frequency is divisible by 3.
2. In π_2 collect those parts that have frequency $\equiv 2 \pmod 3$ such that the number of strictly larger parts is $\not\equiv 2 \pmod 3$. Note that parts of π with frequency belonging to $\{2, 5, \ldots, 3k-4\}$ are exactly the parts accounted in π_2.
3. In π_3 collect those parts that have frequency $\equiv 2 \pmod 3$ such that the number of strictly larger parts is $\equiv 2 \pmod 3$. Clearly, any part appearing in π_3 has frequency at least $3k-1$.
4. In π_4 collect those parts that have frequency $\equiv 1 \pmod 3$ such that the number of strictly larger parts is $\equiv 1 \pmod 3$. Parts of π with frequency belonging to $\{4, 7, \ldots, 3k-2\}$ are exactly the parts accounted in π_4.
5. In π_5 collect those parts that have frequency $\equiv 1 \pmod 3$ such that the number of strictly larger parts is $\not\equiv 1 \pmod 3$. Any part appearing in π_5 has frequency at least $3k+1$.

Retain 2 copies of each part appearing in π_2 and move the rest of the copies to π_1. Retain $3k-1$ copies of each part appearing in π_3 and move the rest of the copies to π_1. Retain 4 copies of each part appearing in π_3 and move the rest of the copies to π_1. Retain $3k+1$ copies of each part appearing in π_3 and move the rest of the copies to π_1. Denote the new π_1 by π_1'. After this, keep only 1 copy of each part appearing in π_2 and π_3, discard the rest of the copies, and call the new partitions π_2' and π_3'. Similarly get π_4' and π_5' by retaining 2 copies of each part in π_4 and π_5, respectively.

Coalesce every tuple of 3 copies of a part j from π_1' into a new part $3j$, and call the new partition π_1'' and keep this aside.

Consider $\mu = \pi_2' + \pi_3' + \pi_4' + \pi_5'$ and map this via inverse of Stockhofe's bijection we used above to obtain a new partition μ' in which no part is a multiple of 3. In μ', send every part $3m_j - 1$ to the part $(3k+3)\,m_j - (3k-1)$ and every part $3m_j - 2$ to $(3k+3)\,m_j - 2$. Call this new partition μ''.

Finally, merge π_1'' and μ''.

We leave it to the reader to convince that this is indeed the inverse map. □

Example 5.2. Letting $k = 1$ in Family 1.1 recovers MacMahon's identity (the second and third conditions on the sum-side are vacuous).

Now we present identities obtained with $k = 2$ which were the ones found by our computer program:

Example 5.3. Take $k = 2$ in Family 1.1.

Product: Parts are $\equiv 0, 2, 3, 6, 7 \pmod 9$.

Sum:

- Difference between adjacent parts is not 1.
- If the difference between adjacent parts is 2 then their sum is $\not\equiv 0 \pmod 6$.
- If the difference between adjacent parts is 4 then their sum is $\not\equiv 2, 4 \pmod 6$.
- Initial conditions are given by a fictitious zero, i.e., the smallest part is not 1 or 4.

Conjugate:

- Difference between adjacent parts is not 1.
- If a part appears exactly twice, then the number of parts bigger than it is $\not\equiv 2 \pmod 3$.
- If a part appears exactly four times, then the number of parts bigger than it is $\equiv 1 \pmod 3$.

Flat form: Forbid $[1, 2; \equiv 1\,(2)]$, $[2, 2; \equiv 0\,(6)]$, $[3, 2; \equiv 2\,(6)]$, and $[3, 2; \equiv 4\,(6)]$.

Recursions: Even though we have provided a proof above, we also provide the following recursions as they will lead to a nice pattern:

$$P_1 = 1, \quad P_2 = \frac{1}{1-q^2}, \quad P_3 = \frac{1}{1-q^3} + \frac{1}{1-q^2} - 1,$$

$$P_{3k} = P_{3k-1} + \frac{q^{3k}}{1-q^{3k}}\left(P_{3k-2} - P_{3k-4} + P_{3k-5}\right),$$

$$P_{3k+1} = P_{3k} + \frac{q^{3k+1}}{1-q^{3k+1}}\left(P_{3k-2} - P_{3k-3} + P_{3k-4}\right),$$

$$P_{3k+2} = P_{3k+1} + \frac{q^{3k+2}}{1-q^{3k+2}} P_{3k}.$$

□

Example 5.4. Take $k = 2$ in Family 1.2.

Product: Parts are $\equiv 0, 3, 5, 6, 7 \pmod 9$.

Sum:

- Difference between adjacent parts is not 1.
- If the difference between adjacent parts is 2 then their sum is $\not\equiv 2 \pmod 6$.
- If the difference between adjacent parts is 4 then their sum is $\not\equiv 0, 4 \pmod 6$.
- Initial conditions are given by a fictitious zero, i.e., the smallest part is not 1, 2 or 4.

Conjugate:

- Difference between adjacent parts is not 1.
- If a part appears exactly twice then the number of parts bigger than it is $\not\equiv 0 \pmod 3$.
- If a part appears exactly four times then the number of parts bigger than it is $\equiv 2 \pmod 3$.

Flat form: Forbid $[1, 2; \equiv 1\,(2)]$, $[2, 2; \equiv 2\,(6)]$, $[3, 2; \equiv 0\,(6)]$, and $[3, 2; \equiv 4\,(6)]$.

Recurrences:

$$P_1 = P_2 = 1, \quad P_3 = P_4 = \frac{1}{1-q^3}, \quad P_5 = \frac{1}{1-q^5} + \frac{1}{1-q^3} - 1,$$

$$P_{3k} = P_{3k-1} + \frac{q^{3k}}{1-q^{3k}} P_{3k-2},$$

$$P_{3k+1} = P_{3k} + \frac{q^{3k+1}}{1-q^{3k+1}} \left(P_{3k-1} - P_{3k-3} + P_{3k-4}\right),$$

$$P_{3k+2} = P_{3k+1} + \frac{q^{3k+2}}{1-q^{3k+2}} \left(P_{3k-1} - P_{3k-2} + P_{3k-3}\right).$$

□

Example 5.5. Take $k = 2$ in Family 1.3.

Product: Parts are $\equiv 0, 2, 3, 4, 6 \pmod 9$.

Sum:

- Difference between adjacent parts is not 1.
- If the difference between adjacent parts is 2, then their sum is $\not\equiv 4 \pmod 6$.
- If the difference between adjacent parts is 4, then their sum is $\not\equiv 0, 2 \pmod 6$.
- Initial conditions are given by a fictitious zero, i.e., the smallest part is not 1.

Conjugate:

- Difference between adjacent parts is not 1.
- If a part appears exactly twice, then the number of parts bigger than it is $\not\equiv 1 \pmod 3$.
- If a part appears exactly four times, then the number of parts bigger than it is $\equiv 0 \pmod 3$.

Flat form: Forbid $[1, 2; \equiv 1\,(2)]$, $[2, 2; \equiv 4\,(6)]$, $[3, 2; \equiv 0\,(6)]$, and $[3, 2; \equiv 2\,(6)]$.

Recurrences: We have the following recursions:

$$
\begin{aligned}
P_1 = 1,\ P_2 &= \frac{1}{1-q^2},\ P_3 = \frac{1-q^5}{(1-q^3)(1-q^2)}, \\
P_4 &= P_3 + \frac{q^4}{(1-q^4)(1-q^2)},\ P_5 = P_4 + \frac{q^5}{1-q^5}P_3, \\
P_{3k} &= P_{3k-1} + \frac{q^{3k}}{1-q^{3k}}\left(P_{3k-3} - P_{3k-4} + P_{3k-5}\right), \\
P_{3k+1} &= P_{3k} + \frac{q^{3k+1}}{1-q^{3k+1}}P_{3k-1}, \\
P_{3k+2} &= P_{3k+1} + \frac{q^{3k+2}}{1-q^{3k+2}}\left(P_{3k} - P_{3k-2} + P_{3k-3}\right).
\end{aligned}
$$

□

Remark 5.6. Note how the recursions in the previous three identities are related by a cyclic shift.

▶ **Family 2**

This is an infinite family, with one identity for every even modulus ≥ 4.

Fix an even $k \geq 1$.

Product: Each part is either even or $\equiv -1 \pmod{2k+2}$.

Sum:

- An odd part $2j+1$ is not immediately adjacent to any of the $2j, 2j-2, \ldots, 2j-2k+2$ (its previous k even numbers).
- Initial conditions are implied by adding a fictitious zero. That is, the smallest part is not equal to $1, 3, \ldots, 2k-1$.

Conjugate: If a part appears exactly 1, or 3, …, or $2k-1$ times, then there are an odd number of parts strictly greater than it.

Euler's celebrated partition identity states that, for any nonnegative integer n, the number of partitions of n into odd parts equals the number of partitions of n into distinct parts. A key ingredient in our work is the bijective proof of this identity given by Sylvester in his classic, colorfully-named treatise on partitions [31]. This may not be the "simplest" proof—or even the easiest bijective proof—but it possesses some properties that will be important for us later. (See the work of Zeilberger [33] for a recursive formulation of the bijection; for more information on partition bijections, see Pak's lucid survey article [26]).

Proof. This and the following few families will be proved using Pak and Postnikov's bijection [27]:

Consider a partition of n counted in the product side, i.e., a partition in which each part is either even or $\equiv -1 \pmod{2k+2}$. First, we break all even parts in halves, that is, for the parts that are congruent to $0 \pmod 2$, replace all parts $2j$ with two copies of the part j.

The remaining parts are all of the form $(2k+2)m_j - 1$ for some positive integers m_j. Replace each of these parts with $2m_j - 1$; we are now considering a partition into odd parts. We send this to a partition into distinct parts [27]. For this new partition $\mu_1 + \mu_2 + \mu_3 + \mu_4 + \cdots$ we replace each odd-indexed part with $2k+1$ copies of that part.

The proof that this procedure gives a partition of correct weight and that this map is a bijection between partitions counted in the product side and the ones counted in the conjugate formulation is exactly as in the proof of Family 1 given above.

Now we produce a candidate for the inverse map. Consider a partition π of weight n satisfying the conditions of the conjugate formulation. Break π into three classes. Collect in π_1 those parts that appear with an even frequency, collect in π_2 those parts that appear with an odd frequency such that the number of parts that are strictly larger is also odd, and collect in π_3 those parts that appear with an odd frequency such that the number of parts that are strictly larger is even. Note that π_2 necessarily contains all those parts of π that appear with an odd frequency $\leq 2k-1$, and any part appearing in π_3 has an frequency at least $2k+1$.

Now, retain one copy of each part appearing in π_2, and move the rest of the copies to π_1. Retain $2k+1$ copies of each part appearing in π_3 and move the rest of the copies (of which there are an even number) to π_1. After this, only retain a single copy of each part appearing in π_3 and discard the rest of the copies. Call the new partitions π_1', π_2' and π_3'.

For π_1', merge two copies of each part j into a new part $2j$, call the new partition π_1'' and keep it aside.

Consider $\mu = \pi_2' + \pi_3'$. It is not hard to see that in μ the distinct parts and parts of odd index are precisely the parts coming from π_3'. Now map μ to a partition with odd parts μ'. In μ' map every odd part $2m_j - 1$ to $(2k+2)m_j - 1$ to obtain a new partition μ''. Finally, merge π_1'' and μ''.

We leave it to the reader to convince herself that this is indeed the inverse map. $\square$

Let us give a specific example. Consider the theorem in the case that $k = 2$. The product side allows parts congruent to $0 \pmod 2$ and $5 \pmod 6$; for example, consider

$$40 + 23 + 14 + 14 + 12 + 11 + 6 + 6 + 6 + 5 + 5.$$

First, we replace all of the even parts $2j$ with two copies of j, obtaining

$$\begin{aligned} 40 + 14 + 14 + 12 + 6 + 6 + 6 \mapsto {} & 20 + 20 + 7 + 7 + 7 + 7 \\ & +6 + 6 + 3 + 3 + 3 + 3 + 3 + 3. \end{aligned}$$

Now consider the remaining odd parts, which are all congruent to $5 \pmod 6$:

$$23 + 11 + 5 + 5.$$

Sending each part of the form $6m_j - 1$ to $2m_j - 1$ produces

$$7 + 3 + 1 + 1.$$

This maps to the following partition with distinct parts:

$$7 + 4 + 1.$$

We now replace each odd-indexed part with 5 copies of itself, producing

$$7 + 7 + 7 + 7 + 7 + 4 + 1 + 1 + 1 + 1 + 1.$$

Now, combining this with the previously obtained parts, we finally get

$$\begin{aligned} &20 + 20 + 7 + 7 + 7 + 7 + 7 + 7 + 7 + 7 + 7 + 6 + 6 + 4 \\ &\quad +3 + 3 + 3 + 3 + 3 + 3 + 1 + 1 + 1 + 1 + 1. \end{aligned}$$

For the inverse map, check that

$$\begin{aligned} \pi_1 &= 20 + 20 + 6 + 6 + 3 + 3 + 3 + 3 + 3 + 3, \\ \pi_2 &= 4, \\ \pi_3 &= 7 + 7 + 7 + 7 + 7 + 7 + 7 + 7 + 7 + 1 + 1 + 1 + 1 + 1. \end{aligned}$$

We get

$$\begin{aligned} \pi_1' &= 20 + 20 + 7 + 7 + 7 + 7 + 6 + 6 + 3 + 3 + 3 + 3 + 3 + 3, \\ \pi_2' &= 4, \\ \pi_3' &= 7 + 1. \end{aligned}$$

We have

$$\pi_2'' = 40 + 14 + 14 + 12 + 6 + 6 + 6.$$

We also have

$$\begin{aligned} \mu &= 7 + 4 + 1, \\ \mu' &= 7 + 3 + 1 + 1 \end{aligned}$$

and

$$\mu'' = 23 + 11 + 5 + 5.$$

Finally, we have

$$\pi'' + \mu'' = 40 + 23 + 14 + 14 + 12 + 11 + 6 + 6 + 6 + 5 + 5.$$

Remark 5.7. All of the families from here until Family 7 will use a very similar procedure to obtain the bijections. We shall only indicate how the proofs differ, leaving the details to the reader.

Example 5.8. Product: Parts are $\equiv 0, 2, 3 \pmod 4$.

Sum:

- An odd part $2j + 1$ is not immediately adjacent to $2j$.
- The smallest part is not 1.

Flat form: Forbid $[1, 2; \equiv 1\,(4)]$.

Proof. This particular identity can be proved quickly using recursions.

$$P_1 = 1,$$
$$P_{2j+1} = \frac{q^{2j+1}}{1-q^{2j+1}} P_{2j-1} + \frac{1}{1-q^{2j}} P_{2j-1} = \frac{1-q^{4j+1}}{(1-q^{2j})(1-q^{2j+1})} P_{2j-1}.$$

Now take the limit as $j \to \infty$. □

Example 5.9. Product: Parts are $\equiv 0, 2, 4, 5 \pmod 6$.

Sum:

- An odd part $2j+1$ is not immediately adjacent to either of $2j$ or $2j-2$.
- Initial conditions are implied by a fictitious zero, i.e., the smallest part is not equal to 1 or 3.

Flat-form: Forbid $[1, 2; \equiv 1\,(4)]$, and $[2, 2; \equiv 3\,(4)]$.

▶ **Family 3**

This is an infinite family with one identity for each even modulus ≥ 4.

Fix $k \geq 1$.

Product: Each part is either even or $\equiv 1 \pmod{2k+2}$.

Sum: An even part $2j$ is forbidden to be adjacent to either of $2j-1, 2j-3, \dots, 2j-2k+1$ (its previous k odd numbers).

Conjugate: If a part appears exactly 1, or 3, ..., or $2k-1$ times, then there are an even number of parts strictly greater than it.

Proof. Consider a partition of n counted in the product side. Then, break all even parts in halves. Now, the remaining parts are all of the form $(2k+2)m_j+1$ for some positive integers m_j. Replace each of these parts with $2m_j+1$; we are now considering a partition into odd parts. Now, send this to a partition into distinct parts. For this new partition $\mu_1 + \mu_2 + \mu_3 + \mu_4 + \cdots$ replace each even-indexed part with $2k+1$ copies of that part. □

Example 5.10. Product: Parts are $\equiv 0, 1, 2 \pmod 4$.

Sum: An even part $2j$ is forbidden to be immediately adjacent to $2j-1$.

Flat form: Forbid $[1, 2; \equiv 3\,(4)]$.

Proof. Let P_j be the generating function of sum sides, with added restriction that the largest part is $\leq j$. Then,

$$P_{2j} = \frac{1}{1-q^{2j}} P_{2j-2} - P_{2j-2} + P_{2j-1}, \quad P_{2j-1} = \frac{1}{1-q^{2j-1}} P_{2j-2}.$$

So, we get

$$P_{2j} = P_{2j-2}\left(\frac{1}{1-q^{2j}} + \frac{1}{1-q^{2j-1}} - 1\right) = P_{2j-2}\left(\frac{1-q^{4j-1}}{(1-q^{2j})(1-q^{2j-1})}\right).$$

Now use $P_2 = \dfrac{1-q^3}{(1-q)(1-q^2)}$ and induct. □

Example 5.11. Product: Parts are $\not\equiv 0, 1, 2, 4 \pmod 6$.

Sum:

- An even part $2j$ is forbidden to be immediately adjacent to $2j-1$ or $2j-3$.

Flat form: Forbid $[1,2;\equiv 3\,(4)]$, and $[2,2;\equiv 1\,(4)]$.

▶ **Family 4**

This is an infinite family, with one identity for every even modulus greater than or 8. For modulus 6, one of the conditions becomes redundant and one gets MacMahon's identity recalled in the Introduction.

Fix $k \geq 1$.

Product: Parts are either even or $\equiv 3 \pmod{2k+6}$.

Sum:

- Difference between adjacent parts is not 1.
- An even part $2j$ is not immediately adjacent to any of $2j-3, \ldots, 2j-2k-1$.
- The initial condition is implied by a fictitious zero. That is, the smallest part is not 1.

Conjugate:

- No part appears exactly once.
- If a part appears exactly 3, or 5, …, or $2k+1$ times then there are an even number of parts strictly greater than it.

Proof. Consider a partition of n counted in the product side. Then, break all even parts in halves. Now, the remaining parts are all of the form $(2k+6)m_j+3$ for some positive integers m_j. Replace each of these parts with $2m_j+1$; we are now considering a partition into odd parts. Now, send this to a partition into distinct parts. For this new partition $\mu_1+\mu_2+\mu_3+\mu_4+\cdots$ replace each odd-indexed part with 3 copies of that part and each even-indexed part with $2k+3$ copies of that part. □

Example 5.12. Product: Parts are $\equiv 0,2,3,4,6 \pmod 8$.

Sum:

- Difference between adjacent parts is not 1.
- An even part $2j$ is not immediately adjacent to $2j-3$.
- The initial condition is implied by a fictitious zero. That is, the smallest part is not 1.

Flat form: Forbid $[1,2;\equiv 1\,(2)]$, and $[2,2;\equiv 1\,(4)]$.

▶ **Family 5**

This is again an infinite family, with one identity for every even modulus ≥ 8. Again, for modulus 6, one of the conditions becomes redundant and we get MacMahon's identity.

Fix $k \geq 1$.

Product: Parts are either even or $\equiv -3 \pmod{2k+6}$.

Sum:

- Difference between adjacent parts is not 1.
- An odd part $2j + 1$ is not allowed to be immediately adjacent to $2j - 2, \ldots, 2j - 2k$.
- Initial conditions are implied by a fictitious zero, i.e., the smallest part is not equal to $1, 3, \ldots, 2k + 1$.

Conjugate:

- No part appears exactly once.
- If a part appears exactly 3, or 5, …, or $2k + 1$ times then there are an odd number of parts strictly greater than it.

Proof. Consider a partition of n counted in the product side. Then, break all even parts in halves. Now, the remaining parts are all of the form $(2k+6)m_j - 3$ for some positive integers m_j. Replace each of these parts with $2m_j - 1$; we are now considering a partition into odd parts. Now, send this to a partition into distinct parts. For this new partition $\mu_1 + \mu_2 + \mu_3 + \mu_4 + \cdots$ replace each even-indexed part with 3 copies of that part and each odd-indexed part with $2k + 3$ copies of that part. □

Example 5.13. Product: Parts are $\equiv 0, 2, 4, 5, 6 \pmod 8$.

Sum:

- Difference between adjacent parts is not 1.
- An odd part $2j + 1$ is not allowed to be immediately adjacent to $2j - 2$.
- The smallest part is not equal to 1 or 3.

Flat form: Forbid $[1, 2; \equiv 1\,(2)]$, and $[2, 2; \equiv 3\,(4)]$.

► Family 6

An infinite family with one identity for every modulus divisible by 4 and ≥ 12. Fix $k \geq 1$.

Product: Parts are even or $\equiv 2k + 5 \pmod{4k + 8}$.

Sum:

- Difference between consecutive parts cannot be $1, 3, \ldots, 2k + 1$.
- An odd part $2j + 1$ cannot be immediately adjacent to $2j - 2k - 2$.
- Initial conditions are implied by a fictitious zero, that is, the smallest part cannot be either of $1, 3, \ldots, 2k + 3$.

Conjugate:

- No part appears exactly 1, 3, …, $2k + 1$ times.
- If a part appears exactly $2k + 3$ times then there are an odd number of parts strictly greater than it.

Proof. Consider a partition of n counted in the product side. Then, break all even parts in halves. Now, the remaining parts are all of the form $(4k + 8)m_j + (2k + 5)$ for some positive integers m_j. Replace each of these parts with $2m_j + 1$; we are now considering a partition into odd parts. Now, send this to a partition into distinct parts. For this new partition $\mu_1 + \mu_2 + \mu_3 + \mu_4 + \cdots$ replace each odd-indexed part with $2k + 5$ copies of that part and replace each even-indexed part with $2k + 3$ copies of that part. □

Example 5.14. Product: Parts are $\equiv 0, 2, 4, 6, 7, 8, 10 \pmod{12}$.
Sum:

- Difference between consecutive parts cannot be 1 or 3.
- An odd part $2j+1$ cannot be immediately adjacent to $2j-4$.
- Initial conditions are implied by a fictitious zero, that is, the smallest part cannot be either of $1, 3, 5$.

Flat form: Forbid $[1, 2; \equiv 1\,(2)]$, $[2, 2; \equiv 1\,(2)]$, and $[3, 2; \equiv 1\,(4)]$.

▶ **Family 7**
An infinite family, with one identity for every modulus divisible by 4 that is ≥ 12.

Fix $k \geq 1$.
Product: Each part is either even or $\equiv 2k+3 \pmod{4k+8}$.
Sum:

- Difference between consecutive parts cannot be $1, 3, \ldots, 2k+1$.
- An even part $2j$ cannot be immediately adjacent to $2j-2k-3$.
- Initial conditions are given by a fictitious zero, that is, the smallest part is not amongst $1, 3, \ldots, 2k+1$.

Conjugate:

- No part appears exactly 1, 3, …, $2k+1$ times.
- If a part appears exactly $2k+3$ times then there are an even number of parts strictly greater than it.

Proof. Consider a partition of n counted in the product side. Then, break all even parts in halves. Now, the remaining parts are all of the form $(4k+8)m_j + (2k+3)$ for some positive integers m_j. Replace each of these parts with $2m_j+1$; we are now considering a partition into odd parts. Now, send this to a partition into distinct parts. For this new partition $\mu_1 + \mu_2 + \mu_3 + \mu_4 + \cdots$ replace each even-indexed part with $2k+5$ copies of that part and replace each odd-indexed part with $2k+3$ copies of that part. $\square$

Example 5.15. Product: Each part is either even or $\equiv 5 \pmod{12}$.
Sum:

- Difference between consecutive parts cannot be 1 or 3.
- An even part $2j$ cannot be immediately adjacent to $2j-5$.
- The smallest part is not 1 or 3.

Flat form: Forbid $[1, 2; \equiv 1\,(2)]$, $[2, 2; \equiv 1\,(2)]$, and $[3, 2; \equiv 3\,(4)]$.

Remark 5.16. Families 2–7 are of a very similar nature. It seems very likely that they can all be incorporated into a grand family and proved together. We leave this to an interested reader.

▶ **Family 8**
Let $k \geq 2$.[1]

[1] This family is incorporated from the article http://arxiv.org/abs/1703.04715 [math.CO].

Product: Each part is either even but $\not\equiv 2 \pmod{4k}$ or odd and $\equiv 1, 2k+1 \pmod{4k}$.

Sum:

– If an odd part $2j+1$ is present, then none of the other parts are equal to any of $2j+1, 2j+2, \ldots, 2j+2k-1$.

Actually, this family is in a sense dual to the following family of identities due to Andrews, some special cases of which were found by our computer program:

Theorem 5.17 (Theorem 3, [2]). *Let $k \geq 2$. Product: Each part is either even but $\not\equiv 4k-2 \pmod{4k}$ or odd and $\equiv 2k-1, 4k-1 \pmod{4k}$.*
Sum:

– *If an odd part $2j+1$ is present, then none of the other parts are equal to any of $2j+1, 2j, \ldots, 2j-2k+3$.*
– *The smallest part is not equal to any of $1, 3, \ldots, 2k-3$.*

Over the past decade or so, there has been a lot of interest in exploring overpartition analogues of classical partition identities (as a small and by no means exhaustive sample, see papers by Chen et al. [9], Lovejoy [22], Corteel and Lovejoy [10] and Dousse [12]). Overpartitions are partitions in which the last occurrence of any part may appear overlined.

The fact that odd parts are not allowed to be repeated (though even parts may be repeated arbitrarily many times) in the identities above suggests that both are actually special cases of an overpartition theorem. We now present an overpartition generalization that can be used to recover Family 8 and Theorem 5.17 upon appropriate specializations.

Theorem 5.18. *For $k \geq 2$, let $A_k(m,n)$ be the number of overpartitions of n with exactly m overlined parts, subject to the following conditions:*

- *If an overlined part $\overline{b}$ appears then all of the non-overlined parts $b, b+1, \ldots, b+k-2$ are forbidden to appear.*
- *If an overlined part $\overline{b}$ appears then all of the overlined parts $\overline{b+1}$, $\overline{b+2}$, $\ldots$, $\overline{b+k-1}$ are forbidden to appear.*

Then,

$$\sum_{m,n\geq 0} A_k(m,n) a^m q^n = \frac{(-aq;q^k)_\infty}{(q;q)_\infty}.$$

Proof. Fix $k \geq 2$. Let $p_j(m,n)$ be the number of overpartitions of n with m overlined parts that satisfy the conditions in Theorem 5.18, with the further restriction that all parts are $\leq j$. Let $r_j(m,n)$ be the number of overpartitions of n counted by $p_j(m,n)$ where $\overline{j}$, $\overline{j-1}$, $\ldots$, $\overline{j-k+2}$ do not appear (that is, the largest possible overlined part is $\overline{j-k+1}$). Then, let

$$P_j(a,q) = \sum_{m,n\geq 0} p_j(m,n) a^m q^n,$$

$$R_j(a,q) = \sum_{m,n\geq 0} r_j(m,n) a^m q^n,$$

and let $P_0 = R_0 = 1$. It is clear that

$$R_\infty(a,q) = P_\infty(a,q) = \sum_{m,n\geq 0} D_k(m,n)a^m q^n.$$

Let

$$F(a,x,q) = \sum_{j\geq 0} R_j(a,q)x^j.$$

Observe that the following recursion and initial conditions are satisfied:

$$\begin{aligned} R_j(a,q) &= \frac{1}{1-q^j}R_{j-1}(a,q) + \frac{aq^{j-k+1}}{1-q^j}R_{j-k}(a,q),\ j\geq k,\\ R_j(a,q) &= \frac{1}{(q;q)_j},\ 0\leq j<k. \end{aligned}$$

Note the following alternate way to write the recursion and the initial conditions:

$$\begin{aligned} R_j(a,q) &= \frac{1}{1-q^j}R_{j-1}(a,q) + \frac{aq^{j-k+1}}{1-q^j}R_{j-k}(a,q),\ j\geq 1,\\ R_0(a,q) &= 1, \quad R_j(a,q) = 0 \text{ for } -k<j<0, \end{aligned}$$

which immediately gives

$$(1-x)F(a,x,q) = F(a,xq,q) + ax^kqF(a,xq,q) = (1+ax^kq)F(a,xq,q).$$

Noting that

$$\lim_{n\to\infty} F(a,xq^n,q) = R_0(a,q) = 1,$$

we obtain

$$F(a,x,q) = \prod_{j\geq 0}\frac{1+ax^kq^{jk+1}}{1-xq^j}.$$

Finally, by Appell's comparison theorem [11, p. 101] we have:

$$R_\infty(a,q) = \lim_{x\to 1}((1-x)F(a,x,q)) = \lim_{x\to 1}\left(\prod_{j\geq 0}\frac{1+ax^kq^{jk+1}}{1-xq^{j+1}}\right) = \frac{\left(-aq;q^k\right)_\infty}{(q;q)_\infty}.$$

□

Now, Family 8 and Theorem 5.17 can be recovered by appropriate specializations. Letting $(a,q)\mapsto\left(q^{-1},q^2\right)$ (that is, we map every nonoverlined part $j\mapsto 2j$ and every overlined part $\overline{j}\mapsto 2j-1$) gives Family 8, while using $(a,q)\mapsto\left(q^{2k-3},q^2\right)$ (now mapping $j\mapsto 2j$ and every overlined part $\overline{j}\mapsto 2j+2k-3$) provides us with Theorem 5.17.

However, many more corollaries can be found. For $k\geq 2$, by choosing $i\in\{0,\ldots,k-1\}$ and letting $(a,q)\mapsto\left(q^{2i-1},q^2\right)$, we get:

Corollary 5.19. *Let $B(n)$ be the number of partitions of a non-negative integer n in which each part is either even but $\not\equiv 4i+2 \pmod{4k}$ or odd and $\equiv 2i+1, 2k+2i+1 \pmod{4k}$. Also, let $C(n)$ be the number of partitions of n in which if an odd part $2j+1$ is present, then none of the other even parts are equal to any of*

$$2j-2i+2, 2j-2i+4, \ldots, 2j+2k-2i-2,$$

none of the other odd parts are equal to any of

$$2j+1, 2j+3, \ldots, 2j+2k-1,$$

and the smallest odd part is at least $2i+1$. Then, $B(n) = C(n)$ for all n.

We leave it to the reader to work out identities related to the specializations $q \mapsto q^t$ for $t > 2$.

► Family 9

This is perhaps the easiest of the families that we came across, but it is interesting nonetheless. The proofs of all of the identities in this family follow the same pattern as in Family 9.2 below.

For every modulus, we have k identities that avoid exactly one congruence class in their product. This can be greatly generalized. We first start with the "base" case.

Fix $k \geq 4$ and let $1 \leq j \leq k$.

Product: Parts are $\not\equiv j \pmod{k}$.
Sum:

- If difference at distance $\lceil k/2 \rceil - 1$ is strictly less than 2, then the sum of these $\lceil k/2 \rceil$ parts is $\not\equiv j \pmod{k}$.
- Initial conditions are implied by adding $\lceil k/2 \rceil - 1$ fictitious zeros.

Flat form: Forbid $[1, \lceil k/2 \rceil; \equiv j\,(k)]$.

▷ **Family 9.1** Here we present the full set of identities for $k = 5$.

1. *Product*: Parts are $\not\equiv 1 \pmod{5}$.
   Sum:
   - If difference at distance 2 is 0 or 1, then the sum of these three parts is $\not\equiv 1 \pmod{5}$
   - The smallest part is at least 2.

   Flat form: Forbid $[1, 3; \equiv 1\,(5)]$.
2. *Product*: Parts are $\not\equiv 2 \pmod{5}$.
   Sum:
   - If difference at distance 2 is 0 or 1, then the sum of these three parts is $\not\equiv 2 \pmod{5}$
   - 1 appears at most once.

   Flat form: Forbid $[1, 3; \equiv 2\,(5)]$.
3. *Product*: Parts are $\not\equiv 3 \pmod{5}$.

Sum:

- If difference at distance 2 is 0 or 1, then the sum of these three parts is $\not\equiv 3 \pmod 5$

Flat form: Forbid $[1, 3; \equiv 3\,(5)]$.

5. *Product*: Parts are $\not\equiv 4 \pmod 5$.
   Sum:
   - If difference at distance 2 is 0 or 1, then the sum of these three parts is $\not\equiv 4 \pmod 5$

   Flat form: Forbid $[1, 3; \equiv 4\,(5)]$.
6. *Product*: Parts are $\not\equiv 5 \pmod 5$.
   Sum:
   - If difference at distance 2 is 0 or 1, then the sum of these three parts is $\not\equiv 5 \pmod 5$

   Flat form: Forbid $[1, 3; \equiv 5\,(5)]$.

Avoiding More Number of Congruence Classes

This can be generalized to avoiding two or more congruence classes. The main idea for that is as follows. Let N be the intended modulus, and let S be a set of congruence classes we wish to avoid on the product. Suppose that we are looking for an identity with the following form:
Product: Parts are $\not\equiv S \pmod N$.
Sum:

- If difference at distance 2 is strictly less than 2, i.e., $\lambda_i - \lambda_{i+2} \leq 1$, then the sum of these parts is $\not\equiv S \pmod N$, i.e., $\lambda_i + \lambda_{i+1} + \lambda_{i+2} \not\equiv S \pmod N$.
- Possibly add fictitious zeros as appropriate.

Then, it appears to us that this can always be done as long as elements of S are *sufficiently spread out*. Below, we show how to do this for a few moduli N and a corresponding set S.

Avoiding 2 Congruence Classes. We wish to let $S = \{i, j\}$ be a pair of integers which will be forbidden residues and let N be a modulus.

We sketch a proof of some mod$-$9s; the proofs of others are similar.

▷ **Family 9.2** A family of Mod$-$9s, $N = 9$.

The set S can be taken to be one of

$$\{0,3\},\quad \{0,4\},\quad \{0,5\},\quad \{0,6\},\quad \{1,6\},\quad \{2,6\},\quad \{3,6\},\quad \{3,7\},\quad \{3,8\},$$

with no fictitious zeros added.

Proof. We show how this works for a few pairs. The pairs $\{0,3\}, \{0,6\}, \{3,6\}$ yield very easy identities. The rest are very mildly challenging. The proofs are similar to Identity 5.10.

For $\{0,4\}$, observe that

$$\begin{aligned} &P_{3n+3}\\ &= \left(1+q^{3n+3}+q^{3n+3}q^{3n+3}\right)\left(\frac{q^{3n+2}}{1-q^{3n+2}}(1+q^{3n+1})+\frac{1}{1-q^{3n+1}}\right)P_{3n}\\ &= \frac{(1-q^{9n+9})(1-q^{9n+4})}{(1-q^{3n+3})(1-q^{3n+2})(1-q^{3n+1})}P_{3n}. \end{aligned}$$

Substituting $P_0 = 1$ and letting $n \to \infty$, we get the result.

For $\{0,5\}$, the recursion changes to

$$\begin{aligned} &P_{3n+3}\\ &= \left(1+q^{3n+3}+q^{3n+3}q^{3n+3}\right)\left(\frac{q^{3n+2}q^{3n+2}}{1-q^{3n+2}}+\frac{1}{1-q^{3n+1}}(1+q^{3n+2})\right)P_{3n}. \end{aligned}$$

For $\{3,7\}$, the recursion changes to

$$\begin{aligned} &P_{3n+4}\\ &= \left(1+q^{3n+4}+q^{3n+4}q^{3n+4}\right)\left(\frac{q^{3n+3}}{1-q^{3n+3}}(1+q^{3n+2})+\frac{1}{1-q^{3n+2}}\right)P_{3n+1}. \end{aligned}$$

For $\{3,8\}$, the recursion changes to

$$\begin{aligned} &P_{3n+4}\\ &= \left(1+q^{3n+4}+q^{3n+4}q^{3n+4}\right)\left(\frac{q^{3n+3}q^{3n+3}}{1-q^{3n+3}}+\frac{1}{1-q^{3n+2}}(1+q^{3n+3})\right)P_{3n+1}. \end{aligned}$$

For $\{3,7\}$ and $\{3,8\}$, we let $P_1 = (1-q^3)/(1-q)$. Note the similarity of recursions of $\{0,4\}$ with $\{3,7\}$ and $\{0,5\}$ with $\{3,8\}$. □

▷ **Family 9.3** A family of Mod$-$10s, $N = 10$. The set S takes the values:

$$\{0,5\},\quad \{3,8\},\quad \{3,9\},\quad \{4,9\},\quad \{5,9\},$$

with no fictitious zeros added.

Note that $\{0,5\},\{3,8\},\{4,9\}$ are from an already discovered family of Mod$-$5s.

▷ **Family 9.4** A family of Mod$-$11s, $N = 11$. The set S takes values:

$$\begin{aligned} \text{With no fictitious zeros:}\quad & \{0,4\},\{0,5\},\{0,6\},\ \{3,7\},\{3,8\},\{3,9\},\\ & \{4,9\},\{4,10\},\ \{5,10\},\ \{6,10\}.\\ \text{With 1 fictitious zero:}\quad & \{2,7\},\{2,8\},\{2,9\}.\\ \text{With 2 fictitious zeros:}\quad & \{1,6\},\{1,7\}. \end{aligned}$$

Avoiding 3 Congruence Classes. We exhibit this with an example.

▷ **Identity 9.5.** We continue working with difference at distance 2 and no fictitious zeros. The set $S = \{0,9,16\}$ of three elements modulo 23 works.

One may find other pairs N, S.

Beyond Difference-at-Distance 2

This idea naturally generalizes to conditions with higher distances, as we show with an example.

▷ **Identity 9.6** Let $S = \{0, 7\}$, $N = 17$.
Product: Parts are $\not\equiv S \pmod{N}$.
Sum:

– If difference at distance 3 is strictly less than 2, that is $\lambda_i - \lambda_{i+3} \leq 1$, then the sum of these parts, that is, $\lambda_i + \lambda_{i+1} + \lambda_{i+2} + \lambda_{i+3} \not\equiv S \pmod{N}$.

And so on for other pairs N, S and with conditions at larger distances....

Remark 5.20. We leave to the interested reader to work out a precise theorem that covers all of these examples.

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Shashank Kanade
Department of Mathematics
University of Denver
2390 S York St
Denver
CO 80208
USA
e-mail: shashank.kanade@du.edu

Debajyoti Nandi
Chennai Mathematical Institute H1
Sipcot IT Park
Siruseri
Tamil Nadu 603103
India
e-mail: debajyoti.nandi@gmail.com

Matthew C. Russell
Department of Mathematics
Rutgers, The State University of New Jersey
110 Frelinghuysen Road
Piscataway
NJ 08854
USA
e-mail: russell2@math.rutgers.edu

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Andrews–Gordon Type Series for Kanade–Russell Conjectures

Dedicated to George E. Andrews for his 80th birthday

Kağan Kurşungöz

Abstract. We construct Andrews–Gordon type positive series as generating functions of partitions satisfying certain difference conditions in six conjectures by Kanade and Russell. Thus, we obtain q-series conjectures as companions to Kanade and Russell's combinatorial conjectures. We construct generating functions for missing partition enumerants as well, without claiming new partition identities.

Mathematics Subject Classification. 05A17, 05A15, 11P84.

Keywords. Partition generating function, Andrews–Gordon identities, Kanade–Russell conjectures.

1. Introduction

In November 2014, Kanade and Russell announced six new partition identities using some computer help [6]. The difference conditions on partitions are inspired by Capparelli's identities [1,5].

The first of the conjectures is given below.

Conjecture 1.1 (The Kanade–Russell conjecture I_1). *The number of partitions of a non-negative integer into parts* $\equiv \pm 1, \pm 3 \pmod 9$ *is the same as the number of partitions with difference at least three at distance two such that if two successive parts differ by at most one, then their sum is divisible by three.*

Here, difference at distance two means the difference between the ith and $(i+2)$th parts. The former condition in the conjecture is a congruence condition, and the latter is a difference condition. For example, $n = 9$ has seven partitions satisfying the first constraint:

$$1+1+\cdots+1, \quad 1+1+\cdots+1+3, \quad 1+1+1+3+3,$$
$$1+1+1+6, \quad 1+8, \quad 3+3+3, \quad 3+6,$$

as well as seven partitions satisfying the second constraint:

$$9, \quad 1+8, \quad 2+7, \quad 3+6, \quad 1+3+5, \quad 4+5, \quad 1+2+6.$$

A quote attributed to the late A.O.L. Atkin asserts that it is often easier to prove identities in the theory of q-series than to discover them. Kanade and Russell's conjectures have been counterexamples, since they evaded proof for more than three years so far. This paper, unfortunately, is no attempt to prove them.

After the preprint of this paper appeared, Bringmann, Jennings-Shaffer and Mahlburg [4] announced proofs of the fifth and sixth conjectures in [6] and more conjectures from [7].

The goal of this paper is to construct Andrews–Gordon type series as generating functions of the partitions in the conjectures. In particular, generating functions for partitions satisfying the difference conditions will be constructed. The Gordon marking of a partition and clusters will be utilized [9].

The next section lists the definitions and a small result that will be used throughout the paper. Section 3 deals with the first four or the "(mod 9)" conjectures and some missing cases. Section 4 treats the last two or the "(mod 12)" conjectures and some missing cases. Section 5 lists alternative generating functions of Sect. 4. We do not assert any partition identities for the missing cases in Sects. 3, 4 and 5. In Sect. 6, we collect some of the constructed series thus far and state q-series conjectures as analytic companions to the Kanade–Russell's combinatorial conjectures. Thanks to [4], some of the formulas will be theorems in Sect. 6. We conclude with some commentary, a few open problems, and some directions for further research in Sect. 7. The appendix by Emre Erol contains a metaphor and explanation for parts of a construction in Sect. 4 and related terminology.

2. Definitions and Preliminary Results

An integer partition λ of a natural number n is a non-decreasing sequence of positive integers that sum up to n:

$$n = \lambda_1 + \lambda_2 + \cdots + \lambda_m,$$
$$0 < \lambda_1 \leq \lambda_2 \leq \cdots \leq \lambda_m.$$

The λ_i's are called parts. The number of parts m is called the length of the partition λ, denoted by $l(\lambda)$. The number being partitioned is the weight of the partition λ, denoted by $|\lambda|$. One could also reverse the weak inequalities and take non-decreasing sequences, but we will stick to this definition for purposes of this note. The point is that reordering the same parts will not give us a new partition. For example, the five partitions of $n = 4$ are

$$4, \quad 1+3, \quad 2+2, \quad 1+1+2, \quad 1+1+1+1.$$

We sometimes allow zeros to appear in the partition. Clearly, they have no contribution to the weight of the partition, but the length changes as we add or take out zeros.

Given a partition λ, if there exists positive integers d and k such that $\lambda_{j+d} - \lambda_j \geq k$ for all $j = 1, 2, \ldots, l(\lambda) - d$, we say that λ *has difference at least k at distance d.*

Many partition identities have the form "the number of partitions of n satisfying condition A = the number of partitions of n satisfying condition B" [3]. We can abbreviate this as $p(n|\text{ cond. } A) = p(n|\text{ cond. } B)$. Any form of the series

$$F(q) = \sum_{n\geq 0} p(n|\text{ cond. } A)q^n$$

is called a *partition generating function*, or $F(q)$ is said to generate $p(n|\text{ cond. } A)$.

The definitions below are taken from [9]. Although they are lengthy, they are included here for self-containment.

Definition 2.1. The *Gordon marking* of a partition λ is an assignment of positive integers (marks) to λ such that parts equal to any given integer a are assigned distinct marks from the set

$$\mathbb{Z}_{>0} \backslash \{r| \ \exists\ r\text{-marked } \lambda_j = a - 1\}$$

such that the smallest possible marks are used first. We can represent the Gordon marking by a two-dimensional array, where the row index counted from bottom to top indicates the mark.

Example 2.2. For the partition

$$\begin{aligned}\lambda = 2 &+ 2 + 3 + 4 + 5 + 6 + 6 + 7 + 9 + 11 \\ &+ 13 + 13 + 15 + 15 + 16 + 17 + 18,\end{aligned}$$

the Gordon marking is

$$\begin{aligned}\lambda = 2_1 &+ 2_2 + 3_3 + 4_1 + 5_2 + 6_1 + 6_3 + 7_2 + 9_1 + 11_1 \\ &+ 13_1 + 13_2 + 15_1 + 15_2 + 16_3 + 17_1 + 18_2,\end{aligned}$$

or

$$\left\{\begin{array}{ccccccccccccccccc} & 3 & & & 6 & & & & & & 16 & & \\ 2 & & & 5 & & 7 & & & 13 & 15 & & & 18 \\ 2 & & 4 & & 6 & & 9 & 11 & 13 & 15 & & 17 & \end{array}\right\}.$$

This last representation of partitions will be used throughout the note.

Definition 2.3. Given a partition λ, let λ_j be an r-marked part such that

(a) there are no $r + 1$ or higher marked parts $= \lambda_j$ or $= \lambda_j + 1$;

(b1) either there is an r_0 marked part $\lambda_{j_0} = \lambda_j - 1$, $r_0 < r$ such that there are no r_0-marked parts $= \lambda_j + 1$, and no $r_0 + 1$ or higher marked parts equal to $\lambda_j - 1$;

(b2) or there are $1, 2, \ldots, (r - 1)$-marked parts $= \lambda_j$ or $= \lambda_j + 1$, and no r-marked parts $= \lambda_j + 2$.

A *forward move of the rth kind* is replacing the r_0-marked λ_{j_0} with an r_0 marked $\lambda_{j_0} + 1$ if (a) and (b1) hold; and replacing the r-marked λ_j with an r-marked $\lambda_j + 1$ if (a) and (b2) hold, but (b1) fails.

Example 2.4. A forward move of the third kind on the 3-marked 16 (in boldface) of the partition in the above example makes the partition

$$\left\{\begin{array}{ccccccccccccc} & 3 & & & 6 & & & & & & \mathbf{16} & & \\ 2 & & & 5 & & 7 & & & 13 & & 16 & & 18 \\ 2 & & 4 & & 6 & & 9 & 11 & 13 & 15 & & 17 & \end{array}\right\}.$$

Definition 2.5. For a partition λ, let $\lambda_j \neq 1$ be an r-marked part such that

(c) there are no $(r+1)$ or greater marked parts that are $= \lambda_j$ or $= \lambda_j + 1$;
(d) there is an $r_0 \leq r$ such that there is an r_0-marked $\lambda_{j_0} = \lambda_j$, but no r_0-marked parts $= \lambda_j - 2$.

Choose the smallest r_0 described in (d). A *backward move of the rth kind* on λ_j is replacing the r_0-marked λ_{j_0} with an r_0-marked $\lambda_{j_0} - 1$.

Example 2.6. A backward move of the third kind on the 3-marked 6 of the last displayed partition makes it

$$\left\{\begin{array}{ccccccccccccc} & 3 & & \mathbf{5} & & & & & & & 16 & & \\ 2 & & & 5 & & 7 & & & 13 & & 16 & & 18 \\ 2 & & 4 & & 6 & & 9 & 11 & 13 & 15 & & 17 & \end{array}\right\}.$$

The 6 becomes 5 (in boldface).

Definition 2.7. An *r-cluster* in $\lambda = \lambda_1 + \lambda_2 + \cdots + \lambda_m$ is a sub-partition $\lambda_{i_1} \leq \lambda_{i_2} \leq \cdots \leq \lambda_{i_r}$ such that λ_{i_j} is j-marked for $j = 1, 2, \ldots, r$, $\lambda_{i_{j+1}} - \lambda_{i_j} = 0$ or 1 for $j = 1, 2, \ldots, r-1$, and there are no $(r+1)$-marked parts $= \lambda_{i_r}$ or $= \lambda_{i_r} + 1$.

Example 2.8.

$$\left\{\begin{array}{ccccccccccccc} & 3 & & & 6 & & & & & & 16 & & \\ 2 & & & 5 & & 7 & & & 13 & 15 & & & 18 \\ 2 & & 4 & & 6 & & 9 & 11 & 13 & 15 & & 17 & \end{array}\right\}$$

has the following clusters:

$$\left\{ \underbrace{\begin{matrix} 3 \\ 2 \\ 2 \end{matrix}}_{\text{a 3-cluster}} \quad \underbrace{\begin{matrix} 6 \\ 5 \\ 4 \end{matrix}}_{\text{a 3-cluster}} \quad \underbrace{\begin{matrix} \\ 7 \\ 6 \end{matrix}}_{\text{a 2-cluster}} \quad \underbrace{\begin{matrix} \\ \\ 9 \end{matrix}}_{\text{a 1-cluster}} \quad \underbrace{\begin{matrix} \\ \\ 11 \end{matrix}}_{\text{a 1-cluster}} \right.$$

$$\left. \underbrace{\begin{matrix} \\ 13 \\ 13 \end{matrix}}_{\text{a 2-cluster}} \quad \underbrace{\begin{matrix} 16 \\ 15 \\ 15 \end{matrix}}_{\text{a 3-cluster}} \quad \underbrace{\begin{matrix} \\ 18 \\ 17 \end{matrix}}_{\text{a 2-cluster}} \right\}$$

When we compare two clusters, not necessarily having the same number of parts, we compare the 1-marked parts in them. The *largest 2-cluster* means the 2-cluster having the largest 1-marked part, etc.

We will also need the following result in Sect. 4.

Proposition 2.9. *The partitions into at most n parts, in which all odd parts are distinct, is generated by* $\dfrac{(-q;q^2)_n}{(q^2;q^2)_n}$.

Proof. By the q-binomial theorem [3],

$$\frac{(-qt;q^2)_\infty}{(t;q^2)_\infty} = \sum_{n\geq 0} \frac{(-q;q^2)_n}{(q^2;q^2)_n} t^n.$$

The right-hand side obviously generates partitions in which no odd part repeats, and the exponent of t accounts for the number of parts, zeros allowed. ☐

Here, and throughout,

$$(a;q)_n = \prod_{j=0}^{n} (1 - aq^{j-1}),$$
$$(a_1, a_2, \ldots, a_k; q)_n = (a_1;q)_n (a_2;q)_n \cdots (a_k;q)_n$$

for $n \in \mathbb{N} \cup \{\infty\}$ and $|q| < 1$.

3. Kanade and Russell's First Four Conjectures and Some Missing Cases

Theorem 3.1 (cf. The Kanade–Russell conjecture I_1). *For $n, m \in \mathbb{N}$, let $kr_1(n,m)$ be the number of partitions of n into m parts with difference at least three at distance two such that if two successive parts differ by at most one, then their sum is divisible by 3. Then*

$$\sum_{n,m\geq 0} kr_1(n,m) q^n x^m = \sum_{n_1,n_2\geq 0} \frac{q^{3n_2^2+n_1^2+3n_1n_2} x^{2n_2+n_1}}{(q;q)_{n_1}(q^3;q^3)_{n_2}}. \tag{3.1}$$

Proof. For any λ enumerated by $kr_1(n,m)$, we will construct a unique triple (β, μ, η) meeting the following criteria:

- β is the base partition into $m = 2n_2 + n_1$ parts having n_2 2-clusters and n_1 1-clusters. β satisfies the difference conditions set forth by $kr_1(n,m)$.
- μ is a partition with n_1 parts (counting zeros).
- η is a partition into multiples of three with n_2 parts (counting zeros).
- $|\lambda| = |\beta| + |\mu| + |\eta|$.

Conversely, given a triple (β, μ, η) as described above, we will construct a unique λ counted by $kr_1(n,m)$, where $m = 2n_2 + n_1$. We will arrange constructions so that they are inverses of each other at each step. This will give a one-to-one correspondence between the said λ and (β, μ, η), yielding

$$\sum_{n,m\geq 0} kr_1(n,m) q^n x^m = \sum_{n_1,n_2\geq 0} q^{|\beta|} x^{l(\beta)} \sum_{\mu,\eta} q^{|\mu|+|\eta|}, \tag{3.2}$$

where β is the partition with n_2 2-clusters, n_1 1-clusters, and having the smallest possible weight. Notice that λ cannot have r-clusters for $r \geq 3$, since the existence of an r-cluster requires the existence of an r-marked part; hence, λ has difference at most one at distance $r-1$.

In building β, we will place the 1- and 2-clusters, which are as small as possible, one after the other without violating the difference conditions. The 2-clusters may look like

$$\left\{ \begin{array}{ccc} & 3k & \\ (\text{parts } \leq 3k-3) & 3k & (\text{parts } \geq 3k+3) \end{array} \right\},$$

or

$$\left\{ \begin{array}{cccc} & & 3k+2 & \\ (\text{parts } \leq 3k-1) & 3k+1 & & (\text{parts } \geq 3k+4) \end{array} \right\},$$

but not

$$\begin{array}{ccccccccc} 3k+1 & 3k+2 & & 3k+1 & & & & 3k+3 & \\ 3k+1, & 3k+2, & 3k & & , \text{ or} & 3k+2 & & & . \end{array}$$

In the first two cases, the sum of two successive displayed parts is divisible by 3. In the last four, it is not.

One can check that the minimal weight of β is attained when all 2-clusters are smaller than the 1-clusters, and all clusters are as small as possible. We will give indications of this fact in the course of the proof. Thus, β is

$$\left\{ \begin{array}{cccccccccc} & 2 & & 5 & & & 3n_2-1 & & & \\ 1 & & 4 & & \cdots & 3n_2-2 & & 3n_2+1 & 3n_2+3 & \\ & & & & & & & & & \cdots \; 3n_2+2n_1-1 \end{array} \right\}. \tag{3.3}$$

Here, $n_1, n_2 \geq 0$. The weight of β is

$$\begin{aligned} |\beta| &= [(1+2)+(4+5)+\cdots+((3n_2-2)+(3n_2-1))] \\ &\quad + [(3n_2+1)+(3n_2+3)+\cdots+(3n_2+2n_1-1)] \\ &= [3+9+\cdots+3(2n_2-1)] + 3n_2n_1 + n_1^2 \\ &= 3n_2^2 + n_1^2 + 3n_2n_1. \end{aligned}$$

Clearly, μ is generated by $1/(q;q)_{n_1}$, and η by $1/(q^3;q^3)_{n_2}$, so that

$$\sum_{n_1,n_2 \geq 0} q^{|\beta|} x^{l(\beta)} \sum_{\mu,\eta} q^{|\mu|+|\eta|} = \sum_{n_1,n_2 \geq 0} \frac{q^{3n_2^2+n_1^2+3n_1n_2} x^{2n_2+n_1}}{(q;q)_{n_1}(q^3;q^3)_{n_2}}. \tag{3.4}$$

Combining (3.2) and (3.4) leads to a proof of the theorem.

Given a triple (β,μ,η), we will first move the ith largest 1-cluster the ith largest part of μ times forward, for $i = 1, 2, \ldots, n_1$, in this order. Then, we move the ith largest 2-cluster $\frac{1}{3}\times$(the ith largest part of η) times forward, for $i = 1, 2, \ldots, n_2$, in this order. This will give us λ. The forward and backward moves on the 2-clusters are not exactly the forward or backward moves of the second kind in Definitions 2.3 and 2.5.

Conversely, given λ, we first determine the number of 2- and 1-clusters, n_2, and n_1, respectively. We first move the ith smallest 2-cluster backward as many times as possible for $i = 1, 2, \ldots, n_2$, in this order, and record the number of moves as $\frac{1}{3}\eta_1$, $\frac{1}{3}\eta_2$, $\ldots$, $\frac{1}{3}\eta_{n_2}$. Then, we move the ith smallest 1-cluster backward as many times as possible for $i = 1, 2, \ldots, n_1$, in this order, and record the number of moves as μ_1, μ_2, $\ldots$, μ_{n_1}. Not only will we have obtained μ and η, but also β in the end.

Notice that we perform the forward and backward moves in the exact reverse order.

Starting with (β, μ, η), we simply add the ith largest part of μ to the ith largest 1-cluster in β. This preserves the difference condition because the 1-clusters were at least two apart to start with, and larger parts are added to larger 1-clusters, keeping or increasing the gaps. We now have the intermediate partition

$$\left\{\begin{array}{l} \begin{array}{cccccc} & 2 & 5 & & & 3n_2-1 \\ 1 & 4 & & \cdots & 3n_2-2 & \end{array} \\ \\ \qquad (\text{parts } \geq 3n_2+1, \text{ all 1-clusters }) \end{array}\right\}. \tag{3.5}$$

This also adds the weight of μ to the weight of β.

We now describe the forward moves on the 2-clusters. There are several cases.

$$\begin{array}{c} \left\{\begin{array}{cccc} & & \mathbf{3k+2} & \\ (\text{parts } \leq 3k-1) & \mathbf{3k+1} & & (\text{parts } \geq 3k+6) \end{array}\right\} \\ \Big\downarrow \text{ one forward move on the displayed 2-cluster} \\ \left\{\begin{array}{ccc} & \mathbf{3k+3} & \\ (\text{parts } \leq 3k-1) & \mathbf{3k+3} & (\text{parts } \geq 3k+6) \end{array}\right\} \end{array} \tag{3.6}$$

Here and elsewhere, we highlight the cluster we move.

$$\begin{array}{c} \left\{\begin{array}{ccc} & \mathbf{3k} & \\ (\text{parts } \leq 3k-3) & \mathbf{3k} & (\text{parts } \geq 3k+4) \end{array}\right\} \\ \Big\downarrow \text{ one forward move on the displayed 2-cluster} \\ \left\{\begin{array}{cccc} & & \mathbf{3k+2} & \\ (\text{parts } \leq 3k-3) & \mathbf{3k+1} & & (\text{parts } \geq 3k+4) \end{array}\right\} \end{array} \tag{3.7}$$

Observe that one forward move adds three to the weight of the intermediate partition. This is why we require parts of η to be multiples of three.

$$\begin{array}{c} \left\{\begin{array}{ccccc} & & \mathbf{3k+2} & & \\ (\text{parts } \leq 3k-1) & \mathbf{3k+1} & & 3k+4 & (\text{parts } \geq 3k+7) \end{array}\right\} \\ \Big\downarrow \text{ one forward move on the displayed 2-cluster} \\ \left\{\begin{array}{ccc} & \mathbf{3k+2} & \\ (\text{parts } \leq 3k-1) & \underbrace{\mathbf{3k+2}\ 3k+4}_{!} & (\text{parts } \geq 3k+7) \end{array}\right\} \text{ (temporarily)} \\ \Big\downarrow \text{ adjustment} \\ \left\{\begin{array}{ccccc} & & \mathbf{3k+5} & & \\ (\text{parts } \leq 3k-1) & 3k+1\ \mathbf{3k+4} & & & (\text{parts } \geq 3k+7) \end{array}\right\} \end{array}$$

Notice that the adjustment does not change the weight, and the terminal configuration satisfies the difference condition if the initial one does. The adjustment here is simply subtracting three from the obstacle, namely, the displayed

1-cluster, and move the 2-cluster one more time forward as in (3.6) or (3.7), as if there are no obstacles.

There are four more cases in which a forward move on a 2-cluster is followed by one or more adjustments. The idea is the same, so we skip the details.

$$\left\{ \begin{array}{cccccc} & & \mathbf{3k+2} & & & \\ (\text{parts } \leq 3k-1) & \mathbf{3k+1} & & 3k+4 \; 3k+6 & (\text{parts } \geq 3k+9) \end{array} \right\}$$

$\downarrow$ one forward move on the displayed 2-cluster, followed by two adjustments

$$\left\{ \begin{array}{ccccc} & & & \mathbf{3k+6} & \\ (\text{parts } \leq 3k-1) & 3k+1 & 3k+3 & \mathbf{3k+6} & (\text{parts } \geq 3k+9) \end{array} \right\},$$

$$\left\{ \begin{array}{cccccc} & & \mathbf{3k+2} & & & \\ (\text{parts } \leq 3k-1) & \mathbf{3k+1} & & 3k+4 \; 3k+6 \; 3k+8 & (\text{parts } \geq 3k+10) \end{array} \right\}$$

$\downarrow$ one forward move on the displayed 2-cluster, followed by three adjustments

$$\left\{ \begin{array}{ccccccc} & & & & & \mathbf{3k+8} & \\ (\text{parts } \leq 3k-1) & 3k+1 & 3k+3 & 3k+5 & \mathbf{3k+7} & & (\text{parts } \geq 3k+10) \end{array} \right\},$$

$$\left\{ \begin{array}{cccc} & \mathbf{3k} & & \\ (\text{parts } \leq 3k-3) & \mathbf{3k} & 3k+3 & (\text{parts } \geq 3k+6) \end{array} \right\}$$

$\downarrow$ one forward move on the displayed 2-cluster, followed by an adjustment

$$\left\{ \begin{array}{cccc} & & \mathbf{3k+2} & \\ (\text{parts } \leq 3k-3) & 3k & \mathbf{3k+2} & (\text{parts } \geq 3k+6) \end{array} \right\},$$

$$\left\{ \begin{array}{ccccc} & \mathbf{3k} & & & \\ (\text{parts } \leq 3k-3) & \mathbf{3k} & 3k+3 & 3k+5 & (\text{parts } \geq 3k+7) \end{array} \right\}$$

$\downarrow$ one forward move on the displayed 2-cluster, followed by two adjustments

$$\left\{ \begin{array}{cccccc} & & & & \mathbf{3k+5} & \\ (\text{parts } \leq 3k-3) & 3k & 3k+2 & \mathbf{3k+4} & & (\text{parts } \geq 3k+7) \end{array} \right\}.$$

The above cases are exclusive, there are no others. One can easily verify that one forward move on the displayed 2-cluster allows at least one forward move on the preceding 2-cluster. Therefore, all parts of η can be realized as forward moves on the 2-clusters, registering the weight of η on the weight of the intermediate partition. In all the above cases, the terminal configurations conform to the difference condition provided that the respective initial configurations do. This is due to the fact that the difference conditions can be checked locally as the differences between successive parts, and as differences at distance two.

The final partition is the λ we have been aiming at. It is enumerated by $kr_1(n, m)$.

Now, given λ counted by $kr_1(n, m)$, having n_2 2-clusters and n_1 1-clusters, so that $m = 2n_2 + n_1$, we will decompose it into the triple (β, μ, η) as described at the beginning of the proof.

We start by moving the smallest 2-cluster backward as many times as necessary to stow it as

$$\left\{ \begin{array}{ccc} & \mathbf{2} & \\ \mathbf{1} & & (\text{parts } \geq 4) \end{array} \right\}.$$

We record the number of moves as $\frac{1}{3}\eta_1$, which gives us the first part of η. If the smallest 2-cluster is already $\begin{smallmatrix} & 2 \\ 1 & \end{smallmatrix}$, we set $\eta_1 = 0$.

We need to describe the backward moves on the 2-clusters. Again, there are several cases.

$$\left\{ \begin{array}{ccc} & \mathbf{3k} & \\ (\text{parts } \leq 3k-4) & \mathbf{3k} & (\text{parts } \geq 3k+3) \end{array} \right\} \tag{3.8}$$

$$\Big\downarrow \text{ one backward move on the displayed 2-cluster}$$

$$\left\{ \begin{array}{cccc} & & \mathbf{3k-1} & \\ (\text{parts } \leq 3k-4) & \mathbf{3k-2} & & (\text{parts } \geq 3k+3) \end{array} \right\},$$

$$\left\{ \begin{array}{cccc} & & \mathbf{3k+2} & \\ (\text{parts } \leq 3k-3) & \mathbf{3k+1} & & (\text{parts } \geq 3k+4) \end{array} \right\} \tag{3.9}$$

$$\Big\downarrow \text{ one backward move on the displayed 2-cluster}$$

$$\left\{ \begin{array}{ccc} & \mathbf{3k} & \\ (\text{parts } \leq 3k-3) & \mathbf{3k} & (\text{parts } \geq 3k+4) \end{array} \right\}.$$

Clearly, one backward move on a 2-cluster decreases the weight of λ by three, which is registered in parts of η. Thus, the parts of η are evidently multiples of 3.

$$\left\{ \begin{array}{ccccc} & & & \mathbf{3k+2} & \\ (\text{parts } \leq 3k-4) & 3k-2 & \mathbf{3k+1} & & (\text{parts } \geq 3k+4) \end{array} \right\}$$

$$\Big\downarrow \text{ one backward move on the displayed 2-cluster}$$

$$\left\{ \begin{array}{ccc} & \mathbf{3k} & \\ (\text{parts } \leq 3k-4) & \underbrace{3k-2\ \mathbf{3k}}_{!} & (\text{parts } \geq 3k+4) \end{array} \right\} \text{ (temporarily)}$$

$$\Big\downarrow \text{ adjustment}$$

$$\left\{ \begin{array}{ccccc} & & \mathbf{3k-1} & & \\ (\text{parts } \leq 3k-4) & \mathbf{3k-2} & & 3k+1 & (\text{parts } \geq 3k+4) \end{array} \right\}$$

Again, the adjustment does not alter the weight of the partition. It only resolves the violation of the difference condition by moving the temporarily problematic 1-cluster three times forward, and the temporarily problematic 2-cluster one time backward as in (3.8) or (3.9) as if there are no obstacles. The terminal partition satisfies the difference conditions if the initial one does. Recall that we assume that the initial partitions always satisfy the respective difference conditions.

There are four more cases. We omit the intermediate steps, since they are completely analogous to the above case.

$$\left\{ \begin{array}{ccccc} & & & \mathbf{3k} & \\ (\text{parts } \leq 3k-7) & 3k-5 & 3k-3 & \mathbf{3k} & (\text{parts } \geq 3k+3) \end{array} \right\}$$

$$\Big\downarrow \text{ one backward move on the displayed 2-cluster, followed by two adjustments}$$

$$\left\{ \begin{array}{cccccc} & & \mathbf{3k-4} & & & \\ (\text{parts } \leq 3k-7) & \mathbf{3k-5} & & 3k-2 & 3k & (\text{parts } \geq 3k+3) \end{array} \right\},$$

$$\left\{ \begin{array}{ccccccc} & & & & & \mathbf{3k+2} & \\ (\text{parts } \leq 3k-7) & 3k-5 & 3k-3 & 3k-1 & \mathbf{3k+1} & & (\text{parts } \geq 3k+4) \end{array} \right\}$$

$$\Big\downarrow \text{ one backward move on the displayed 2-cluster, followed by three adjustments}$$

$$\left\{\begin{array}{ccccc} & \mathbf{3k-4} & & & \\ (\text{parts } \le 3k-7)\ \mathbf{3k-5} & & 3k-2\ 3k\ 3k+2\ (\text{parts } \ge 3k+4) \end{array}\right\},$$

$$\left\{\begin{array}{c} \mathbf{3k} \\ (\text{parts } \le 3k-6)\ 3k-3\ \mathbf{3k}\ (\text{parts } \ge 3k+3) \end{array}\right\}$$

$\downarrow$ one backward move on the displayed 2-cluster, followed by an adjustment

$$\left\{\begin{array}{c} \mathbf{3k-3} \\ (\text{parts } \le 3k-6)\ \mathbf{3k-3}\ 3k\ (\text{parts } \ge 3k+3) \end{array}\right\},$$

$$\left\{\begin{array}{cc} & \mathbf{3k+2} \\ (\text{parts } \le 3k-6)\ 3k-3\ 3k-1\ \mathbf{3k+1} & (\text{parts } \ge 3k+4) \end{array}\right\}$$

$\downarrow$ one backward move on the displayed 2-cluster, followed by two adjustments

$$\left\{\begin{array}{c} \mathbf{3k-3} \\ (\text{parts } \le 3k-6)\ \mathbf{3k-3}\ 3k\ 3k+2\ (\text{parts } \ge 3k+4) \end{array}\right\}.$$

The above cases exhaust all possibilities. One can verify that the 2-cluster succeeding the displayed one may be moved at least once backward after the described backward move. Once the smallest 2-cluster is stowed as $\begin{smallmatrix} & 2 \\ 1 & \end{smallmatrix}$, we continue with the next smallest 2-cluster. We move it backward as many times as possible and place it as $\begin{smallmatrix} & 5 \\ 4 & \end{smallmatrix}$, recording the number of moves as $\frac{1}{3}\eta_2$. Then, continue with the next smallest 2-cluster, etc., obtaining η. The above discussion ensures that $\eta_1 \le \eta_2 \le \cdots \le \eta_{n_2}$.

The careful reader will have noticed that the respective cases for the backward moves and the forward moves on the 2-clusters have swapped initial and terminal configurations. The forward and backward moves are inverses of each other in this sense.

Once the 2-clusters are lined up as in (3.5) and we have η, we subtract μ_1 from the smallest 1-cluster to make it $3n_2+1$, μ_2 from the next smallest to make it $3n_2+3$, etc. This way, we will have constructed μ. Because the successive 1-clusters are at least two apart by the Gordon marking, $\mu_1 \le \mu_2 \le \cdots \le \mu_{n_1}$ Subtracting μ_i from the ith smallest 1-cluster is nothing but performing μ_i backward moves on it. The forward and backward moves on the 1-clusters are obviously inverses of each other.

The remaining partition is (3.3), namely, the base partition β.

This justifies (3.2) and, therefore, concludes the proof. □

As in other similar proofs, one can make the forward and backward moves on the 1- or 2-clusters exact opposites of each other, together with the temporary rule breaking in the middle. However, we find the descriptions in the proofs more appealing.

Example 3.2. Using the notation in the above proof, we will work in the forward direction, and construct the partition λ having $n_1 = 3$ 1-clusters, $n_2 = 2$ 2-clusters, with $\mu = 0+1+1$, and $\eta = 3+6$. We start with β is in the form (3.3):

$$\beta = \left\{\begin{array}{cccccc} & 2 & & 5 & & \\ 1 & & 4 & & 7\ 9\ \mathbf{11} \end{array}\right\}.$$

Applying μ first, we obtain

$$\left\{\begin{array}{ccccccc} & 2 & & \mathbf{5} & & & \\ 1 & & \mathbf{4} & & 7 & 10 & 12 \end{array}\right\}.$$

Then, we continue with incorporating η, first $\frac{1}{3}\times$ its largest part as forward moves on the largest 2-cluster:

$\downarrow$ the first forward move on the larger 2-cluster

$$\left\{\begin{array}{ccccc} & 2 & \mathbf{6} & & \\ 1 & & \underbrace{\mathbf{6}\ 7}_{!} & 10 & 12 \end{array}\right\}$$

$\downarrow$ adjustment

$$\left\{\begin{array}{cccccc} & 2 & & & \mathbf{8} & & \\ 1 & & 4 & \mathbf{7} & & 10 & 12 \end{array}\right\}$$

$\downarrow$ one more forward move on the larger 2-cluster

$$\left\{\begin{array}{ccccc} & 2 & & \mathbf{9} & \\ 1 & & 4 & \underbrace{\mathbf{9}\ 10}_{!} & 12 \end{array}\right\}$$

$\downarrow$ adjustment

$$\left\{\begin{array}{ccccc} & 2 & & & \mathbf{11} \\ 1 & & 4 & 7 & \underbrace{\mathbf{10}\quad\quad 12}_{!} \end{array}\right\}$$

$\downarrow$ adjustment

$$\left\{\begin{array}{cccccc} & \mathbf{2} & & & & 12 \\ \mathbf{1} & & 4 & 7 & 9 & 12 \end{array}\right\}$$

This finishes the $\frac{1}{3}\eta_2 = 2$ forward moves on the larger 2-cluster. We continue with $\frac{1}{3}\eta_1 = 1$ forward move on the smaller 2-cluster.

$$\left\{\begin{array}{cccc} \mathbf{3} & & & 12 \\ \underbrace{\mathbf{3}\ 4}_{!} & 7 & 9 & 12 \end{array}\right\}$$

$\downarrow$ adjustment

$$\lambda = \left\{\begin{array}{cccccc} & & \mathbf{5} & & & 12 \\ 1 & \mathbf{4} & & 7 & 9 & 12 \end{array}\right\}$$

As expected,

$$|\beta| + |\mu| + |\eta| = 39 + 2 + 9 = 50 = |\lambda|.$$

Theorem 3.3 (cf. The Kanade–Russell conjecture I_2). *For $n, m \in \mathbb{N}$, let $kr_2(n, m)$ be the number of partitions of n into m parts with smallest part at least two, and difference at least three at distance two such that if two successive parts differ by at most one, then their sum is divisible by three. Then*

$$\sum_{n,m\geq 0} kr_2(n,m)q^n x^m = \sum_{n_1,n_2\geq 0} \frac{q^{3n_2^2+3n_2+n_1^2+n_1+3n_1n_2}x^{2n_2+n_1}}{(q;q)_{n_1}(q^3;q^3)_{n_2}}. \tag{3.10}$$

Proof. The proof is completely analogous to that of Theorem 3.1, except that we have to use two different base partitions β for the cases $n_1 = 0$ and $n_1 > 0$. When $n_1 = 0$, the base partition is clearly

$$\left\{\begin{array}{cccc} 3 & 6 & & 3n_2 \\ 3 & 6 & \cdots & 3n_2 \end{array}\right\}, \tag{3.11}$$

with weight $3n_2^2 + 3n_2$. If, however, $n_1 > 0$, that is, there is at least one 1-cluster, the seemingly obvious choice

$$\left\{\begin{array}{cccccccc} 3 & 6 & & 3n_2 & & & & \\ 3 & 6 & \cdots & 3n_2 & 3n_2+3 & 3n_2+5 & \cdots & 3n_2+2n_1+1 \end{array}\right\} \tag{3.12}$$

does not have minimal weight. Moreover, one can never obtain a partition counted by $kr_2(n, m)$ containing the part 2 this way. The correct base partition in this case is

$$\left\{\begin{array}{cccccccccccc} & & 5 & & 8 & & & 3n_2+2 & & & & \\ 2 & 4 & & 7 & & \cdots & 3n_2+1 & & 3n_2+4 & 3n_2+6 & \cdots & 3n_2+2n_1 \end{array}\right\}, \tag{3.13}$$

for $n_1 > 0$. One can check that (3.13) has smaller weight than (3.12), and that any other lineup of 2- and 1-clusters results in a greater weight. (3.13) has weight $3n_2^2 + 3n_2 + n_1^2 + n_1 + 3n_2n_1$, the $n_1 = 0$ case of which yields the weight of (3.11).

There is one more twist before we leave the rest of the proof to the reader. We need to discuss how the smallest 1-cluster can move forward. Recall that in the proof of Theorem 3.1, in order for the smallest one cluster to move forward, each of the other 1-clusters must have moved forward at least once. It is the same here, so we assume that all but the smallest 1-clusters, if any, have moved in (3.13). This yields the configuration below.

$$\left\{\begin{array}{cccccccccccc} & & 5 & & 8 & & & 3n_2+2 & & & & \\ \mathbf{2} & 4 & & 7 & & \cdots & 3n_2+1 & & 3n_2+5 & 3n_2+7 & \cdots & 3n_2+2n_1+1 \end{array}\right\}$$

Now we want to move the smallest 1-cluster forward once. This will entail prestidigitation of the smallest 1-cluster through the 2-clusters (please see Sect. 7 and the Appendix).

$$\left\{\begin{array}{ccccccccccc} & 5 & & 8 & & & 3n_2+2 & & & & \\ \underbrace{\mathbf{3}\ \ 4}_{!} & & 7 & & \cdots & 3n_2+1 & & 3n_2+5 & 3n_2+7 & \cdots & 3n_2+2n_1+1 \end{array}\right\}$$

$$\Big\downarrow \text{ adjustment}$$

$$\left\{\begin{array}{cccccccccc} 3 & & 8 & & & 3n_2+2 & & & & \\ 3 & \underbrace{\mathbf{6}\ \ 7}_{!} & & \cdots & 3n_2+1 & & 3n_2+5 & 3n_2+7 & \cdots & 3n_2+2n_1+1 \end{array}\right\}$$

$$\Big\downarrow n_2 - 1 \text{ more adjustments in a similar fashion}$$

$$\left\{\begin{array}{cccccccccc} 3 & 6 & & 3n_2 & & & & & & \\ 3 & 6 & \cdots & 3n_2 & \mathbf{3n_2} + \mathbf{3} & 3n_2+5 & 3n_2+7 & \cdots & 3n_2+2n_1+1 \end{array}\right\},$$

incidentally arriving at (3.12), the weight of which is exactly n_1 more than that of (3.13), for this reason.

As in the proof of Theorem 3.1, the backward moves on the 2-clusters make the intermediate partition

$$\left\{\begin{array}{llll} 3\ 6 & & 3n_2 & \\ 3\ 6 & \cdots & 3n_2 & (\text{parts } \geq 3n_2+3,\ \text{all 1-clusters }) \end{array}\right\}.$$

We first move the smallest 1-cluster so as to bring it back to $3n_2+3$, recording the number of moves as $\mu_1 - 1$. Now the intermediate partition looks like

$$\left\{\begin{array}{lllll} 3\ 6 & & 3n_2 & & \\ 3\ 6 & \cdots & 3n_2 & \mathbf{3n_2+3} & (\text{parts } \geq 3n_2+5,\ \text{all 1-clusters }) \end{array}\right\}.$$

The final backward move on the smallest 1-cluster will again entail prestidigitation of the smallest 1-cluster through the 2-clusters.

$$\left\{\begin{array}{llll} 3\ 6 & & 3n_2 & \\ 3\ 6 & \cdots & \underbrace{3n_2\ \mathbf{3n_2+2}}_{!} & (\text{parts } \geq 3n_2+5) \end{array}\right\}$$

$$\Big\downarrow \text{ adjustment}$$

$$\left\{\begin{array}{llllll} 3\ 6 & & 3n_2-3 & & 3n_2+2 & \\ 3\ 6 & \cdots & \underbrace{3n_2-3\ \mathbf{3n_2-1}}_{!} & 3n_2+1 & & (\text{parts } \geq 3n_2+5) \end{array}\right\}$$

$$\Big\downarrow \text{ after } n_2-1 \text{ more adjustments of similar sort}$$

$$\left\{\begin{array}{llllll} & 5 & 8 & & 3n_2+2 & \\ \mathbf{2}\ 4 & & 7 & \cdots\ 3n_2+1 & & (\text{parts } \geq 3n_2+5) \end{array}\right\}.$$

As far as the lineup of the smallest 1-cluster and all the 2-clusters is concerned, the initial and terminal partitions are swapped in the forward and the backward moves. Also, notice that this extra move on the smallest 1-cluster opens room for the larger 1-clusters to move backward at least once more. The remaining parts of the proof are completely analogous to those parts of the proof of Theorem 3.1. $\square$

Theorem 3.4 (cf. The Kanade–Russell conjecture I_3). *For $n, m \in \mathbb{N}$, let $kr_3(n, m)$ be the number of partitions of n into m parts with smallest part at least three, and difference at least three at distance two such that if two successive parts differ by at most one, then their sum is divisible by three. Then*

$$\sum_{n,m\geq 0} kr_3(n,m) q^n x^m = \sum_{n_1,n_2\geq 0} \frac{q^{3n_2^2+3n_2+n_1^2+2n_1+3n_1n_2} x^{2n_2+n_1}}{(q;q)_{n_1}(q^3;q^3)_{n_2}}. \tag{3.14}$$

Proof. The proof of Theorem 3.1 applies mutatis mutandis. The only difference being the base partition β:

$$\left\{\begin{array}{llllll} 3\ 6 & & 3n_2 & & & \\ 3\ 6 & \cdots & 3n_2 & 3n_2+3\ \ 3n_2+5 & \cdots & 3n_2+2n_1+1 \end{array}\right\}.$$

It is (3.12) and has weight $3n_2^2+3n_2+n_1^2+2n_1+3n_1n_2$. This weight is minimal among all partitions having n_2 2-clusters, n_1 1-clusters, and satisfying the difference conditions imposed by $kr_3(n, m)$. $\square$

Theorem 3.5 (cf. The Kanade–Russell conjecture I_4). *For $n, m \in \mathbb{N}$, let $kr_4(n, m)$ be the number of partitions of n into m parts with smallest part at least two, and difference at least three at distance two such that if two successive parts differ by at most one, then their sum is $\equiv 2 \pmod 3$. Then*

$$\sum_{n,m \geq 0} kr_4(n,m) q^n x^m = \sum_{n_1, n_2 \geq 0} \frac{q^{3n_2^2 + 2n_2 + n_1^2 + n_1 + 3n_1 n_2} x^{2n_2 + n_1}}{(q;q)_{n_1} (q^3;q^3)_{n_2}}. \tag{3.15}$$

Proof. We observe that if we take a partition counted by $kr_1(n, m)$ and add 1 to all parts, the smallest parts becomes at least two. Also, the 2-clusters, the only pair of parts whose pairwise difference is at most one, become

$$\left\{\begin{array}{ccc} & 3k+1 & \\ (\text{parts } \leq 3k-2) & 3k+1 & (\text{parts } \geq 3k+4) \end{array}\right\}$$

and

$$\left\{\begin{array}{cccc} & & 3k+3 & \\ (\text{parts } \leq 3k) & 3k+2 & & (\text{parts } \geq 3k+5) \end{array}\right\},$$

instead of

$$\left\{\begin{array}{ccc} & 3k & \\ (\text{parts } \leq 3k-3) & 3k & (\text{parts } \geq 3k+3) \end{array}\right\}$$

and

$$\left\{\begin{array}{cccc} & & 3k+2 & \\ (\text{parts } \leq 3k-1) & 3k+1 & & (\text{parts } \geq 3k+4) \end{array}\right\},$$

respectively. Therefore, the sum of parts of the displayed 2-clusters becomes $\equiv 2 \pmod 3$, conforming to the definition of $kr_4(n, m)$.

Conversely, a partition enumerated by $kr_4(n, m)$ can only have 1- or 2-marked parts in its Gordon marking. Therefore, such a partition can have r-clusters for $r = 1, 2$, but not for $r \geq 3$. Because the 2-clusters consist of a pair of parts with difference zero or one, they can be

$$\begin{array}{c} 3k \\ 3k \end{array}, \begin{array}{c} 3k+1 \\ 3k+1 \end{array}, \begin{array}{c} 3k+2 \\ 3k+2 \end{array},$$

$$\begin{array}{cc} & 3k+1 \\ 3k & \end{array}, \begin{array}{cc} & 3k+2 \\ 3k+1 & \end{array}, \text{ or } \begin{array}{cc} & 3k+3 \\ 3k+2 & \end{array}.$$

Only the second and the sixth ones have sums $\equiv 2 \pmod 3$; therefore, only such 2-clusters can occur in the said partition. Because all parts are at least two we will not lose any parts, nor do we need to redo the Gordon marking when we subtract one from all parts. This operation makes the partition satisfy the conditions of $kr_1(n, m)$. Therefore, we have $kr_4(n, m) = kr_1(n + m, m)$, yielding the theorem. $\square$

We can now turn our attention to the missing cases of partitions defined similarly to $kr_1(n, m)$–$kr_4(n, m)$. It turns out that only two such cases need justification like the proofs of Theorems 3.1, 3.3, and 3.4, and the remaining ones can be obtained via shifts as in the proof of Theorem 3.5. Although Kanade and Russell's machinery in [6] does not give nice single infinite products, hence nice partition identities for these missing cases, it is possible to write generating functions for them such as the Andrews–Gordon identities [2].

Theorem 3.6. *For $n, m \in \mathbb{N}$, let $kr_{3-1}(n, m)$ be the number of partitions of n into m parts with smallest part at least two, and difference at least three at distance two such that if two successive parts differ by at most one, then their sum is $\equiv 2 \pmod 3$. Then*

$$\sum_{n,m\geq 0} kr_{3-1}(n,m) q^n x^m = \sum_{n_1,n_2\geq 1} \frac{q^{3n_2^2+6n_2+n_1^2+3n_1+3n_1n_2-1} x^{2n_2+n_1}}{(q;q)_{n_1}(q^3;q^3)_{n_2}} + \sum_{n_2\geq 0} \frac{q^{3n_2^2+6n_2} x^{2n_2}}{(q^3;q^3)_{n_2}} + \sum_{n_1\geq 1} \frac{q^{n_1^2+2n_1} x^{n_1}}{(q;q)_{n_1}}.$$

Proof. The idea of the proof is a direct extension of the proof of Theorem 3.3 based on the proof of Theorem 3.1. The necessity of separate sums is in fact the necessity of different types of base partitions β for various constellations of the 2- and 1-clusters. Observe that the ranges of the three sums ($n_1, n_2 \geq 1$; $n_1 = 0$, $n_2 \geq 0$; $n_1 \geq 1$, $n_2 = 0$) form a set partition of the expected natural range $n_1, n_2 \geq 0$. Recall that n_r is the number of the r-clusters of the partition at hand for $r = 1, 2$.

The base partition for the case $n_1, n_2 \geq 1$ is

$$\left\{ \begin{array}{cccccccc} & 6 & 9 & & 3n_3+3 & & & \\ 3 & 6 & 9 & \cdots & 3n_3+3 & 3n_3+6 \;\; 3n_3+8 & \cdots & 3n_3+2n_1+4 \end{array} \right\},$$

with weight $3n_2^2 + 6n_2 + n_1^2 + 3n_1 + 3n_1n_2 - 1$. Clearly, there are no 1-clusters greater than the 2-clusters if $n_1 = 1$.

When $n_1 = 0$ and $n_2 \geq 0$, the base partition β is

$$\left\{ \begin{array}{ccccc} & 5 & 8 & & 3n_3+2 \\ 4 & 7 & \cdots & 3n_3+1 & \end{array} \right\},$$

with weight $3n_2^2 + 6n_2$. It is the empty partition if $n_2 = 0$.

Finally, if $n_2 = 0$ and $n_1 \geq 1$, the base partition β is

$$\left\{ 3 \quad 5 \quad \cdots \quad 2n_1+1 \right\},$$

with weight $n_1^2 + 2n_1$. We do not want to double count the empty partition here, hence $n_1 \geq 1$.

Without much difficulty, one can verify that the above βs are partitions with minimal weight having specified numbers of 1- and 2-clusters (n_1 and n_2, respectively), while satisfying the difference conditions set forth by $kr_{3-1}(n, m)$. $\square$

One can play with the $\pmod 3$ condition on sums and adjust the lower limit for the smallest part to populate the list. Theorems 3.1, 3.3, 3.4, 3.5 and 3.6 are exclusive to obtain the respective series as generating functions by means of shifts on parts. We present two more examples.

Theorem 3.7. *For $n, m \in \mathbb{N}$, let us define the partition enumerants below.*

$kr_1^b(n,m)$ is the number of partitions of n into m parts with difference at least three at distance two such that if two successive parts differ by at most one, then their sum is $\equiv 1 \pmod 3$.

$kr_{4-2}^b(n,m)$ is the number of partitions of n into m parts with at most one occurrence of the part 1, and difference at least three at distance two such that if two successive parts differ by at most one, then their sum is $\equiv 2 \pmod 3$.

Then

$$\sum_{n,m\geq 0} kr_1^b(n,m) q^n x^m = \sum_{n_1,n_2\geq 0} \frac{q^{3n_2^2+n_2+n_1^2+3n_1n_2} x^{2n_2+n_1}}{(q;q)_{n_1}(q^3;q^3)_{n_2}},$$

and

$$\sum_{n,m\geq 0} kr_{4-2}^b(n,m) q^n x^m = \sum_{n_1,n_2\geq 1} \frac{q^{3n_2^2+2n_2+n_1^2+n_1+3n_1n_2-1} x^{2n_2+n_1}}{(q;q)_{n_1}(q^3;q^3)_{n_2}} + \sum_{n_2\geq 0} \frac{q^{3n_2^2+2n_2} x^{2n_2}}{(q^3;q^3)_{n_2}} + \sum_{n_1\geq 1} \frac{q^{n_1^2} x^{n_1}}{(q;q)_{n_1}}.$$

Proof. It suffices to see that $kr_1^b(n+m,m) = kr_2(n,m)$, and that $kr_{4-2}^b(n+2m,m) = kr_{3-1}(n,m)$. Then, the results become corollaries of Theorems 3.3 and 3.6, respectively. □

We conclude this section with one last example.

Theorem 3.8. *For $n, m \in \mathbb{N}$, let $kr_{1-1}^b(n,m)$ be the number of partitions of n into m parts with at most one occurrence of the part 2, and with difference at least three at distance two such that if two successive parts differ by at most one, then their sum is $\equiv 1 \pmod 3$. Then*

$$\sum_{n,m\geq 0} kr_{1-1}^b(n,m) q^n x^m = \sum_{\substack{n_1\geq 0\\ n_2\geq 1}} \frac{q^{3n_2^2+4n_2+n_1^2+n_1+3n_1n_2} x^{2n_2+n_1}}{(q;q)_{n_1}(q^3;q^3)_{n_2}} + \sum_{\substack{n_1\geq 0\\ n_2\geq 1}} \frac{q^{3n_2^2+4n_2+(n_1+1)^2+3n_1n_2} x^{2n_2+n_1+1}}{(q;q)_{n_1}(q^3;q^3)_{n_2}} + \sum_{n_1\geq 0} \frac{q^{n_1^2} x^{n_1}}{(q;q)_{n_1}}.$$

The enumerant $kr_{1-1}^b(n,m)$ is brought to our attention by Alexander Berkovich. It is unusual in the sense that the number of occurrences is not restricted for the smallest admissible part, but for a larger one. We include it here to demonstrate the fact that the method may treat extra conditions on the parts $\leq M$ for any fixed positive integer M on top of the general difference conditions.

Proof. The proof is reminiscent of that of Theorem 3.6. We need base partitions β for several cases. Below, λ is a partition enumerated by $kr_{1-1}^b(n,m)$, and n_r is the number of r-clusters for $r = 1, 2$.

(i) λ has no 2-clusters, i.e. $n_2 = 0$,
(ii) λ has at least one 2-cluster, but no 1's,
(iii) λ has at least one 2-cluster, and a 1.

In case **(i)**, the base partition β obviously is

$$\left\{ 1\ 3 \cdots 2n_1 - 1 \right\},$$

with weight n_1^2.

In case **(ii)**, the base partitions β are

$$\left\{ \begin{matrix} & 4 & & 7 & & & 3n_2+1 \\ 3 & & 6 & & \cdots & 3n_2 & \end{matrix} \right\}$$

when $n_1 = 0$,

$$\left\{ \begin{matrix} & 5\ 8 & & 3n_2+2 \\ 2 & 5\ 8 & \cdots & 3n_2+2 \end{matrix} \right\} \tag{3.16}$$

when $n_1 = 1$,

$$\left\{ \begin{matrix} & 7 & & 10 & & 3n_2+4 & & & & \\ 2\ 4\ 6 & & 9 & & \cdots\ 3n_2+3 & & 3n_2+6 & 3n_2+8 & \cdots & 3n_2+2n_1 \end{matrix} \right\} \tag{3.17}$$

when $n_1 \geq 2$. The weights of all three partitions above are $3n_2^2 + 4n_2 + n_1^2 + n_1 + 3n_2 n_1$. In (3.16), the initial forward move on the smallest 1-cluster, and in (3.17), the initial forward moves on the two smallest 1-clusters involve prestidigitating the said 1-clusters through the 2-clusters, if any.

In case **(iii)**, the base partition is

$$\left\{ \begin{matrix} & 4 & & 7 & & 3n_2+1 & & & & \\ 1\ 3 & & 6 & & \cdots\ 3n_2 & & 3n_2+3 & 3n_2+5 & \cdots & 3n_2+2n_1+1 \end{matrix} \right\}. \tag{3.18}$$

Here, we leave the part 1 where it is, and set n_1 = the number of 1-clusters except the part 1. In other words, we do not perform any forward moves on the part 1. ☐

Remark 3.9. An anonymous referee commented that an equivalent formula to Theorem 3.8 is that

$$\sum_{n,m\geq 0} kr_{1-1}^b(n,m) q^n x^m = \sum_{m,n\geq 0} \frac{q^{Q(m,n)+2m+4n}(1+xq)}{(q;q)_m (q^3;q^3)_n} x^{2n+m} + \sum_{m,n\geq 0} \frac{q^{Q(m,n)+2+3m+7n} x^{1+2n+m}}{(q;q)_m (q^3;q^3)_n}, \tag{3.19}$$

where $Q(m,n) = m^2 + 3mn + 3n^2$.

One can verify it by considering partitions with

(a) smallest part > 2,
(b) smallest part $= 1$,
(c) smallest part $= 2$.

Yet a third way to obtain another alternative is to exclude the partitions counted by $kr_1^b(n,m)$ which have the 2-cluster $\begin{matrix}2\\2\end{matrix}$ using $kr_{3-1}(n,m)$. However, we do not favor inclusion–exclusion in this note.

Example 3.10. Following the notation in the section so far, we will decode the partition λ enumerated by $kr^b_{1-1}(62, 7)$ below into (β, μ, η):

$$\left\{\begin{array}{cccccc} & & \mathbf{7} & & & 14 \\ 1 & \mathbf{6} & & 9 & 11 & 14 \end{array}\right\}.$$

Obviously, we are in the case **(iii)** of the above proof. λ has $n_2 = 2$ 2-clusters, $n_1 = 2$ 1-clusters, and a 1. We stow the smaller 2-cluster first and record η_1 as three times the performed number of moves:

$\downarrow$ one backward move on the smaller 2-cluster

$$\left\{\begin{array}{ccccc} & \mathbf{5} & & & 14 \\ 1 & \mathbf{5} & 9 & 11 & 14 \end{array}\right\}$$

$\downarrow$ one more backward move on the smaller 2-cluster

$$\left\{\begin{array}{cccccc} & & 4 & & & \mathbf{14} \\ 1 & 3 & & 9 & 11 & \mathbf{14} \end{array}\right\}$$

At this point, we have $\eta_1 = 6$.

$\downarrow$ one backward move on the larger 2-cluster

$$\left\{\begin{array}{cccccc} & & 4 & & & \mathbf{13} \\ 1 & 3 & & 9 & \multicolumn{2}{c}{\underbrace{11\ \mathbf{12}\qquad}_{!}} \end{array}\right\}$$

$\downarrow$ adjustment

$$\left\{\begin{array}{cccccc} & & 4 & & \mathbf{11} & \\ 1 & 3 & & \multicolumn{2}{c}{\underbrace{9\ \mathbf{11}}_{!}} & 14 \end{array}\right\}$$

$\downarrow$ adjustment

$$\left\{\begin{array}{ccccccc} & & 4 & & \mathbf{10} & & \\ 1 & 3 & & \mathbf{9} & & 12 & 14 \end{array}\right\}$$

$\downarrow$ two more backward moves on the larger 2-cluster

$$\left\{\begin{array}{ccccccc} & & 4 & & 7 & & \\ 1 & 3 & & 6 & & \mathbf{12} & 14 \end{array}\right\}$$

Now we have $\eta = 6 + 9$. Decoding the backward moves on the 1-clusters is easier. It is obvious that $\mu = 3 + 3$ and once we perform that many backward moves on the respective 1-clusters, we arrive at (3.18).

$$\left\{\begin{array}{ccccccc} & & 4 & & 7 & & \\ 1 & 3 & & 6 & & 9 & 11 \end{array}\right\}$$

The sums of weights also check

$$|\lambda| = 62 = 41 + 6 + 15 = |\beta| + |\mu| + |\eta|.$$

4. Kanade and Russell's Conjectures I_5 and I_6 and Some Missing Cases

Theorem 4.1 (cf. The Kanade–Russell Conjecture I_5). *For $m, n \in \mathbb{N}$, let $kr_5(m,n)$ be the number of partitions of n into m parts, with at most one occurrence of the part 1, and difference at least three at distance three such that is parts at distance two differ by at most 1, then their sum, together with the intermediate part, is $\equiv 1 \pmod 3$. Then*

$$\begin{aligned}\sum_{m,n\geq 0} kr_5(n,m)q^n x^m &= \sum_{n_1,n_2,n_3\geq 0} \frac{q^{(9n_3^2+5n_3)/2+2n_2^2+n_2+n_1^2}(-q;q^2)_{n_2}}{(q;q)_{n_1}(q^2;q^2)_{n_2}(q^3;q^3)_{n_3}} \\ &\quad \times q^{6n_3n_2+3n_3n_1+2n_2n_1}x^{3n_3+2n_2+n_1}. \end{aligned} \tag{4.1}$$

Proof. Throughout the proof, n_r will denote the number of r-clusters for $r = 1, 2, 3$. λ will denote a partition enumerated by $kr_5(n,m)$. We will follow the idea of proof in Theorem 3.1, but there are more intricacies. Construction of the base partition is a major part.

The base partition when $n_1 > 0$ is

$$\left\{\begin{array}{llllll} & & & & & 2n_2+4 \\ 2 & 4 & & 2n_2 & 2n_2+3 & \\ 1 & 3 & \cdots\ 2n_2-1 & 2n_2+1 & 2n_2+3 & \end{array}\right.$$

$$\begin{array}{llll} & 2n_2+7 & & 2n_2+3n_3+1 \\ 2n_2+6 & & 2n_2+3n_3 & \\ 2n_2+6 & & \cdots\ 2n_2+3n_3 & \end{array}$$

$$\left.\begin{array}{l} \\ 2n_2+3n_3+3 \;\; 2n_2+3n_3+5 \cdots\ 2n_2+3n_3+2n_1-1 \end{array}\right\}, \tag{4.2}$$

and when $n_1 = 0$ it is

$$\left\{\begin{array}{lllll} & & & & 2n_2+3 \\ 2 & 4 & & 2n_2 \;\; 2n_2+2 & \\ 1 & 3 & \cdots\ 2n_2-1 & \;\; 2n_2+2 & \end{array}\right.$$

$$\begin{array}{llll} & 2n_2+6 & & 2n_2+3n_3 \\ 2n_2+5 & & 2n_2+3n_3-1 & \\ 2n_2+5 & & \cdots\ 2n_2+3n_3-1 & \end{array}$$

$$\left.\begin{array}{l} \\ 2n_2+3n_3+2 \;\; 2n_2+3n_3+4 \cdots\ 2n_2+3n_3+2n_1-2 \end{array}\right\}. \tag{4.3}$$

The weight of both of them is $(9n_3^2+5n_3)/2+2n_2^2+n_2+n_1^2+6n_3n_2+3n_3n_1+2n_2n_1$.

We have to argue that this is indeed the partition counted by $kr_5(n,m)$ having n_r r-clusters for $r = 1, 2, 3$ and minimal weight.

If λ has a 3-marked part k, then there is a 2-marked part k or $k-1$, and a 1-marked part k or $k-1$. There can be no other parts equal to k or $k-1$ because of the difference at least three at distance three condition. For the same reason, the succeeding smallest part can be at least $k+2$, and the preceding smallest part can be at most $k-2$. Among the three possibilities for the 3-clusters,

$$\begin{matrix} & k \\ k-1 & \\ k-1 & \end{matrix}\ , \qquad \begin{matrix} & k \\ & k \\ k-1 & \end{matrix} \quad \text{and} \quad \begin{matrix} k \\ k \\ k \end{matrix}\ ,$$

which all have difference at most 1 at distance two, the only one satisfying the sum condition, i.e. the sum of the parts, together with the middle part $\equiv 1 \pmod 3$ is

$$\left\{ \begin{matrix} & & k & \\ & k-1 & & \\ (\text{parts } \leq k-3) & k-1 & & (\text{parts } \geq k+2) \end{matrix} \right\}.$$

Therefore, all 3-clusters are of this form. The preceding cluster can be at most $\begin{matrix} & k-3 \\ k-4 & \\ k-4 & \end{matrix}$, and the succeeding cluster can be at least $\begin{matrix} & k+3 \\ k+2 & \\ k+2 & \end{matrix}$. Also, a 3-cluster in λ can be $\begin{matrix} & 3 \\ 2 & \\ 2 & \end{matrix}$, but not $\begin{matrix} & 2 \\ 1 & \\ 1 & \end{matrix}$, because at most one occurrence of the part 1 is allowed. This shows that if a base partition consists of 3-clusters only, it will be

$$\left\{ \begin{matrix} & 3 & & 6 & & & 3n_3 \\ 2 & & 5 & & & 3n_3-1 & \\ 2 & & 5 & & \cdots & 3n_3-1 & \end{matrix} \right\}.$$

For a moment, suppose that there are no 3-clusters in λ. Equivalently, there are no 3-marked parts. The 2-clusters will look like $\begin{matrix} & k \\ k-1 & \end{matrix}$ or $\begin{matrix} k \\ k \end{matrix}$. Two successive 2-clusters may look like

$$\left\{ \begin{matrix} & & k & & k+2 & \\ \cdots & k-1 & & k+1 & & \cdots \end{matrix} \right\}$$

or

$$\left\{ \begin{matrix} & k & & k+3 & \\ \cdots & k & k+2 & & \cdots \end{matrix} \right\},$$

but not

$$\left\{ \begin{matrix} & k & k+2 & \\ \cdots & k & k+2 & \cdots \end{matrix} \right\}.$$

In the last instance, the difference at least three at distance three condition is violated.

1-clusters preceding or succeeding a 2-cluster may look like

$$\left\{\begin{array}{cccccccc} & & & & k & & & \\ \cdots & k-5 & k-3 & k-1 & & k+1 & k+3 & \cdots \end{array}\right\}$$

or

$$\left\{\begin{array}{ccccccc} & & & k & & & \\ \cdots & k-4 & k-2 & k & k+2 & k+4 & \cdots \end{array}\right\}.$$

Recall that if 1-clusters have pairwise difference 1, they become 2-clusters. Or an instance such as

$$\left\{\begin{array}{ccccc} & & & k & \\ \cdots & k-2 & k-1 & & \cdots \end{array}\right\}$$

requires redefinition of the Gordon marking, hence the clusters as

$$\left\{\begin{array}{cccc} & & k-1 & \\ \cdots & k-2 & & k\cdots \end{array}\right\},$$

or even create a 3-cluster.

Therefore, a base partition consisting only of 1- and 2-clusters looks like

$$\left\{\begin{array}{cccccccccc} & 2 & & 4 & & & 2n_2 & & & \\ 1 & & 3 & & \cdots & 2n_2-1 & & 2n_2+1 & 2n_2+3 & \cdots & 2n_2+2n_1-1 \end{array}\right\}.$$

Having 2-clusters greater than 1-clusters will only increase the weight. One way to see this is that the 1-marked parts can be $1, 3, \ldots, 2k-1$ for the least weight. The introduction of the 2-marked parts will form 2-clusters. $2, 4, \ldots, 2l$ is the least addendum to the weight. We recall once again that a second occurrence of 1 is not allowed. This covers the cases $n_1 = 0$ or $n_2 = 0$ as well.

The remaining cases are the coexistence of 3-clusters, and 1- and 2-clusters. We will examine the cases $n_1 = 0$, $n_2, n_3 > 0$, and $n_1, n_3 > 0$, $n_2 \geq 0$ separately, for reasons that will become clear in the course.

It is clear that each cluster should have as small parts as possible in a base partition to ensure minimum weight. Therefore, we will only focus on the relative placement of the clusters. The naive guess is to place 3-clusters first, followed by 2-clusters, and then the 1-clusters. For example,

$$\left\{\begin{array}{ccccccccc} & 3 & & 6 & & & & & \\ 2 & & 5 & & 8 & & 11 & & \\ 2 & & 5 & & 8 & 10 & & 12 & 14 \end{array}\right\}$$

has weight 86. However

$$\left\{\begin{array}{cccccccccc} & & & & & 7 & & 10 & & \\ & 2 & & 4 & 6 & & 9 & & & \\ 1 & & 3 & & 6 & & 9 & & 12 & 14 \end{array}\right\}$$

has weight 83, while

$$\left\{\begin{array}{cccccccccc} & & & & & & 8 & & 11 & \\ & 2 & & 4 & & 7 & & 10 & & \\ 1 & & 3 & & 5 & 7 & & 10 & & 12 \end{array}\right\}$$

has weight 80. Having been experienced, one tries

$$\left\{\begin{array}{cccccccccc} & & & & & & & 10 & & 13 \\ & 2 & & 4 & & & 9 & & 12 & \\ 1 & & 3 & & 5 & 7 & 9 & & 12 & \end{array}\right\},$$

but the weight becomes 87. The naive guess has another problem, we will come back to it during the implementation of the forward moves.

The general case is similarly treated. One should keep in mind that the 2-clusters should precede the 1-clusters in the base partition as discussed above, so the relative places of the 3-clusters are to be decided. One can also verify that placing 1- or 2-clusters between two 3-clusters increases the weight. In summary, depending on the existence of 1-clusters, the base partition will be (4.3) or (4.2).

Next, we argue that any λ enumerated by $kr_5(n, m)$ having n_r r-clusters for $r = 1, 2, 3$ corresponds to a quadruple (β, μ, η, ν) such that

- β is one of the base partitions (4.3) or (4.2), depending on $n_1 = 0$ or $n_1 > 0$, respectively,
- μ is a partition with n_1 parts (counting zeros),
- η is a partition with n_2 parts (counting zeros) where no odd part repeats,
- ν is a partition into multiples of three with n_3 parts (counting zeros),
- $|\lambda| = |\beta| + |\mu| + |\eta| + |\nu|$.

If, say, μ has less than n_1 positive parts, we simply write $\mu_1 = \mu_2 = \cdots = \mu_s = 0$. That is, the first so many parts of μ are declared zero. Recall that we agreed to write the smaller parts first in a partition. If μ is the empty partition, then all parts of it are zero. η and ν are treated likewise. This will give us

$$\begin{aligned} &\sum_{m,n\geq 0} kr_5(n,m) q^n x^m \\ &\quad = \sum_{n_1,n_2,n_3\geq 0} q^{|\beta|} x^{l(\beta)} \sum_{\beta,\mu,\eta,\nu} q^{|\mu|+|\eta|+|\nu|} \\ &\quad = \sum_{n_1,n_2,n_3\geq 0} \underbrace{q^{(9n_3^2+5n_3)/2+2n_2^2+n_2+n_1^2+6n_3n_2+3n_3n_1+2n_2n_1} x^{3n_3+2n_2+n_1}}_{\text{generating } \beta} \cdots \\ &\qquad \times \underbrace{\frac{1}{(q;q)_{n_1}}}_{\text{generating } \mu} \underbrace{\frac{(-q;q^2)_{n_2}}{(q^2;q^2)_{n_2}}}_{\text{generating } \eta} \underbrace{\frac{1}{(q^3;q^3)_{n_3}}}_{\text{generating } \nu}, \end{aligned} \tag{4.4}$$

proving the theorem. We used Proposition 2.9 in the generation of η.

Given a quadruple (β, μ, η, ν) as described above, we will obtain λ in a series of forward moves.

(a) The ith largest 1-cluster in β is moved forward the ith largest part of μ times for $i = 1, 2, \ldots, n_1$, in this order.

(b) The ith largest 2-cluster in the obtained intermediate partition is moved forward the ith largest part of η times for $i = 1, 2, \ldots, n_2$, in this order.

(c) The ith largest 3-cluster in the obtained intermediate partition is moved forward $\frac{1}{3}\times$(the ith largest part of ν) times for $i = 1, 2, \ldots, n_3$, in this order.

Conversely, given λ, we will obtain the quadruple (β, μ, η, ν) by performing backward moves on the 3-, 2-, and 1-, clusters in the exact reverse order. Finally, we will argue that the forward moves and the backward moves on the r-clusters are inverses of each other for $r = 1, 2, 3$, and that the moves honor the difference conditions defining $kr_5(n, m)$.

The forward and backward moves on the 3-clusters are not exactly forward and backward moves of the third kind in the sense of Definitions 2.3 and 2.5. However, the forward and backward moves on the 2-clusters are forward or backward moves of the second kind, with one exception. The exception is described in due course.

We start with the forward moves. When β has at least one 1-cluster, i.e. $n_1 > 0$, the smallest 1-cluster is smaller than the 3-clusters For $i = 1, 2, \ldots, n_1 - 1$, we simply add the ith largest part of μ to the ith largest 1-cluster. This only increases the pairwise difference of the 1-clusters, so the difference conditions are retained. If $\mu_1 > 0$, observe that the $(n_1 - 1)$th 1-cluster, if it exists, is moved forward μ_2 times. Therefore, it is now equal to $2n_2 + 3n_3 + 3 + \mu_2 \geq 2n_2 + 3n_3 + 3 + \mu_1$. The first forward move on the smallest 1-cluster $2n_2 + 1$ entails a prestidigitation through the 3-clusters as described below.

$$
\left\{
\begin{array}{llllll}
 & & & 2n_2+4 & & 2n_2+7 \\
 & 2n_2 & 2n_2+3 & & 2n_2+6 & \\
\cdots\ 2n_2-1 & \mathbf{2n_2+1} & 2n_2+3 & & 2n_2+6 & \cdots
\end{array}
\right.
$$
$$
\left.
\begin{array}{ll}
\quad 2n_2+3n_3+1 & \\
2n_2+3n_3 & \\
2n_2+3n_3 & (\text{ 1-clusters } \geq 2n_2+3n_3+3+\mu_1)
\end{array}
\right\}
$$
$$
\Big\downarrow \ \text{1 forward move on the 1-cluster } 2n_2+1
$$
$$
\left\{
\begin{array}{llll}
 & & & 2n_2+4 \\
 & 2n_2 & & 2n_2+3 \\
\cdots & 2n_2-1 & \multicolumn{2}{l}{\underbrace{\mathbf{2n_2+2}\ \ 2n_2+3}_{!}}
\end{array}
\right.
$$
$$
\left.
\begin{array}{lll}
 & 2n_2+7 & \qquad 2n_2+3n_3+1 \\
2n_2+6 & & 2n_2+3n_3 \\
2n_2+6 & & \cdots\ 2n_2+3n_3 \\
 & & \\
\multicolumn{3}{l}{(\text{ 1-clusters } \geq 2n_2+3n_3+3+\mu_1)}
\end{array}
\right\} \text{(temporarily)}
$$

Here, the ! symbol signifies the violation of the difference condition at the indicated place. As usual, we highlight the cluster(s) that is (are) being moved.

$$\Big\downarrow \text{ adjustment}$$

$$\left\{\begin{array}{llll} & & & 2n_2+3 \\ & 2n_2 & 2n_2+2 & \\ \cdots\ 2n_2-1 & & 2n_2+2 & \end{array}\right.$$

$$\left.\begin{array}{l} \begin{array}{lllll} & & 2n_2+7 & & 2n_2+3n_3+1 \\ & 2n_2+6 & & 2n_2+3n_3 & \\ \multicolumn{3}{l}{\underbrace{\mathbf{2n_2+5}\ 2n_2+6\qquad\qquad}_{!}} & \cdots\ 2n_2+3n_3 & \end{array} \\ \\ \qquad (\text{ 1-clusters } \geq 2n_2+3n_3+3+\mu_1) \end{array}\right\}\text{(temporarily)}$$

$$\Big\downarrow \text{ after a total of } n_3 \text{ similar adjustments}$$

$$\left\{\begin{array}{llllll} & & & 2n_2+3 & & 2n_2+6 \\ & 2n_2 & 2n_2+2 & & 2n_2+5 & \\ \cdots\ 2n_2-1 & & 2n_2+2 & & 2n_2+5 & \cdots \end{array}\right.$$

$$\left.\begin{array}{l} \begin{array}{lll} & 2n_2+3n_3 & \\ 2n_2+3n_3-1 & & \\ 2n_2+3n_3-1 & & \mathbf{2n_2+3n_3+2} \end{array} \\ \\ (\text{ 1-clusters } \geq 2n_2+3n_3+3+\mu_1) \end{array}\right\}$$

Notice that the adjustments do not alter the weight. When the 1-cluster encounters a 3-cluster, temporarily violating the difference condition, they switch places like in a *puss-in-the-corner* game. Three is added to the 1-cluster, and each part in the 3-cluster is decreased by one, therefore preserving the total weight. The process is repeated if there is another 3-cluster ahead.

We still need to add $\mu_1 - 1$ to the 1-cluster $2n_2 + 3n_2 + 2$, making it $2n_2 + 3n_3 + \mu_1 + 1$, respecting the difference condition in the configuration

$$\left\{\begin{array}{llllll} & & & 2n_2+3 & & 2n_2+6 \\ & 2n_2 & 2n_2+2 & & 2n_2+5 & \\ \cdots\ 2n_2-1 & & 2n_2+2 & & 2n_2+5 & \cdots \end{array}\right.$$

$$\left.\begin{array}{lll} & 2n_2+3n_3 & \\ 2n_2+3n_3-1 & & \\ 2n_2+3n_3-1 & & (\text{ 1-clusters } \geq 2n_2+3n_3+1+\mu_1) \end{array}\right\}$$

for $\mu_1 > 0$. In case $\mu_1 = 0$, i.e. μ has less than n_1 positive parts, the smallest 1-cluster stays in its original place at this stage.

Next, the forward moves on the 2-clusters are implemented. The ith largest 2-cluster is moved the ith largest part of η times forward. For each positive part of η, we will prestidigitate the 2-clusters through the 3-clusters as follows:

$$
\left\{
\begin{array}{llllll}
 & & & & & 2n_2+3 \\
 & & 2n_2-2 & & \mathbf{2n_2} & 2n_2+2 \\
\cdots & 2n_2-3 & & \mathbf{2n_{2-1}} & & 2n_2+2
\end{array}
\right.
$$

$$
\left.
\begin{array}{llll}
 & 2n_2+6 & & 2n_2+3n_3 \\
2n_2+5 & & 2n_2+3n_3-1 & \\
2n_2+5 & & \cdots\ 2n_2+3n_3-1 & \\
 & & & \\
 & \text{(parts } \geq 2n_2+3n_3+2) & &
\end{array}
\right\}
$$

$$
\Big\downarrow \text{ 1 forward move on the 2-cluster } \begin{array}{ll} & 2n_2 \\ 2n_2-1 & \end{array}
$$

$$
\left\{
\begin{array}{lllll}
 & & & & 2n_2+3 \\
 & & 2n_2-2 & \mathbf{2n_2} & 2n_2+2 \\
\cdots & 2n_2-3 & & \underbrace{\mathbf{2n_2} \quad 2n_2+2 \qquad\quad}_{!} &
\end{array}
\right.
$$

$$
\left.
\begin{array}{llll}
 & 2n_2+6 & & 2n_2+3n_3 \\
2n_2+5 & & 2n_2+3n_3-1 & \\
2n_2+5 & & \cdots\ 2n_2+3n_3-1 & \\
 & & & \\
\text{(parts } \geq 2n_2+3n_3+2) & & &
\end{array}
\right\} \text{(temporarily)}
$$

$$
\Big\downarrow \text{ adjustment}
$$

$$
\left\{
\begin{array}{lllll}
 & & & & 2n_2+1 \\
 & & 2n_2-2 & 2n_2 & \\
\cdots & 2n_2-3 & & 2n_2 &
\end{array}
\right.
$$

$$
\left.
\begin{array}{llll}
 & 2n_2+6 & & 2n_2+3n_3 \\
\mathbf{2n_{2+3}} & 2n_2+5 & 2n_2+3n_3-1 & \\
\underbrace{\mathbf{2n_{2+3}} \quad 2n_2+5 \qquad\quad}_{!} & & \cdots\ 2n_2+3n_3-1 & \\
 & & & \\
 & \text{(parts } \geq 2n_2+3n_3+2) & &
\end{array}
\right\} \text{(temporarily)}
$$

$$
\Big\downarrow \text{ after } n_3-1 \text{ adjustments of the same kind}
$$

$$
\left\{
\begin{array}{lllllll}
 & & & 2n_2+1 & & 2n_2+4 & \\
 & & 2n_2-2\ 2n_2 & & 2n_2+3 & & \\
\cdots & 2n_2-3 & 2n_2 & & 2n_2+3 & & \cdots
\end{array}
\right.
$$

$$
\left.
\begin{array}{lll}
 & 2n_2+3n_3-2 & \\
2n_2+3n_3-3 & & \mathbf{2n_2+3n_3} \\
2n_2+3n_3-3 & & \mathbf{2n_2+3n_3} \ \text{(parts } \geq 2n_2+3n_3+2)
\end{array}
\right\}
$$

At this point, the parts $\geq 2n_2 + 3n_3 + 2$ are all 1-clusters, so the difference conditions are met. The initial move on each of the so many largest 2-clusters for each nonzero part of η is this prestidigitation of the 2-clusters through the 3-clusters. After this initial move, the remaining moves are performed as in the construction of the series side of Andrews–Gordon identities [8].

There is one more condition on the collective forward moves on the 2-clusters. η cannot have repeated odd parts. In other words, two successive 2-clusters cannot be moved the same odd number of times forward. Let us see why this violates the difference condition.

Assume, on the contrary, that each of the two consecutive 2-clusters is to be moved $2r+1$ times forward. After the initial prestidigitation through the 3-clusters, the 2-clusters will be

$$\left\{\begin{array}{lllll} & & k-2 & & \\ & k-3 & & \mathbf{k\ k+2} & \\ \cdots & k-3 & & \underbrace{\mathbf{k\ k+2}}_{!} & (\text{parts } \geq k+4, \text{ all 1- or 2-clusters }) \end{array}\right\}.$$

Then, the 2-clusters violating the difference at least three at distance three condition will be double moved forward r times each, each pair of double moves retaining the violation as

$$\left\{\begin{array}{lll} & \mathbf{k\ k+2} & \\ \cdots & \underbrace{\mathbf{k\ k+2}}_{!} & \cdots \end{array}\right\} \longrightarrow \left\{\begin{array}{lll} & \mathbf{k+1\ k+3} & \\ \cdots & \underbrace{\mathbf{k+1\ k+3}}_{!} & \cdots \end{array}\right\},$$

or

$$\left\{\begin{array}{llll} & \mathbf{k\ k+2} & & \\ \cdots & \underbrace{\mathbf{k\ k+2}}_{!} & k+4 & \cdots \end{array}\right\} \longrightarrow \left\{\begin{array}{llll} & & \mathbf{k+2\ k+4} & \\ \cdots & k & \underbrace{\mathbf{k+2\ k+4}}_{!} & \cdots \end{array}\right\}.$$

In the latter possibility, the 2-clusters encountered a 1-cluster on the way.

However, the same even number of forward moves will leave the clusters as

$$\left\{\begin{array}{llllll} & & \mathbf{k+1} & & \mathbf{k+3} & \\ \cdots & \mathbf{k} & & \mathbf{k+2} & & \cdots \end{array}\right\},$$

conforming to the difference condition. Or, one extra move on the larger cluster will yield

$$\left\{\begin{array}{llll} & \mathbf{k} & \mathbf{k+3} & \\ \cdots & \mathbf{k\ k+2} & & \cdots \end{array}\right\},$$

again honoring the difference condition.

Thus, after the implementation of μ and η as forward moves on the 1- and 2-clusters, the intermediate partition looks like

$$\left\{\begin{array}{llllllll} & & & & & & 2s_2+3 & \\ & 2 & & 4 & & 2s_2 & 2s_2+2 & \\ 1 & & 3 & & \cdots\ 2s_2-1 & & 2s_2+2 & \\ & & & & & & & \\ & & & 2s_2+6 & & & & \mathbf{2s_2+3n_3} \\ & & 2s_2+5 & & & \mathbf{2s_2+3n_{3-1}} & & \\ & & 2s_2+5 & & \cdots & \mathbf{2s_2+3n_3-1} & & \\ & & & & & & & \\ & & & (\text{parts } \geq 2s_2+3n_3+2, \text{ all 1- or 2-clusters}) & & & & \end{array}\right\},$$

for $s_2 \geq 0$, or

$$\left\{\begin{array}{llllllll} & & & & & & 2s_2+4 \\ 2 & 4 & & 2s_2 & & 2s_2+3 & \\ 1 & 3 & \cdots\, 2s_2-1 & & 2s_2+1 & 2s_2+3 & \end{array}\right.$$

$$\begin{array}{llll} & 2s_2+7 & & \mathbf{2s_2+3n_{3+1}} \\ 2s_2+6 & & \mathbf{2s_2+3n_3} & \\ 2s_2+6 & & \cdots\, \mathbf{2s_2+3n_3} & \end{array}$$

$$\left.(\text{parts } \geq 2s_2+3n_3+3, \text{ all 1- or 2-clusters})\right\},$$

again, for $s_2 \geq 0$. Both of the above satisfy the difference conditions. The former possibly has a sediment, i.e. unmoved 2-clusters if $s_2 > 0$. The latter has a sediment consisting of a 1-cluster, and if $s_2 > 0$, some 2-clusters as well. The presence of unmoved 1- or 2-clusters, namely, sediments, indicates that μ or η, respectively, have some zeros.

It remains to move the ith largest 3-cluster $\frac{1}{3}\times$(the ith largest part of ν) times forward. Recall that ν consists of multiples of three. The forward moves on the 3-clusters can be visualized in the following exclusive cases, each adding three to the weight of the partition. In each case, we assume that the initial configuration satisfies the necessary difference conditions.

$$\left\{\begin{array}{llll} & & \mathbf{k+1} & \\ & \mathbf{k} & & \\ (\text{parts } \leq k-2) & \mathbf{k} & & (\text{parts } \geq k+4) \end{array}\right\}$$
$$\Big\downarrow \text{ 1 forward move on the displayed 3-cluster}$$
$$\left\{\begin{array}{llll} & & \mathbf{k+2} & \\ & \mathbf{k+1} & & \\ (\text{parts } \leq k-2) & \mathbf{k+1} & & (\text{parts } \geq k+4) \end{array}\right\}$$

Above, the part $k-2$ cannot repeat if it occurs, since we assumed that the initial configuration satisfies the difference conditions. $k+4$ may occur up to twice, but not thrice.

$$\left\{\begin{array}{lllll} & & \mathbf{k+1} & & \\ & \mathbf{k} & & & \\ (\text{parts } \leq k-2) & \mathbf{k} & & k+3 & (\text{parts } \geq k+5) \end{array}\right\}$$
$$\Big\downarrow \text{ 1 forward move on the displayed 3-cluster}$$
$$\left\{\begin{array}{lll} & & \mathbf{k+2} \\ & \mathbf{k+1} & \\ (\text{parts } \leq k-2) & \multicolumn{2}{l}{\underbrace{\mathbf{k+1} \qquad\quad k+3}_{!}} \end{array}\right.$$
$$\left.(\text{parts } \geq k+5)\right\} (\text{temporarily})$$
$$\Big\downarrow \text{ adjustment}$$
$$\left\{\begin{array}{lllll} & & & \mathbf{k+3} & \\ & & \mathbf{k+2} & & \\ (\text{parts } \leq k-2) & k & \mathbf{k+2} & & (\text{parts } \geq k+5) \end{array}\right\}$$

Above, again, the part $k-2$ can occur only once. $k+5$ may occur twice, but not thrice.

$$
\left\{
\begin{array}{lccccccc}
 & & \mathbf{k+1} & & & & & \\
 & \mathbf{k} & & & k+4 & & k+6 & \\
(\text{parts } \leq k-2) & \mathbf{k} & & k+3 & & k+5 & &
\end{array}
\right.
$$

$$
\left.
\begin{array}{lccc}
 & k+2s+2 & & \\
\cdots k+2s+1 & & & (\text{parts } \geq k+2s+4)
\end{array}
\right\}
$$

$\downarrow$ 1 forward move on the displayed 3-cluster

$$
\left\{
\begin{array}{lccccc}
 & & \mathbf{k+2} & & & \\
 & \mathbf{k+1} & & k+4 & & k+6 \\
(\text{parts } \leq k-2) & \underbrace{\mathbf{k+1} \quad\quad k+3}_{!} & & & k+5 &
\end{array}
\right.
$$

$$
\left.
\begin{array}{lccc}
 & k+2s+2 & & \\
\cdots k+2s+1 & & & (\text{parts } \geq k+2s+4)
\end{array}
\right\} (\text{temporarily})
$$

$\downarrow$ adjustment

$$
\left\{
\begin{array}{lccccccc}
 & & & \mathbf{k+4} & & & & \\
 & & k+1 & \mathbf{k+3} & & & k+6 & \\
(\text{parts } \leq k-2) & k & & \underbrace{\mathbf{k+3} \quad\quad k+5 \quad\quad}_{!} & & & & \cdots
\end{array}
\right.
$$

$$
\left.
\begin{array}{lccc}
 & k+2s+2 & & \\
k+2s+1 & & & (\text{parts } \geq k+2s+4)
\end{array}
\right\} (\text{temporarily})
$$

$\downarrow$ after $s-1$ similar adjustments

$$
\left\{
\begin{array}{lccccc}
 & & k+1 & & k+3 & \\
(\text{parts } \leq k-2) & k & & k+2 & & \cdots
\end{array}
\right.
$$

$$
\begin{array}{lccc}
 & & & \mathbf{k+2s+2} \\
 & k+2s-1 & \mathbf{k+2s+1} & \\
k+2s-2 & & \mathbf{k+2s+1} &
\end{array}
$$

$$
\left.
\begin{array}{c}
 \\
(\text{parts } \geq k+2s+4)
\end{array}
\right\}
$$

for $s \geq 1$. Again, if $k-2$ occurs in the above configuration, it cannot repeat. $k+2s-4$ may repeat up to twice. The adjustments do not alter the weight. The adjustments are switching places of the 3- and 2-clusters when they are too close together. There are three other cases summarized below. They are very similar to the ones already explained, so we omit the details.

$$\left\{\begin{array}{lccccccc} & & \mathbf{k+1} & & & & \\ & \mathbf{k} & & & k+4 & & k+6 \\ (\text{parts } \le k-2) & \mathbf{k} & & k+3 & & k+5 & \end{array}\right.$$

$$\left.\begin{array}{lccc} & & k+2s+2 & \\ \cdots\ k+2s+1 & & & k+2s+3 \quad (\text{parts } \ge k+2s+5) \end{array}\right\}$$

$\downarrow$ 1 forward move on the displayed 3-cluster, followed by adjustments

$$\left\{\begin{array}{lccccc} & & k+1 & & k+3 & \\ (\text{parts } \le k-2) & k & & k+2 & & \cdots \end{array}\right.$$

$$\begin{array}{ccccc} & & & & \mathbf{k+2s+3} \\ & k+2s-1 & & \mathbf{k+2s+2} & \\ k+2s-2 & & k+2s & \mathbf{k+2s+2} & \end{array}$$

$$\left.(\text{parts } \ge k+2s+5)\right\}$$

for $s \ge 0$.

$$\left\{\begin{array}{lccccccc} & & \mathbf{k+1} & & & & & \\ & \mathbf{k} & & k+3 & & k+6 & & k+8 \\ (\text{parts } \le k-2) & \mathbf{k} & & k+3 & k+5 & & k+7 & \end{array}\right.$$

$$\left.\begin{array}{lcc} & k+2s+2 & \\ \cdots\ k+2s+1 & & (\text{parts } \ge k+2s+4) \end{array}\right\}$$

$\downarrow$ 1 forward move on the displayed 3-cluster, followed by adjustments

$$\left\{\begin{array}{lcccccc} & k & & k+3 & & k+5 & \\ (\text{parts } \le k-2) & k & k+2 & & k+4 & & \cdots \end{array}\right.$$

$$\begin{array}{cccc} & & & \mathbf{k+2s+2} \\ & k+2s-1 & \mathbf{k+2s+1} & \\ k+2s-2 & & \mathbf{k+2s+1} & \end{array}$$

$$\left.(\text{parts } \ge k+2s+4)\right\}$$

for $s \ge 1$, the case $s = 1$ giving an empty streak after the smallest displayed 2-cluster.

$$
\left\{
\begin{array}{lcccccccc}
 & & \mathbf{k+1} & & & & & \\
 & \mathbf{k} & & k+3 & & k+6 & & k+8 \\
(\text{parts } \leq k-2) & \mathbf{k} & & k+3 & k+5 & & k+7 &
\end{array}
\right.
$$

$$
\left.
\begin{array}{ccccc}
 & & k+2s+2 & & \\
\cdots & k+2s+1 & & k+2s+3 & (\text{parts } \geq k+2s+5)
\end{array}
\right\}
$$

$\downarrow$ 1 forward move on the displayed 3-cluster, followed by adjustments

$$
\left\{
\begin{array}{lcccccc}
 & k & & k+3 & & k+5 & \\
(\text{parts } \leq k-2) & k & k+2 & & k+4 & & \cdots
\end{array}
\right.
$$

$$
\begin{array}{ccccc}
 & & & & \mathbf{k+2s+3} \\
 & k+2s-1 & & \mathbf{k+2s+2} & \\
k+2s-2 & & k+2s & \mathbf{k+2s+2} &
\end{array}
$$

$$
\left.
(\text{parts } \geq k+2s+5)
\right\}
$$

for $s \geq 1$. In the above three respective cases, $k+2s+4$ or $k+2s+5$ may repeat up to twice. None of the cases may $k-2$ repeat without violating the difference conditions in the initial configuration.

It is routine to check that in all of the above forward moves on the 3-cluster, the preceding cluster, if any, may also move forward at least once. This concludes the construction of λ enumerated by $kr_5(n,m)$, given (β, μ, η, ν).

The reverse part of the construction is the decomposition of λ into the quadruple (β, μ, η, ν) as described above. First, we determine the number or r-clusters n_r for $r = 1, 2, 3$ in λ.

We will first move the smallest 3-cluster, if any, backward so many times, and call the number of required moves $\frac{1}{3} \times \nu_1$, where ν_1 is the smallest part of ν. ν_1 will clearly be a multiple of three. Each backward move on this cluster will deduct three from the weight of λ, and the same amount will be registered as the weight of ν.

λ may start with either of the following sediments.

$$
\left\{
\begin{array}{cccccc}
 & 2 & 3 & & 2s & \\
1 & & 4 & \cdots \ 2s-1 & & (\text{parts } \geq 2s+2)
\end{array}
\right\},
$$

or

$$
\left\{
\begin{array}{ccccccc}
 & 2 & 3 & & 2s & & \\
1 & & 4 & \cdots \ 2s-1 & & 2s+1 & (\text{parts } \geq 2s+3)
\end{array}
\right\},
$$

for $s \geq 0$, the case $s = 0$ corresponding to having no 2-clusters in the sediments. In the above two events, the backward moves on the smallest 3-cluster will stow it as

$$
\left\{
\begin{array}{cccccccc}
 & & & & & & \mathbf{2s+3} & \\
 & 2 & 3 & & 2s & \mathbf{2s+2} & & \\
1 & & 4 & \cdots \ 2s-1 & & \mathbf{2s+2} & & (\text{parts } \geq 2s+5)
\end{array}
\right\},
$$

or

$$
\left\{\begin{array}{cccccccccc}
 & & & & & & & & \mathbf{2s+4} & \\
 & 2 & 3 & & & 2s & & \mathbf{2s+3} & & \\
1 & & & 4 & \cdots 2s-1 & & 2s+1 & \mathbf{2s+3} & & (\text{parts } \geq 2s+6)
\end{array}\right\},
$$

respectively. If the smallest 3-cluster is already one of the displayed ones above, we declare $\nu_1 = 0$.

Let us describe the backward moves and adjustments in the exclusive cases below. Then, we will argue that the 3-cluster cannot go further back.

$$
\left\{\begin{array}{cccc}
 & & \mathbf{k+1} & \\
 & \mathbf{k} & & \\
(\text{parts } \leq k-3) & \mathbf{k} & & (\text{parts } \geq k+3)
\end{array}\right\}
$$

$$
\Big\downarrow \text{ 1 backward move on the displayed 3-cluster}
$$

$$
\left\{\begin{array}{cccc}
 & & \mathbf{k} & \\
 & \mathbf{k-1} & & \\
(\text{parts } \leq k-3) & \mathbf{k-1} & & (\text{parts } \geq k+3)
\end{array}\right\}
$$

Above, $k-3$ will be assumed to not repeat, so that the difference conditions are met in the terminal configuration. However, $k-3$ may very well repeat without violating the difference conditions in the initial configuration. That case will be treated below. $k+3$ may repeat up to twice.

$$
\left\{\begin{array}{ccccc}
 & & & \mathbf{k+1} & \\
 & & \mathbf{k} & & \\
(\text{parts } \leq k-4) & k-2 & \mathbf{k} & & (\text{parts } \geq k+3)
\end{array}\right\}
$$

$$
\Big\downarrow \text{ 1 forward move on the displayed 3-cluster}
$$

$$
\left\{\begin{array}{ccc}
 & \mathbf{k} & \\
 & \mathbf{k-1} & \\
(\text{parts } \leq k-4) & \underbrace{k-2\ \mathbf{k-1}}_{!} & (\text{parts } \geq k+3)
\end{array}\right\} (\text{temporarily})
$$

$$
\Big\downarrow \text{ adjustment}
$$

$$
\left\{\begin{array}{ccccc}
 & & \mathbf{k-1} & & \\
 & \mathbf{k-2} & & & \\
(\text{parts } \leq k-4) & \mathbf{k-2} & & k+1 & (\text{parts } \geq k+3)
\end{array}\right\}
$$

Observe that the adjustment does not change the weight of the partition. Again, we assume that $k-4$ is not repeated, so that the difference condition is not violated in the terminal configuration. The case of repeating $(k-4)$'s will be treated below. $k+3$ may repeat up to twice, but not thrice.

$$\left\{\begin{array}{lllll} & & k-2s & & k-2s+2 \\ (\text{parts } \leq k-2s-3) & k-2s-1 & & k-2s+1 & \end{array}\right.$$
$$\left.\begin{array}{llll} & & \mathbf{k+1} & \\ & k-2 & \mathbf{k} & \\ \cdots k-3 & & \mathbf{k} & (\text{parts } \geq k+3) \end{array}\right\}$$

$$\Big\downarrow \text{ 1 backward move on the displayed 3-cluster}$$

$$\left\{\begin{array}{lllll} & & k-2s & & k-2s+2 \\ (\text{parts } \leq k-2s-3) & k-2s-1 & & k-2s+1 & \end{array}\right.$$
$$\left.\begin{array}{ll} & \qquad\qquad\quad \mathbf{k} \\ & \qquad k-2 \; \mathbf{k-1} \\ \underbrace{\cdots k-3 \qquad\quad \mathbf{k-1}}_{!} & (\text{parts } \geq k+3) \end{array}\right\} \text{(temporarily)}$$

$$\Big\downarrow \text{ adjustment}$$

$$\left\{\begin{array}{lllll} & & k-2s & & k-2s+2 \\ (\text{parts } \leq k-2s-3) & k-2s-1 & & k-2s+1 & \end{array}\right.$$
$$\begin{array}{ll} & \qquad\qquad\qquad \mathbf{k-2} \\ & \qquad k-4 \; \mathbf{k-3} \qquad\qquad k+1 \\ \cdots & \underbrace{k-5 \qquad\quad \mathbf{k-3} \qquad\qquad k}_{!} \end{array}$$
$$\left.\begin{array}{l} \\ (\text{parts } \geq k+3) \end{array}\right\} \text{(temporarily)}$$

$$\Big\downarrow \text{ after } s-1 \text{ similar adjustments}$$

$$\left\{\begin{array}{lllll} & & \mathbf{k-2s} & & \\ & \mathbf{k-2s-1} & & & k-2s+3 \\ (\text{parts } \leq k-2s-3) & \mathbf{k-2s-1} & & k-2s+2 & \end{array}\right.$$
$$\left.\begin{array}{lllll} & k-2s+5 & & k+1 & \\ k+2s+4 & & \cdots k & & (\text{parts } \geq k+3) \end{array}\right\}$$

for $s \geq 1$. Here, again, we will assume that $k-2s-3$ does not repeat, so that the terminal configuration conforms to the difference conditions set forth by $kr_5(n,m)$. $k+3$ may repeat up to twice. As before, the adjustments do not alter the weight. The three cases below are very similar to the last one. They cover the cases of repeated smaller parts as well. We leave the details to the reader.

$$\left\{\begin{array}{lllll} & & k-2s-1 & & k-2s+1\\ (\text{parts } \le k-2s-4) & k-2s-2 & & k-2s & \end{array}\right.$$

$$\left.\begin{array}{lllll} & & & \mathbf{k+1} & \\ & k-3 & \mathbf{k} & & \\ \cdots k-4 & & k-2\ \mathbf{k} & & (\text{parts } \ge k+3)\end{array}\right\}$$

$\downarrow$ 1 backward move on the displayed 3-cluster, followed by adjustments

$$\left\{\begin{array}{lll} & & \mathbf{k-2s-1}\\ & \mathbf{k-2s-2} & \\ (\text{parts } \le k-2s-4) & \mathbf{k-2s-2} & \end{array}\right.$$

$$\begin{array}{lllll} & k-2s+2 & & k-2s+4 & \\ k-2s+1 & & k-2s+3 & & \cdots \end{array}$$

$$\left.\begin{array}{llll} & k & & \\ k-1 & & k+1 & (\text{parts } \ge k+3)\end{array}\right\}$$

for $s \ge 1$.

$$\left\{\begin{array}{llll} & k-2s-1 & & k-2s+2\\ (\text{parts } \le k-2s-4) & k-2s-1 & k-2s+1 & \end{array}\right.$$

$$\left.\begin{array}{lllll} & & & & \mathbf{k+1}\\ & k-2s+4 & & k-3\ \mathbf{k} & \\ k-2s+3 & & \cdots k-2 & \mathbf{k} & \\ & & (\text{parts } \ge k+3) & & \end{array}\right\}$$

$\downarrow$ 1 backward move on the displayed 3-cluster, followed by adjustments

$$\left\{\begin{array}{llll} & & \mathbf{k-2s} & \\ & \mathbf{k-2s-1} & & k-2s+2\\ (\text{parts } \le k-2s-4) & \mathbf{k-2s-1} & & k-2s+2 \end{array}\right.$$

$$\left.\begin{array}{llllll} & k-2s+5 & & k-2s+7 & & k+1\\ k-2s+4 & & k-2s+6 & & \cdots k & \\ & & (\text{parts } \ge k+3) & & & \end{array}\right\}$$

for $s \ge 1$, the case $s = 1$ giving an empty streak after the smallest displayed 2-cluster.

$$
\left\{
\begin{array}{lllll}
 & k-2s-2 & & k-2s+1 & \\
(\text{parts } \leq k-2s-4) & k-2s-2 & k-2s & &
\end{array}
\right.
$$

$$
\begin{array}{llllll}
 & & & & & \mathbf{k+1} \\
 & k-2s+3 & & k-3 & & \mathbf{k} \\
k-2s+2 & & \cdots\ k-4 & & k-2 & \mathbf{k}
\end{array}
$$

$$
\left.
\begin{array}{r}
\\
(\text{parts } \geq k+3)
\end{array}
\right\}
$$

$\downarrow$ 1 backward move on the displayed 3-cluster, followed by adjustments

$$
\left\{
\begin{array}{llll}
 & & \mathbf{k-2s-1} & \\
 & \mathbf{k-2s-2} & & k-2s+1 \\
(\text{parts } \leq k-2s-4) & \mathbf{k-2s-2} & & k-2s+1
\end{array}
\right.
$$

$$
\begin{array}{llllll}
 & k-2s+4 & & k-2s+6 & & k \\
k-2s+3 & & k-2s+5 & & \cdots\ k-1 &
\end{array}
$$

$$
\left.
\begin{array}{r}
\\
k+1 \quad (\text{parts } \geq k+3)
\end{array}
\right\}
$$

for $s \geq 1$. Above, $k+3$ may repeat twice, but not thrice. In none of the respective three cases above, do $k-2s-4$ or $k-2s-3$ repeat, if they occur. Notice that the omitted cases of repetition are taken care of by the last two cases.

Again, it is routine to verify that one backward move on a 3-cluster allows at least one move on the succeeding 3-cluster.

Once we complete the backward moves on the smallest 3-cluster, we repeat the same process for the next smallest, and move it backward as far as it can go, recording the number of moves as $\frac{1}{3} \times \nu_2$, $\frac{1}{3} \times \nu_3$, $\ldots$, $\frac{1}{3} \times \nu_{n_3}$. This will give us the partition ν with n_3 parts (counting zeros) into multiples of three. The intermediate partition looks like

$$
\left\{
\begin{array}{lllllll}
 & & & & & & 2s+3 \\
 & 2 & & 4 & & 2s \ 2s+2 & \\
1 & & 3 & & \cdots\ 2s-1 & \ \ 2s+2 &
\end{array}
\right.
$$

$$
\begin{array}{llll}
 & 2s+6 & & 2s+3n_3 \\
2s+5 & & 2s+3n_3-1 & \\
2s+5 & & \cdots\ 2s+3n_3-1 &
\end{array}
$$

$$
\left.
\begin{array}{r}
\\
(\text{parts } \geq 2s+3n_3+2, \text{ all 1- or 2-clusters })
\end{array}
\right\}
\tag{4.5}
$$

for $s \geq 0$, $s = 0$ being the case of no 2-clusters smaller than the 3-clusters, or

$$
\left\{
\begin{array}{llllllll}
 & & & & & & & 2s+4 \\
 & 2 & & 4 & & 2s & 2s+3 & \\
1 & & 3 & & \cdots\ 2s-1 & \ \ 2s+1 & 2s+3 &
\end{array}
\right.
$$

$$\left.\begin{array}{cccc}
 & 2s+7 & & 2s+3n_3+1 \\
2s+6 & & 2s+3n_3 & \\
2s+6 & & \cdots\ 2s+3n_3 & \\
 & & & \\
\multicolumn{4}{c}{(\text{parts } \geq 2s+3n_3+3, \text{ all 1- or 2-clusters })}
\end{array}\right\} \tag{4.6}$$

for $s \geq 0$. If one or more 3-clusters were in the indicated places, we would have set $\eta_1 = 0$, $\eta_2 = 0$, …, as many as necessary.

Notice that the cases for the backward moves on the 3-clusters are inverses of the cases for the forward moves on the 3-clusters, in their respective order, after necessary shifts of all parts. The rulebreaking in the middle temporary cases are slightly different; however, the initial cases become the terminal cases and vice versa. We find the given descriptions more intuitive.

For a moment, suppose we wanted to move the smallest 3-cluster backward one more time, and do some adjustments so as to retain the difference conditions imposed by $kr_5(n, m)$, in the intermediate partition (4.5).

$$\left\{\begin{array}{ccccccccc}
 & & & & & & \mathbf{2s+3} & & \\
 & 2 & & 4 & & 2s\ \mathbf{2s+2} & & & \\
1 & & 3 & & \cdots\ 2s-1 & \mathbf{2s+2} & & & (\text{parts } \geq 2s+5)
\end{array}\right\}$$

$\downarrow$ 1 backward move on the displayed 3-cluster, followed by adjustments

$$\left\{\begin{array}{ccccccc}
 & \mathbf{2} & & & & & \\
\mathbf{1} & & 5 & 7 & & 2s & \\
\mathbf{1} & 4 & & 6 & \cdots\ 2s-1 & & (\text{parts } \geq 2s+5)
\end{array}\right\}$$

This creates two occurrences of 1's, which is forbidden by the conditions of $kr_5(n, m)$, and shows us that the 3-clusters are indeed as small as they can be.

Now, in either (4.5) or (4.6), we continue with implementing the backward moves on the 2-clusters. In either configuration, if $s > 0$, we set $\eta_1 = \eta_2 = \cdots = \eta_s = 0$. This is because the smallest s 2-clusters are already minimal. They cannot be moved further back. We then move the $(s+1)$th smallest 2-cluster using the backward moves of the second kind (Definition 2.5), bringing it to

$$\left\{\begin{array}{ccccccccc}
 & & & & & 2s+3 & & 2s+6 & \\
 & 2 & & 4 & 2s\ \ 2s+2 & & 2s+5 & & \\
1 & & 3 & \cdots\ 2s-1 & 2s+2 & & 2s+5 & & \cdots
\end{array}\right.$$

$$\left.\begin{array}{ccc}
 & 2s+3n_3 & \\
2s+3n_3-1 & & \mathbf{2s+3n_3+2} \\
2s+3n_3-1 & & \mathbf{2s+3n_3+2} \\
 & & \\
\multicolumn{3}{c}{(\text{parts } \geq 2s+3n_3+4, \text{ all 1- or 2-clusters })}
\end{array}\right\},$$

or

$$\left\{\begin{array}{cccccccc}
 & & & & & & & 2s+4 \\
 & 2 & & 4 & & 2s & 2s+3 & \\
1 & & 3 & & \cdots\ 2s-1 & 2s+1 & 2s+3 &
\end{array}\right.$$

$$\left.\begin{array}{lllll}
 & 2s+7 & & 2s+3n_3+1 & \\
2s+6 & & 2s+3n_3 & & \mathbf{2s+3n_3+3} \\
2s+6 & & \cdots\, 2s+3n_3 & & \mathbf{2s+3n_3+3} \\
 & & & & \\
\multicolumn{5}{l}{(\text{parts } \geq 2s+3n_3+5, \text{ all 1- or 2-clusters })}
\end{array}\right\} .$$

We record the number of required moves as $\eta_{s+1} - 1$. If $n_3 > 0$, the final backward move involves prestidigitating the 2-cluster through the 3-clusters as follows. After one more backward move of the second kind on the $(s+1)$th smallest 2-cluster, say, in the former configuration,

$$\left.\begin{array}{l}
\left\{\begin{array}{llllllll}
 & & & & & 2s+3 & & 2s+6 \\
 & 2 & 4 & & 2s\ 2s+2 & & 2s+5 & \\
1 & 3 & \cdots & 2s-1 & 2s+2 & & 2s+5 & \cdots
\end{array}\right. \\
\underbrace{\begin{array}{llll}
 & 2s+3n_3 & & \\
2s+3n_3-1 & & & \mathbf{2s+3n_3+2} \\
2s+3n_3-1 & & \mathbf{2s+3n_3+1} &
\end{array}}_{!} \\
 \\
(\text{parts } \geq 2s+3n_3+4, \text{ all 1- or 2-clusters })
\end{array}\right\} \text{(temporarily)}$$

$$\Big\downarrow \text{ adjustment}$$

$$\left.\begin{array}{l}
\left\{\begin{array}{llllllll}
 & & & & & 2s+3 & & 2s+6 \\
 & 2 & 4 & & 2s\ 2s+2 & & 2s+5 & \\
1 & 3 & \cdots & 2s-1 & 2s+2 & & 2s+5 & \cdots
\end{array}\right. \\
\underbrace{\begin{array}{llll}
 & 2s+3n_3-3 & & \\
2s+3n_3-4 & & & \mathbf{2s+3n_3-1} \\
2s+3n_3-4 & & \mathbf{2s+3n_3-2} &
\end{array}}_{!} \\
\begin{array}{lll}
 & 2s+3n_3+2 & \\
2s+3n_3+1 & & \\
2s+3n_3+1 & & (\text{parts } \geq 2s+3n_3+4)
\end{array}
\end{array}\right\} \text{(temporarily)}$$

$$\Big\downarrow \text{ after } n_3-1 \text{ similar adjustments}$$

$$\left.\begin{array}{l}
\left\{\begin{array}{llllll}
 & & & & & 2s+5 \\
 & 2 & 4 & & \mathbf{2s+2}\ 2s+4 & \\
1 & 3 & \cdots\ \mathbf{2s+1} & & 2s+4 &
\end{array}\right. \\
\begin{array}{lllll}
 & 2s+8 & & & 2s+3n_3+2 \\
2s+7 & & & 2s+3n_3+1 & \\
2s+7 & & & \cdots\ 2s+3n_3+1 &
\end{array} \\
 \\
(\text{parts } \geq 2s+3n_3+4, \text{ all 1- or 2-clusters })
\end{array}\right\} ,$$

and the $(s+1)$st 2-cluster is stowed in its proper place. This determines η_{s+1}, which is positive. The second case is almost the same except that the Gordon marking has to be updated after the final adjustment. We repeat the process and record $\eta_{s+2}, \eta_{s+3}, \ldots, \eta_{n_2}$. We note that the total weight of λ and η remains constant, because any drop in the weight of λ is registered in η in the same amount, thanks to the definition of the backward move of the second kind, namely, Definition 2.5.

At this point, we should justify the fact that η cannot have repeated odd parts. Initially, and after any moves followed by a streak of adjustments, λ has satisfied the difference conditions given by $kr_5(n, m)$. Also, the moves on the 2- and 3-clusters are performed in the exact reverse order. As we showed in the forward moves on the 2-clusters, any repeated odd part in η will result in a violation of the said difference conditions. Moreover, the violation precisely occurs when η has repeated odd parts. Thus, η as constructed above cannot have repeated odd parts.

So far, the intermediate partition looks like

$$\left\{ \begin{array}{cccccccc} & & & & & & & 2n_2+3 \\ & 2 & & 4 & & 2n_2 & 2n_2+2 & \\ 1 & & 3 & & \cdots\ 2n_2-1 & & 2n_2+2 & \end{array} \right.$$

$$\begin{array}{ccccc} & 2n_2+6 & & & 2n_2+3n_3 \\ 2n_2+5 & & & 2n_2+3n_3-1 & \\ 2n_2+5 & & \cdots & 2n_2+3n_3-1 & \end{array}$$

$$\left. (\text{parts } \geq 2n_2+3n_3+2, \text{ all 1-clusters }) \right\}, \tag{4.7}$$

or

$$\left\{ \begin{array}{cccccccc} & & & & & & & 2n_2+4 \\ & 2 & & 4 & & 2n_2 & & 2n_2+3 \\ 1 & & 3 & & \cdots\ 2n_2-1 & & 2n_2+1 & 2n_2+3 \end{array} \right.$$

$$\begin{array}{ccccc} & 2n_2+7 & & & 2n_2+3n_3+1 \\ 2n_2+6 & & & 2n_2+3n_3 & \\ 2n_2+6 & & \cdots & 2n_2+3n_3 & \end{array}$$

$$\left. (\text{parts } \geq 2n_2+3n_3+3, \text{ all 1-clusters }) \right\}, \tag{4.8}$$

where n_2, n_3, or both, are possibly zero.

In (4.8), we simply start by setting $\mu_1 = 0$, because the smallest 1-cluster is already as small as it can be. It cannot be moved further back without vanishing or messing up the Gordon marking, therefore changing at least one of n_1, n_2 or n_3.

In (4.7), we first subtract the necessary amount from the smallest 1-cluster and record the necessary number of moves as $\mu_1 - 1$. The partition becomes

$$\left\{\begin{array}{cccccc} & & & & & 2n_2+3 \\ & 2 & 4 & & 2n_2 \; 2n_2+2 & \\ 1 & 3 & \cdots \; 2n_2-1 & & 2n_2+2 & \end{array}\right.$$

$$\begin{array}{cccccc} & 2n_2+6 & & & 2n_2+3n_3 & \\ 2n_2+5 & & & 2n_2+3n_3-1 & & \\ 2n_2+5 & & \cdots & 2n_2+3n_3-1 & & \mathbf{2n_2+3n_3+2} \end{array}$$

$$\left.\begin{array}{c} \\ (\text{parts } \geq 2n_2+3n_3+4, \text{ all 1-clusters }) \end{array}\right\} .$$

We then perform one more deduction on the smallest 1-cluster, followed by prestidigitating that 1-cluster through the 3-clusters, hence obtaining μ_1.

$$\left\{\begin{array}{cccccc} & & & & & 2n_2+3 \\ & 2 & 4 & & 2n_2 \; 2n_2+2 & \\ 1 & 3 & \cdots \; 2n_2-1 & & 2n_2+2 & \end{array}\right.$$

$$\begin{array}{cccc} & 2n_2+6 & & \\ 2n_2+5 & & & \\ 2n_2+5 & & \cdots & \underbrace{\begin{array}{ccc} & 2n_2+3n_3 & \\ 2n_2+3n_3-1 & & \\ 2n_2+3n_3-1 & & \mathbf{2n_2+3n_3+1} \end{array}}_{!} \end{array}$$

$$\left.\begin{array}{c} \\ (\text{parts } \geq 2n_2+3n_3+4, \text{ all 1-clusters }) \end{array}\right\}$$

$$\Big\downarrow \text{ adjustment}$$

$$\left\{\begin{array}{cccccc} & & & & & 2n_2+3 \\ & 2 & 4 & & 2n_2 \; 2n_2+2 & \\ 1 & 3 & \cdots \; 2n_2-1 & & 2n_2+2 & \end{array}\right.$$

$$\begin{array}{cccc} & 2n_2+6 & & \\ 2n_2+5 & & & \\ 2n_2+5 & & \cdots & \underbrace{\begin{array}{ccc} & 2n_2+3n_3-3 & \\ 2n_2+3n_3-4 & & \\ 2n_2+3n_3-4 & & \mathbf{2n_2+3n_3-2} \end{array}}_{!} \end{array}$$

$$\left.\begin{array}{ccc} & 2n_2+3n_3+1 & \\ 2n_2+3n_3 & & \\ 2n_2+3n_3 & & (\text{parts } \geq 2n_2+3n_3+4, \text{ all 1-clusters }) \end{array}\right\}$$

$$\Big\downarrow \text{ after } n_3-1 \text{ similar adjustments}$$

$$\left\{\begin{array}{ccccccc} & & & & & & 2n_2+4 \\ & 2 & 4 & & 2n_2 & 2n_2+3 & \\ 1 & 3 & \cdots \; 2n_2-1 & & \mathbf{2n_2+1} & 2n_2+3 & \end{array}\right.$$

$$\begin{array}{ccccc} & 2n_2+7 & & & 2n_2+3n_3+1 \\ 2n_2+6 & & & 2n_2+3n_3 & \\ 2n_2+6 & & \cdots & 2n_2+3n_3 & \end{array}$$

$$\left.\begin{array}{c} \\ (\text{parts } \geq 2n_2+3n_3+4 \text{ all 1-clusters }) \end{array}\right\} ,$$

arriving at (4.8) with $\mu_1 > 0$.

We continue with subtracting μ_i from the ith smallest 1-cluster for $i = 2, 3, \ldots, n_1$ in the given order, to obtain the base partition β as (4.2). Because the pairwise difference of 1-clusters are at least two, we immediately get $\mu_2 \leq \mu_3 \leq \cdots \leq \mu_{n_1}$. To see that $\mu_1 \leq \mu_2$, simply notice that without the final backward move involving the prestidigitation of the smallest 1-cluster through the 3-clusters, we would have $\mu_1 - 1 \leq \mu_2 - 1 \leq \cdots \leq \mu_{n_1} - 1$. If there were no 1-clusters, we would have stopped at (4.7), which incidentally would have been the base partition β, and declare μ the empty partition. This yields the quadruple (β, μ, η, ν) we have been looking for, given λ counted by $kr_5(n, m)$, and concludes the proof. $\square$

Example 4.2. Following the notation in the proof of Theorem 4.1, let us take the base partition β having $n_1 = 3$ 1-clusters, $n_2 = 2$ 2-clusters, and $n_3 = 2$ 3-clusters. Assume that $\mu = 1 + 1 + 1$, $\eta = 0 + 5$, and $\nu = 3 + 9$.

$$\beta = \left\{ \begin{array}{ccccccccccc}
 & & & & & & 8 & & 11 & & \\
 & 2 & & 4 & & 7 & & 10 & & & \\
1 & & 3 & & 5 & 7 & & 10 & & 13 & \mathbf{15}
\end{array} \right\}$$

The weight of β is 96.

We first incorporate μ_3 and μ_2 on the two largest 1-clusters, which are simple additions.

$$\left\{ \begin{array}{ccccccccccc}
 & & & & & & 8 & & 11 & & \\
 & 2 & & 4 & & 7 & & 10 & & & \\
1 & & 3 & & \mathbf{5} & 7 & & 10 & & 14 & 16
\end{array} \right\}$$

We then perform the $\mu_1 = 1$ forward move on the smallest 1-cluster, and watch it being prestidigitated through the 3-clusters.

$$\left\{ \begin{array}{cccccccccc}
 & & & & & 8 & & 11 & & \\
 & 2 & & 4 & 7 & & 10 & & & \\
1 & & 3 & & \underbrace{\mathbf{6}\ 7}_{!} & & 10 & & 14 & 16
\end{array} \right\}$$

$$\Big\downarrow \text{ adjustment}$$

$$\left\{ \begin{array}{cccccccccc}
 & & & & & 7 & & 11 & & \\
 & 2 & & 4 & 6 & & 10 & & & \\
1 & & 3 & & 6 & & \underbrace{\mathbf{9}\ 10}_{!} & & 14 & 16
\end{array} \right\}$$

$$\Big\downarrow \text{ adjustment}$$

$$\left\{ \begin{array}{ccccccccccc}
 & & & & & 7 & & 10 & & & \\
 & 2 & & 4 & 6 & & 9 & & & & \\
1 & & 3 & & 6 & & 9 & & \mathbf{12} & 14 & 16
\end{array} \right\}$$

This completes the incorporation of μ as forward moves on the 1-clusters. Next, we turn to $\eta = 0 + 5$. The larger 2-cluster will be moved 5 times forward. The first of those moves will involve prestidigitation through the 3-clusters. The smaller 2-cluster will stay put, thanks to η_1 being zero.

$$\downarrow \text{ the first move forward on the larger 2-cluster}$$

$$\left\{\begin{array}{cccccccccccc} &&&&&7&&&10&&& \\ &2&\mathbf{4}&&6&&&9&&&& \\ 1&&\mathbf{4}&&6&&&9&&12&14&16 \\ &&\multicolumn{3}{c}{\underbrace{\hphantom{4\ 6}}_{!}}&&&&&&& \end{array}\right\}$$

$$\downarrow \text{ adjustment}$$

$$\left\{\begin{array}{cccccccccccc} &&&5&&&&&10&&& \\ &2&4&&&\mathbf{7}&&9&&&& \\ 1&&4&&&\mathbf{7}&&9&&12&14&16 \\ &&&&&\multicolumn{4}{c}{\underbrace{\hphantom{7\ 9\ \ }}_{!}}&&& \end{array}\right\}$$

$$\downarrow \text{ adjustment}$$

$$\left\{\begin{array}{cccccccccccc} &&&5&&&8&&&&& \\ &2&4&&&7&&&\mathbf{10}&&& \\ 1&&4&&&7&&&\mathbf{10}&12&14&16 \end{array}\right\}$$

$$\downarrow \text{ four more moves on the larger 2-cluster}$$

$$\left\{\begin{array}{cccccccccccc} &&&5&&&\mathbf{8}&&&&& \\ &2&4&&&\mathbf{7}&&&&&14& \\ 1&&4&&&\mathbf{7}&&&10&12&14&16 \end{array}\right\}$$

Finally, we use $\nu = 3 + 9$ to move the larger 3-cluster $\frac{1}{3}\nu_2 = 3$ times forward, and then the smaller 3-cluster $\frac{1}{3}\nu_1 = 1$ times forward.

$\downarrow$ the first forward move on the larger 3-cluster

$$\left\{\begin{array}{cccccccccccccccc} & & & & 5 & & & & \mathbf{9} & & & & & & & \\ & 2 & & 4 & & & & \mathbf{8} & & & & & & 14 & & \\ 1 & & & 4 & & & & \multicolumn{3}{c}{\underbrace{\mathbf{8}\quad 10}_{!}} & & 12 & & 14 & & 16 \end{array}\right\}$$

$\downarrow$ adjustment

$$\left\{\begin{array}{cccccccccccccccc} & & & & 5 & & & & & \mathbf{10} & & & & & & \\ & 2 & & 4 & & & & & \mathbf{9} & & & & & 14 & & \\ 1 & & & 4 & & & \mathbf{7} & & \mathbf{9} & & & 12 & & 14 & & 16 \end{array}\right\}$$

$\downarrow$ the second forward move on the larger 3-cluster

$$\left\{\begin{array}{cccccccccccccccc} & & & & 5 & & & & & & \mathbf{11} & & & & & \\ & 2 & & 4 & & & & & & \mathbf{10} & & & & 14 & & \\ 1 & & & 4 & & & 7 & & & \multicolumn{3}{c}{\underbrace{\mathbf{10}\quad 12}_{!}} & & 14 & & 16 \end{array}\right\}$$

$\downarrow$ adjustment

$$\left\{\begin{array}{cccccccccccccccc} & & & & 5 & & & & & & & \mathbf{12} & & & & \\ & 2 & & 4 & & & & & & & \mathbf{11} & & & 14 & & \\ 1 & & & 4 & & & 7 & & 9 & & \mathbf{11} & & & 14 & & 16 \end{array}\right\}$$

$\downarrow$ the third, and the last, forward move on the larger 3-cluster

$$\left\{\begin{array}{cccccccccccccccc} & & & & 5 & & & & & & & & \mathbf{13} & & & \\ & 2 & & 4 & & & & & & & & \mathbf{12} & & 14 & & \\ 1 & & & 4 & & & 7 & & 9 & & & \multicolumn{3}{c}{\underbrace{\mathbf{12}\quad 14}_{!}} & & 16 \end{array}\right\}$$

$\downarrow$ adjustment

$$\left\{\begin{array}{cccccccccccccccc} & & & & 5 & & & & & & & & & & \mathbf{15} & \\ & 2 & & 4 & & & & & & & 11 & & & \mathbf{14} & & \\ 1 & & & 4 & & & 7 & & 9 & & 11 & & & \multicolumn{3}{c}{\underbrace{\mathbf{14}\quad 16}_{!}} \end{array}\right\}$$

$\downarrow$ adjustment

$$\left\{\begin{array}{cccccccccccccccc} & & & & \mathbf{5} & & & & & & & & & & & 16 \\ & 2 & & \mathbf{4} & & & & & & & 11 & & & & 15 & \\ 1 & & & \mathbf{4} & & & 7 & & 9 & & 11 & & 13 & & 15 & \end{array}\right\}$$

$\downarrow$ one forward move on the smaller 3-cluster

$$\left\{\begin{array}{cccccccccccccccc} & & & & & \mathbf{6} & & & & & & & & & & 16 \\ & 2 & & & \mathbf{5} & & & & & & 11 & & & & 15 & \\ 1 & & & & \multicolumn{3}{c}{\underbrace{\mathbf{5}\quad 7}_{!}} & & 9 & & 11 & & 13 & & 15 & \end{array}\right\}$$

$\downarrow$ adjustment

$$\lambda = \left\{\begin{array}{cccccccccccccccc} & & & & & & \mathbf{7} & & & & & & & & & 16 \\ & 2 & & & & \mathbf{6} & & & & & 11 & & & & 15 & \\ 1 & & & 4 & & \mathbf{6} & & & 9 & & 11 & & 13 & & 15 & \end{array}\right\}$$

The weight of λ, as expected is 116. λ has the sediment $\begin{smallmatrix} & 2 \\ 1 & \end{smallmatrix}$, for the sole unmoved 2-clusters:

$$|\lambda| = 116 = 96 + 3 + 5 + 12 = |\beta| + |\mu| + |\eta| + |\nu|.$$

Theorem 4.3 (cf. The Kanade–Russell conjecture I_6). *For $n, m \in \mathbb{N}$, let $kr_6(n, m)$ be the number of partitions of n into m parts with smallest part at least 2, at most one appearance of the part 2, and difference at least three at distance three such that if parts at distance two differ by at most one, then their sum, together with the intermediate part, is $\equiv 2 \pmod 3$. Then*

$$\sum_{m,n\geq 0} kr_6(n,m)q^n x^m = \sum_{n_1,n_2,n_3\geq 0} \frac{q^{(9n_3^2+7n_3)/2+2n_2^2+3n_2+n_1^2+n_1}}{(q;q)_{n_1}(q^2;q^2)_{n_2}(q^3;q^3)_{n_3}} \times (-q;q^2)_{n_2} q^{6n_3n_2+3n_3n_1+2n_2n_1} x^{3n_3+2n_2+n_1}. \quad (4.9)$$

Proof. The proof is a simpler version of the proof of Theorem 4.1. There is only one type of base partition β.

$$\left\{ \begin{array}{ccccccc} & 3 & 6 & & 3n_3 & & \\ & 3 & 6 & & 3n_3 & & 3n_3+3 \\ 2 & & 5 & \cdots\ 3n_3-1 & & 3n_3+2 & \end{array} \right.$$

$$\begin{array}{ccc} & 3n_3+5 & & 3n_3+2n_2+1 \\ 3n_3+4 & & \cdots\ 3n_3+2n_2 & \end{array}$$

$$\left. 3n_3+2n_2+2\ \ 3n_3+2n_2+4\ \cdots\ \ 3n_3+2n_2+2n_1 \right\}$$

This partition has the minimum weight among all enumerated by $kr_6(n, m)$, having n_r r-clusters for $r = 1, 2, 3$. Here, any n_r may be zero. Clearly, the only possible 3-clusters are

$$\left\{ \begin{array}{ccc} & k+1 & \\ & k+1 & \\ (\text{parts } \leq k-2) & k & (\text{parts } \geq k+3) \end{array} \right\}.$$

The rest of the proof is the same as that of Theorem 4.1. One does not even need to prestidigitate the 1- or 2- clusters through the 3-clusters. $\square$

We now write the generating functions for some similarly described enumerants, which are not listed in [6] because they did not yield nice infinite products, hence partition identities. In their proofs, we indicate the extra details only.

Theorem 4.4. *For $n, m \in \mathbb{N}$, let $kr^c_{1-2}(n, m)$ be the number of partitions of n into m parts with difference at least three at distance three such that if parts at distance two differ by at most one, then their sum, together with the intermediate part, is $\equiv 1 \pmod 3$. Then*

$$
\begin{aligned}
&\sum_{m,n\geq 0} kr^{c}_{1-2}(n,m) q^n x^m \\
&\quad = \sum_{\substack{n_1,n_3\geq 0\\ n_2>0}} \frac{q^{(9n_3^2-n_3)/2+2n_2^2+n_2+n_1^2}}{(q;q)_{n_1}(q^2;q^2)_{n_2}(q^3;q^3)_{n_3}}\,(1+q)(-q;q^2)_{n_2-1} \\
&\qquad \times \quad q^{6n_3n_2+3n_3n_1+2n_2n_1-1}\ x^{3n_3+2n_2+n_1} \\
&\qquad + \sum_{n_1,n_3\geq 0} \frac{q^{(9n_3^2-n_3)/2+n_1^2+3n_3n_1} x^{3n_3+n_1}}{(q;q)_{n_1}(q^3;q^3)_{n_3}}
\end{aligned}
\tag{4.10}
$$

$$
\begin{aligned}
&\quad = \sum_{n_1,n_2,n_3\geq 0} \frac{q^{(9n_3^2-n_3)/2+2n_2^2+n_2+n_1^2}}{(q;q)_{n_1}(q^2;q^2)_{n_2}\ (q^3;q^3)_{n_3}}\,(-1/q;q^2)_{n_2} \\
&\qquad \times q^{6n_3n_2+3n_3n_1+2n_2n_1}\ x^{3n_3+2n_2+n_1}.
\end{aligned}
\tag{4.11}
$$

Remark 4.5. Notice that no λ enumerated by $kr^{c}_{1-2}(n,m)$ can have three occurrences of 1.

Proof of Theorem 4.4. We will show (4.10) only. (4.11) follows by standard algebraic manipulations.

The proof is similar to the proof of Theorem 4.1. Two separate series are for two separate base partitions for the cases $n_1, n_3 \geq 0, n_2 > 0$, and $n_1, n_3 \geq 0, n_2 = 0$. Here, again, n_r is the number of r-clusters for $r = 1, 2, 3$ of the partition at hand.

In case $n_2 > 0$, the base partition β is

$$
\left\{
\begin{array}{cccccccccc}
 & 2 & & 5 & & & 3n_3-1 & & & \\
1 & & 4 & & & 3n_3-2 & & 3n_3+1 & & 3n_3+4 \\
1 & & 4 & & \cdots & 3n_3-2 & & 3n_3+1 & 3n_3+3 &
\end{array}
\right.
$$

$$
\begin{array}{ccccc}
 & 3n_3+6 & & & 3n_3+2n_2 \\
3n_3+5 & & & \cdots\ 3n_3+2n_2-1 &
\end{array}
$$

$$
\left.
\begin{array}{ccc}
\\
\\
3n_3+2n_2+1 & 3n_3+2n_2+3 \ \cdots & 3n_3+2n_2+2n_1-1
\end{array}
\right\},
\tag{4.12}
$$

with weight $(9n_3^2 - n_3)/2 + 2n_2^2 + n_2 + n_1^2 + 6n_3n_2 + 3n_3n_1 + 2n_2n_1 - 1$.

When $n_2 = 0$, the base partition is

$$
\left\{
\begin{array}{ccccccccccc}
 & 2 & & 5 & & & 3n_3-1 & & & & \\
1 & & 4 & & & 3n_3-2 & & & & & \\
1 & & 4 & & \cdots & 3n_3-2 & & 3n_3+1 & 3n_3+3 & \cdots & 3n_3+2n_1-1
\end{array}
\right\},
\tag{4.13}
$$

with weight $(9n_3^2 - n_3)/2 + n_1^2 + 3n_3n_1$. This is not the $n_2 = 0$ case of (4.12).

The novelty in (4.12) is that the smallest 2-cluster $\begin{array}{c}3n_3+1\\3n_3+1\end{array}$ has an extra move forward. If that extra move is made, then the 2-clusters in the resulting partition can be treated as in the proof of Theorem 4.1. Without this extra move, we only have $n_2 - 1$ 2-clusters to move forward.

In a partition λ enumerated by $kr^c_{1-2}(n,m)$, we check if there is a sediment of the form

$$\left\{\begin{array}{cccccccc} & 2 & & 5 & & 3s-1 & & \\ 1 & & 4 & & 3s-2 & & 3s+1 & \\ 1 & & 4 & & \cdots\, 3s-2 & & 3s+1 & (\text{parts } \geq 3s+3) \end{array}\right\}$$

for $s \geq 0$ to tell the cases apart.

The partition accounting for the forward or backward moves on the 2-clusters is generated by

$$\frac{(-q;q^2)_{n_2-1}}{(q^2;q^2)_{n_2-1}} + q\frac{(-q;q^2)_{n_2}}{(q^2;q^2)_{n_2}} = \frac{(1+q)(-q;q^2)_{n_2-1}}{(q^2;q^2)_{n_2}}$$

for $n_2 \geq 1$. The factor q in the second term is for the extra move. For $n_2 = 0$, it is simply 1, the empty partition.

The rest of the proof is the same as the proof of Theorem 4.1, except that prestidigitating 1- or 2-clusters through the 3-clusters is not necessary. $\square$

Example 4.6. Following the notation of the proof of the above theorem, let

$$\lambda = \left\{\begin{array}{ccccccccc} & & & \mathbf{7} & & & & & 16 \\ & 2 & & \mathbf{6} & & 11 & & 15 & \\ 1 & & 4 & \mathbf{6} & 9 & 11 & 13 & 15 & \end{array}\right\}.$$

This is one of the partitions we encountered before. We will examine it once more as a partition satisfying the conditions of $kr^c_{1-2}(116,13)$. λ as such has no sediments; therefore, the initial forward move was applied to the smallest 2-cluster, and η has two parts.

We begin by decoding ν through the backward moves on the 3-clusters, the smallest first.

$\big\downarrow$ one backward move on the smallest 3-cluster

$$\left\{\begin{array}{cccccccc} & & \mathbf{6} & & & & & 16 \\ & 2 & \mathbf{5} & & 11 & & 15 & \\ 1 & \underbrace{4\ \mathbf{5}}_{!} & & 9 & 11 & 13 & 15 & \end{array}\right\}$$

$\big\downarrow$ adjustment

$$\left\{\begin{array}{ccccccccc} & & \mathbf{5} & & & & & & 16 \\ & 2\ \mathbf{4} & & & & 11 & & 15 & \\ 1 & \mathbf{4} & & 7 & 9 & 11 & 13 & 15 & \end{array}\right\}$$

$\big\downarrow$ one more backward move on the smallest 3-cluster

$$\left\{\begin{array}{ccccccccc} & & \mathbf{4} & & & & & & 16 \\ & 2\ \mathbf{3} & & & & 11 & & 15 & \\ \multicolumn{2}{c}{\underbrace{1\quad \mathbf{3}}_{!}} & & 7 & 9 & 11 & 13 & 15 & \end{array}\right\}$$

$\big\downarrow$ adjustment

$$\left\{\begin{array}{ccccccccc} & 2 & & & & & & & \mathbf{16} \\ 1 & & 5 & & & 11 & & \mathbf{15} & \\ 1 & 4 & & 7 & 9 & 11 & 13 & \mathbf{15} & \end{array}\right\}$$

The smallest 3-cluster has been stowed after two backward moves on it, thus, $\nu_1 = 3 \cdot 2 = 6$.

$$\Big\downarrow \text{ one backward move on the larger 3-cluster}$$

$$\left\{\begin{array}{cccccccccccccccc} & 2 & & & & & & & & & & & & & \mathbf{15} \\ 1 & & & & 5 & & & & & & 11 & & & \mathbf{14} \\ 1 & & & 4 & & & 7 & & 9 & & 11 & & \multicolumn{2}{c}{\underbrace{13\ \mathbf{14}}_{!}} \end{array}\right\}$$

$$\Big\downarrow \text{ adjustment}$$

$$\left\{\begin{array}{cccccccccccccccc} & 2 & & & & & & & & & & & & \mathbf{14} \\ 1 & & & & 5 & & & & & & 11 & & \mathbf{13} \\ 1 & & & 4 & & & 7 & & 9 & & \multicolumn{3}{c}{\underbrace{11\ \mathbf{13}}_{!}} & & & 16 \end{array}\right\}$$

$$\Big\downarrow \text{ adjustment}$$

$$\left\{\begin{array}{cccccccccccccccc} & 2 & & & & & & & & & & \mathbf{12} \\ 1 & & & & 5 & & & & & & \mathbf{11} & & & 14 \\ 1 & & & 4 & & & 7 & & 9 & & \mathbf{11} & & & 14 & & 16 \end{array}\right\}$$

$$\Big\downarrow \text{ one more backward move on the larger 3-cluster}$$

$$\left\{\begin{array}{cccccccccccccccc} & 2 & & & & & & & & & \mathbf{11} \\ 1 & & & & 5 & & & & & \mathbf{10} & & & & 14 \\ 1 & & & 4 & & & 7 & & \multicolumn{2}{c}{\underbrace{9\ \mathbf{10}}_{!}} & & & & 14 & & 16 \end{array}\right\}$$

$$\Big\downarrow \text{ adjustment}$$

$$\left\{\begin{array}{cccccccccccccccc} & 2 & & & & & & & & \mathbf{10} \\ 1 & & & & 5 & & & & \mathbf{9} & & & & & 14 \\ 1 & & & 4 & & & 7 & & \mathbf{9} & & & 12 & & 14 & & 16 \end{array}\right\}$$

$$\Big\downarrow \text{ one more backward move on the larger 3-cluster}$$

$$\left\{\begin{array}{cccccccccccccccc} & 2 & & & & & & & \mathbf{9} \\ 1 & & & & 5 & & & \mathbf{8} & & & & & & 14 \\ 1 & & & 4 & & & \multicolumn{2}{c}{\underbrace{7\ \mathbf{8}}_{!}} & & & & 12 & & 14 & & 16 \end{array}\right\}$$

$$\Big\downarrow \text{ adjustment}$$

$$\left\{\begin{array}{cccccccccccccccc} & 2 & & & & & & \mathbf{8} \\ 1 & & & & 5 & & \mathbf{7} & & & & & & & 14 \\ 1 & & & 4 & & & \mathbf{7} & & & 10 & & 12 & & 14 & & 16 \end{array}\right\}$$

$$\Big\downarrow \text{ one more backward move on the larger 3-cluster}$$

$$\left\{\begin{array}{cccccccccccccccc} & 2 & & & & & \mathbf{7} \\ 1 & & & & 5 & \mathbf{6} & & & & & & & & 14 \\ 1 & & & \multicolumn{3}{c}{\underbrace{4\ \mathbf{6}}_{!}} & & & & 10 & & 12 & & 14 & & 16 \end{array}\right\}$$

$$\Big\downarrow \text{ adjustment}$$

$$\left\{\begin{array}{cccccccccccccccc} & 2 & & & 5 \\ 1 & & & 4 & & & & \mathbf{8} & & & & & & 14 \\ 1 & & & 4 & & & \mathbf{7} & & & 10 & & 12 & & 14 & & 16 \end{array}\right\}$$

At this point, we deduce that $\nu_2 = 3 \cdot 4 = 12$. Also, looking at the smallest 2-cluster, $\eta_1 = 0$ can be seen. Because with one more backward move on the smallest 2-cluster, the intermediate partition becomes

$$\left\{ \begin{matrix} & 2 & & 5 & & & & & \\ 1 & & 4 & & 7 & & & \mathbf{14} & \\ 1 & & 4 & & 7 & 10 & 12 & \mathbf{14} & 16 \end{matrix} \right\}.$$

This must be the extra move.

$$\Big\downarrow \text{ five backward moves on the larger 2-cluster}$$

$$\left\{ \begin{matrix} & 2 & & 5 & & & & & \\ 1 & & 4 & & 7 & & 10 & & & \\ 1 & & 4 & & 7 & 9 & & \mathbf{12} & 14 & 16 \end{matrix} \right\}$$

This yields $\eta_2 = 5$. Finally, it is clear that $\mu = 1 + 1 + 1$, so that the partition becomes (4.12).

$$\left\{ \begin{matrix} & 2 & & 5 & & & & & \\ 1 & & 4 & & 7 & & 10 & & & \\ 1 & & 4 & & 7 & 9 & & 11 & 13 & 15 \end{matrix} \right\}$$

In other words, the base partition for $n_2 > 0$. The weight of λ is indeed

$$|\lambda| = 116 = 89 + 3 + (1 + 5) + 18 = |\beta| + |\mu| + (\text{extra move} + |\eta|) + |\nu|.$$

Theorem 4.7. *For $n, m \in \mathbb{N}$, let $kr^c_{2-2}(n, m)$ be the number of partitions of n into m parts with difference at least three at distance three such that if parts at distance two differ by at most one, then their sum, together with the intermediate part, is $\equiv 2 \pmod 3$. Then*

$$\begin{aligned} &\sum_{m,n \geq 0} kr^c_{2-2}(n, m) q^n x^m \\ &\quad = \sum_{\substack{n_1, n_3 \geq 0 \\ n_2 > 0}} \frac{q^{(9n_3^2 + n_3)/2 + 2n_2^2 + n_2 + n_1^2}}{(q; q)_{n_1} (q^2; q^2)_{n_2} (q^3; q^3)_{n_3}} (1 + q)(-q; q^2)_{n_2 - 1} \\ &\qquad \times q^{6n_3 n_2 + 3n_3 n_1 + 2n_2 n_1 - 1} x^{3n_3 + 2n_2 + n_1} \\ &\qquad + \sum_{n_1, n_3 \geq 0} \frac{q^{(9n_3^2 + n_3)/2 + n_1^2 + 3n_3 n_1} x^{3n_3 + n_1}}{(q; q)_{n_1} (q^3; q^3)_{n_3}} \qquad (4.14) \\ &\quad = \sum_{n_1, n_2, n_3 \geq 0} \frac{q^{(9n_3^2 + n_3)/2 + 2n_2^2 + n_2 + n_1^2}}{(q; q)_{n_1} (q^2; q^2)_{n_2} (q^3; q^3)_{n_3}} \\ &\qquad \times q^{6n_3 n_2 + 3n_3 n_1 + 2n_2 n_1} x^{3n_3 + 2n_2 + n_1}. \qquad (4.15) \end{aligned}$$

Remark 4.8. A partition enumerated by $kr^c_{2-2}(n, m)$ may contain the 2-cluster $\begin{matrix} 1 \\ 1 \end{matrix}$, but not the 3-clusters $\begin{matrix} & 2 \\ 1 & \\ 1 & \end{matrix}$ or $\begin{matrix} 1 \\ 1 \\ 1 \end{matrix}$, so it can have up to two occurrences of 1.

Proof of Theorem 4.7. (4.15) follows from (4.14) by standard algebraic manipulations, so we demonstrate (4.14) only.

The proof is very similar to the proof of Theorem 4.4. The two base partitions are the following:

$$\left\{\begin{array}{lllllll} & 4 & 7 & & 3n_3+1 & & \\ 1 & 4 & 7 & & 3n_3+1 & & 3n_3+4 \\ 1\ \ 3 & 6 & & \cdots\ 3n_3 & & 3n_3+3 & \end{array}\right.$$

$$\begin{array}{llll} & 3n_3+6 & & 3n_3+2n_2 \\ 3n_3+5 & & \cdots\ 3n_3+2n_2-1 & \end{array}$$

$$\left.\begin{array}{llll} \\ \\ 3n_3+2n_2+1 & 3n_3+2n_2+3 & \cdots & 3n_3+2n_2+2n_1-1 \end{array}\right\}, \tag{4.16}$$

whose weight is $(9n_3^2+n_3)/2+2n_2^2+n_2+n_1^2+6n_3n_2+3n_3n_1+2n_2n_1-1$, for $n_1, n_3 \geq 0, n_2 > 0$.

$$\left\{\begin{array}{llllllll} & 2 & 5 & & 3n_3-1 & & & \\ & 2 & 5 & & 3n_3-1 & & & \\ 1 & & 4 & \cdots\ 3n_3-2 & & 3n_3+1 & 3n_3+3\ \cdots & 3n_3+2n_1-1 \end{array}\right\},$$

whose weight is $(9n_3^2+n_3)/2+n_1^2+3n_3n_1$, for $n_1, n_3 \geq 0$. This is not the case $n_2 = 0$ of (4.16).

The smallest 2-cluster in (4.16) has one extra move forward to enter the game, which entails a prestidigitation through the 3-clusters, and making (4.16) into

$$\left\{\begin{array}{llllll} & 2 & 5 & & 3n_3-1 & \\ & 2 & 5 & & 3n_3-1 & \qquad 3n_3+2 \\ 1 & & 4 & \cdots\ 3n_3-2 & \qquad 3n_3+1 & \end{array}\right.$$

$$\begin{array}{lllll} & 3n_3+4 & & 3n_3+2n_2 & \\ 3n_3+3 & & \cdots\ 3n_3+2n_2-1 & & 3n_3+2n_2+1 \end{array}$$

$$\left.\begin{array}{lll} \\ \\ 3n_3+2n_2+3 & \cdots & 3n_3+2n_2+2n_1-1 \end{array}\right\}.$$

To tell the cases in which this extra move is made or not apart, we simply check if λ contains the 2-cluster $\begin{smallmatrix}1\\1\end{smallmatrix}$ as a sediment or not. $\square$

Theorem 4.9. *For $n, m \in \mathbb{N}$, let $kr_{2-1}^c(n,m)$ be the number of partitions of n into m parts with at most one occurrence of the part 1, and difference at least three at distance three such that if parts at distance two differ by at most one, then their sum, together with the intermediate part, is $\equiv 2 \pmod 3$. Then*

$$\sum_{m,n\geq 0} kr_{2-1}^c(n,m)q^n x^m$$

$$= \sum_{n_1,n_2,n_3\geq 0} \frac{q^{(9n_3^2+n_3)/2+2n_2^2+n_2+n_1^2}}{(q;q)_{n_1}(q^2;q^2)_{n_2}(q^3;q^3)_{n_3}} \times (-q;q^2)_{n_2}\, q^{6n_3n_2+3n_3n_1+2n_2n_1}\, x^{3n_3+2n_2+n_1}. \tag{4.17}$$

Proof. It suffices to observe that $kr^c_{2-1}(n+m,m) = kr_6(n,m)$. Then, the result becomes a corollary of Theorem 4.3. □

By means of shifts of all parts of a partition, one can put restrictions on the size of the smallest part and its number of occurrences. Then, the generating functions of such partitions may be obtained as corollaries of Theorems 4.1, 4.3, 4.4, 4.7 and 4.9.

5. Alternative Series for Kanade and Russell's Conjectures I_5 and I_6

In [10], it has been shown that

$$\sum_{n\geq 0} \frac{q^{n^2}(-q;q^2)_n x^n}{(q^2;q^2)_n} = \sum_{n_1,n_2\geq 0} \frac{q^{4n_2^2+(3n_1^2-n_1)/2+4n_2n_1} x^{2n_2+n_1}}{(q;q)_{n_1}(q^4;q^4)_{n_2}}. \tag{5.1}$$

Using this formula in (4.1), (4.9), (4.10), (4.14) and (4.17), and a little q-series algebra will yield the following:

$$\begin{aligned}\sum_{m,n\geq 0} kr_5(n,m)q^n x^m &= \sum_{n_1,n_2,n_3\geq 0} \frac{q^{(9n_3^2+5n_3)/2+2n_2^2+n_2+n_1^2}}{(q;q)_{n_1}(q^2;q^2)_{n_2}(q^3;q^3)_{n_3}} \\ &\quad\times (-q;q^2)_{n_2}\, q^{6n_3n_2+3n_3n_1+2n_2n_1}\, x^{3n_3+2n_2+n_1} \\ &= \sum_{n_1,m_2,n_3,m_4\geq 0} \frac{q^{8m_4^2+2m_4+(9n_3^2+5n_3)/2+(5m_2+m_2)/2+n_1^2}}{(q;q)_{n_1}(q;q)_{m_2}(q^3;q^3)_{n_3}(q^4;q^4)_{m_4}} \\ &\quad\times q^{12m_4n_3+8m_4m_2+4m_4n_1+6n_3m_2+3n_3n_1+2m_2n_1}\, x^{4m_4+3n_3+2m_2+n_1},\end{aligned} \tag{5.2}$$

$$\begin{aligned}\sum_{m,n\geq 0} kr_6(n,m)q^n x^m &= \sum_{n_1,n_2,n_3\geq 0} \frac{q^{(9n_3^2+7n_3)/2+2n_2^2+3n_2+n_1^2+n_1}}{(q;q)_{n_1}(q^2;q^2)_{n_2}(q^3;q^3)_{n_3}} \\ &\quad\times (-q;q^2)_{n_2}\, q^{6n_3n_2+3n_3n_1+2n_2n_1}\, x^{3n_3+2n_2+n_1} \\ &= \sum_{n_1,m_2,n_3,m_4\geq 0} \frac{q^{8m_4^2+6m_4+(9n_3^2+7n_3)/2+(5m_2^2+5m_2)/2+n_1^2+n_1}}{(q;q)_{n_1}(q;q)_{m_2}(q^3;q^3)_{n_3}(q^4;q^4)_{m_4}} \\ &\quad\times q^{12m_4n_3+8m_4m_2+4m_4n_1+6n_3m_2+3n_3n_1+2m_2n_1}\, x^{4m_4+3n_3+2m_2+n_1},\end{aligned} \tag{5.3}$$

$$\begin{aligned}\sum_{m,n\geq 0} kr^c_{1-2}(n,m)q^n x^m &= \sum_{n_1,n_2,n_3\geq 0} \frac{q^{(9n_3^2-n_3)/2+2n_2^2+n_2+n_1^2}}{(q;q)_{n_1}(q^2;q^2)_{n_2}(q^3;q^3)_{n_3}} \\ &\quad\times (-1/q;q^2)_{n_2}\, q^{6n_3n_2+3n_3n_1+2n_2n_1}\, x^{3n_3+2n_2+n_1} \\ &= \sum_{n_1,m_2,n_3,m_4\geq 0} \frac{q^{8m_4^2+2m_4+(9n_3^2-n_3)/2+(5m_2^2+m_2)/2+n_1^2}}{(q;q)_{n_1}(q;q)_{m_2}(q^3;q^3)_{n_3}(q^4;q^4)_{m_4}}\end{aligned} \tag{5.4}$$

$$\times\, (1 + x^2 q^{8m_4+6n_3+4m_2+2n_1+2})$$
$$\times\, q^{12m_4n_3+8m_4m_2+4m_4n_1+6n_3m_2+3n_3n_1+2m_2n_1}\ x^{4m_4+3n_3+2m_2+n_1},$$

$$\sum_{m,n\geq 0} kr^c_{2-2}(n,m) q^n x^m = \sum_{n_1,n_2,n_3\geq 0} \frac{q^{(9n_3^2+n_3)/2+2n_2^2+n_2+n_1^2}}{(q;q)_{n_1}(q^2;q^2)_{n_2}(q^3;q^3)_{n_3}}$$
$$\times\, (-1/q;q^2)_{n_2}\ q^{6n_3n_2+3n_3n_1+2n_2n_1}\ x^{3n_3+2n_2+n_1}$$
$$= \sum_{n_1,m_2,n_3,m_4\geq 0} \frac{q^{8m_4^2+2m_4+(9n_3^2+n_3)/2+(5m_2^2+m_2)/2+n_1^2}}{(q;q)_{n_1}(q;q)_{m_2}(q^3;q^3)_{n_3}(q^4;q^4)_{m_4}} \tag{5.5}$$
$$\times\, (1 + x^2 q^{8m_4+6n_3+4m_2+2n_1+2})$$
$$\times\, q^{12m_4n_3+8m_4m_2+4m_4n_1+6n_3m_2+3n_3n_1+2m_2n_1}\ x^{4m_4+3n_3+2m_2+n_1},$$

$$\sum_{m,n\geq 0} kr^c_{2-1}(n,m) q^n x^m = \sum_{n_1,n_2,n_3\geq 0} \frac{q^{(9n_3^2+n_3)/2+2n_2^2+n_2+n_1^2}}{(q;q)_{n_1}(q^2;q^2)_{n_2}(q^3;q^3)_{n_3}}$$
$$\times\, (-q;q^2)_{n_2}\ q^{6n_3n_2+3n_3n_1+2n_2n_1}\ x^{3n_3+2n_2+n_1}$$
$$= \sum_{n_1,m_2,n_3,m_4\geq 0} \frac{q^{8m_4^2+2m_4+(9n_3^2+n_3)/2+(5m_2+m_2)/2+n_1^2}}{(q;q)_{n_1}(q;q)_{m_2}(q^3;q^3)_{n_3}(q^4;q^4)_{m_4}} \tag{5.6}$$
$$\times\, q^{12m_4n_3+8m_4m_2+4m_4n_1+6n_3m_2+3n_3n_1+2m_2n_1}\ x^{4m_4+3n_3+2m_2+n_1}.$$

The combinatorics of the new formulas is as follows. We focus on the 2-clusters only, as the incorporation of the 1- and 3-clusters in the discussion is routine. The 2-clusters are lined up as

$$\left\{\begin{array}{cccccc} & 2 & & 4 & & 2n_2 \\ 1 & & 3 & & \cdots\ 2n_2-1 & \end{array}\right\}.$$

Then, we set $n_2 = 2m_4 + m_2$ for $m_2, m_4 \in \mathbb{N}$ and move the ith largest 2-cluster $m_2 - i$ times forward for $i = 1, 2, \ldots, m_2$.

$$\left\{\begin{array}{ccccccccccc} & 2 & & 4 & & 4m_4 & & 4m_4+2 & 4m_4+4 & & 4m_4+7 \\ 1 & & 3 & & \cdots\ 4m_4-1 & & 4m_4+1 & & 4m_4+4 & 4m_4+6 & \cdots \end{array}\right\}$$

Next, we declare the consecutive 2-clusters

$$\begin{array}{cccc} & 2 & & 4 \\ 1 & & 3 & \end{array},\quad \begin{array}{cccc} & 6 & & 8 \\ 5 & & 7 & \end{array},\ \cdots,$$
$$\begin{array}{cccc} & 4m_4-2 & & 4m_4 \\ 4m_4-3 & & 4m_4-1 & \end{array}$$

*2-cluster pair*s, and the others *individual 2-cluster*s. One forward move on an individual 2-cluster still adds one to the total weight, but one forward move on a 2-cluster pair adds four.

The procession of 2-cluster pairs through individual 2-clusters is defined similar to movement of pairs in [10, Sect. 3]. The procession of 2-cluster pairs through 1-clusters or prestidigitation of 2-cluster pairs through the 3-clusters is defined in the obvious way.

6. q-Series Versions of Kanade–Russell Conjectures

Given a partition counter, say $kr_1(n,m)$ in Theorem 3.1, we define

$$KR_1(n) = \sum_{m \geq 0} kr_1(m,n).$$

Then, we have the following relation between the generating functions:

$$\sum_{n \geq 0} KR_1(n) q^n = \left. \sum_{n,m \geq 0} kr_1(m,n) x^m q^n \right|_{x=1}.$$

In other words, substituting $x = 1$ renders the track of number of parts ineffective.

Using this idea in the respective theorems above gives the following conjectured q-series identities, in conjunction with [6].

Conjecture 6.1.

$$\frac{1}{(q,q^3,q^6,q^8;q^9)_\infty} \stackrel{?}{=} \sum_{n_1,n_2 \geq 0} \frac{q^{3n_2^2+n_1^2+3n_1n_2}}{(q;q)_{n_1}(q^3;q^3)_{n_2}}, \tag{6.1}$$

$$\frac{1}{(q^2,q^3,q^6,q^7;q^9)_\infty} \stackrel{?}{=} \sum_{n_1,n_2 \geq 0} \frac{q^{3n_2^2+3n_2+n_1^2+n_1+3n_1n_2}}{(q;q)_{n_1}(q^3;q^3)_{n_2}}, \tag{6.2}$$

$$\frac{1}{(q^3,q^4,q^5,q^6;q^9)_\infty} \stackrel{?}{=} \sum_{n_1,n_2 \geq 0} \frac{q^{3n_2^2+3n_2+n_1^2+2n_1+3n_1n_2}}{(q;q)_{n_1}(q^3;q^3)_{n_2}}, \tag{6.3}$$

$$\frac{1}{(q^2,q^3,q^5,q^8;q^9)_\infty} \stackrel{?}{=} \sum_{n_1,n_2 \geq 0} \frac{q^{3n_2^2+2n_2+n_1^2+n_1+3n_1n_2}}{(q;q)_{n_1}(q^3;q^3)_{n_2}}, \tag{6.4}$$

$$\begin{aligned} \frac{1}{(q,q^4,q^6,q^7;q^9)_\infty} &\stackrel{?}{=} \sum_{m,n \geq 0} \frac{q^{Q(m,n)+2m+4n}(1+q)}{(q;q)_m(q^3;q^3)_n} \\ &\quad + \sum_{m,n \geq 0} \frac{q^{Q(m,n)+2+3m+7n}}{(q;q)_m(q^3;q^3)_n}. \end{aligned} \tag{6.5}$$

The relation (6.1) is a combination of (3.1) and [6, I_1], (6.2) of (3.10) and [6, I_2], (6.3) of (3.14) and [6, I_3], (6.4) of (3.15) and [6, I_4]. (6.5) is the $x = 1$ case of (3.19), and added here upon the request of the anonymous referee. Thanks to [4], the conjectures became theorems for the fifth and the sixth conjectures in [6].

Theorem 6.2.

$$\frac{1}{(q,q^3,q^4,q^6,q^7,q^{10},q^{11};q^{12})_\infty} = \sum_{n_1,n_2,n_3 \geq 0} \frac{q^{(9n_3^2+5n_3)/2+2n_2^2+n_2+n_1^2+6n_3n_2+3n_3n_1+2n_2n_1}(-q;q^2)_{n_2}}{(q;q)_{n_1}(q^2;q^2)_{n_2}(q^3;q^3)_{n_3}}$$

$$= \sum_{n_1,m_2,n_3,m_4\geq 0} \frac{q^{8m_4^2+2m_4+(9n_3^2+5n_3)/2+(5m_2^2+m_2)/2+n_1^2}}{(q;q)_{n_1}(q;q)_{m_2}(q^3;q^3)_{n_3}(q^4;q^4)_{m_4}} \times q^{12m_4n_3+8m_4m_2+4m_4n_1+6n_3m_2+3n_3n_1+2m_2n_1}, \tag{6.6}$$

$$\frac{1}{(q^2,q^3,q^5,q^6,q^7,q^8,q^{11};q^{12})_\infty} = \sum_{n_1,n_2,n_3\geq 0} \frac{q^{(9n_3^2+7n_3)/2+2n_2^2+3n_2+n_1^2+n_1+6n_3n_2+3n_3n_1+2n_2n_1}(-q;q^2)_{n_2}}{(q;q)_{n_1}(q^2;q^2)_{n_2}(q^3;q^3)_{n_3}}$$
$$= \sum_{n_1,m_2,n_3,m_4\geq 0} \frac{q^{8m_4^2+6m_4+(9n_3^2+7n_3)/2+(5m_2^2+5m_2)/2+n_1^2+n_1}}{(q;q)_{n_1}(q;q)_{m_2}(q^3;q^3)_{n_3}(q^4;q^4)_{m_4}} \times q^{12m_4n_3+8m_4m_2+4m_4n_1+6n_3m_2+3n_3n_1+2m_2n_1}. \tag{6.7}$$

The relation (6.6) is a combination of (4.1), (5.2), [6, I_5], and [4, Sect. 4.9] and (6.7) of (4.9), (5.4), [6, I_6], and [4, Sect. 4.10].

7. Comments and Further Work

The series constructed in this paper are different from the series constructed in [7]. The approach is different, as well.

The usage of Gordon marking in the proof of Theorem 3.1, or other theorems in Sect. 3 does not make them immensely easier. One can simply declare, say, in Theorem 3.1, $[3k, 3k]$ or $[3k+1, 3k+2]$ admissible pairs, other parts singletons, and imitate the proofs in [10].

However, Gordon marking is vital in the proof of Theorem 4.1, or other theorems in Sects. 4 and 5; and it is prudent to have all Kanade–Russell conjectures together. Without Gordon marking, the proof of Theorem 4.1 becomes more tedious than it already is.

Normally, an r-cluster cannot go through an s-cluster if $s \geq r$ [8]. The *prestidigitation* is an exception without which the proofs are longer and less elegant, if not impossible (please see the Appendix).

Unfortunately, in Sects. 4 and 5, one cannot make the sum condition on the 3-clusters $\equiv 0 \pmod 3$ instead of $\equiv 1 \pmod 3$ or $\equiv 2 \pmod 3$. It is not possible to define forward or backward moves compatible with both Gordon marking and the given difference conditions.

For instance, let $kr^c_{3-3}(n, m)$ be the number of partitions of n into m parts with difference at least three at distance three such that if the difference at distance two is at most one, then the sum of those parts, together with the intermediate part, is divisible by three. The 3-clusters in a partition λ enumerated by $kr^c_{3-3}(n, m)$ must be of the form

$$\left\{ \begin{array}{c} k \\ k \\ (\text{parts } \leq k-3)\ k\ (\text{parts } \geq k+3) \end{array} \right\}.$$

One simply cannot make a forward move on the 3-cluster in the partition below.

$$\left\{\begin{array}{ll} \mathbf{1} & \\ \mathbf{1} & \\ \mathbf{1} & 4 \end{array}\right\} \xrightarrow{\text{move}} \left\{\begin{array}{c} \mathbf{2} \\ \mathbf{2} \\ \underbrace{\mathbf{2}\ 4}_{!} \end{array}\right\} \xrightarrow{\text{adjustment}} \left\{\begin{array}{c} \mathbf{3} \\ \mathbf{3} \\ \underbrace{1\ \mathbf{3}}_{!} \end{array}\right\}$$

The violation of the difference condition persists after the adjustment. To resolve it, we should either compromise the invariance of the number of r-clusters for fixed r, or define some other kind of moves. In short, the $\equiv 0 \pmod 3$ case cannot be treated with the machinery developed in this paper.

It should be possible to incorporate differences at distance four, so that 4-clusters enter the stage. However, such a venture is not advisable before we have partition identities, or conjectures, pertaining to difference at distance four as natural extensions of Kanade–Russell conjectures [6].

A windfall would be the construction of evidently positive multiple series using similar ideas for the new classes of partitions described in [7]. Not all series in [7] are evidently positive.

Of course, the biggest open problem is the proof of Kanade–Russell conjectures. Using the series constructed here or in [7], and Bailey pairs, will it be possible to give at least an analytic proof of the conjectures? A good starting point might be [11].

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Appendix

Now, let us try to visualize this process with a metaphor.

Imagine a person walking into a fancy cupcake store to taste the delicacies that he heard so much about from his colleagues at work. The cupcakes are neatly arranged in a large display case with one shelf over another. Each shelf has different kinds of cupcakes put into boxes of different sizes. There is certain logic to the way the boxes are displayed. The shelves have boxes with three cupcakes at the first two rows followed by a box with a single cupcake or two cupcakes at the back of the shelves.

The hypothetical cupcake enthusiast starts gazing colorful cupcakes of various types until his eyes are fixated towards a single box with a single cupcake in it. The box is located behind two bigger boxes with three cupcakes in each at a middle shelve as per the logic of display and there is hardly any space for one to grab the box with the single cupcake from the back of the shelf. The cupcake enthusiast is certain of his choice and makes a move towards the box in the back to grab it. The shop owner at the register sees the customer's move and immediately interrupts him: 'I am afraid you can't move the box at the back of the self without my help sir! It's impossible for you to squeeze your hand through the narrow space between the shelves without ruining the cupcakes.

The cupcake enthusiast stops for a brief moment, listens to the shop owner's warning and then he confidently keeps moving towards the box with the single cupcake behind the two larger boxes with three cupcakes in each. He thrusts his hand towards the narrow middle shelf and magic happens in the blink of an eye. The customer is able bring both the single-size box and the single cupcake of his choice to the front of the shelf albeit separately. The customer turned out to be a prestidigitator and performed some masterly sleight-of-hand. He retrieved the single cupcake of his choice by relocating it through the two other boxes with three cupcakes. The cupcake was swiftly put in an out of these larger boxes and united at the very front of the shelf with its original box in the end. The impossible became possible under this rare circumstance that allowed different cake to be put in and out of the boxes of three.

The shop owner was awed. He asked if the same trick could be done with another middle shelf that had a box with two cupcakes at the back as well. The cupcake enthusiast tried his trick there too and it worked again! The box of two and the cupcakes are separately delivered to the front while the to cupcakes got in an out of the boxes of three. Not only that, he was able to put back all the boxes that he retrieved from the back of the middle shelf to their original places reversing his trick. The shopkeeper, now amused, decided to offer his cupcakes free of charge to the customer.

Needless to say, each cupcake represents individual numbers and each box represents a free cluster of a particular size in this metaphor. I can only hope that the 'prestidigitator cupcake enthusiast's proof of his 'sleight-of-hand' would also prove to be as 'amusing' for his fellow mathematicians in real life as it does in the metaphor.

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Kağan Kurşungöz
Faculty of Engineering and Natural Sciences
Sabancı University
İstanbul
Turkey
e-mail: kursungoz@sabanciuniv.edu

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A Generalization of Schröter's Formula

To George Andrews, on his 80th Birthday

James Mc Laughlin

Abstract. We prove a generalization of Schröter's formula to a product of an arbitrary number of Jacobi triple products. It is then shown that many of the well-known identities involving Jacobi triple products (for example the Quintuple Product Identity, the Septuple Product Identity, and Winquist's Identity) all then follow as special cases of this general identity. Various other general identities, for example certain expansions of $(q;q)_\infty$ and $(q;q)_\infty^k$, $k \geq 3$, as combinations of Jacobi triple products, are also proved.

Mathematics Subject Classification. Primary 33D15; Secondary 11F27, 11B65.

Keywords. Schröter's formula, Jacobi triple product identity, Quintuple product identity, Septuple product identity, Winquist's identity.

1. Foreword

I was happy to receive the email sent by the organizers of the Combinatory Analysis 2018 conference, reminding attendees that there would be a "Special Issue of the Annals of Combinatorics to honor George Andrews at the occasion of passing the milestone age of 80", and soliciting papers with "new or unpublished work relating to the mathematical interests of George Andrews".

I had previously done some work on extending Schröter's identity for a product of two Jacobi triple products to a product of arbitrarily many such products. When this email sent to conference attendees arrived from the organizers, it spurred me to complete the proof of the identity that I had found. The topic, Jacobi triple products, is certainly one that is frequently found in

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the papers of Professor Andrews, so I was happy to submit this paper to the conference proceedings.

2. Introduction

The Jacobi triple product identity, first proved by Jacobi [8], is one of the fundamental identities in q-series. It may be written (see, for example, [5, Equation (II.28), p. 357]) as

$$\sum_{n=-\infty}^{\infty} q^{n^2} z^n = \left(-qz, \frac{-q}{z}, q^2; q^2\right)_\infty. \tag{2.1}$$

For space saving reasons, we will occasionally use the notation

$$\langle a; q^{2j} \rangle_\infty$$

to denote the triple product $(a, q^{2j}/a, q^{2j}; q^{2j})_\infty$.

This paper is concerned with identities in which products of Jacobi triple products are expanded into sums and products of other triple products. A well-known example of such an identity is the quintuple product identity (see Cooper's excellent paper [3] for the history of this identity and a survey of its various proofs). This identity may be written as

$$\begin{aligned}&(z, q^2/z, q^2; q^2)_\infty (q^2 z^2, q^2/z^2; q^4)_\infty \\ &\quad = (-q^2 z^3, -q^4/z^3, q^6; q^6)_\infty - z(-q^4 z^3, -q^2/z^3, q^6; q^6)_\infty.\end{aligned} \tag{2.2}$$

Remark 2.1. Strictly speaking, the factor $(q^2 z^2, q^2/z^2; q^4)_\infty$ is not a triple product, but becomes so if we multiply on the left side by $(q^4; q^4)_\infty$, while the right side remains a sum/product combination of triple products if we multiply on the right side by the equivalent $(q^4, q^8, q^{12}; q^{12})_\infty$. A similar situation will hold for other identities in the paper.

A second example is given by the Septuple Product Identity, first found by Hirschhorn [6] [in fact, Hirschhorn found a two-parameter extension from which the Septuple Product Identity follows upon setting $a = -z/q$ and $b = -z^2/q$ in Eq. (2.1) on page 32], and re-discovered by Farkas and Kra [4]:

$$\begin{aligned}&\left(z, \frac{q^2}{z}, q^2; q^2\right)_\infty \left(-z, \frac{-q^2}{z}, q^2; q^2\right)_\infty \left(z, \frac{q^4}{z}, q^4; q^4\right)_\infty \\ &\quad = (q^2, q^2, q^4; q^4)_\infty (q^8, q^{12}, q^{20}; q^{20})_\infty \\ &\qquad \times \left\{ \left(q^4 z^5, \frac{q^{16}}{z^5}, q^{20}; q^{20}\right)_\infty + z^3 \left(q^{16} z^5, \frac{q^4}{z^5}, q^{20}; q^{20}\right)_\infty \right\} \\ &\qquad - (q^2, q^2, q^4; q^4)_\infty (q^4, q^{16}, q^{20}; q^{20})_\infty \\ &\qquad \times \left\{ z \left(q^8 z^5, \frac{q^{12}}{z^5}, q^{20}; q^{20}\right)_\infty + z^2 \left(q^{12} z^5, \frac{q^8}{z^5}, q^{20}; q^{20}\right)_\infty \right\}.\end{aligned}$$

A third example is contained in Winquist's Identity (Winquist [9]):

$$\left(a, \frac{q^2}{a}, b, \frac{q^2}{b}, ab, \frac{q^2}{ab}, \frac{a}{b}, \frac{q^2 b}{a}, q^2, q^2, q^2, q^2; q^2\right)_\infty = (q^2, q^2; q^2)_\infty$$
$$\times \left[\left(a^3, \frac{q^6}{a^3}, q^6; q^6\right)_\infty \left\{ \left(b^3 q^2, \frac{q^4}{b^3}, q^6; q^6\right)_\infty - b\left(b^3 q^4, \frac{q^2}{b^3}, q^6; q^6\right)_\infty \right\} \right.$$
$$\left. - \frac{a}{b}\left(b^3, \frac{q^6}{b^3}, q^6; q^6\right)_\infty \left\{ \left(a^3 q^2, \frac{q^4}{a^3}, q^6; q^6\right)_\infty - a\left(a^3 q^4, \frac{q^2}{a^3}, q^6; q^6\right)_\infty \right\} \right].$$

In the present paper, we prove an expansion for a product of k ($k \geq 3$) Jacobi triple products in terms of sums of products of other Jacobi triple products (Theorem 2.2 below), and then show that all of the identities above, and also various other identities, follow as special cases. The main theorem of the paper is the following.

Theorem 2.2. *Let $k \geq 1$ be a positive integer and let $n_1, n_2, \ldots, n_k$, N be positive integers, such that $N = \operatorname{lcm}(n_1, n_2, \ldots, n_k)$, or a multiple thereof. For ease of notation, for $1 \leq i \leq k$, set $u_i := N/n_i$, $v_i := u_1 + u_2 + \cdots + u_i$, and $w_i := v_i + 1$. Let z, a, a_1, a_2, …, a_k be non-zero complex numbers and suppose $|q| < 1$. Then*

$$\left\langle -q^N az; q^{2N}\right\rangle_\infty \prod_{i=1}^{k} \left\langle -q^{n_i} a_i z; q^{2n_i}\right\rangle_\infty$$
$$= \sum_{j_1=0}^{v_1} \sum_{j_2=0}^{v_2} \cdots \sum_{j_k=0}^{v_k} z^{j_k} q^{n_1 j_1^2 + n_2(j_2-j_1)^2 + n_3(j_3-j_2)^2 + \cdots + n_k(j_k - j_{k-1})^2}$$
$$\times\, a_1^{j_1} a_2^{j_2-j_1} a_3^{j_3-j_2} \cdots a_k^{j_k-j_{k-1}} \left\langle -q^{n_1+N+2n_1 j_1} \frac{a_1}{a}; q^{2(n_1+N)}\right\rangle_\infty$$
$$\times \prod_{i=2}^{k} \left\langle -\, q^{n_i(w_i w_{i-1} + 2w_i j_{i-1} - 2w_{i-1} j_i)} \frac{a\, a_1^{u_1} a_2^{u_2} \cdots a_{i-1}^{u_{i-1}}}{a_i^{w_{i-1}}}; q^{2n_i w_i w_{i-1}} \right\rangle_\infty$$
$$\times \left\langle -q^{N w_k + 2N j_k} a_1^{u_1} a_2^{u_2} \cdots a_k^{u_k} a z^{w_k}; q^{2N w_k}\right\rangle_\infty. \tag{2.3}$$

Observe that the expansion (2.3) provides a w_k-dissection of the left side into powers of z that lie in arithmetic progressions modulo w_k.

The quintuple product identity, the septuple product identity and Winquist's identity, and others all follow as special cases of the above identity. Other applications include expansions of Ramanujan theta functions, or powers of these, as sums and products of Jacobi triple products. As an example of one of these latter identities, we have that, for an arbitrary integer $k \geq 3$:

$$(q;q)_\infty^k$$
$$= \sum_{j_1=0}^{1} \cdots \sum_{j_{k-1}=0}^{k-1} (-1)^{j_{k-1}} q^{3(j_1^2 + j_2^2 + \cdots + j_{k-2}^2) + j_{k-1}(3j_{k-1}+1)/2 - 3(j_1 j_2 + \cdots + j_{k-2} j_{k-1})}$$

$$\times \left\langle -q^{3+3j_1}; q^6 \right\rangle_\infty \left\langle (-1)^{k+1} q^{2k+3j_{k-1}}; q^{3k} \right\rangle_\infty$$
$$\times \prod_{i=2}^{k-1} \left\langle -q^{3i(i+1)/2+3(i+1)j_{i-1}-3ij_i}; q^{3i(i+1)} \right\rangle_\infty .$$

Remark 2.3. Cao [2] proves a quite general theorem (Theorem 1.4) which also exhibits a product of arbitrarily many Jacobi triple products as a sum containing other Jacobi triple products. Cao's theorem is more general in the sense that it allows for a greater variety of expansions, but, in full generality, appears more restrictive in the sense that for such an identity, it must also be shown that the entries of a certain associated matrix satisfy certain conditions. We had initially thought that the result in the present paper was independent of Cao's result. However, it was pointed out by one of the referees that the key induction step in the proof of our Theorem 2.2, namely, Corollary 3.10, is actually a special case of Corollary 2.2 in Cao's paper [2], and thus, that our result in Theorem 2.2 could have been derived from the results in Cao's paper [2], by following the appropriate path and making the appropriate specializations.

3. Extensions of Schröter's Identity

Before coming to the main theorem and its consequences, we first consider Schröter's identity, and also state some elementary extensions. The methods of proof will also preview the methods used to prove the main theorem in the next section.

Schröter's identity (see [1, p. 111])

$$\begin{aligned} &\left\langle -q^{n_1} a; q^{2n_1} \right\rangle_\infty \left\langle -q^{n_2} b; q^{2n_2} \right\rangle_\infty \\ &= \sum_{j=0}^{n_1+n_2-1} q^{n_1 j^2} a^j \left\langle \frac{-q^{n_1+n_2+2n_1 j} a}{b}; q^{2(n_1+n_2)} \right\rangle_\infty \\ &\quad \times \left\langle -q^{(n_1+n_2+2j)n_1 n_2} a^{n_2} b^{n_1}; q^{2(n_1+n_2)n_1 n_2} \right\rangle_\infty , \end{aligned} \tag{3.1}$$

first appeared in Schröter's 1854 dissertation.

In Lemma 3.3, we introduce a variable z as a "book-keeping" device by replacing a with az and b with bz, and also give a proof of a slight extension of Schröter's identity by introducing an integer variable m into the summation range (Schröter's original identity is the case $m = 1$ of the identity in Lemma 3.3), and then use this result in conjunction with Lemma 3.1 to derive a more general extension.

We begin by recalling the following well-known elementary identity.

Lemma 3.1. *Let p be a positive integer and let q and z be complex numbers with $z \neq 0$ and $|q| < 1$. Then*

$$\left\langle -qz; q^2 \right\rangle_\infty = \sum_{j=0}^{p-1} q^{j^2} z^j \left\langle -q^{p^2+2pj} z^p; q^{2p^2} \right\rangle_\infty . \tag{3.2}$$

Proof. The proof follows directly from (2.1), upon breaking the sum on the left side into p sums, in each of which the exponents n all lie in the same arithmetic progression modulo p, and then applying (2.1) to each sum. $\square$

Before coming to the extension of Schröter's identity, we also need a preliminary lemma.

Lemma 3.2. *If c is a non-zero complex number, n is any positive integer, j is any integer, and $|q| < 1$, then*

$$(cq^{n+2jn}, q^{n-2jn}/c; q^{2n})_\infty = (cq^n, q^n/c; q^{2n})_\infty \left(\frac{-1}{c}\right)^j \frac{1}{q^{nj^2}}. \tag{3.3}$$

Proof. The statement is clearly true if $j = 0$. If $j > 0$, then

$$(cq^{n+2jn}, q^{n-2jn}/c; q^{2n})_\infty = (cq^n, q^n/c; q^{2n})_\infty \frac{(q^{n-2jn}/c; q^{2n})_j}{(cq^n; q^{2n})_j},$$

and the result follows for $j > 0$ upon applying the identity (see [5, Identity (I.8), p. 351]):

$$(aq^{-j}; q)_j = (q/a; q)_j \left(\frac{-a}{q}\right)^j q^{-j(j-1)/2}$$

to $(q^{n-2jn}/c; q^{2n})_j$ (with $a = q^n/c$ and q replaced with q^{2n}). The result for $j < 0$ follows from the $j > 0$ case, after replacing j with $-j$ and c with $1/c$. $\square$

Lemma 3.3. (An extension of Schröter's Identity) *Let a, b, and z be non-zero complex numbers, let q be a complex number with $|q| < 1$, and let $m \geq 1$ be an integer. Then*

$$\begin{aligned}
&\left\langle -q^{n_1}az; q^{2n_1}\right\rangle_\infty \left\langle -q^{n_2}bz; q^{2n_2}\right\rangle_\infty \\
&= \sum_{j=0}^{m(n_1+n_2)-1} q^{n_1 j^2} (az)^j \left\langle \frac{-q^{n_1+n_2+2n_1 j}a}{b}; q^{2(n_1+n_2)}\right\rangle_\infty \\
&\quad \times \left\langle -q^{(m(n_1+n_2)+2j)n_1 n_2 m} a^{mn_2} b^{mn_1} z^{m(n_1+n_2)}; q^{2(n_1+n_2)n_1 n_2 m^2}\right\rangle_\infty .
\end{aligned} \tag{3.4}$$

Proof. After two applications of the Jacobi triple product identity

$$\begin{aligned}
F_1(z) &:= \left\langle -q^{n_1}az; q^{2n_1}\right\rangle_\infty \left\langle -q^{n_2}bz; q^{2n_2}\right\rangle_\infty \\
&= \sum_{m_1, m_2 \in \mathbb{Z}} q^{n_1 m_1^2} (az)^{m_1} \frac{q^{n_2 m_2^2}}{(bz)^{m_2}} \\
&= \sum_{t, m_2 \in \mathbb{Z}} (az)^t q^{n_1(t^2+2tm_2+m_2^2)} \frac{q^{n_2 m_2^2} a^{m_2}}{b^{m_2}} \qquad (t = m_1 - m_2) \\
&= \sum_{t, m_2 \in \mathbb{Z}} (az)^t q^{n_1 t^2} q^{(n_1+n_2)m_2^2} \left(\frac{aq^{2n_1 t}}{b}\right)^{m_2}
\end{aligned}$$

$$= \sum_{t\in\mathbb{Z}} (az)^t q^{n_1 t^2} \left\langle \frac{-q^{n_1+n_2+2n_1 t}a}{b}; q^{2(n_1+n_2)} \right\rangle_\infty .$$

Now, set $t = m(n_1+n_2)r + j$, $r \in \mathbb{Z}$, $0 \le j \le m(n_1+n_2)-1$, and apply Lemma 3.2 (with n_1+n_2, $n_1 mr$ and $-ab^{-1}q^{2n_1 j}$ instead of n, j and c, respectively) to the triple products to get

$$\begin{aligned} F_1(z) = &\sum_{j=0}^{m(n_1+n_2)-1} q^{n_1 j^2}(az)^j \left\langle \frac{-q^{n_1+n_2+2n_1 j}a}{b}; q^{2(n_1+n_2)} \right\rangle_\infty \\ &\times \sum_{r\in\mathbb{Z}} q^{m^2(n_1+n_2)n_1 n_2 r^2} (a^{n_2} b^{n_1} z^{n_1+n_2} q^{2n_1 n_2 j})^{mr}. \end{aligned}$$

The result follows after one further application of the Jacobi triple product identity. □

Theorem 3.4. (A second extension of Schröter's Identity) *Let a, b and z be non-zero complex numbers, let q be a complex number with $|q| < 1$, and let $m \ge 1$, $n_1 \ge 1$, $n_2 \ge 1$ and $p \ge 1$ be integers. Then*

$$\begin{aligned} &\left\langle -q^{n_1}az; q^{2n_1} \right\rangle_\infty \left\langle -q^{n_2}bz; q^{2n_2} \right\rangle_\infty \\ &= \sum_{j_1=0}^{m(n_1+p^2 n_2)-1} \sum_{j_2=0}^{p-1} q^{n_1 j_1^2 + j_2^2 n_2} a^{j_1} b^{j_2} z^{j_1+j_2} \\ &\quad \times \left\langle \frac{-q^{n_1+p^2 n_2 + 2n_1 j_1 - 2p j_2 n_2}a}{b^p z^{p-1}}; q^{2(n_1+p^2 n_2)} \right\rangle_\infty \\ &\quad \times \Big\langle -a^{mp^2 n_2} b^{mn_1 p} z^{mp(n_1+pn_2)} q^{mn_1 pn_2 \left(2j_1 p + 2j_2 + mp\left(n_1+p^2 n_2\right)\right)}; \\ &\qquad q^{2m^2 n_1 p^2 n_2 \left(n_1+p^2 n_2\right)} \Big\rangle_\infty . \end{aligned} \tag{3.5}$$

Proof. Apply Lemma 3.1 to the product $\left(-q^{n_2}bz, -q^{n_2}/bz, q^{2n_2}; q^{2n_2}\right)_\infty$, and then apply Lemma 3.3 to each pair of triple products in the resulting expression. □

Remark 3.5. In Lemma 3.3 and Theorem 3.4, the presence of the z variable is not actually necessary, as it could be absorbed into the a and b variables, without affecting the generality of the result. However, its usefulness derives from the fact that the right side of (3.4) provides a $m(n_1+n_2)$-dissection of the left side into $m(n_1+n_2)$ functions in each of which the powers of z all lie in the same arithmetic progression modulo $m(n_1+n_2)$. For this reason, we retain the variable z in Theorem 3.4, and elsewhere throughout the paper (see, for example, the proof of Corollary 4.9, where this dissection proves useful).

We note that in the case where $n_1 | n_2$, there exists a second family of expansions that do not come directly from Theorem 3.4.

Corollary 3.6. *Let a, b z, q, m, p, n_1, and n_2 be as in Theorem 3.4, with the additional requirement that $n_1|n_2$. Then*

$$\left\langle -q^{n_1}az;q^{2n_1}\right\rangle_\infty \left\langle -q^{n_2}bz;q^{2n_2}\right\rangle_\infty \tag{3.6}$$

$$= \sum_{j_1=0}^{m(1+p^2n_2/n_1)-1} \sum_{j_2=0}^{p-1} q^{n_1j_1^2+n_2j_2^2}a^{j_1}b^{j_2}z^{j_1+j_2}$$
$$\times \left\langle \frac{-q^{n_1+p^2n_2+2n_1j_1-2pj_2n_2}a}{b^pz^{p-1}};q^{2(n_1+p^2n_2)}\right\rangle_\infty$$
$$\times \Bigg\langle -a^{mp^2n_2/n_1}b^{mp}z^{mp(1+pn_2/n_1)}q^{mpn_2\left(2j_1p+2j_2+mp\left(1+p^2n_2/n_1\right)\right)};$$
$$q^{2m^2p^2n_2\left(1+p^2n_2/n_1\right)}\Bigg\rangle_\infty. \tag{3.7}$$

Proof. Write

$$\left\langle -q^{n_2}bz;q^{2n_2}\right\rangle_\infty = \left\langle -(q^{n_1})^{n_2/n_1}bz;(q^{n_1})^{2n_2/n_1}\right\rangle_\infty,$$

and then apply Theorem 3.4, with n_1 replaced with 1, n_2 replaced with n_2/n_1 and q replaced with q^{n_1}. □

Theorem 3.4 is more general in the sense that it holds also when $n_1 \nmid n_2$. However, when $n_1|n_2$, Corollary 3.6 is actually the stronger result as it implies Theorem 3.4 in this case (replace m with mn_1 in Corollary 3.6).

Remark 3.7. (1) Cao ([2, Theorem 2.3, Equation (2.50)]), using a different approach, has given a generalized Schröter's formula for a product of two Jacobi triple products, a formula which implies our identity (3.4), but not (3.5) (or at least not without additional transformations).

(2) Even though no applications of (3.4), (3.5) and (3.6) are given in the present paper with $m > 1$ or $p > 1$, they are included for the sake of completeness.

We note that the quintuple product identity also follows from Schröter's Theorem. The quintuple product identity is usually written in the form:

$$(-z,-q/z,q;q)_\infty(qz^2,q/z^2;q^2)_\infty = \sum_{n=-\infty}^{\infty}(-1)^nq^{n(3n-1)/2}z^{3n}(1+zq^n).$$

Upon replacing q with q^2, z with $-z$, using the Jacobi triple product identity to sum the resulting series on the right side, this identity may be restated in the form given in (2.2), and we show that it follows from Corollary 3.6.

Corollary 3.8. *Let z be a non-zero complex number, and suppose $|q| < 1$. Then, (2.2) holds.*

Proof. In (3.4), set $m = 1$, $n_2 = 2$, $a = -1/q^3$, $b = -1/z^3$, and (2.2) follows after some simple manipulations and using the fact that $(q^4,q^8,q^{12};q^{12})_\infty = (q^4;q^4)_\infty$. □

Schröter's theorem and its various extensions for a product of two Jacobi triple products naturally lead to the following question. Given a product of k ($k \geq 3$, k an integer) Jacobi triple products:

$$F(z) := \prod_{i=1}^{k} \left\langle -q^{n_i} a_i z; q^{2n_i} \right\rangle_\infty ,$$

and we M-dissect $F(z)$ by writing

$$F(z) = \sum_{j=0}^{M-1} z^j F_j(z^M),$$

for some integer M; can an explicit representation of each $F_j(z^M)$ be given? We gave an affirmative answer to this question in Theorem 2.2, and prove this theorem in the next section, in the case where each $n_i|N$ (with $M = N/n_1 + \cdots + N/n_k + 1$).

To this end, an identity which follows from a special case of the next identity, due to Cao [2], is needed. We state this result of Cao in terms of q-products, rather than using Ramanujan's theta function $f(a, b)$, as Cao did.

Proposition 3.9. (Cao, Corollary 2.2, [2]) *If* $|ab| < 1$ *and* $(cd) = (ab)^{k_1 k_2}$, *where both* k_1 *and* k_2 *are positive integers, then*

$$\begin{aligned}(-a, -b, ab; ab)_\infty (-c, -d, cd; cd)_\infty &= \sum_{r=0}^{k_1+k_2-1} (ab)^{r^2/2} \left(\frac{a}{b}\right)^{r/2} \\ &\times \left((ab)^{k_1^2/2+k_1 r} \left(\frac{a}{b}\right)^{k_1/2} c, (ab)^{k_1^2/2-k_1 r} \left(\frac{b}{a}\right)^{k_1/2} d, (ab)^{k_1^2} cd; (ab)^{k_1^2} cd\right)_\infty \\ &\times \left((ab)^{k_2^2/2+k_2 r} \left(\frac{a}{b}\right)^{k_2/2} d, (ab)^{k_2^2/2-k_2 r} \left(\frac{b}{a}\right)^{k_2/2} c, (ab)^{k_2^2} cd; (ab)^{k_2^2} cd\right)_\infty .\end{aligned} \tag{3.8}$$

The special case that is needed may be stated as follows.

Corollary 3.10. *Let* j' *be an integer, and let* n, N, *and* w' *be positive integers such that* $n|N$. *Let* a, e, z *and* q *be non-zero complex numbers with* $|q| < 1$. *Define*

$$u := \frac{N}{n}, \quad w := u + w', \quad v := w - 1.$$

Then

$$\begin{aligned}&\left\langle -ezq^n; q^{2n} \right\rangle_\infty \left\langle -az^{w'} q^{Nw'+2j'N}; q^{2Nw'} \right\rangle_\infty \\ &= \sum_{j=0}^{v} e^{j-j'} q^{n(j-j')^2} z^{j-j'} \times \left\langle \frac{-a}{e^{w'}} q^{nw'w+2n(j'w-jw')}; q^{2nw'w} \right\rangle_\infty \\ &\quad \times \left\langle -ae^u z^w q^{2jN+Nw}; q^{2Nw} \right\rangle_\infty .\end{aligned} \tag{3.9}$$

Proof. In Proposition 3.9, set $a = ezq^n$, $b = q^n/(ez)$, $c = q^{N(w'-2j')}/(az^{w'})$, $d = az^{w'}q^{N(w'+2j')}$, $k_1 = u$, and $k_2 = w'$. Then, it can be seen that $(cd) = (ab)^{k_1k_2}$, and that the left side of (3.8) becomes the left side of (3.9). The right side of (3.8) becomes

$$\sum_{r=0}^{v} q^{nr^2}(ez)^r\langle -ae^u z^w q^{N(w+2r+2j')}; q^{2Nw}\rangle_\infty \\ \times \left\langle -\frac{e^{w'}}{a} q^{nww'+2n(w'r-j'w)+2nj'w'}; q^{2nww'}\right\rangle_\infty .$$

The result follows upon, in turn, replacing r with $r - j'$ (so that the interval of summation is also changed to one of another w consecutive integers), using the division algorithm (with r and w) to write each of the resulting new r values in the form $r = mw + j$ for some integers j and m with $0 \le j \le w-1$, and finally applying (3.3) to each of the terms in the resulting sum. $\square$

4. Main Result and Its Implications

We now come to the proof of the main result of the paper. The proof is essentially a simple induction argument using identities (3.6) and (3.9).

Remark 4.1. It should be pointed out that attempting to iterate Schröter's original identity (the case $m = 1$ of the identity in Lemma 3.3) does not appear to easily lead to any result similar to that in Theorem 2.2.

Proof of Theorem 2.2. The proof is by induction on k. If $k = 1$, then (2.3) becomes

$$\langle -q^N az; q^{2N}\rangle_\infty \langle -q^{n_1}a_1 z; q^{2n_1}\rangle_\infty \\ = \sum_{j_1=0}^{v_1} z^{j_1} q^{n_1 j_1^2} a_1^{j_1} \left\langle -q^{n_1+N+2n_1j_1}\frac{a_1}{a}; q^{2(n_1+N)}\right\rangle_\infty \\ \times \langle -q^{Nw_1+2Nj_1}a_1^{u_1}az^{w_1}; q^{2Nw_1}\rangle_\infty .$$

However, this is simply identity (3.6), with $n_2 = N$, $b = a$, $m = p = 1$, upon recalling that $v_1 = u_1 = N/n_1$ and $w_1 = v_1 + 1$.

Suppose (2.3) holds for $k = 1, 2, \ldots, r$. Now, consider the left side of (2.3) with $k = r+1$. Employing the $k = r$ case on the first r Jacobi triple products, we have

$$\langle -q^N az; q^{2N}\rangle_\infty \prod_{i=1}^{r+1} \langle -q^{n_i}a_i z; q^{2n_i}\rangle_\infty \\ = \sum_{j_1=0}^{v_1}\sum_{j_2=0}^{v_2}\cdots\sum_{j_r=0}^{v_r} z^{j_r} q^{n_1j_1^2+n_2(j_2-j_1)^2+n_3(j_3-j_2)^2+\cdots+n_r(j_r-j_{r-1})^2} \\ \times a_1^{j_1}a_2^{j_2-j_1}a_3^{j_3-j_2}\cdots a_r^{j_r-j_{r-1}} \left\langle -q^{n_1+N+2n_1j_1}\frac{a_1}{a}; q^{2(n_1+N)}\right\rangle_\infty$$

$$\times \prod_{i=2}^{r} \left\langle -q^{n_i(w_i w_{i-1}+2w_i j_{i-1}-2w_{i-1}j_i)} \frac{a\, a_1^{u_1} a_2^{u_2} \cdots a_{i-1}^{u_{i-1}}}{a_i^{w_{i-1}}}; q^{2n_i w_i w_{i-1}} \right\rangle_\infty$$
$$\times \left\langle -q^{Nw_r+2Nj_r} a_1^{u_1} a_2^{u_2} \cdots a_r^{u_r} a z^{w_r}; q^{2Nw_r} \right\rangle_\infty \left\langle -q^{n_{r+1}} a_{r+1} z; q^{2n_{r+1}} \right\rangle_\infty . \tag{4.1}$$

Identity (3.9) is now applied to the final two triple products on the right side of (4.1) above. In this identity, n is replaced with n_{r+1}, j' with j_r, j with j_{r+1}, e with a_{r+1}, a with $aa_1^{u_1}a_2^{u_2}\cdots a_r^{u_r}$ and w' with w_r. Hence, in the notation of Theorem 2.2, u takes the value $N/n_{r+1} = u_{r+1}$, w takes the value $u + w' = u_{r+1} + w_r = w_{r+1}$, and v takes the value $w - 1 = w_{r+1} - 1 = v_{r+1}$. After these substitutions are made, then (3.9) gives that

$$\left\langle -q^{n_{r+1}} a_{r+1} z; q^{2n_{r+1}} \right\rangle_\infty \left\langle -q^{Nw_r+2Nj_r} a_1^{u_1} a_2^{u_2} \cdots a_r^{u_r} a z^{w_r}; q^{2Nw_r} \right\rangle_\infty$$
$$= \sum_{j_{r+1}=0}^{v_{r+1}} a_{r+1}^{j_{r+1}-j_r} q^{n_{r+1}(j_{r+1}-j_r)^2} z^{j_{r+1}-j_r}$$
$$\times \left\langle -q^{n_{r+1}(w_{r+1}w_r+2w_{r+1}j_r-2w_r j_{r+1})} \frac{a\, a_1^{u_1} a_2^{u_2} \cdots a_r^{u_r}}{a_{r+1}^{w_r}}; q^{2n_{r+1}w_{r+1}w_r} \right\rangle_\infty$$
$$\times \left\langle -q^{Nw_{r+1}+2Nj_{r+1}} a_1^{u_1} a_2^{u_2} \cdots a_r^{u_r} a_{r+1}^{u_{r+1}} a z^{w_{r+1}}; q^{2Nw_{r+1}} \right\rangle_\infty . \tag{4.2}$$

The substitution of the right side of (4.2) into (4.1) to replace the left side of (4.2) gives that (2.3) holds for $k = r + 1$, and thus by induction that it is true for all integers $k \geq 1$. This concludes the proof of Theorem 2.2. □

Corollary 4.2. *Let z, a_1, a_2, …, and a_k be non-zero complex numbers and suppose $|q| < 1$. Then*

$$\prod_{i=1}^{k} \left\langle -qa_i z; q^2 \right\rangle_\infty$$
$$= \sum_{j_1=0}^{1} \sum_{j_2=0}^{2} \cdots \sum_{j_{k-1}=0}^{k-1} z^{j_{k-1}} q^{j_1^2+(j_2-j_1)^2+\cdots+(j_{k-1}-j_{k-2})^2}$$
$$\times a_1^{j_1} a_2^{j_2-j_1} \cdots a_{k-1}^{j_{k-1}-j_{k-2}} \left\langle -q^{2+2j_1} \frac{a_1}{a_k}; q^4 \right\rangle_\infty$$
$$\times \left\langle -q^{k+2j_{k-1}} a_1 a_2 \cdots a_k z^k; q^{2k} \right\rangle_\infty$$
$$\times \prod_{i=2}^{k-1} \left\langle -q^{i(i+1)+2(i+1)j_{i-1}-2ij_i} \frac{a_k a_1 a_2 \cdots a_{i-1}}{a_i^i}; q^{2i(i+1)} \right\rangle_\infty . \tag{4.3}$$

Proof. Replace k with $k - 1$ in Theorem 2.2, and then set $a = a_k$, $n_1 = n_2 = \cdots = n_{k-1} = N = 1$, so that each $u_i = 1$, $v_i = i$ and $w_i = i + 1$. □

Remark 4.3. (1) Note that the sum (4.3) contains $k!$ terms, each with k Jacobi triple products.

(2) Since the left side of (4.3) is invariant under any permutation of the numbers $a_1, \ldots, a_k$, so is the right side.

(3) The appearance of z in (4.3) is essentially a "book-keeping" device, as without loss of generality, each a_i could be replaced with a_i/z (or, equivalently, set $z = 1$).
(4) Upon setting each $a_i = 1$, we get an expression for $(-qz, -q/z, q^2; q^2)_\infty^k$ $(k \geq 3)$:

$$\begin{aligned}
&\left\langle -qz; q^2 \right\rangle_\infty^k \\
&\quad = \sum_{j_1=0}^{1} \sum_{j_2=0}^{2} \cdots \sum_{j_{k-1}=0}^{k-1} z^{j_{k-1}} q^{j_1^2+(j_2-j_1)^2+\cdots+(j_{k-1}-j_{k-2})^2} \left\langle -q^{2+2j_1}; q^4 \right\rangle_\infty \\
&\qquad \times \left\langle -q^{k+2j_{k-1}} z^k; q^{2k} \right\rangle_\infty \prod_{i=2}^{k-1} \left\langle -q^{i(i+1)+2(i+1)j_{i-1}-2ij_i}; q^{2i(i+1)} \right\rangle_\infty .
\end{aligned} \tag{4.4}$$

This identity provides expansions in terms of triple products for Ramanujan's theta functions, $f(-q) = \langle q; q^3 \rangle_\infty = (q; q)_\infty$, $\phi(q) = \langle -q; q^2 \rangle_\infty$ and $\psi(q) = \langle -q; q^4 \rangle_\infty$. We give one example:

Corollary 4.4. *If* $|q| < 1$, *then*

$$\begin{aligned}
&f(-q)^k = (q; q)_\infty^k \\
&\quad = \sum_{j_1=0}^{1} \cdots \sum_{j_{k-1}=0}^{k-1} (-1)^{j_{k-1}} q^{3(j_1^2+j_2^2+\cdots+j_{k-2}^2)+j_{k-1}(3j_{k-1}+1)/2-3(j_1j_2+j_2j_3+\cdots+j_{k-2}j_{k-1})} \\
&\quad \times \left\langle -q^{3+3j_1}; q^6 \right\rangle_\infty \left\langle (-1)^{k+1} q^{2k+3j_{k-1}}; q^{3k} \right\rangle_\infty \\
&\quad \times \prod_{i=2}^{k-1} \left\langle -q^{3i(i+1)/2+3(i+1)j_{i-1}-3ij_i}; q^{3i(i+1)} \right\rangle_\infty .
\end{aligned} \tag{4.5}$$

Proof. In (4.4), replace q with $q^{3/2}$, and let $z = -q^{1/2}$. $\square$

Remark 4.5. The squares in the exponent of q that precede the infinite products in (4.4) have been multiplied out and the terms rearranged, to make it more explicit that, after the replacement of q with $q^{3/2}$, the new exponent is, indeed, integral.

As a second illustration of (2.3), we exhibit the $k = 3$ case of the identity explicitly, and show that it implies the quintuple product identity. This identity was also stated [2, Equation (3.2)] by Cao.

Corollary 4.6. (Extended Quintuple Product Identity) *If* $a, b, c, z \neq 0$ *and* $|q| < 1$, *then*

$$\begin{aligned}
&\left\langle -qaz; q^2 \right\rangle_\infty \left\langle -qbz; q^2 \right\rangle_\infty \left\langle -qcz; q^2 \right\rangle_\infty = \left\langle -\frac{q^2 a}{c}; q^4 \right\rangle_\infty \\
&\left\{ \left\langle -\frac{q^6 ac}{b^2}; q^{12} \right\rangle_\infty \left\langle -q^3 abcz^3; q^6 \right\rangle_\infty + qbz \left\langle -\frac{q^2 ac}{b^2}; q^{12} \right\rangle_\infty \left\langle -q^5 abcz^3; q^6 \right\rangle_\infty \right.
\end{aligned}$$

$$+ q^4 b^2 z^2 \left\langle -\frac{ac}{q^2b^2}; q^{12} \right\rangle_\infty \langle -q^7 abcz^3; q^6 \rangle_\infty \Bigg\} + \frac{q^2 a}{b} \left\langle -\frac{q^4 a}{c}; q^4 \right\rangle_\infty$$
$$\Bigg\{ \left\langle -\frac{q^{12}ac}{b^2}; q^{12} \right\rangle_\infty \langle -q^3 abcz^3; q^6 \rangle_\infty + \frac{bz}{q} \left\langle -\frac{q^8 ac}{b^2}; q^{12} \right\rangle_\infty \langle -q^5 abcz^3; q^6 \rangle_\infty$$
$$+ b^2 z^2 \left\langle -\frac{q^4 ac}{b^2}; q^{12} \right\rangle_\infty \langle -q^7 abcz^3; q^6 \rangle_\infty \Bigg\}. \tag{4.6}$$

Proof. This follows after some slight rearrangement of terms in (4.3), after substituting $k = 3$ and letting $a_1 = a$, $a_2 = b$, and $a_3 = c$. □

Recall from (2.2) that the quintuple product identity may be written as:

$$(z, q^2/z, qz, q/z, -qz, -q/z, q^2, q^2, q^2; q^2)_\infty$$
$$= (q^2; q^2)_\infty^2 \{(-q^2 z^3, -q^4/z^3, q^6; q^6)_\infty - z(-q^4 z^3, -q^2/z^3, q^6; q^6)_\infty\}.$$

However, this follows from (4.6) upon setting $a = -1$, $b = -1/q$ and $c = 1$, after some elementary infinite product manipulations.

The septuple product identity also follows from Theorem 2.2, but the proof is less direct, as it also needs Schröter's identity. We will prove the septuple product identity in the following form.

Corollary 4.7. (Septuple Product Identity) *Let z and q be complex numbers, with $z \neq 0$ and $|q| < 1$. Then*

$$\langle z; q^2 \rangle_\infty \langle -z; q^2 \rangle_\infty \langle z; q^4 \rangle_\infty$$
$$= \langle q^2; q^4 \rangle_\infty \langle q^8; q^{20} \rangle_\infty \left\{ \langle q^4 z^5; q^{20} \rangle_\infty + z^3 \langle q^{16} z^5; q^{20} \rangle_\infty \right\}$$
$$- \langle q^2; q^4 \rangle_\infty \langle q^4; q^{20} \rangle_\infty \left\{ z \langle q^8 z^5; q^{20} \rangle_\infty + z^2 \langle q^{12} z^5; q^{20} \rangle_\infty \right\}. \tag{4.7}$$

Proof. In (2.3), set $k = 2$, $n_1 = n_2 = 1$, and $N = 2$ (so that $u_1 = u_2 = 2$, $u_3 = 1$, $v_1 = 2, v_2 = 4$, $w_1 = 3$, and $w_2 = 5$), $z = 1$, $a_1 = -z/q$, $a_2 = z/q$, and $a_3 = -z/q^2$. Then, the left side of (2.3) becomes the left side of (4.7), and the right side of (2.3) becomes

$$\sum_{j_2=0}^{4} q^{j_2^2 - j_2} z^{j_2} \langle q^{4+4j_2} z^5; q^{20} \rangle_\infty$$
$$\times \sum_{j_1=0}^{2} q^{2j_1^2 - 2j_1 j_2} (-1)^{j_1} \langle -q^{4+2j_1}; q^6 \rangle \langle_\infty q^{14-6j_2+10j_1}; q^{30} \rangle_\infty.$$

It is easy to show that the inner sum is zero in the case $j_2 = 4$ and that proving (4.7) then comes down to proving the pair of identities:

$$\langle q^2; q^4 \rangle_\infty \langle q^8; q^{20} \rangle_\infty = \langle -q^4; q^6 \rangle_\infty \langle q^{14}; q^{30} \rangle_\infty$$
$$- q^2 \langle -1; q^6 \rangle_\infty \langle q^{24}; q^{30} \rangle_\infty - q^2 \langle -q^2; q^6 \rangle_\infty \langle q^4; q^{30} \rangle_\infty, \tag{4.8}$$

$$\langle q^2; q^4 \rangle_\infty \langle q^4; q^{20} \rangle_\infty = \langle -1; q^6 \rangle_\infty \langle q^{18}; q^{30} \rangle_\infty$$

$$-\langle -q^2;q^6\rangle_\infty\langle q^8;q^{30}\rangle_\infty - q^2\langle -q^2;q^6\rangle_\infty\langle q^{28};q^{30}\rangle_\infty. \tag{4.9}$$

Let $g(m,n) := q^{2n^2+10m^2+2m}(-1)^{m+n}$. We use the Jacobi triple product identity to write the infinite product on the left side of identity (4.8) as an infinite series. We next use a method similar to that of Hirschhorn [7] to first sum diagonally, and then divide the diagonal sums into six congruence classes. This gives

$$\begin{aligned}\langle q^2;q^4\rangle_\infty\langle q^8;q^{20}\rangle_\infty &= \sum_{m=-\infty}^{\infty}\sum_{n=-\infty}^{\infty} g(m,n) && (4.10)\\ &= \sum_{k=-\infty}^{\infty}\sum_{m+n=k} g(m,n)\\ &= \sum_{j=0}^{5}\sum_{r=-\infty}^{\infty}\sum_{s=-\infty}^{\infty} g(s-r,5s+r+j)\\ &= \sum_{j=0}^{5}(-1)^{6s+j}q^{2j^2}\sum_{r=-\infty}^{\infty}\sum_{s=-\infty}^{\infty} q^{12r^2+(4j-2)r+60s^2+(20j+2)s}. && (4.11)\end{aligned}$$

We similarly expand the right side of identity (4.8) to get

$$\sum_{u,v=-\infty}^{\infty} q^{3u^2+u+15v^2+v}(-1)^v - q^2\sum_{u,v=-\infty}^{\infty} q^{3u^2+3u+15v^2+9v}(-1)^v - q^2\sum_{u,v=-\infty}^{\infty} q^{3u^2+u+15v^2+11v}(-1)^v.$$

We next expand each the three sums into four sums by setting $u = 2r$ and $2r+1$ and $v = 2s$ and $2s+1$. By comparing the resulting 12 sums with the expression (4.10) (after possibly replacing r with $r \pm 1$ and/or s with $s \pm 1$ in some cases), it can be seen that proving identity (4.8) now depends on proving that

$$\begin{aligned}&q^6\langle -q^{10};q^{24}\rangle_\infty\langle -q^{22};q^{120}\rangle_\infty - q^4\langle -q^2;q^{24}\rangle_\infty\langle -q^{38};q^{120}\rangle_\infty\\ &\quad + q^8\langle -q^6;q^{24}\rangle_\infty\langle -q^{18};q^{120}\rangle_\infty - q^2\langle -q^6;q^{24}\rangle_\infty\langle -q^{42};q^{120}\rangle_\infty\\ &\quad - q^{14}\langle -q^{10};q^{24}\rangle_\infty\langle -q^2;q^{120}\rangle_\infty + q^2\langle -q^2;q^{24}\rangle_\infty\langle -q^{58};q^{120}\rangle_\infty = 0.\end{aligned}$$

However, this follows from Schröter's identity (3.1), by setting $n_1 = 1$, $n_2 = 5$, $a = -1/q$, and $b = -q^4$, and then replacing q with q^2.

The proof of the second identity (4.9) proceeds similarly, except at the end, it depends on proving the identity:

$$\begin{aligned}&-\langle -q^{10};q^{24}\rangle_\infty\langle -q^{46};q^{120}\rangle_\infty - q^{12}\langle -q^6;q^{24}\rangle_\infty\langle -q^6;q^{120}\rangle_\infty\\ &\quad + q^{10}\langle -q^2;q^{24}\rangle_\infty\langle -q^{24};q^{120}\rangle_\infty + q^4\langle -q^{10};q^{24}\rangle_\infty\langle -q^{26};q^{120}\rangle_\infty\\ &\quad + \langle -q^6;q^{24}\rangle_\infty\langle -q^{54};q^{120}\rangle_\infty - q^4\langle -q^2;q^{24}\rangle_\infty\langle -q^{34};q^{120}\rangle_\infty = 0.\end{aligned}$$

This also follows from Schröter's identity (3.1), the only difference being that this time we set $b=-q^2$, before replacing q with q^2. □

Winquist's identity may also be derived from Theorem 2.2. As with the proof of the septuple product identity, our proof needs several applications of Schröter's identity (3.1), rather than following directly from the theorem. We prove Winquist's identity in the following form.

Corollary 4.8. (Winquist's Identity) *Let a and b be non-zero complex numbers and q a complex number with $|q|<1$. Then*

$$\begin{aligned}&\left\langle a;q^2\right\rangle_\infty \left\langle b;q^2\right\rangle_\infty \left\langle ab;q^2\right\rangle_\infty \left\langle \frac{a}{b};q^2\right\rangle_\infty \\ &\quad= (q^2,q^2;q^2)_\infty\Big[\left\langle a^3;q^6\right\rangle_\infty \{\left\langle b^3q^2;q^6\right\rangle_\infty - b\left\langle b^3q^4;q^6\right\rangle_\infty\} \\ &\qquad - \frac{a}{b}\left\langle b^3;q^6\right\rangle_\infty \{\left\langle a^3q^2;q^6\right\rangle_\infty - a\left\langle a^3q^4;q^6\right\rangle_\infty\}\Big]. \end{aligned} \tag{4.12}$$

Proof. In (2.3), set $k=3$, $n_1=n_2=1=n_3=N=1$ (so that $u_1=u_2=u_3=1$, $v_1=1, v_2=2$, $v_3=3$, $w_1=2$, $w_2=3$ and $w_3=4$), $z=1$, $a_1=-a/q$, $a_2=-ab/q$, $a_3=-b/q$, and $a=-a/(bq)$. Then, the left side of (2.3) becomes the left side of (4.12), and the right side of (2.3) becomes, after some slight manipulation:

$$\begin{aligned}&\sum_{j_1=0}^{1}\sum_{j_2=0}^{2} q^{2j_1^2+2j_2^2-2j_1j_2}\left(\frac{1}{b}\right)^{j_1} a^{j_2}\left\langle -bq^{2+2j_1};q^4\right\rangle_\infty \left\langle \frac{-q^{6+6j_1-4j_2}}{b^3};q^{12}\right\rangle_\infty \\ &\quad\times\sum_{j_3=0}^{3}\left(\frac{-b}{q^{2j_2+1}}\right)^{j_3} q^{j_3^2}\left\langle -q^{12+6j_3}\frac{b^3}{q^{8j_2}a^3};q^{24}\right\rangle_\infty \left\langle -q^{2j_3}a^3b;q^8\right\rangle_\infty .\end{aligned}$$

Next, we apply Schröter's identity (3.1), with $n_1=1$, $n_2=3$, a replaced with $-b/q^{2j_2+1}$ and b with $-q^{3-2j_2}/a^3$ to get that the inner sum over j_3 is $\left\langle b/q^{2j_2};q^2\right\rangle_\infty \left\langle q^{6-2j_2}/a^3;q^6\right\rangle_\infty$. Thus, the sum above is equal to

$$\begin{aligned}&\sum_{j_1=0}^{1}\sum_{j_2=0}^{2} q^{2j_1^2+2j_2^2-2j_1j_2}\left(\frac{1}{b}\right)^{j_1} a^{j_2}\left\langle -bq^{2+2j_1};q^4\right\rangle_\infty \left\langle \frac{-q^{6+6j_1-4j_2}}{b^3};q^{12}\right\rangle_\infty \\ &\quad\times\left\langle q^{2j_2}a^3;q^6\right\rangle_\infty \left\langle bq^{-2j_2};q^2\right\rangle_\infty .\end{aligned}$$

By comparison with the right side of (4.12), the result will follow if the following three identities hold:

$$\begin{aligned}&(q^2,q^2;q^2)_\infty\{\left\langle b^3q^2;q^6\right\rangle_\infty - b\left\langle b^3q^4;q^6\right\rangle_\infty\} \\ &\quad=\left\langle b;q^2\right\rangle_\infty\left\{\left\langle -bq^2;q^4\right\rangle_\infty\left\langle -b^3q^6;q^{12}\right\rangle_\infty+\frac{q^2}{b}\left\langle -bq^4;q^4\right\rangle_\infty\left\langle -b^3;q^{12}\right\rangle_\infty\right\}, \\ &(q^2,q^2;q^2)_\infty\left\langle b^3;q^6\right\rangle_\infty \\ &\quad=\left\langle b;q^2\right\rangle_\infty\{b^2\left\langle -bq^2;q^4\right\rangle_\infty\left\langle -b^3q^{10};q^{12}\right\rangle_\infty+b\left\langle -bq^4;q^4\right\rangle_\infty\left\langle -b^3q^4;q^{12}\right\rangle_\infty\} \\ &\quad=\left\langle b;q^2\right\rangle_\infty\{\left\langle -bq^2;q^4\right\rangle_\infty\left\langle -b^3q^2;q^{12}\right\rangle_\infty+b^2\left\langle -bq^4;q^4\right\rangle_\infty\left\langle -b^3q^8;q^{12}\right\rangle_\infty\}.\end{aligned} \tag{4.13}$$

We apply Schröter's identity (3.1) again, with $n_1 = 1$, $n_2 = 2$, a replaced with $-b/q$ and, respectively, b kept as b and replaced with bq^2, to get that

$$\left\langle b;q^2\right\rangle_\infty \left\langle -bq^2;q^4\right\rangle_\infty = (q^2;q^2)_\infty \left\{\left\langle -b^3q^4;q^{12}\right\rangle_\infty - b\left\langle -b^3q^8;q^{12}\right\rangle_\infty\right\},$$
$$\left\langle b;q^2\right\rangle_\infty \left\langle -bq^4;q^4\right\rangle_\infty = (q^2;q^2)_\infty \left\{b^{-1}\left\langle -b^3q^2;q^{12}\right\rangle_\infty - b\left\langle -b^3q^{10};q^{12}\right\rangle_\infty\right\}.$$

After inserting the expressions on the right above in Eq. (4.12), the proof of Winquist's identity will follow if it can be shown that the following three identities hold:

$$\begin{aligned}&(q^2;q^2)_\infty \left\langle b^3;q^6\right\rangle_\infty\\ &\quad= \left\langle -b^3q^4;q^{12}\right\rangle_\infty \left\langle -b^3q^2;q^{12}\right\rangle_\infty - b^3\left\langle -b^3q^8;q^{12}\right\rangle_\infty \left\langle -b^3q^{10};q^{12}\right\rangle_\infty,\end{aligned} \tag{4.14}$$

$$\begin{aligned}&(q^2;q^2)_\infty \left\langle b^3q^2;q^6\right\rangle_\infty\\ &\quad= \left\langle -b^3q^4;q^{12}\right\rangle_\infty \left\langle -b^3q^6;q^{12}\right\rangle_\infty - q^2\left\langle -b^3q^{10};q^{12}\right\rangle_\infty \left\langle -b^3;q^{12}\right\rangle_\infty,\end{aligned} \tag{4.15}$$

$$\begin{aligned}&(q^2;q^2)_\infty \left\langle b^3q^4;q^6\right\rangle_\infty\\ &\quad= \left\langle -b^3q^8;q^{12}\right\rangle_\infty \left\langle -b^3q^6;q^{12}\right\rangle_\infty - \frac{q^2}{b^3}\left\langle -b^3q^2;q^{12}\right\rangle_\infty \left\langle -b^3;q^{12}\right\rangle_\infty.\end{aligned} \tag{4.16}$$

Once again appealing to Schröter's identity (3.1), with $n_1 = 1$, $n_2 = 1$, and q replaced with q^3, we get that

$$\begin{aligned}&\left\langle -aq^3;q^6\right\rangle_\infty \left\langle -bq^3;q^6\right\rangle_\infty\\ &\quad= \left\langle -\frac{a}{b}q^6;q^{12}\right\rangle_\infty \left\langle -abq^6;q^{12}\right\rangle_\infty + aq^3\left\langle -\frac{a}{b}q^{12};q^{12}\right\rangle_\infty \left\langle -abq^{12};q^{12}\right\rangle_\infty.\end{aligned} \tag{4.17}$$

Identities (4.14), (4.15), and (4.16) follow upon replacing (a,b) in identity (4.17) by, respectively, $(-b^3/q^3,-q)$, $(-1/q,-b^3/q)$ and $(-1/(b^3q),-q)$. This completes the proof of Winquist's identity. ☐

It is also possible to use identity (2.3) to derive an expression for $(q;q)_\infty^k$ that is different from that given in Corollary 4.4.

Corollary 4.9. *Let $k \geq 3$ be an integer, let $\omega = \exp(2\pi i/k)$, and suppose $|q| < 1$. Then*

$$\begin{aligned}&(q;q)_\infty^k\\ &\quad= \left(q^k;q^k\right)_\infty \sum_{j_1=0}^{1}\sum_{j_2=0}^{2}\cdots\sum_{j_{k-2}=0}^{k-2} q^{(j_1^2+j_2^2+\cdots+j_{k-2}^2)-(j_1j_2+j_2j_3+\cdots+j_{k-3}j_{k-2})}\\ &\qquad\times \omega^{-j_1-j_2-j_3-\cdots-j_{k-2}}\left\langle -q^{1+j_1}\omega;q^2\right\rangle_\infty \left\langle (-1)^k q^{k(k-1)/2+kj_{k-2}};q^{k(k-1)}\right\rangle_\infty\\ &\qquad\times \prod_{i=2}^{k-2}\left\langle -q^{i(i+1)/2+(i+1)j_{i-1}-ij_i}\omega^{-i(i+1)/2};q^{i(i+1)}\right\rangle_\infty.\end{aligned} \tag{4.18}$$

Proof. In (4.3), replace z with $-z$ and set $a_i = \omega^i$, $1 \leq i \leq k$, so that the left side becomes $(q^kz^k, q^k/z^k;q^{2k})_\infty \left(q^2;q^2\right)_\infty^k$. Since all the powers of z on the left side have exponent $\equiv 0 \pmod{k}$, each of the multiple sums on the right side with j_{k-1} fixed, $1 \leq j_{k-1} \leq k-1$, is identically zero, so that the only non-zero sum is the one with $j_{k-1} = 0$. With the given values for the a_i, it is clear that $a_1/a_k = \omega$, each $a_i/a_{i+1} = 1/\omega$, $a_1 a_2 \cdots a_k = (-1)^{k-1}$ and

$$\frac{a_k a_1 a_2 \cdots a_{i-1}}{a_i^i} = \omega^{-i(i+1)/2}.$$

The result follows after canceling the $(q^k z^k, q^k/z^k; q^{2k})_\infty$ factor on each side, separating off the $k-1$ term in the sum on the right side of the equation that follows from (4.3), and finally replacing q^2 with q. □

It is also possible to derive expressions for $(q;q)_\infty$ as combinations of Jacobi triple products from (2.3).

Corollary 4.10. *Let $k \geq 3$ be an integer; suppose $|q| < 1$. Then*

$$\begin{aligned}(q;q)_\infty = &\frac{1}{(q^{2k+1};q^{2k+1})_\infty^{k-1}} \sum_{j_1=0}^{1} \sum_{j_2=0}^{2} \cdots \sum_{j_{k-1}=0}^{k-1} (-1)^{j_{k-1}} \\ &\times q^{(2k+1)(j_1^2+(j_2-j_1)^2+\cdots+(j_{k-1}-j_{k-2})^2)/2-j_1-j_2-\cdots-j_{k-2}+(k-3/2)j_{k-1}} \\ &\times \left\langle -q^{(2k+1)(1+j_1)-k+1}; q^{2(2k+1)} \right\rangle_\infty \\ &\times \left\langle (-1)^{k+1} q^{k(3k+1)/2+(2k+1)j_{k-1}}; q^{(2k+1)k} \right\rangle_\infty \\ &\times \prod_{i=2}^{k-1} \left\langle -q^{k(i(i+1)+1)+(2k+1)((i+1)j_{i-1}-ij_i)}; q^{(2k+1)i(i+1)} \right\rangle_\infty. \end{aligned} \tag{4.19}$$

Proof. In (4.3), replace q with $q^{(2k+1)/2}$ and set $z = -1$, $a_1 = q^{1/2}$, $a_2 = q^{3/2}$, …, $a_k = q^{k-1/2}$. The left side of the identity then becomes $(q;q)_\infty$ $(q^{2k+1};q^{2k+1})_\infty^{k-1}$, and after some simple algebra on the resulting right side of (4.3), the result follows after dividing both sides of this new expression by the factor $(q^{2k+1};q^{2k+1})_\infty^{k-1}$. □

In a similar vein, the following identity holds.

Corollary 4.11. *Let $k \geq 3$ be an integer, and suppose $|q| < 1$. Then*

$$\begin{aligned}(q;q)_\infty = &\frac{1}{(q^k;q^k)_\infty (q^{2k};q^{2k})_\infty^{k-2}} \sum_{j_1=0}^{1} \sum_{j_2=0}^{2} \cdots \sum_{j_{k-1}=0}^{k-1} (-1)^{j_{k-1}} \\ &\times q^{k(j_1^2+(j_2-j_1)^2+\cdots+(j_{k-1}-j_{k-2})^2)-j_1-j_2-\cdots-j_{k-2}+(k-1)j_{k-1}} \\ &\times \left\langle -q^{2k(1+j_1)+1}; q^{4k} \right\rangle_\infty \left\langle (-1)^{k+1} q^{k(3k-1)/2+2kj_{k-1}}; q^{2k^2} \right\rangle_\infty \\ &\times \prod_{i=2}^{k-1} \left\langle -q^{(2k-1)i(i+1)/2+2k((i+1)j_{i-1}-ij_i)}; q^{2ki(i+1)} \right\rangle_\infty. \end{aligned} \tag{4.20}$$

Proof. This time in (4.3), replace q with q^k and set $z = -1$, $a_1 = q$, $a_2 = q^2$, …, $a_{k-1} = q^{k-1}$, $a_k = 1$. The left side of the identity then becomes $(q;q)_\infty$ $(q^k;q^k)_\infty$ $(q^{2k};q^{2k})_\infty^{k-2}$, and the result follows after dividing both sides by $(q^k;q^k)_\infty (q^{2k};q^{2k})_\infty^{k-2}$. □

5. Concluding Remarks

Theorem 2.2 has the restriction that each n_i satisfies $n_i|N$. Using Theorem 2.1 in Cao's paper [2], it is possible to drop this restriction and derive an expansion of a product of an arbitrary number of Jacobi triple products in terms of sums of products of other Jacobi triple products. However, it would be hard to find a general formula such as (2.3) above.

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James Mc Laughlin
Mathematics Department
West Chester University
25 University Avenue
West Chester
PA 19383
USA
e-mail: jmclaughlin@wcupa.edu;
jmclaughlin2@wcupa.edu

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A Truncated Theta Identity of Gauss and Overpartitions into Odd Parts

Dedicated to Professor George E. Andrews

Mircea Merca, Chun Wang and Ae Ja Yee

Abstract. We examine two truncated series derived from a classical theta identity of Gauss. As a consequence, we obtain two infinite families of inequalities for the overpartition function $\overline{p_o}(n)$ counting the number of overpartitions into odd parts. We provide partition-theoretic interpretations of these results.

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Keywords. Overpartitions, Recurrences, Theta series.

1. Introduction

An overpartition of a positive integer n is a partition of n in which the first occurrence of a part of each size may be overlined [10]. For example, there are 8 overpartitions of 3:

$$3,\ \bar{3},\ 2+1,\ \bar{2}+1,\ 2+\bar{1},\ \bar{2}+\bar{1},\ 1+1+1 \text{ and } \bar{1}+1+1.$$

Let $\overline{p}(n)$ be the number of overpartitions of n. Then the generating function of $\overline{p}(n)$ is

$$\sum_{n=0}^{\infty} \overline{p}(n) q^n = \frac{(-q;q)_\infty}{(q;q)_\infty}.$$

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Here and throughout this paper, we use the following customary q-series notation:

$$(a;q)_n = \begin{cases} 1, & \text{for } n=0, \\ (1-a)(1-aq)\cdots(1-aq^{n-1}), & \text{for } n>0; \end{cases}$$

$$(a;q)_\infty = \lim_{n\to\infty}(a;q)_n;$$

$$\begin{bmatrix} n \\ k \end{bmatrix} = \begin{cases} \dfrac{(q;q)_n}{(q;q)_k(q;q)_{n-k}}, & \text{if } 0 \le k \le n, \\ 0, & \text{otherwise.} \end{cases}$$

Andrews and Merca [2] considered Euler's pentagonal number theorem and proved a truncated theorem on partitions. Subsequently, Guo and Zeng [12] considered the following identity of Gauss

$$1 + 2\sum_{n=1}^{\infty}(-1)^n q^{n^2} = \frac{(q;q)_\infty}{(-q;q)_\infty}, \tag{1.1}$$

and they proved a new truncated theorem on overpartitions. Namely, for $k \ge 1$,

$$\frac{(-q;q)_\infty}{(q;q)_\infty}\left(1 + 2\sum_{j=1}^{k}(-1)^j q^{j^2}\right) = 1 + (-1)^k \sum_{n=k+1}^{\infty} \frac{(-q;q)_k(-1;q)_{n-k}q^{(k+1)n}}{(q;q)_n} \begin{bmatrix} n-1 \\ k-1 \end{bmatrix}. \tag{1.2}$$

As a consequence of this result, they derived the following inequality for $\overline{p}(n)$:

$$(-1)^k \left(\overline{p}(n) + 2\sum_{j=1}^{k}(-1)^j \overline{p}(n-j^2)\right) \ge 0, \tag{1.3}$$

with strict inequality if $n \ge (k+1)^2$. Very recently, Andrews and Merca [3] provided the following revision of (1.2):

$$\frac{(-q;q)_\infty}{(q;q)_\infty}\left(1 + 2\sum_{j=1}^{k}(-1)^j q^{j^2}\right) = 1 + 2(-1)^k \frac{(-q;q)_k}{(q;q)_k} \sum_{j=0}^{\infty} \frac{q^{(k+1)(k+j+1)}(-q^{k+j+2};q)_\infty}{(1-q^{k+j+1})(q^{k+j+2};q)_\infty}. \tag{1.4}$$

From this identity, they immediately deduced an interpretation of the sum in the inequality (1.3) considering $\overline{M}_k(n)$, the number of overpartitions of n in which the first part larger than k appears at least $k+1$ times:

$$(-1)^k \left(\overline{p}(n) + 2\sum_{j=1}^{k}(-1)^j \overline{p}(n-j^2)\right) = \overline{M}_k(n), \tag{1.5}$$

for $n, k \ge 1$. Shortly after that, Ballantine et al. [4] gave a combinatorial proof of this interpretation.

Other recent investigations on the truncated theta series can be found in several papers by Chan et al. [7], Chern [9], He et al. [13], Kolitsch [15], Kolitsch and Burnette [16], Mao [18,19], Merca [20], Wang and Yee [22–24], and Yee [25].

In this paper, we consider overpartitions into odd parts and shall prove similar results. Let $\overline{p_o}(n)$ be the number of overpartitions into odd parts. Then its generating function is

$$\sum_{n=0}^{\infty} \overline{p_o}(n) q^n = \frac{(-q;q^2)_\infty}{(q;q^2)_\infty}. \tag{1.6}$$

This expression first appeared in the following series-product identity

$$\sum_{n=0}^{\infty} \frac{(-1;q)_n q^{n(n+1)/2}}{(q;q)_n} = \frac{(-q;q^2)_\infty}{(q;q^2)_\infty},$$

which was given by Lebesgue [17] in 1840. More recently, the generating function of $\overline{p_o}(n)$ appeared in the works of Bessenrodt [5], Santos and Sills [21]. Various arithmetic properties of $\overline{p_o}(n)$ have been investigated later by Chen [8], Hirschhorn and Sellers [14].

In analogy with the truncated identities in (1.2) and (1.4), we have two symmetrical results on $\overline{p_o}(n)$.

Theorem 1.1. *For a positive integer k,*

$$\frac{(-q;q^2)_\infty}{(q;q^2)_\infty}\left(1+2\sum_{j=1}^{k}(-1)^j q^{j^2}\right)$$
$$= 1 + 2\sum_{j=1}^{\infty}(-1)^j q^{2j^2} + 2(-1)^k q^{(k+1)^2}(-q;q^2)_\infty \sum_{j=0}^{\infty} \frac{q^{(2k+2j+3)j}}{(q^2;q^2)_j (q;q^2)_{k+j+1}}$$

and

$$\frac{(-q;q^2)_\infty}{(q;q^2)_\infty}\left(1+2\sum_{j=1}^{k}(-1)^j q^{2j^2}\right)$$
$$= 1 + 2\sum_{j=1}^{\infty} q^{j^2} + 2(-1)^k q^{2(k+1)^2}(-q;q^2)_\infty^2 \sum_{j=0}^{\infty} \frac{q^{2(2k+2j+3)j}}{(q^4;q^4)_j (q^2;q^4)_{k+j+1}}.$$

We can deduce the following results where $\delta_{i,j}$ is the Kronecker delta function.

Corollary 1.2. *Let k and n be positive integers.*

(a) *For $n \geq (k+1)^2$,*

$$(-1)^k\left(\overline{p_o}(n) + 2\sum_{j=1}^{k}(-1)^j \overline{p_o}(n-j^2) - (-1)^{\lfloor\sqrt{n/2}\rfloor}\cdot 2\delta_{n,2\lfloor\sqrt{n/2}\rfloor^2}\right) \geq 2.$$

(b) *For $n < (k+1)^2$,*

$$\overline{p_o}(n) + 2\sum_{j=1}^{k}(-1)^j \overline{p_o}(n-j^2) = (-1)^{\lfloor\sqrt{n/2}\rfloor}\cdot 2\delta_{n,2\lfloor\sqrt{n/2}\rfloor^2}.$$

(c) *For* $n \geq 2(k+1)^2$,

$$(-1)^k \left(\overline{p_o}(n) + 2\sum_{j=1}^{k} (-1)^j \overline{p_o}(n-2j^2) - 2\delta_{n,\lfloor \sqrt{n} \rfloor^2} \right) \geq 2.$$

(d) *For* $n < 2(k+1)^2$,

$$\overline{p_o}(n) + 2\sum_{j=1}^{k} (-1)^j \overline{p_o}(n-2j^2) = 2\delta_{n,\lfloor \sqrt{n} \rfloor^2}.$$

We remark that the last relation of this corollary provides an efficient algorithm for computing the function $\overline{p_o}(n)$.

The rest of this paper is organized as follows. We will first prove Theorem 1.1 in Sect. 2. In Sect. 3, we will provide a combinatorial interpretation of the right-hand side of each identity in Theorem 1.1.

2. Proof of Theorem 1.1

To prove the theorem, we consider the Gauss hypergeometric series

$${}_2\phi_1\left(\begin{matrix} a, b \\ c \end{matrix}; q, z\right) = \sum_{n=0}^{\infty} \frac{(a;q)_n (b;q)_n}{(q;q)_n (c;q)_n} z^n$$

and the second identity by Heine's transformation of ${}_2\phi_1$ series [11, (III.2)], namely

$${}_2\phi_1\left(\begin{matrix} a, b \\ c \end{matrix}; q, z\right) = \frac{(c/b;q)_\infty (bz;q)_\infty}{(c;q)_\infty (z;q)_\infty} {}_2\phi_1\left(\begin{matrix} abz/c, b \\ bz \end{matrix}; q, c/b\right). \tag{2.1}$$

We first prove the first identity in Theorem 1.1. By Gauss' identity (1.1), we can write the left-hand side as follows:

$$\begin{aligned}
&\frac{(-q;q^2)_\infty}{(q;q^2)_\infty} \left(1 + 2\sum_{j=1}^{k} (-1)^j q^{j^2}\right) \\
&\quad = \frac{(q^2;q^2)_\infty}{(-q^2;q^2)_\infty} - 2\frac{(-q;q^2)_\infty}{(q;q^2)_\infty} \sum_{j=k+1}^{\infty} (-1)^j q^{j^2} \\
&\quad = 1 + 2\sum_{j=1}^{\infty} (-1)^j q^{2j^2} + 2(-1)^k q^{(k+1)^2} \frac{(-q;q^2)_\infty}{(q;q^2)_\infty} \sum_{j=0}^{\infty} (-1)^j q^{j^2+2j(k+1)} \\
&\quad = 1 + 2\sum_{j=1}^{\infty} (-1)^j q^{2j^2} + 2(-1)^k q^{(k+1)^2} \frac{(-q;q^2)_\infty}{(q;q^2)_\infty} \lim_{\tau \to 0} {}_2\phi_1\left(\begin{matrix} q^2, \frac{q^{2k+3}}{\tau} \\ 0 \end{matrix}; q^2, \tau\right) \\
&\quad = 1 + 2\sum_{j=1}^{\infty} (-1)^j q^{2j^2} \\
&\qquad + 2(-1)^k q^{(k+1)^2} \frac{(-q;q^2)_\infty}{(q;q^2)_\infty} \lim_{\tau \to 0} \frac{(q^{2k+3};q^2)_\infty}{(\tau;q^2)_\infty} \sum_{j=0}^{\infty} \frac{(-1)^j \tau^j q^{j^2+j} (\frac{q^{2k+3}}{\tau};q^2)_j}{(q^2;q^2)_j (q^{2k+3};q^2)_j}
\end{aligned}$$

$$= 1 + 2\sum_{j=1}^{\infty}(-1)^j q^{2j^2} + 2(-1)^k q^{(k+1)^2}\frac{(-q;q^2)_\infty}{(q;q^2)_{k+1}}\sum_{j=0}^{\infty}\frac{q^{2j^2+(2k+3)j}}{(q^2;q^2)_j(q^{2k+3};q^2)_j}$$
$$= 1 + 2\sum_{j=1}^{\infty}(-1)^j q^{2j^2} + 2(-1)^k q^{(k+1)^2}(-q;q^2)_\infty\sum_{j=0}^{\infty}\frac{q^{(2k+2j+3)j}}{(q^2;q^2)_j(q;q^2)_{k+j+1}},$$

where the fourth equality follows from (2.1).

The proof of the second identity is similar to the proof of the first one. With q replaced by q^2, the Gauss identity (1.1) becomes

$$1 + 2\sum_{n=1}^{k}(-1)^n q^{2n^2} = \frac{(q^2;q^2)_\infty}{(-q^2;q^2)_\infty} - 2\sum_{n=k+1}^{\infty}(-1)^n q^{2n^2}.$$

Multiplying both sides of this identity by the generating function of $\overline{p_o}(n)$, we get

$$\frac{(-q;q^2)_\infty}{(q;q^2)_\infty}\left(1 + 2\sum_{j=1}^{k}(-1)^j q^{2j^2}\right)$$
$$= \frac{(-q;-q)_\infty}{(q;-q)_\infty} - 2\frac{(-q;q^2)_\infty}{(q;q^2)_\infty}\sum_{j=k+1}^{\infty}(-1)^j q^{2j^2}$$
$$= 1 + 2\sum_{j=1}^{\infty} q^{j^2} + 2(-1)^k q^{2(k+1)^2}\frac{(-q;q^2)_\infty}{(q;q^2)_\infty}\sum_{j=0}^{\infty}(-1)^j q^{2j^2+4j(k+1)}$$
$$= 1 + 2\sum_{j=1}^{\infty} q^{j^2} + 2(-1)^k q^{2(k+1)^2}\frac{(-q;q^2)_\infty(q^{4k+6};q^4)_\infty}{(q;q^2)_\infty}$$
$$\times\sum_{j=0}^{\infty}\frac{q^{4j^2+2(2k+3)j}}{(q^4;q^4)_j(q^{4k+6};q^4)_j} \quad \text{by (2.1)}$$
$$= 1 + 2\sum_{j=1}^{\infty} q^{j^2} + 2(-1)^k q^{2(k+1)^2}\frac{(-q;q^2)_\infty(-q^{2k+3};q^2)_\infty}{(q;q^2)_{k+1}}$$
$$\times\sum_{j=0}^{\infty}\frac{q^{4j^2+2(2k+3)j}}{(q^4;q^4)_j(q^{4k+6};q^4)_j}$$
$$= 1 + 2\sum_{j=1}^{\infty} q^{j^2} + 2(-1)^k q^{2(k+1)^2}(-q;q^2)_\infty$$
$$\times\sum_{j=0}^{\infty}\frac{q^{4j^2+2(2k+3)j}(-q^{2k+2j+3};q^2)_\infty}{(q^4;q^4)_j(q;q^2)_{k+j+1}}$$
$$= 1 + 2\sum_{j=1}^{\infty} q^{j^2} + 2(-1)^k q^{2(k+1)^2}(-q;q^2)_\infty^2\sum_{j=0}^{\infty}\frac{q^{2(2k+2j+3)j}}{(q^4;q^4)_j(q^2;q^4)_{k+j+1}}.$$

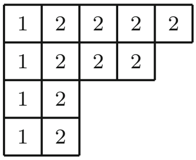

FIGURE 1. The 2-modular Ferrers graph of $9 + 7 + 3 + 3$

3. Partitions Arising from Theorem 1.1

In this section, we will explain what partitions are generated by the right-hand sides of the identities in Theorem 1.1. We first recall some necessary definitions.

For a partition λ, we denote the sum of all parts of λ by $|\lambda|$. The Ferrers graph of a partition λ is a graphical representation of λ whose ith row has as many boxes as the ith part λ_i. Such a graph is called a Ferrers graph of shape λ.

For a positive integer k, any positive integer n can be uniquely written as $ka + s$ with $a \geq 0$ and $1 \leqslant s \leqslant k$. The k-modular partitions are a modification of the Ferrers graph so that n is represented by a row of a boxes with k in each of them and one box with s in it. This notion was first introduced by MacMahon [1, p. 13]. For instance, Fig. 1 shows the 2-modular Ferrers graph of the partition $9+7+3+3$ with shape $5+4+2+2$. Here, we put boxes with 1 in the first column for convenience.

Another combinatorial notion needed is m-Durfee rectangles. For a non-negative integer m, define an m-rectangle to be a rectangle whose width minus its height is m. For a Ferrers graph of shape λ, define the m-Durfee rectangle to be the largest m-rectangle which fits in the graph [6]. When $m = 0$, the m-Durfee rectangle becomes the Durfee square of a partition. In Fig. 1, the 2-Durfee rectangle of the partition is the rectangle of size 2×4.

For a fixed $k \geq 1$ and any $n \geq 0$, define $M_{o,k}(n)$ to be the number of partitions of n into odd parts such that all odd numbers less than or equal to $2k + 1$ occur as parts at least once and the parts below the $(k + 2)$-Durfee rectangle in the 2-modular graph are strictly less than the width of the rectangle. For instance, let $k = 2$. Then the partition $11+11+7+7+5+3+1$ is counted by $M_{o,2}(45)$. However, the partition $11 + 11 + 11 + 5 + 3 + 3 + 1$ is not counted by $M_{o,2}(41)$, because its 4-Durfee rectangle is of size 2×6 and the third part of length 11 that goes below the Durfee rectangle forms a row of length 6.

Theorem 3.1. *For a fixed $k \geq 1$,*

$$\sum_{n=0}^{\infty} M_{o,k}(n)q^n = q^{(k+1)^2} \sum_{j=0}^{\infty} \frac{q^{(2k+2j+3)j}}{(q^2;q^2)_j(q;q^2)_{k+j+1}}.$$

Proof. For a partition counted by $M_{o,k}(n)$, assume that its $(k + 2)$-Durfee rectangle is of size $j \times (k + 2 + j)$. By the Durfee rectangle, the 2-modular Ferrers graph can be divided into three parts, namely the Durfee rectangle, the parts below the rectangle and the parts to the right of the rectangle. Then, the weight of the Durfee rectangle is $j(2(k+j+2)-1)$. Also, it follows from the definition of $M_{o,k}(n)$, the parts below the rectangle and the parts to the right of the rectangle are generated by $q^{(k+1)^2}/(q;q^2)_{k+j+1}$ and $1/(q^2;q^2)_j$, respectively. Here, $q^{(k+1)^2}$ accounts for all the odd numbers between 1 and $2k+1$. Therefore, we can see that the summand on the right-hand side in the statement generates partitions counted by $M_{o,k}(n)$ whose $(k+2)$-Durfee rectangle is of size $j \times (k+2+j)$. □

Corollary 3.2. *For $k \geq 1$ and $n \geq (k+1)^2$,*

$$(-1)^k \left(\overline{p_o}(n) + 2\sum_{j=1}^{k} (-1)^j \overline{p_o}(n-j^2) - (-1)^{\lfloor \sqrt{n/2} \rfloor} \cdot 2\delta_{n,2\lfloor \sqrt{n/2} \rfloor^2} \right)$$
$$= 2\overline{M_{o,k}}(n),$$

where $\overline{M_{o,k}}(n)$ counts overpartitions of n into odd parts in which the non-overlined parts form a partition counted by $M_{o,k}(n-a)$, a is the sum of overlined parts, and

$$(-1)^k \left(\overline{p_o}(n) + 2\sum_{j=1}^{k} (-1)^j \overline{p_o}(n-2j^2) - 2\delta_{n,\lfloor \sqrt{n} \rfloor^2} \right) = 2N_{o,k}(n),$$

where $N_{o,k}(n)$ counts triples (λ, μ, ν) such that λ and μ are partitions into distinct odd parts and ν is a partition counted by $M_{o,k}\big((n-|\lambda|-|\mu|)/2\big)$.

Proof. The statements easily follow from Theorems 1.1 and 3.1, so we omit the details. □

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Mircea Merca
Academy of Romanian Scientists
Ilfov 3, Sector 5
Bucharest
Romania
e-mail: mircea.merca@profinfo.edu.ro

Chun Wang
Department of Mathematics
East China Normal University
500 Dongchuan Road
Shanghai 200241
People's Republic of China
e-mail: wangchunmath@outlook.com

Ae Ja Yee
Department of Mathematics
The Pennsylvania State University
University Park PA16802
USA
e-mail: auy2@psu.edu

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Combinatory Classes of Compositions with Higher Order Conjugation

Dedicated to George Andrews on the occasion of his 80th birthday

Augustine O. Munagi

Abstract. We consider certain classes of compositions of numbers based on the recently introduced extension of conjugation to higher orders. We use generating functions and combinatorial identities to provide enumeration results for compositions possessing conjugates of a given order. Working under some popular themes in the theory, we show that results for these compositions specialize to standard results in a natural way. We also give a generalization of MacMahon's identities for inverse-conjugate compositions and discuss inverse-reciprocal compositions.

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Keywords. Composition, Conjugate, Generating function, Inverse-conjugate, Reciprocal.

1. Introduction

The classical development of the theory of compositions by MacMahon [8] has recently been extended to compositions with conjugates of higher orders (see Munagi [10]). The latter provides a new classification of the set of compositions of a positive integer n by extension of standard conjugation.

In this sequel, we will highlight certain classes of compositions possessing conjugates of a prescribed order with an emphasis on their enumeration and some identities which they satisfy. As in [10], this presentation will be largely expository and self-contained.

The subject of compositions has engaged the attention of George E. Andrews on several occasions. Early in his career, Andrews edited the huge collected works of MacMahon, clarifying several proofs while providing leading commentaries on open-ended problems [7]. Andrews subsequently published some interesting articles on compositions (e.g., [2–5]) besides a concise treatment of the subject in his book [1].

2. Preliminaries

A composition of a positive integer n is an ordered partition of n; that is, any sequence of positive integers $(n_1, \ldots, n_k)$, such that $n_1 + \cdots + n_k = n$.

The number of compositions of n is known to be $c(n) = 2^{n-1}$. Our textbook references for this study consist of Andrews [1], MacMahon [7], and Heubach-Mansour [6]. More information on methods of obtaining the conjugate of a composition may be found in [9,11].

Compositions will also be expressed in the *symbolic* notation by replacing every maximal string of 1s of length x by 1^x, where two adjacent *big parts* (i.e., parts > 1)are assumed to be separated by 1^0. Thus, for example, $(5, 1, 1, 8, 2, 1, 7, 1, 1, 1, 1) = (5, 1^2, 8, 1^0, 2, 1, 7, 1^4)$ has 8 symbolic parts and 11 (actual) parts.

Without loss of generality, consider a composition of n of the form:

$$C = (1^{a_1}, b_1, 1^{a_2}, b_2, \ldots, 1^{a_v}, b_v),\ a_1 > 0, a_i \geq 0, i > 1,\ b_i > 1,\ \forall\, i. \tag{2.1}$$

The classical definition of the conjugate, denoted by C', is as follows:

$$C' = (a_1 + 1, 1^{b_1 - 2}, a_2 + 2, 1^{b_2 - 2}, \ldots, a_v + 2, 1^{b_v - 1}). \tag{2.2}$$

This is now the conjugate of *order* 1 of C. Thus, for instance, the conjugate of $(5, 1^2, 8, 1^0, 2, 1, 7, 1^4)$ is $(1^4, 4, 1^6, 2, 1^0, 3, 1^5, 5)$.

Given an integer $t > 0$, the *conjugate of order* t is given by

$$C^{(t)} = (a_1 + t, 1^{b_1 - 2t}, a_2 + 2t, 1^{b_2 - 2t}, \ldots, a_v + 2t, 1^{b_v - t}). \tag{2.3}$$

A composition C is said to be t-*conjugable* if and only if $C^{(t)} \neq \emptyset$. This non-void condition of course implies that no purported big part is less than $2t$ and no string of 1s has negative length. Thus, to be t-conjugable, the composition in (2.1) also requires $b_i \geq 2t$, $1 \leq i < v$ and $b_v > t$.

The permissible sets of parts are depicted in the sketch below for non-trivial compositions of n, that is, compositions $C \neq (n)$, (1^n). These two compositions are m-conjugable for all integers $m > 0$:

$$\underbrace{-\text{BOUNDARY}-}_{\text{part} \in \{1, t+1, t+2, \ldots\}} \mid \underbrace{- - \text{INTERIOR} - -}_{\text{parts} \in \{1, 2t, 2t+1, \ldots\}} \mid \underbrace{-\text{BOUNDARY}-}_{\text{part} \in \{1, t+1, t+2, \ldots\}}$$

The higher order conjugates are denoted successively, for $t = 1, 2, 3, \ldots$, by C', C'', C''', $\ldots$, $C^{(m)}$, $\ldots$.

For example, if $C = (1^3, 8, 1, 6, 1^2, 5)$, then $C''' = (3+3, 1^{8-6}, 1+6, 1^{6-6}, 2+6, 1^{5-3}) = (6, 1^2, 7, 1^0, 8, 1^2)$.

If C is t-conjugable, then $(C^{(t)})^{(t)} = C$. Therefore, t-conjugation is an involution.

We briefly summarize the essential consequences from [10] that are most relevant to this study and uncover a few new items.

The line graph $\mathrm{LG}(C)$ of C consists of n equal line segments separated by $n - 1$ gaps in which two adjacent parts a, b are demarcated by placing a dot after a segments. For example, when $C = (3, 1, 1, 2, 4, 1)$, the $\mathrm{LG}(C)$ is $- - - \cdot - \cdot - \cdot - - \cdot - - - - \cdot -$.

The line graph of $C^{(t)}$ is obtained from $\mathrm{LG}(C)$ with the following rule:

Place a dot in any gap that previously had no dot except the first $t-1$ gaps immediately before or after a previous dot.

If the *length* or the number of parts $\ell(C)$ of C is k, then it may be deduced from $\mathrm{LG}(C)$ that

$$\ell(C') = n - k + 1. \tag{2.4}$$

The *reciprocal* of $C = (\ldots, 1^{a_u}, b_v, 1^{a_{u+1}}, b_{v+1}, \ldots)$, denoted by $r(C)$, is the composition obtained by replacing each string 1^{a_i} with a_i and each part $b_i > 1$ with 1^{b_i}, i.e., $r(C) = (\ldots, a_u, 1^{b_v}, a_{u+1}, 1^{b_{v+1}}, \ldots)$. If $r(C) \neq \emptyset$, we call C a *reciprocal composition*. The definition simply demands that $a_i, b_i \geq 2$ for all i. The reciprocal composition may be viewed as a "zero-conjugate", since it corresponds to $C^{(0)}$ in (2.3).

There is a recursive conjugation rule [10, p. 10]:

$$C^{(t)} = r(C^{(t-u)})^{(u)}, \quad 0 \leq u \leq t. \tag{2.5}$$

What is the length of $r(C)$? When C is changed to $r(C)$ each symbolic part of C is 'flipped' independently. If C has b symbolic parts with $\ell(C) = k$, then both $LG(C)$ and $LG(r(C))$ have $b-1$ fixed dots in between symbolic parts. It follows that $LG(r(C))$ may be obtained by inserting $b-1$ extra dots into $LG(C')$ to give a total of $n-k+b-1$ dots. Hence

$$\ell(r(C)) = n - k + b. \tag{2.6}$$

Proposition 2.1. *Let C be a composition of n with k parts, $1 < k < n$. Then*

$$\ell(C^{(t)}) = n - k - (t-1)b + t,$$

where b is the number of symbolic parts of C.

Proof. The proof is by induction on t based on (2.4) and (2.6).

Assume that the proposition holds for some $t > 0$, and consider $C^{(t+1)}$. Since $C^{(t+1)} \neq \emptyset$, we have, by (2.5), that $r(C^{(t)}) \neq \emptyset$. Let $m_t = \ell(C^{(t)})$. Then, using (2.6), we obtain

$$\ell(r(C^{(t)})) = n - m_t + b.$$

Therefore

$$\ell(C^{(t+1)}) = \ell(r(C^{(t)})') = n - (n - m_t + b) + 1 = m_t - b + 1;$$

that is:

$$\ell(C^{(t+1)}) = (n - k - (t-1)b + t) - b + 1 = n - k - tb + t + 1.$$

Hence, the result holds for $t+1$. The proof is complete. □

Finally, we recall the generating function for the number $c_t(n,k)$ of t-conjugable compositions into k parts with

$$c_t(n) = c_t(n,1) + \cdots + c_t(n,n).$$

The construct is

$$(x + x^{t+1} + x^{t+2} + \cdots)(x + x^{2t} + x^{2t+1} + \cdots)^{k-2}(x + x^{t+1} + x^{t+2} + \cdots) :$$

$$\sum_{n=k}^{\infty} c_t(n,k)x^n = \frac{(x - x^2 + x^{t+1})^2(x - x^2 + x^{2t})^{k-2}}{(1-x)^k}, \ k > 1.$$

Hence

$$\sum_{n=1}^{\infty} c_t(n)x^n = \frac{x(1 + x - x^t)}{1 - x - x^t}.$$

Explicitly

$$c_t(n) = 2\sum_{i=0}^{\lfloor \frac{n-2}{t} \rfloor} \binom{n-2-(t-1)i}{i}, \tag{2.7}$$

where $c_t(1) = 1$, $c_t(n) = 2$, $t \geq n-1$.

3. Special Classes of Compositions

The four basic shapes are given by $1c1, 1c2, 2c1$, and $2c2$, where $1c1$ represents compositions with first part 1 and last part 1, $1c2$ represents compositions with first part 1 and last part > 1, and so forth.

Let $c_t(n, k \mid P)$ be the number of t-conjugable k-compositions of n that satisfy condition P, and $c_t(n \mid P) = \sum_k c_t(n, k \mid P)$.

The lower case letter c in each enumeration function will be replaced with the upper case C to denote the corresponding enumerated set.

It is proved in [11] that

$$c_1(n \mid 1c1) = c_1(n \mid 1c2) = c_1(n \mid 2c1) = c_1(n \mid 2c2). \tag{3.1}$$

By applying t-conjugation, for any $t > 0$, we still have

$$c_t(n \mid 1c1) = c_t(n \mid 2c2),$$
$$c_t(n \mid 1c2) = c_t(n \mid 2c1).$$

However, we find that

$$c_t(n \mid 1c1) \neq c_t(n \mid 1c2), \ t > 1.$$

This inequality arises, mainly because $t + 2 < 3t - 1 \iff t > 1$.

Lemma 3.1. *The set $C_t(N \mid 1c2)$ of t-conjugable $1c2$-compositions of n admits the following decomposition:*

$$C_t(N \mid \textit{last part} = t+1 \ \textit{or} \geq 3t) \bigcup C_t(N \mid t+2 \leq \textit{last part} \leq 3t-1).$$

Note that if $t = 1$, then $C_t(N \mid t+2 \leq \text{last part} \leq 3t-1) = \emptyset$. Hence, the following result extends the first equality in (3.1) to t-conjugable compositions for all $t > 0$.

Theorem 3.2. *Let $t > 0$ be an integer.*

$$c_t(n \mid 1c1) = c_t(n - t + 1 \mid 1c2, \textit{last part} = t+1 \ \textit{or} \geq 3t).$$

Proof. Let

$$C = (1, c_2, \ldots, c_{k-1}, 1) \in C_t(n \mid 1c1).$$

Then, apply the map

$$C \longmapsto (1, c_2, \ldots, c_{k-1} + t) \in C_t(n - t + 1 \mid 1c2, \text{last part } = t + 1 \text{ or } \geq 3t).$$

Conversely, if

$$E = (1, e_2, \ldots, e_k) \in C_t(n - t + 1 \mid 1c2, \text{last part } = t + 1 \text{ or } \geq 3t),$$

then

$$E \longmapsto \begin{cases} (1, e_2, \ldots, e_{k-1}, 1^2), & e_k = t + 1, \\ (1, e_2, \ldots, e_{k-1}, e_k - t, 1), & e_k \geq 3t. \end{cases}$$

□

The following reduction identities hold.

Theorem 3.3. *Let r, t be positive integers with $t > 1$. Then, we have*

1. $c_t(n \mid 1c1, rparts > 1) = c_{t-1}(n - 2r \mid 1c1, rparts > 1)$;
2. $c_t(n \mid 1c2, rparts > 1) = c_{t-1}(n - 2r + 1 \mid 1c2, rparts > 1)$;
3. $c_t(n \mid 2c2, rparts > 1) = c_{t-1}(n - 2r + 2 \mid 2c2, rparts > 1)$.

Proof. The proof follows from the respective observations:

1. $(1^{a_1}, b_1, 1^{a_2}, \ldots, b_r, 1^{a_{r+1}}) \in C_t(n)$
   $\iff (1^{a_1}, b_1 - 2, 1^{a_2}, \ldots, b_r - 2, 1^{a_{r+1}}) \in C_{t-1}(n - 2r)$;
2. $(1^{a_1}, b_1, 1^{a_2}, \ldots, 1^{a_r}, b_r) \in C_t(n)$
   $\iff (1^{a_1}, b_1 - 2, 1^{a_2}, \ldots, 1^{a_r}, b_r - 1) \in C_{t-1}(n - 2r + 1)$.
3. $(b_1, 1^{a_1}, b_2, \ldots, 1^{a_{r-1}}, b_r) \in C_t(n)$
   $\iff (b_1 - 1, 1^{a_1}, b_2 - 2, \ldots, 1^{a_{r-1}}, b_r - 1) \in C_{t-1}(n - 2r + 2)$. □

The following assertion is an extension of Theorem 3.3 (put $v = t + 1$). It may be proved by subtracting $v - t$ and $2(v - t)$ from each boundary big part and from each interior big part, respectively.

Corollary 3.4. *Let r, t, v be positive integers, $v \geq t$. Then, we have*

1. $c_v(n \mid 1c1, rparts > 1) = c_t(n - 2r(v - t) \mid 1c1, rparts > 1)$;
2. $c_v(n \mid 1c2, rparts > 1) = c_t(n - (2r - 1)(v - t) \mid 1c2, rparts > 1)$;
3. $c_v(n \mid 2c2, rparts > 1) = c_t(n - (2r - 2)(v - t) \mid 2c2, rparts > 1)$.

We remark that the four basic shapes admit substantial refinements that may lead to the discovery of further combinatorial identities. Given integers $1 < x \leq y$, let $[x, y]$ represent any boundary part b of a composition with $x \leq b \leq y$, and write $[x]$ for $[x, x]$. Then, the shape $1c2$, for instance, assumes the following further variants (besides those obtained by reversals of the orders of parts of compositions):

$$1c[t + 1],\ 1c[t + 1, 2t - 1],\ 1c[t + 1, n],\ 1c[2t],\ 1c[2t, n],\ t > 1. \tag{3.2}$$

Similarly, the shape $2c2$ assumes at least 15 additional forms I_1cI_2, where I_1 and I_2 represent any two of the intervals appearing in (3.2).

4. Compositions with Bounded Part-Sizes

We first consider compositions in which every part-size is at least an integer v. Obviously, $c_t(n \mid \text{parts} \geq 1) = c_t(n)$. Therefore, we assume $t+1 \leq v \leq 2t$. Then

$$\sum_{n=1}^{\infty} c_t(n,k \mid \text{parts} \geq v)x^n$$

is given by

$$(x^v + x^{v+1} + \cdots)(x^{2t} + x^{2t+1} + \cdots)^{k-2}(x^v + x^{v+1} + \cdots) = \frac{x^{2v+2t(k-2)}}{(1-x)^k}$$

$$\Longrightarrow \sum_{n=1}^{\infty} c_t(n \mid \text{parts} \geq v)x^n = \frac{x^v}{1-x} + \frac{x^{2v}}{(1-x)^2}\sum_{k\geq 2}\frac{x^{2t(k-2)}}{(1-x)^{k-2}}$$
$$= \frac{x^v - x^{v+1} - x^{2t+v} + x^{2v}}{(1-x)(1-x-x^{2t})}.$$

Therefore

$$c_t(n \mid \text{parts} \geq v) = \sum_k \binom{n-2v-(2t-1)(k-2)+1}{k-1}.$$

Note that this formula is not valid for $t=1$ if $v < 2t$.

When the part-sizes are bounded from above, we consider two cases of $c_t(n,k \mid \text{parts} \leq u)$.

Case I: $u \geq 2t$. Then

$$h_t(x,n,u,k) := \sum_{n=2}^{\infty} c_t(n,k \mid \text{parts} \leq u)x^n$$

is given by

$$h_t(x,n,u,k) = (x + x^{t+1} + \cdots + x^u)(x + x^{2t} + \cdots + x^u)^{k-2}(x + x^{t+1} + \cdots + x^u)$$
$$= \frac{(x - x^2 + x^{t+1} - x^{u+1})^2}{(1-x)^2}\cdot\frac{(x - x^2 + x^{2t} - x^{u+1})^{k-2}}{(1-x)^{k-2}}.$$

Since one-part compositions are conjugable,

$$h_t(x,n,u) := \sum_{n=1}^{\infty} c_t(n \mid \text{parts} \leq u)x^n$$

becomes

$$h_t(x,n,u) = \frac{x(1-x^u)}{1-x} + \sum_{k\geq 2} h_t(x,n,u,k)$$
$$= \frac{x(1-x^u)(1-2x+x^2-x^{2t}+x^{u+1}) + x^2(1-x+x^t-x^u)^2}{(1-x)(1-2x+x^2-x^{2t}+x^{u+1})}.$$

In particular, since $h_1(x,n,2)$ counts compositions into 1s and 2s, it should be equivalent to the Fibonacci sequence, $F_n = F_{n-1} + F_{n-2}$, $n > 2$, where $F_1 = F_2 = 1$ (see [9, p. 4]):

$$h_1(x,n,2) = \frac{x(1+x)}{1-x-x^2} = 1 + \sum_{n=2}^{\infty} F_{n+1}x^n.$$

Case II: $t+1 \le u \le 2t-1$. This class concerns compositions with no interior big parts. We have

$$\begin{aligned} h_t(x,n,u,k) &= (x + x^{t+1} + \cdots + x^u)(x)^{k-2}(x + x^{t+1} + \cdots + x^u) \\ &= \frac{(x - x^2 + x^{t+1} - x^{u+1})^2 x^{k-2}}{(1-x)^2}, \quad k > 1. \end{aligned}$$

Hence

$$h_t(x,n,u) = \frac{x(1-x)^2(1-x^u) + x^2(1-x+x^t-x^u)^2}{(1-x)^3}, \quad t > 1.$$

When $u = 2t-1$, the following formula may be verified by direct enumeration:

$$c_t(n \mid \text{parts} \le 2t-1) = \begin{cases} 1, & n \ge 1; \\ 2, & 2 \le n \le t+1; \\ 2(n-t), & t+2 \le n \le 2t-1; \\ 2t-1, & 2t \le n \le 2t+1; \\ 2t-1+\binom{n-2t}{2}, & 2t+2 \le n \le 3t; \\ t^2 - \binom{4t-1-n}{2}, & 3t+1 \le n \le 4t-3; \\ t^2, & n \ge 4t-2. \end{cases}$$

The material in this section may be enlarged by bounding only interior or only boundary part-sizes. The numerous composition types encountered in Sect. 3 may also be studied under the same constraints. Analogous recommendations apply to Sects. 5 and 6, as well.

5. Compositions Avoiding a Part-Size

The enumerator here is $c_t(n,k \mid \text{parts} \ne v)$, the number of compositions with k parts that do not contain v as a part. Then, $c_t(n,k \mid \text{parts} \ne 1) = c_t(n,k \mid \text{parts} \ge t+1)$ (cf. Sect. 4). This leaves two intervals to consider.
Case I: $v \ge 2t$. Then

$$g_t(x,n,v,k) := \sum_{n=2}^{\infty} c_t(n,k \mid \text{parts} \ne v)x^n$$

is given by

$$\begin{aligned} g_t(x,n,v,k) &= (x + x^{t+1} + \cdots - x^v)(x + x^{2t} + \cdots - x^v)^{k-2}(x + x^{t+1} + \cdots - x^v) \\ &= \frac{((1-x)(x-x^v) + x^{t+1})^2((1-x)(x-x^v)+x^{2t})^{k-2}}{(1-x)^k}. \end{aligned}$$

Thus

$$g_t(x,n,v) := \sum_{n=1}^{\infty} c_t(n \mid \text{parts} \neq v)x^n$$

is given by

$$\begin{aligned} g_t(x,n,v) &= g_t(x,n,v,1) + \sum_{k\geq 2} g_t(x,n,v,k) \\ &= \frac{x - x^v + x^{v+1}}{1-x} + \frac{(x - x^2 - x^v + x^{v+1} + x^{t+1})^2}{(1-x)(1-2x+x^2+x^v-x^{v+1}-x^{2t})}. \end{aligned}$$

In particular, we recover the generating function for "compositions of n with no occurrence of k" [6]:

$$g_1(x,n,v) = \frac{x - x^v + x^{v+1}}{1-2x+x^v-x^{v+1}}, \ v > 0.$$

Case II: $t+1 < v \leq 2t-1$. This time, the interior parts remain unrestricted. We have:

$$\begin{aligned} g_t(x,n,v,k) &= (x + x^{t+1} + \cdots - x^v)(x + x^{2t} + \cdots)^{k-2}(x + x^{t+1} + \cdots - x^v) \\ &= \frac{((1-x)(x-x^v) + x^{t+1})^2(x - x^2 + x^{2t})^{k-2}}{(1-x)^k}. \end{aligned}$$

Therefore, in the usual manner, we see that $g_t(x,n,v)$ is given by:

$$\frac{(x - x^v + x^{v+1})(1-2x+x^2-x^{2t}) + (x - x^2 - x^v + x^{v+1} + x^{t+1})^2}{(1-x)(1-2x+x^2-x^{2t})}.$$

The simplest case $v = t+1$ is noteworthy:

$$g_t(x,n,t+1) = \frac{(x - x^{t+1} + x^{t+2})(1 - x - x^{2t} + x^{t+2})}{(1-x)(1-2x+x^2-x^{2t})}, \quad t > 1.$$

6. Compositions with Parts in a Residue Class

We first consider compositions into odd parts. Knowledge of the parity of t is required.

Theorem 6.1. *Let* $t \equiv r \pmod 2$, $r = 1, 2$. *Then*

$$c_t(x,r) = \sum_{n=1}^{\infty} c_t(n \mid \textit{odd parts})x^n$$

is given by

$$c_t(x,r) = \frac{x(1 - x - x^2 + x^3 - x^{2t+1}) + (x - x^3 + x^{t-r+3})^2}{(1-x^2)(1-x-x^2+x^3-x^{2t+1})}. \tag{6.1}$$

Proof. Denote the generating function of $c_t(n,k \mid \text{odd parts})$ by $c_t(x,r,k)$. Since

$$t \equiv r \pmod 2 \implies t - r + 3 \equiv 1 \pmod 2$$

and $t-r+3>t$, we have:

$$\begin{aligned}c_t(x,r,k) &= (x+x^{t-r+3}+x^{t-r+5}+\cdots)(x+x^{2t+1}+x^{2t+3}+\cdots)^{k-2}\\ &\quad\times(x+x^{t-r+3}+x^{t-r+5}+\cdots)\\ &= \frac{(x-x^3+x^{t-r+3})^2(x-x^3+x^{2t+1})^{k-2}}{(1-x^2)^k}, \quad k>1.\end{aligned}$$

Hence

$$c_t(x,r)=\sum_{k\geq 1}c_t(x,r,k)=\frac{x}{1-x^2}+\frac{(x-x^3+x^{t-r+3})^2}{(1-x^2)(1-x-x^2+x^3-x^{2t+1})},$$

which simplifies to (6.1). □

Therefore, the generating function for the number of t-conjugable compositions into odd parts, when t is odd, is given by:

$$c_t(x,1)=\frac{x(1-x-x^2+x^3-x^{2t+1})+(x-x^3+x^{t+2})^2}{(1-x^2)(1-x-x^2+x^3-x^{2t+1})}. \tag{6.2}$$

Expectedly, the case $t=1$ affirms that standard compositions into odd parts are enumerated by the Fibonacci numbers:

$$c_1(x,1)=\frac{x}{1-x-x^2}=\sum_{n=1}^{\infty}F_nx^n.$$

When t is even, we have:

$$c_t(x,2)=\frac{x-x^4+2x^{t+2}}{1-x-x^2+x^3-x^{2t+1}}. \tag{6.3}$$

However, now, the case $t=1$ gives

$$c_1(x,2)=\frac{x(1+2x^2-x^3)}{1-x-x^2}.$$

The explicit form is given by

$$[x^n]c_1(x,2)=4F_{n-2}, \quad n>3.$$

6.1. Higher Moduli

Consider the more general enumerator $c_t(n,k \mid \text{parts} \equiv 1 \pmod m)$ with generating function denoted by $c_t(x,r,m,k)$, where $t\equiv r \pmod m$, $1\leq r\leq m$, $m>1$.

(1) For boundary parts, the parity of t implies that $t-r+1+m\equiv 1 \pmod m$. This gives

$$(x+x^{t-r+1+m}+x^{t-r+1+2m}+\cdots).$$

(2a) For interior parts, first assume that $1\leq r\leq \lfloor\frac{m+1}{2}\rfloor$. Then

$$2t\equiv 2r \pmod m \implies 2t-2r+m+1\equiv 1 \pmod m,$$

where $2t-2r+m+1\geq 2t$ is minimal. Thus, the interior segment is

$$(x+x^{2t-2r+1+m}+x^{2t-2r+1+2m}+x^{2t-2r+1+3m}+\cdots).$$

(2b) If $\lfloor \frac{m+3}{2} \rfloor \leq r \leq m$, then
$$2t \equiv 2r \pmod{m} \implies 2t - 2r + 2m + 1 \equiv 1 \pmod{m},$$
where $2t - 2r + 2m + 1 > 2t$ is minimal. Thus, the interior segment is
$$(x + x^{2t-2r+1+2m} + x^{2t-2r+1+3m} + x^{2t-2r+1+4m} + \cdots).$$

Using parts (1) and (2a), we obtain, for $k > 1$:
$$\begin{aligned} c_t(x, r, m, k) &= (x + x^{t-r+1+m} + x^{t-r+1+2m} + \cdots)^2 \\ &\quad \times (x + x^{2t-2r+1+m} + x^{2t-2r+1+2m} + x^{2t-2r+1+3m} + \cdots)^{k-2} \\ &= \frac{(x - x^{m+1} + x^{t-r+1+m})^2}{(1-x^m)^2} \left(\frac{x - x^{m+1} + x^{2t-2r+1+m}}{1-x^m} \right)^{k-2}. \end{aligned}$$

Similarly, parts (1) and (2b) give:
$$c_t(x, r, m, k) = \frac{(x - x^{m+1} + x^{t-r+1+m})^2}{(1-x^m)^2} \left(\frac{x - x^{m+1} + x^{2t-2r+1+2m}}{1-x^m} \right)^{k-2}.$$

It is now a routine matter to compute
$$\sum_{n=0}^{\infty} c(n \mid \text{parts} \equiv 1 \pmod{m}) x^n = \frac{x}{(1-x^m)} + \sum_{k \geq 2} c_t(x, r, m, k)$$
in each case and establish the following theorem.

Theorem 6.2. *Let* $t \equiv r \pmod{m}$, $1 \leq r \leq m$. *Then*
$$c_t(x, r, m) = \sum_{n=0}^{\infty} c(n \mid \textit{parts} \equiv 1 \pmod{m}) x^n$$
is given by
$$c_t(x, r, m) = \frac{x(1 - x - x^m + x^{m+1} - x^{2t-2r+1+m}) + (x - x^{m+1} + x^{t-r+1+m})^2}{(1-x^m)(1 - x - x^m + x^{m+1} - x^{2t-2r+1+m})}, \tag{6.4}$$
where $1 \leq r \leq \lfloor \frac{m+1}{2} \rfloor$, *and*
$$c_t(x, r, m) = \frac{x(1 - x - x^m + x^{m+1}) + (x - x^{m+1})(x - x^{m+1} + 2x^{t-r+1+m})}{(1-x^m)(1 - x - x^m + x^{m+1} - x^{2t-2r+1+2m})}, \tag{6.5}$$
where $\lfloor \frac{m+3}{2} \rfloor \leq r \leq m$.

We illustrate Theorem 6.2 with few examples. The case $t = 1 = r$ gives the familiar generating function for compositions with parts $\equiv 1 \pmod{m}$:
$$c_1(x, 1, m) = \frac{x}{1 - x - x^m}.$$
When $m = 2$, the two parts of the theorem reduce to Eqs. (6.2) and (6.3) respectively.

When $m = 3$, Eq. (6.4) gives

$$c_t(x, r, 3) = \frac{x(1 - x - x^3 + x^4 - x^{2t-2r+4}) + (x - x^4 + x^{t-r+4})^2}{(1 - x^3)(1 - x - x^3 + x^4 - x^{2t-2r+4})}, \quad 1 \le r \le 2,$$

while Eq. (6.5) implies that

$$c_t(x, 3, 3) = \frac{x - x^5 + 2x^{t+2}}{1 - x - x^3 + x^4 - x^{2t+1}}.$$

We remark that the structure of these generating functions suggests the existence of interesting recurrence relations requiring challenging combinatorial proofs. We leave such explorations to the interested reader.

7. Inverse-t-Conjugate Compositions

The *inverse* of a composition C, denoted by $\overline{C}$, is the composition obtained by reversing the order of the parts of C. It is called *self-inverse* if $C = \overline{C}$.

It is clear that $\overline{C}^{(t)} = \overline{C^{(t)}}$ for any $t > 0$.

A t-conjugable composition C is said to be *inverse-t-conjugate* if $\overline{C} = C^{(t)}$.

For example, let $t = 3$ with $C = (1^2, 9, 1, 7, 1^3, 5)$. Then

$$C''' = (5, 1^3, 7, 1, 9, 1^2) = \overline{C}.$$

Since the conjugation operation turns strings of 1s into big parts and vice versa, it is clear that an inverse-t-conjugate composition is necessarily of type $1c2$ or $2c1$, that is, of the following form up to inversion:

$$C = (1^{a_1}, b_1, \ldots, 1^{a_r}, b_r),\ b_i \ge 2t, 1 \le i < r, b_r > t. \tag{7.1}$$

Therefore, the number b of symbolic parts of C is exactly twice the number of big parts, $b = 2r$. Thus, with $\ell(C) = k$, Proposition 2.1 gives:

$$k = n - k - (t - 1)(2r) + t,$$

that is,

$$n = 2k + 2r(t - 1) - t \equiv t \pmod 2.$$

Therefore, inverse-t-conjugate compositions are defined only for weights with the same parity as t. In other words, there is no inverse-t-conjugate composition of even weight if t is odd and vice versa. For example, it is well known that inverse-1-conjugate compositions exist only for odd weights, see [8, 11].

Note that $(1^s, t + s)$ and $(t + s, 1^s)$ are both inverse-t-conjugate compositions of $n = 2s + t,\ s > 0$. These two forms may be extended (modulo $2t$) to the following distinct long forms, up to inversion:

$$C = (1^t, 2t, \ldots, 2t, 2t); \tag{7.2}$$

$$C = (1^u, 2t, \ldots, 2t, t + u), \quad 1 \le u < t. \tag{7.3}$$

Indeed with $n = t + 2s$, if t divides s, then $n = t + (2t)r, r > 0$, which has a composition of the type (7.2). Otherwise, $n = t + (2t)(r - 1) + 2u, r > 0, u < t$, which has a composition of the type (7.3).

The conjugate of the composition C in (7.1) is

$$C^{(t)} = (a_1 + t, 1^{b_1-2t}, a_2 + 2t, 1^{b_2-2t}, \ldots, a_r + 2t, 1^{b_r-t}).$$

Thus, the conditions for C to be inverse-t-conjugate are

$$b_r = a_1 + t, \quad b_{r-1} = a_2 + 2t, \quad b_{r-2} = a_3 + 2t, \quad \ldots, \quad b_1 = a_r + 2t.$$

Hence, we have proved:

Lemma 7.1. *An inverse-t-conjugate composition C (or its inverse) has the form:*

$$C = (1^{b_r-t}, b_1, 1^{b_{r-1}-2t}, b_2, \ldots, 1^{b_2-2t}, b_{r-1}, 1^{b_1-2t}, b_r), \quad b_i \geq 2t. \quad (7.4)$$

Direct summation of the parts of C in (7.4) gives:

$$n = 2(b_1 + \cdots + b_r) - (r-1) \cdot 2t - t \equiv t \pmod 2,$$

as expected.

7.1. A Structure Theorem

In this subsection, we obtain a structure theorem that will reveal the inherent self-complementary property of inverse-t-conjugate compositions. We will need the following algebraic operations.

Let $A = (a_1, \ldots, a_i)$ and $B = (b_1, \ldots, b_j)$ be compositions.

1. Define the concatenation of A and B by $A|B = (a_1, \ldots, a_i, b_1, \ldots, b_j)$. In particular, for a nonnegative integer c, we have $A|(c) = (A, c)$, $(c)|A = (c, A)$ and $A|(1^c) = (A, 1^c)$, $(1^c)|A = (1^c, A)$ with $(A, 0) = (0, A) = A$.
2. Define the *join* of A and B by $A \uplus B = (a_1, \ldots, a_{i-1}, a_i + b_1, b_2, \ldots, b_j)$.
3. Define a unary operation $\#$ to change t copies of 1 at the boundary into a part of size t: $(\ldots, a, 1^u)^{\#} = (\ldots, a, 1^{u-t}, t)$ and ${}^{\#}(1^u, b\ldots) = (t, 1^{u-t}, b\ldots)$ for $u \geq t$.
   If $v < t$, then $(\ldots, b, 1^v)^{\#} = \emptyset$, and if $b > 1$, then $(\ldots, 1^v, b)^{\#} = (\ldots, 1^v, b)$.

The following rules are easily verified:

R1. $(A|B)' = A' \uplus B'$ and $A'|B' = (A \uplus B)'$.

R2. $\overline{A|B} = \overline{B}|\overline{A}$ and $\overline{A \uplus B} = \overline{B} \uplus \overline{A}$.

R3. $\overline{A^{\#}} = {}^{\#}\overline{A}$ and $\overline{A}^{\#} = \overline{{}^{\#}A}$.

Now, observe the following relation between the left-most and right-most segments of (7.4):

$$\overline{(1^{b_r-t}, b_1, \ldots, b_j, 1^{b_{r-j}-2t})} = (b_{r-j} - t, 1^{b_j-2t}, b_{r-j+1}, \ldots, 1^{b_1-2t}, b_r)^{(t)}, \quad (7.5)$$

with $b_{r-j} > 2t$, $0 \leq j \leq u$, $u \in \left\{\left\lfloor \frac{r}{2} \right\rfloor, \left\lfloor \frac{r+1}{2} \right\rfloor\right\}$.

Therefore, if $|C| = 2s + t$, it is possible for the weight of either side of (7.5) to be exactly s. Then

$$\begin{aligned} C &= (1^{b_r-t}, b_1, \ldots, b_j, 1^{b_{r-j}-2t}) \mid (t) \uplus (b_{r-j} - t, 1^{b_j-2t}, b_{r-j+1}, \ldots, 1^{b_1-2t}, b_r) \\ &= (1^{b_r-t}, b_1, \ldots, b_j, 1^{b_{r-j}-2t}) \mid (t) \uplus \overline{(1^{b_r-t}, b_1, \ldots, b_j, 1^{b_{r-j}-2t})}^{(t)}, \end{aligned}$$

where the last equality is obtained by conjugating (7.5).

When $b_{r-j} = 2t$, the right-hand side of (7.5) is not t-conjugable if $t > 1$. This is remedied with the $^{\#}$ operation, since

$$(t, 1^{b_j - 2t}, b_{r-j+1}, \dots) = {}^{\#}(1^{b_j - t}, b_{r-j+1}, \dots).$$

Thus, we obtain

$$\begin{aligned} C &= (1^{b_r - t}, b_1, \dots, b_j, 1^0) \mid (t) \uplus {}^{\#}(1^{b_j - t}, b_{r-j+1}, \dots, 1^{b_1 - 2t}, b_r) \\ &= (1^{b_r - t}, b_1, \dots, b_j) \mid (t) \uplus {}^{\#}\overline{(1^{b_r - t}, b_1, \dots, b_j)}^{(t)}. \end{aligned}$$

Hence, we deduce that C has the form:

$$C = A \mid (t) \uplus {}^{\#}\overline{A}^{(t)}.$$

The gist of the foregoing discussion is summarized in the following theorem.

Theorem 7.2. *Let $C = (c_1, \dots, c_k)$ be an inverse-t-conjugate composition of $n = 2s + t$, $s > 0$, or its inverse. Then, there is an index j, such that $c_1 + \cdots + c_j = s$ and $c_{j+1} + \cdots + c_k = s + t$, where*

$$\overline{(c_1, \dots, c_j)} = \begin{cases} (c_{j+1} - t, c_{j+2}, \dots, c_k)^{(t)}, & c_j = 1,\ c_{j+1} > 2t, \\ {}^{\#}(1^t, c_{j+2}, \dots, c_k)^{(t)}, & c_j > 1,\ c_{j+1} = 2t. \end{cases} \tag{7.6}$$

Thus, C can be written in the form:

$$C = A \mid (t) \uplus {}^{\#}\overline{A}^{(t)}, \tag{7.7}$$

where A is a t-conjugable composition.

Remark 7.3. Theorem 7.2 implies that the sequence of partial sums of an inverse-t-conjugate composition of $2s + t$ contains either s or $s + t$, but not both. One may use rules R2 and R3 to obtain the corresponding dissection when $t + s$ is encountered:

$$\overline{C} = \overline{A \mid (t) \uplus {}^{\#}\overline{A}^{(t)}} = \overline{(t) \uplus {}^{\#}\overline{A}^{(t)}} \mid \overline{A} = \overline{{}^{\#}\overline{A}^{(t)}} \uplus \overline{(t)} \mid \overline{A} = (A^{(t)})^{\#} \uplus (t) \mid \overline{A}.$$

Thus, on replacing A with $A^{(t)}$, we have, generally:

$$C = A^{\#} \uplus (t) \mid \overline{A}^{(t)}. \tag{7.8}$$

For example, consider the following inverse-3-conjugate compositions of 27:

- $(1^2, 9, 1, 7, 1^3, 5) = (1^2, 9, 1) \mid (3) \uplus (4, 1^3, 5) = (1^2, 9, 1) \mid (3) \uplus \overline{(1^2, 9, 1)}'''$.
- $(1^3, 9, 6, 1^3, 6) = (1^3, 9) \mid (3) \uplus (3, 1^3, 6) = (1^3, 9) \mid (3) \uplus {}^{\#}\overline{(1^3, 9)}'''$.

The inverses of these compositions may be dissected using (7.8):

- $(5, 1^3, 7, 1, 9, 1^2) = (5, 1^3, 4) \uplus (3) \mid (1, 9, 1^2) = (5, 1^3, 4) \uplus (3) \mid \overline{(5, 1^3, 4)}'''$.
- $(6, 1^3, 6, 9, 1^3) = (6, 1^6)^{\#} \uplus (3) \mid (9, 1^3) = (6, 1^6)^{\#} \uplus (3) \mid \overline{(6, 1^6)}'''$.

Corollary 7.4. *Let C be an inverse-t-conjugate composition of $2s + t$ given by (7.4). Then, exactly one of the following assertions is true:*

(i) *There is a unique composition A of s, such that $C = A \mid \overline{(A, 1^t)}^{(t)}$.*

(ii) *There is a unique composition* $B = (c_1, \ldots, c_j)$ *of* $s+t$ *with* $c_j \geq 2t$, *such that*

$$C = (c_1, \ldots, c_{j-1}, 2t) \mid \overline{(c_1, \ldots, c_{j-1}, 1^t)}^{(t)}$$

or

$$C = (c_1, \ldots, c_j) \mid \overline{(c_1, \ldots, c_j - t)}^{(t)}, \quad c_j > 2t.$$

Proof. Part (i) is obtained by eliminating the operations $\uplus$ and $^{\#}$ from Eq. (7.7), or by noting that Eq. (7.5) implies an expression of C as the concatenation of two compositions:

$C = (1^{b_r - t}, b_1, \ldots, b_j, 1^{b_{r-j}-2t}) \mid (b_{r-j}, 1^{b_j - 2t}, b_{r-j+1}, \ldots, 1^{b_1 - 2t}, b_r)$. Thus

$$C = (1^{b_r - t}, b_1, \ldots, b_j, 1^{b_{r-j}-2t}) \mid \overline{(1^{b_r - t}, b_1, \ldots, b_j, 1^{b_{r-j}-t})}^{(t)}$$

which gives (i).

The form (ii) follows easily from Eq. (7.8). □

7.2. Extending MacMahon's Bijections

In Section IV, Chapter 1 of [8], MacMahon gives a combinatorial proof of the following theorem in which "inversely conjugate" means inverse-1-conjugate.

Theorem 7.5 (MacMahon).

(i) *The self-inverse compositions of the number* $2n-1$ *are enumerated by* $c(n) = 2^{n-1}$.

(ii) *There is a one-to-one correspondence between inversely conjugate compositions of the number* $2n-1$ *and those which are self-inverse.*

We will extend this theorem in the following way:

Theorem 7.6. *The following sets of compositions are equinumerous:*

(I) *Inverse-t-conjugate compositions of* $2(m-1)+t$.

(II) *Self-inverse* t*-conjugable compositions of* $2m-1$.

(III) t*-conjugable compositions of* m.

Proof. Let the corresponding sets be denoted by:

$$(I)\colon IC_t(2(m-1)+t) \quad (II)\colon SI_t(2m-1) \quad (III)\colon C_t(m).$$

The proof will be given in the order: $(I) \Rightarrow (II) \Rightarrow (III) \Rightarrow (I)$.

$(I) \Rightarrow (II)$: Let

$$C = (c_1, \ldots, c_k) \in IC_t(2m+t-2).$$

Then, by Theorem 7.2, the partial sums of C contain exactly one of $m-1$ and $m+t-1$.

In the first case, C is of the type (7.7) with (i) in Corollary 7.4. Thus, we set

$$C \mapsto A|\overline{(A,1)} = A|(1)|\overline{A} \in SI_t(2m-1).$$

In the second case, C belongs to the type (7.8) with (ii) in Corollary 7.4. Let

$$C = (c_1, \ldots, c_k) = (c_1, \ldots, c_{j-1}, c_j - t) \uplus (t)|(c_{j+1}, \ldots, c_k).$$

Then, we set

$$C \mapsto (c_1, \ldots, c_{j-1})|(2(c_j - t) + 1)|\overline{(c_1, \ldots, c_{j-1})} \in SI_t(2m-1).$$

Note that $2(c_j - t) + 1 \geq 2t + 1$, since $c_j \geq 2t$.
$(II) \Rightarrow (III)$: Let $C \in SI_t(2m-1)$. Then, C has the form (i):

$$C = (b_1, \cdots, b_r)|(1)|\overline{(b_1, \ldots, b_r)}$$

or (ii)

$$C = B|(d)|\overline{B},$$

where $d \geq 2t + 1$ is odd.

For (i), we set

$$C \mapsto (b_1, \ldots, b_r, 1) \in C_t(m).$$

Note that if $r > 1$, then $b_r \geq 2t$, and this (interior) size is preserved in the image of C.

For (ii), we set

$$C \mapsto \left(B, \frac{d+1}{2}\right) \in C_t(m).$$

This image is t-conjugable, since $\frac{d+1}{2} \geq t + 1$.
$(III) \Rightarrow (I)$: Let $E = (e_1, \ldots, e_r) \in C_t(m)$. We invoke Corollary 7.4 in the assignments below.

If $e_r = 1$, then

$$E \mapsto (e_1, \ldots, e_{r-1})|\overline{(e_1, \ldots, e_{r-1}, 1^t)}^{(t)} \in IC_t(2m + t - 2).$$

If $e_r > t + 1$, then

$$E \mapsto (e_1, \ldots, e_r + t - 1)|\overline{(e_1, \ldots, e_r - 1)}^{(t)} \in IC_t(2m + t - 2).$$

If $e_r = t + 1$, then

$$E = (e_1, \ldots, e_{r-1}, t + 1) \mapsto (e_1 \ldots, e_{r-1}, 2t)|\overline{(e_1, \ldots, e_{r-1}, 1^t)}^{(t)}.$$

□

As an illustration of the three maps in the proof of Theorem 7.6, consider $m = 14$ and $t = 4$. Then, we have $IC_4(30) = 52 = SI_4(27) = C_4(14)$. Some of the correspondences are given in Table 1.

TABLE 1. The maps in the proof of Theorem 7.6 when $m = 14$ and $t = 4$

$IC_4(30)$	$\to$	$SI_4(27)$	$\to$	$C_4(14)$
$(1^5, 8, 8, 9)$	$\mapsto$	$(1^5, 8, 1, 8, 1^5)$	$\mapsto$	$(1^5, 8, 1)$
$(1^2, 11, 8, 1^3, 6)$	$\mapsto$	$(1^2, 11, 1, 11, 1^2)$	$\mapsto$	$(1^2, 11, 1)$
$(1, 8, 1^4, 12, 5)$	$\mapsto$	$(1, 8, 1^9, 8, 1)$	$\mapsto$	$(1, 8, 1^5)$
$(1, 9, 1^3, 11, 1, 5)$	$\mapsto$	$(1, 9, 1^7, 9, 1)$	$\mapsto$	$(1, 9, 1^4)$
$(5, 1, 11, 1^3, 9, 1)$	$\mapsto$	$(5, 1, 15, 1, 5)$	$\mapsto$	$(5, 1, 8)$
$(5, 12, 1^4, 8, 1)$	$\mapsto$	$(5, 17, 5)$	$\mapsto$	$(5, 9)$
$(6, 1^3, 8, 11, 1^2)$	$\mapsto$	$(6, 1^3, 9, 1^3, 6)$	$\mapsto$	$(6, 1^3, 5)$
$(9, 8, 8, 1^5)$	$\mapsto$	$(9, 9, 9)$	$\mapsto$	$(9, 5)$
$\cdots\cdots$	$\mapsto$	$\cdots\cdots$	$\mapsto$	$\cdots\cdots$

8. Inverse-Reciprocal Compositions

If $C = (1^{a_1}, b_1, \ldots, 1^{a_v}, b_v)$ with $r(C) \neq \emptyset$, then since $a_1 + 1 > 2$ and $a_i + 2 \geq 4$, it follows that $C' = (a_1 + 1, 1^{b_1 - 2}, \ldots, a_v + 2, 1^{b_v - 1}) \in C_2(n)$. We have the following.

Proposition 8.1 (Munagi [10]). *The number of reciprocal compositions of $n > 1$ is equal to the number of 2-conjugable compositions of n. This number is $2F_{n-1}$.*

A composition C is *inverse-reciprocal* if and only if $r(C) = \overline{C}$. For example, inverse-reciprocal compositions of 14 include $(1^7, 7)$, $(1^5, 2, 1^2, 5)$ and $(1^2, 2, 1^3, 3, 1^2, 2)$.

Using the foregoing results on inverse-t-conjugates, we obtain a 6-way identity.

Corollary 8.2. *The following sets are equinumerous for any integer $n > 1$:*

(i) *Reciprocal compositions of n.*
(ii) *2-conjugable compositions of n.*
(iii) *Inverse-2-conjugate compositions of $2n$.*
(iv) *Self-inverse 2-conjugable compositions of $2n - 1$.*
(v) *Inverse-reciprocal compositions of $2n$.*
(vi) *Inverse-1-conjugate compositions of $2n + 1$ without 2s.*
The common number of compositions in each set is $2F_{n-1}$.

Proof. (i)$\Leftrightarrow$(ii) is a restatement of Proposition 8.1.

(ii)$\Leftrightarrow$ (iii) $\Leftrightarrow$(iv) is proved as the $t = 2$ case of Theorem 7.6.

(iii)$\Leftrightarrow$(v): The 1-conjugate of an inverse-reciprocal composition is inverse-2-conjugate. To see this, note that an inverse-reciprocal composition C (or its inverse) has the form:

$$C = (1^{b_v}, b_1, 1^{b_{v-1}}, b_2, \ldots, 1^{b_2}, b_{v-1}, 1^{b_1}, b_v).$$

Then

$$C' = (b_v + 1, 1^{b_1 - 2}, b_{v-1} + 2, \ldots, b_2 + 2, 1^{b_{v-1} - 2}, b_1 + 2, 1^{b_v - 1}).$$

However

$$(C')'' = (1^{b_v-1}, b_1+2, 1^{b_{v-1}-2}, \ldots, 1^{b_2-2}, b_{v-1}+2, 1^{b_1-2}, b_v+1) = \overline{C'}.$$

(v)⇔(vi): Let C be an inverse-1-conjugate composition of $2n+1$ without 2s. Then, by (7.4), we have:

$$C = (1^{b_r-1}, b_1, 1^{b_{r-1}-2}, b_2, \ldots, 1^{b_1-2}, b_r),\ b_i > 2.$$

Consider a composition E obtained from C by subtracting 1 from each b_i and adding 1 to the length of each interior string of 1s; that is:

$$E = (1^{b_r-1}, b_1-1, 1^{b_{r-1}-1}, b_2-1, \ldots, 1^{b_1-1}, b_r-1).$$

Note that $r(E) \neq \emptyset$, since each $b_i > 2$, and $r(E) = \overline{E}$. Since the weight of E is given by $wt(E) = wt(C) - 1 = 2n$, it is an inverse-reciprocal composition of $2n$. The transition from C to E is clearly reversible. □

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Augustine O. Munagi
School of Mathematics
University of the Witwatersrand
Wits
Johannesburg 2050
South Africa
e-mail: augustine.munagi@wits.ac.za

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Marking and Shifting a Part in Partitions

Kathleen O'Hara and Dennis Stanton

Abstract. Refined versions, analytic and combinatorial, are given for classical integer partition theorems. The examples include the Rogers–Ramanujan identities, the Göllnitz–Gordon identities, Euler's odd = distinct theorem, and the Andrews–Gordon identities. Generalizations of each of these theorems are given where a single part is "marked" or weighted. This allows a single part to be replaced by a new larger part, "shifting" a part, and analogous combinatorial results are given in each case. Versions are also given for marking a sum of parts.

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1. Introduction

Many integer partition theorems can be restated as an analytic identity, as a sum equal to a product. One such example is the first Rogers–Ramanujan identity

$$\frac{1}{\prod_{k=0}^{\infty}(1-q^{5k+1})(1-q^{5k+4})} = 1+\sum_{k=1}^{\infty} q^{k^2}\frac{1}{(1-q)(1-q^2)\cdots(1-q^k)}. \tag{1.1}$$

MacMahon's combinatorial version of (1.1) uses integer partitions. The left side is the generating function for all partitions whose parts are congruent to 1 or 4 mod 5. The factor $1/(1-q^9)$ on the left side allows an arbitrary number of 9's in an integer partition. If we "mark" or weight the 9 by a w, the factor $1/(1-q^9)$ is replaced by

$$\frac{1}{1-wq^9}.$$

One may ask how the right side is modified upon marking a part, and whether a refined combinatorial interpretation exists.

The result is known [9, (2.2)], and there is a refined combinatorial version. The key to the combinatorial result is that the terms in the sum side are positive as power series in q and w.

Theorem 1.1. *Let $M \geq 1$ be any integer congruent to* 1 *or* 4 $\mod 5$. *Then*

$$\frac{1-q^M}{1-wq^M}\frac{1}{\prod_{k=0}^{\infty}(1-q^{5k+1})(1-q^{5k+4})} = 1 + q\frac{1+q+\cdots+q^{M-2}+wq^{M-1}}{1-wq^M} + \sum_{k=2}^{\infty} q^{k^2}\frac{1+q+\cdots+q^{M-1}}{1-wq^M}\frac{1}{(1-q^2)(1-q^3)\cdots(1-q^k)}.$$

Here is a combinatorial version of Theorem 1.1.

Theorem 1.2. *Let M be a positive integer which is congruent to* 1 *or* 4 $\mod 5$. *Then, the number of partitions of n into parts congruent to* 1 *or* 4 $\mod 5$ *with exactly k M's is equal to the number of partitions λ of n with difference at least* 2 *and*

1. *if λ has one part, then $\lfloor n/M \rfloor = k$,*
2. *if λ has at least two parts, then $\lfloor (\lambda_1 - \lambda_2 - 2)/M \rfloor = k$.*

The purpose of this paper is to give the analogous results for several other classical partition theorems: the Göllnitz–Gordon identities, Euler's odd = distinct theorem, and the Andrews–Gordon identities. The main engine, Proposition 3.1, may be applied to many other single sum identities. The results obtained here by marking a part are refinements of the corresponding classical results.

We shall also consider "shifting" a part, for example replacing all 9's by 22's in (1.1). This is replacing the factor

$$\frac{1}{1-q^9} \quad \text{by} \quad \frac{1}{1-q^{22}}.$$

We shall see that the set of partitions enumerated by the sum side is an explicit subset of the partitions in the original identity.

Finally in Sect. 6, we consider marking a sum of parts. We can extend Theorem 1.2 to allow other values of M, for example $M = 7$, by marking the partition $6 + 1$. See Corollary 6.8.

We use the standard notation,

$$(A;q)_k = \prod_{j=0}^{k-1}(1-Aq^j), \quad [M]_q = \frac{1-q^M}{1-q}.$$

If the base q is understood, we may write $(A;q)_k$ as $(A)_k$.

2. The Rogers–Ramanujan Identities

In this section, we give prototypical examples for the Rogers–Ramanujan identities.

First, we state a marked version of the second Rogers–Ramanujan identity, which follows from Proposition 3.1.

Theorem 2.1. *Let $M \geq 2$ be any integer congruent to* 2 *or* 3 mod 5*. Then*

$$\frac{1-q^M}{1-wq^M}\frac{1}{(q^2;q^5)_\infty(q^3;q^5)_\infty} = 1+q^2\frac{[M-2]_q+wq^{M-2}+q^{M-1}}{1-wq^M} + \sum_{k=2}^{\infty} q^{k^2+k}\frac{[M]_q}{1-wq^M}\frac{1}{(q^2;q)_{k-1}}.$$

Here is a combinatorial version of Theorem 2.1.

Theorem 2.2. *Let M be a positive integer which is congruent to* 2 *or* 3 mod 5*. Then, the number of partitions of n into parts congruent to* 2 *or* 3 mod 5 *with exactly k M's is equal to the number of partitions λ of n with difference at least* 2*, no* 1*'s, and*

1. *if λ has one part, then $n = Mk+j$, $2 \leq j \leq M-1$, or $j=0$ or $j=M+1$,*
2. *if λ has at least two parts, then $\lfloor(\lambda_1-\lambda_2-2)/M\rfloor = k$.*

Proof. We simultaneously prove Theorems 1.2 and 2.2. We need to understand the combinatorics of the replacement in the k-th term on the sum side

$$\frac{1}{1-q} \to \frac{[M]_q}{1-wq^M} = \sum_{p=0}^{\infty} q^p w^{\lfloor p/M\rfloor}. \tag{2.1}$$

In the classical Rogers–Ramanujan identities, the factor $1/(1-q)$ represents the difference in the first two parts after the double staircase has been removed. This is the second case of each theorem. $\square$

Example 2.3. Let $k = 2$, $M = 7$, and $n = 22$. The equinumerous sets of partitions for Theorem 2.2 are

$$\{(8,7,7),(7,7,3,3,2),(7,7,2,2,2,2)\} \leftrightarrow \{(22),(20,2),(19,3)\}.$$

Equivalent combinatorial versions of Theorems 1.2 and 2.2 may be given (see [9, Theorem 2, Theorem 3]). This time, the terms $k \geq M$ of the sum side are considered, and the replacement considered is

$$\frac{1}{1-q^M} \to \frac{1}{1-wq^M},$$

namely, the part M is marked on the sum side. We need notation for when a double staircase is removed from a partition with difference at least two.

Definition 2.4. For any partition λ with k parts whose difference of parts is at least 2, let λ^* denote the partition obtained upon removing the double staircase $(2k-1, 2k-3, \ldots, 1)$ from λ, and reading the result by columns.

For any partition λ with k parts and no 1's whose difference of parts is at least 2, let λ^{**} denote the partition obtained upon removing the double staircase $(2k, 2k-2, \ldots, 2)$ from λ, and reading the result by columns.

Theorem 2.5. *Let M be a positive integer which is congruent to* 1 *or* 4 mod 5*. Then, the number of partitions of n into parts congruent to* 1 *or* 4 mod 5 *with exactly k M's is equal to the number of partitions λ of n with difference at least* 2 *and*

1. *if* λ *has one part, then* $\lfloor n/M \rfloor = k$,
2. *if* λ *has between two and* $M-1$ *parts, then* $\lfloor (\lambda_1 - \lambda_2 - 2)/M \rfloor = k$,
3. *if* λ *has at least* M *parts, then* λ^* *has exactly* k M*'s.*

Example 2.6. Let $k = 2$, $M = 4$, and $n = 24$. The equinumerous sets of partitions for Theorem 2.5 are

$$\{(16,4^2),(14,4^2,1^2),(11,4^2,1^5),(9,6,4^2,1),(9,4^2,1^7),(6,6,4^2,1^4),$$
$$(6,4^2,1^{10}),(4^2,1^{16})\} \leftrightarrow$$
$$\{(9,7,5,3),(18,5,1),(17,6,1),(17,5,2),(16,6,2),(16,5,3),(17,7),(18,6)\}.$$

Theorem 2.7. *Let* M *be a positive integer which is congruent to* 2 *or* $3 \mod 5$. *Then, the number of partitions of* n *into parts congruent to* 2 *or* $3 \mod 5$ *with exactly* k M*'s is equal to the number of partitions* λ *of* n *with difference at least* 2*, no* 1*'s and*

1. *if* λ *has one part, then* $n = Mk + j$*, where* $2 \le j \le M-1$*, or* $j = 0$ *or* $j = M+1$,
2. *if* λ *has between two and* $M-1$ *parts, then* $\lfloor (\lambda_1 - \lambda_2 - 2)/M \rfloor = k$,
3. *if* λ *has at least* M *parts, then* λ^{**} *has exactly* k M*'s.*

3. A General Expansion

In this section, we give a general expansion, Proposition 3.1, for marking a single part.

Many partition identities have a sum side of the form

$$\sum_{j=0}^{\infty} \frac{\alpha_j}{(q;q)_j},$$

where α_j has non-negative coefficients as a power series in q.

These include

1. the Rogers–Ramanujan identities, $\alpha_j = q^{j^2}$ or q^{j^2+j},
2. Euler's odd=distinct theorem, $\alpha_j = q^{\binom{j+1}{2}}$,
3. the Göllnitz–Gordon identities, q replaced by q^2, $\alpha_j = q^{j^2}(-q;q^2)_j$,
4. all partitions by the largest part, $\alpha_j = q^j$,
5. all partitions by Durfee square, $\alpha_j = q^{j^2}/(q;q)_j$.

A part of size M may be marked in general using the next proposition.

Proposition 3.1. *For any positive integer* M*, if* $\alpha_0 = 1$,

$$\frac{1-q^M}{1-wq^M} \sum_{j=0}^{\infty} \frac{\alpha_j}{(q;q)_j} = 1 + \frac{\alpha_1 [M]_q - q^M + wq^M}{1-wq^M} + \sum_{j=2}^{\infty} \frac{[M]_q}{1-wq^M} \frac{\alpha_j}{(q^2;q)_{j-1}}.$$

As long as α_1 has the property that

$$\alpha_1 [M]_q - q^M$$

is a positive power series in q, the right side has a combinatorial interpretation.

There are two possible elementary combinatorial interpretations. For any $j \geq 2$, the factor

$$\frac{[M]_q}{1-wq^M} = \sum_{p=0}^{\infty} q^p w^{[p/M]}$$

replaces $1/(1-q)$, which accounts for parts of size 1 in a partition. This is a weighted form of the number of 1's.

The second interpretation holds for terms with $j \geq M$. Here

$$\frac{[M]_q}{1-wq^M}\frac{1}{(q^2;q)_{j-1}} = \frac{1}{(1-q)\cdots(1-q^{M-1})(1-wq^M)(1-q^{M+1})\cdots(1-q^j)}.$$

In this case, the part of size M is marked by w.

For a particular combinatorial application of Proposition 3.1, one must realize what the denominator factors $(1-q)$ and $(1-q^M)$ represent on the sum side. For example, in the first Rogers–Ramanujan identity, these factors account for 1's and M's in λ^*. Since

$$(\#1's \text{ in } \lambda^*) = \lambda_1 - \lambda_2 - 2,$$

the two interpretations are Theorems 1.2 and 2.5.

3.1. Distinct Parts

Choosing $\alpha_j = q^{\binom{j+1}{2}}$ in Proposition 3.1 gives distinct partitions, which by Euler's theorem are equinumerous with partitions into odd parts. Here is the marked version.

Corollary 3.2. *For any odd positive integer M,*

$$\frac{1}{(1-q)(1-q^3)\cdots(1-q^{M-2})(1-wq^M)(1-q^{M+2})\cdots} = 1 + \frac{q+q^2+\cdots+q^{M-1}+wq^M}{1-wq^M} + \sum_{j=2}^{\infty}\frac{q^{\binom{j+1}{2}}}{(q^2;q)_{j-1}}\frac{[M]_q}{1-wq^M}.$$

Definition 3.3. For any partition λ with j distinct parts, let λ^{St} be the partition obtained upon removing a staircase $(j, j-1, \ldots, 1)$ from λ, and reading the result by columns.

Example 3.4. If $\lambda = (8,7,3,1)$, then $\lambda^{St} = (3,2,2,2)$.

Here is the combinatorial version of Corollary 3.2, generalizing Euler's theorem.

Theorem 3.5. *For any odd positive integer M, the number of partitions of n into odd parts with exactly k parts of size M is equal to the number of partitions λ of n into distinct parts such that*

1. *if λ has one part, then $\lfloor n/M \rfloor = k$,*
2. *if λ has at least two parts, then $\lfloor (\lambda_1 - \lambda_2 - 1)/M \rfloor = k$.*

Example 3.6. Let $k = 2$, $M = 5$, and $n = 18$. The equinumerous sets of partitions for Theorem 3.5 are

$$\{(7,5,5,1),(5,5,3,3,1,1),(5,5,3,1^5),(5,5,1^8)\}$$
$$\leftrightarrow \{(16,2),(15,3),(15,2,1),(14,3,1)\}.$$

Proposition 3.7. *There is an M-version of the Sylvester "fishhook" bijection which proves Theorem 3.5.*

Proof. Let FH be the fishhook bijection from partitions with distinct parts to partitions with odd parts. If $FH(\lambda) = \mu$, it is known that the number of 1's in μ is $\lambda_1 - \lambda_2 - 1$, except for $FH(n) = 1^n$. This proves Theorem 3.5 if $M = 1$, and FH is the bijection for $M = 1$.

For the M-version, $M > 1$, let λ have distinct parts. For $\lambda = n$ a single part, define the M-version by $FH^M(n) = (M^k, 1^{n-kM})$, which has k parts of size M. Otherwise, λ has at least two parts, and

$$kM \le \lambda_1 - \lambda_2 - 1 \le (k+1)M - 1.$$

Let θ be the partition with distinct parts where λ_1 has been reduced by kM,

$$0 \le \theta_1 - \theta_2 - 1 \le M - 1.$$

Finally put $\gamma = FH(\theta)$, and note that γ has at most $M - 1$ 1's.

There are two cases. If γ has no parts of size M, define $FH^M(\lambda) = \gamma \cup M^k$, so that $FH^M(\lambda)$ is a partition with odd parts, exactly k parts of size M, and at most $M - 1$ 1's.

If γ has $r \ge 1$ parts of size M, change all of them to rM 1's to obtain γ' with at least M 1's. Then put $FH^M(\lambda) = \gamma' \cup M^k$, so that $FH^M(\lambda)$ is a partition with odd parts, exactly k parts of size M, and at least M 1's. ☐

Theorem 3.8. *For any odd positive integer M, the number of partitions of n into odd parts with exactly k parts of size M, is equal to the number of partitions λ of n into distinct parts such that*

1. *if λ has one part, then $\lfloor n/M \rfloor = k$,*
2. *if λ has between two and $M - 1$ parts, then $\lfloor (\lambda_1 - \lambda_2 - 1)/M \rfloor = k$,*
3. *if λ has at least M parts, then λ^{St} has exactly k M's.*

Example 3.9. Let $k = 2$, $M = 3$, and $n = 18$. The equinumerous sets of partitions for Theorem 3.8 are

$$\{(11,3^2,1),(9,3^2,1^3),(7,5,3^2),(7,3^2,1^5),(5^2,3^2,1^2),(5,3^2,1^7),(3^2,1^{12})\}$$
$$\leftrightarrow \{(7,6,4,1),(8,5,4,1),(11,4,3),(10,5,3),(9,6,3),(8,7,3),(13,5)\}.$$

3.2. Göllnitz–Gordon Identities

The Göllnitz–Gordon identities are (see [1,5,6])

$$\sum_{n=0}^{\infty} q^{n^2} \frac{(-q;q^2)_n}{(q^2;q^2)_n} = \frac{1}{(q;q^8)_\infty (q^4;q^8)_\infty (q^7;q^8)_\infty}, \tag{3.1}$$

$$\sum_{n=0}^{\infty} q^{n^2+2n} \frac{(-q;q^2)_n}{(q^2;q^2)_n} = \frac{1}{(q^3;q^8)_\infty (q^4;q^8)_\infty (q^5;q^8)_\infty}. \tag{3.2}$$

We apply Proposition 3.1 with q replaced by q^2, M replaced by $M/2$, and $\alpha_j = q^{j^2}(-q;q^2)_j$ to obtain the next result.

Corollary 3.10. *Let M be a positive integer. Then*

$$\frac{1-q^M}{1-wq^M}\frac{1}{(q;q^8)_\infty(q^4;q^8)_\infty(q^7;q^8)_\infty} = 1 + \frac{q[M-1]_q + wq^M}{1-wq^M} + \sum_{j=2}^{\infty} q^{j^2}\frac{[M]_q}{1-wq^M}\frac{(-q^3;q^2)_{j-1}}{(q^4;q^2)_{j-1}}.$$

We used

$$\frac{q(1+q)[M/2]_{q^2} - q^M + wq^M}{1-wq^M} = \frac{q[M-1]_q + wq^M}{1-wq^M}$$

to simplify the second term in the sum in Corollary 3.10. Note that the numerator has positive coefficients, and thus a simple combinatorial interpretation.

Here is the combinatorial restatement [6, Theorem 2] of the first Göllnitz–Gordon identity.

Theorem 3.11. *The number of partitions of n into parts congruent to $1, 4$, or $7 \mod 8$ is equal to the number of partitions of n into parts whose difference is at least 2, and greater than 2 for consecutive even parts.*

For the combinatorial version of Corollary 3.10, we need to recall why the sum side of (3.1) is the generating function for the restricted partitions with difference at least 2. In particular, we must identify what the denominator factor $1-q$ represents in the sum side.

Suppose λ is such a partition with j parts. This is equivalent to showing that the generating function for λ^* is

$$\frac{(-q;q^2)_j}{(q^2;q^2)_j} = \frac{1+q}{1-q^2}\frac{(-q^3;q^2)_{j-1}}{(q^4;q^2)_{j-1}}. \tag{3.3}$$

The partition $\mu = \lambda - (2j-1, 2j-3, \ldots, 1)$ has at most j parts, and the odd parts of μ are distinct. The column read version $\lambda^* = \mu^t$ can be built in the following way. Take arbitrary parts from sizes $j, j-1, \ldots, 1$ with even multiplicity, whose generating function is $1/(q^2;q^2)_j$. The rows now have even length. Then, choose a subset of the odd integers $1+0, 2+1, \ldots, j+(j-1)$. For each such odd part $k+(k-1)$ add columns of length k and $k-1$. This keeps all rows even, except the k-th row which is odd, and distinct.

We see that the factor $(1+q)/(1-q^2) = 1/(1-q)$ in (3.3) accounts for 1's in λ^*. In Corollary 3.10, this quotient is replaced by

$$\frac{1+q}{1-q^2} \to \frac{[M]_q}{1-wq^M} = \sum_{p=0}^{\infty} q^p w^{[p/M]}.$$

There is one final opportunity for a 1 to appear in λ^*: when $3 = 2+1$ is chosen as an odd part. This occurs only when the second part of λ is even.

Theorem 3.12. *Let M be a positive integer which is congruent to* $1, 4$ *or* 7 mod 8. *The number of partitions of $n \geq 1$ into parts congruent to* $1, 4$ *or* 7 mod 8 *with exactly k M's, is equal to the number of partitions λ of n into parts whose difference is at least 2, and greater than 2 for consecutive even parts such that*

1. *if λ has a single part, then $[n/M] = k$,*
2. *if λ has at least two parts and the second part of λ is even, then*
$$\lfloor (\lambda_1 - \lambda_2 - 3)/M \rfloor = k,$$
3. *if λ has at least two parts and the second part of λ is odd, then*
$$\lfloor (\lambda_1 - \lambda_2 - 2)/M \rfloor = k.$$

Example 3.13. Let $k = 3$, $M = 7$, and $n = 31$. The equinumerous sets of partitions for Theorem 3.12 are

$$\begin{aligned}&\{(9,7,7,7,1),(7,7,7,4,4,1,1),(7,7,7,4,1^6),(7,7,7,1^{10})\}\\ &\quad \leftrightarrow \{(30,1),(29,2),(28,3),(27,3,1)\}.\end{aligned}$$

Note that $\lambda = (27, 4)$ is not allowed because the second part of λ is even.

For the second Göllnitz–Gordon identity, the version of Corollary 3.10 is

$$\begin{aligned}&\frac{1-q^M}{1-wq^M}\frac{1}{(q^3;q^8)_\infty(q^4;q^8)_\infty(q^5;q^8)_\infty}\\ &\quad = 1 + \frac{q^3+\cdots+q^{M-1}+wq^M+q^{M+1}+q^{M+2}}{1-wq^M}\\ &\quad\quad + \sum_{j=2}^{\infty} q^{j^2+2j}\frac{[M]_q}{1-wq^M}\frac{(-q^3;q^2)_{j-1}}{(q^4;q^2)_{j-1}}.\end{aligned} \tag{3.4}$$

Here is the combinatorial refinement of [6, Theorem 3].

Theorem 3.14. *Let M be a positive integer which is congruent to* $3, 4$ *or* 5 mod 8. *The number of partitions of $n \geq 1$ into parts congruent to* $3, 4$ *or* 5 mod 8 *with exactly k M's, is equal to the number of partitions λ of n into parts whose difference is at least 2, greater than 2 for consecutive even parts, smallest part at least 3, such that*

1. *if λ has a single part, then $n = Mk$, or $n = Mk + j$, $3 \leq j \leq M + 2$, $j \neq M$,*
2. *if λ has at least two parts and the second part of λ is even, then*
$$\lfloor (\lambda_1 - \lambda_2 - 3)/M \rfloor = k,$$
3. *if λ has at least two parts and the second part of λ is odd, then*
$$\lfloor (\lambda_1 - \lambda_2 - 2)/M \rfloor = k.$$

4. An Andrews–Gordon Version

The Andrews–Gordon identities are

Theorem 4.1. *If* $0 \le a \le k$, *then*

$$\frac{(q^{k+1-a}, q^{k+2+a}, q^{2k+3}; q)_\infty}{(q;q)_\infty} = \sum_{n_1 \ge n_2 \ge \cdots \ge n_k \ge 0} \frac{q^{n_1^2+n_2^2+\cdots+n_k^2+n_{k+1-a}+\cdots+n_k}}{(q)_{n_1-n_2}\cdots(q)_{n_{k-1}-n_k}(q)_{n_k}}.$$

The Rogers–Ramanujan identities are the cases $k=1$, $a=0,1$.

Because Theorem 4.1 has a multisum instead of a single sum, we cannot apply Proposition 3.1. Nonetheless, the same idea can be applied to obtain a marked version of Theorem 4.1.

Let F_k^a denote the right-side multisum of Theorem 4.1 for $0 \le a \le k$, and let $F_k^a = F_k^0$ for $a < 0$. So we have

$$F_k^a = F_{k-1}^{a-1} + \sum_{n_1 \ge n_2 \ge \cdots \ge n_k \ge 1} \frac{q^{n_1^2+n_2^2+\cdots+n_k^2+n_{k+1-a}+\cdots+n_k}}{(q)_{n_1-n_2}\cdots(q)_{n_{k-1}-n_k}(q)_{n_k}}.$$

Multiplying by $\frac{1-q^M}{1-wq^M}$ yields

$$\frac{1-q^M}{1-wq^M}F_k^a = \frac{1-q^M}{1-wq^M}F_{k-1}^{a-1}$$
$$+ \sum_{n_1 \ge n_2 \ge \cdots \ge n_k \ge 1} \frac{q^{n_1^2+n_2^2+\cdots+n_k^2+n_{k+1-a}+\cdots+n_k}}{(q)_{n_1-n_2}\cdots(q)_{n_{k-1}-n_k}(q^2;q)_{n_k-1}} \frac{[M]_q}{1-wq^M},$$

which, upon iterating, is the following weighted version of the Andrews–Gordon identities.

Theorem 4.2. *For* $0 \le a \le k$, *let* M *be any positive integer not congruent to* 0, $\pm(k+1-a)$ *modulo* $2k+3$. *Then,*

$$\frac{1-q^M}{1-wq^M}\frac{(q^{k+1-a}, q^{k+2+a}, q^{2k+3}; q)_\infty}{(q;q)_\infty} = 1 + A + \sum_{n_1=2}^{\infty} \frac{q^{n_1^2+B}}{(q^2;q)_{n_1-1}} \frac{[M]_q}{1-wq^M}$$
$$+ \sum_{r=2}^{k} \sum_{n_1 \ge n_2 \ge \cdots \ge n_r \ge 1} \frac{q^{n_1^2+n_2^2+\cdots+n_r^2+n_{k+1-a}+\cdots+n_r}}{(q)_{n_1-n_2}\cdots(q)_{n_{r-1}-n_r}(q^2;q)_{n_r-1}} \frac{[M]_q}{1-wq^M},$$

where

1. *for* $0 \le a < k$, $\quad B = 0$, $\quad A = q([M-1]_q + wq^{M-1})/(1-wq^M)$,
2. *for* $a = k$, $\quad B = n_1$, $\quad A = q^2([M-2]_q + wq^{M-2} + q^{M-1})/(1-wq^M)$.

For a combinatorial version of Theorem 4.2, we use Andrews' Durfee dissections, and $(k+1, k+1-a)$-admissible partitions, see [2].

Definition 4.3. Let k be a positive integer and $0 \le a \le k$. A partition λ is called $(k+1, k+1-a)$-admissible if λ may be dissected by $r \le k$ successive Durfee rectangles, moving down, of sizes

$$n_1 \times n_1, \ldots, n_{k-a} \times n_{k-a},\ (n_{k-a+1}+1) \times n_{k-a}, \ldots, (n_r+1) \times n_r$$

such that the $(n_1 + n_2 + \cdots + n_{k-a+i} + i)$-th part of λ is n_{k-a+i}, for $1 \leq i \leq r-(k-a)$.

Note that $r \leq k-a$ is allowed, in which case all of the Durfee rectangles are squares. Also, the parts of λ to the right of the Durfee rectangles are not constrained, except at the last row of the non-square Durfee rectangle, where it is empty.

Example 4.4. Suppose $k = 3$ and $a = 2$. Then $\lambda = 91$ is not $(4,2)$-admissible: the Durfee square has size $n_1 = 1$, but the next Durfee rectangle of size 2×1 does not exist, so the second part cannot be covered if $r \geq 2$.

Theorem 2 in [2] interprets Theorem 4.1.

Proposition 4.5. *The generating function for all partitions which are* $(k+1, k+1-a)$*-admissible is given by the sum in Theorem* 4.1*.*

We need to understand the replacement

$$\frac{1}{(q)_{n_r}} = \frac{1}{(1-q)(q^2;q)_{n_r-1}} \to \frac{1}{(q^2;q)_{n_r-1}} \frac{[M]_q}{1-wq^M}$$

in the factor $(q)_{n_r}$ to give a combinatorial version of Theorem 4.2.

First, we recall [2] that if the sizes of the Durfee rectangles are fixed by $n_1, n_2, \ldots, n_r$, then the generating function for the partitions which have this Durfee dissection is

$$\frac{1}{(q)_{n_1}} \prod_{j=1}^{r-1} \begin{bmatrix} n_j \\ n_{j+1} \end{bmatrix}_q = \frac{1}{(q)_{n_r}} \prod_{j=1}^{r-1} \frac{1}{(q)_{n_j - n_{j+1}}}.$$

(A simple bijection for this fact is given in [7].) Upon multiplying by

$$\frac{1-q^M}{1-wq^M},$$

we have

$$\frac{[M]_q}{1-wq^M} \frac{1}{(q^2;q)_{n_1-1}} \prod_{j=1}^{r-1} \begin{bmatrix} n_j \\ n_{j+1} \end{bmatrix}_q = \frac{[M]_q}{1-wq^M} \frac{1}{(q^2;q)_{n_r-1}} \prod_{j=1}^{r-1} \frac{1}{(q)_{n_j-n_{j+1}}}.$$

Consider the factor $1/(q)_{n_1}$, which accounts for the portion of the partition to the right of the first Durfee rectangle of λ. In this factor, we are replacing

$$\frac{1}{1-q} \to \frac{[M]_q}{1-wq^M}.$$

As before, the M 1's in the columns to the right of the first Durfee rectangle are weighted by w. These 1's are again a difference in the first two parts of λ.

Putting these pieces together, the following result is a combinatorial restatement of Theorem 4.2 (Table 1).

TABLE 1. Theorem 4.6 when $n = 10, a = 2, k = 3, M = 3$

Partition without 2,7,9	# of 3's	(4,2)-Admissible partition	Value of j
10	0	10	3
811	0	61,111	1
64	0	421,111	0
631	1	322,111	0
61,111	0	331,111	0
55	0	811	2
541	0	6211	1
5311	1	5311	0
511,111	0	5221	1
4411	0	22,222	0
433	2	4411	0
43,111	1	4321	0
4,111,111	0	82	2
3331	3	73	1
331,111	2	64	0
31,111,111	1	55	0
1,111,111,111	0	433	0

Theorem 4.6. *Fix integers a, k, M satisfying $0 \le a \le k$ and $M \not\equiv 0, \pm(k+1-a) \mod 2k+3$. The number of partitions of n into parts not congruent to 0, $\pm(k+1-a) \mod 2k+3$ with exactly j M's, is equal to the number of partitions λ of n which are $(k+1, k+1-a)$-admissible with $r \le k$ Durfee rectangles of sizes*

$$n_1 \times n_1, \ldots, n_{k-a} \times n_{k-a}, (n_{k-a+1} + 1) \times n_{k-a}, \ldots, (n_r + 1) \times n_r$$

of the following form:

1. *if $r = n_1 = 1$, and $0 \le a < k$, then λ is a single part of size Mj, $Mj + 1, \ldots, Mj + (M - 1)$,*
2. *if $r = n_1 = 1$, and $a = k$, then $\lambda = (\lambda_1, 1)$ has size Mj, $Mj+2, \ldots, Mj+(M-1)$, or $Mj + (M + 1)$,*
3. *if $n_1 = 1$ and $r \ge 2$, then $\lfloor(\lambda_1 - n_1)/M\rfloor = j$,*
4. *if $n_1 \ge 2$, then $\lfloor(\lambda_1 - \lambda_2)/M\rfloor = j$.*

5. Shifting a Part

The weighted versions allow one to shift a part. For example in the first Rogers–Ramanujan identity, what happens if parts of size 11 are replaced by parts of size 28? All we need to do is to choose $M = 11$ and $x = q^{17}$ in Theorem 1.1.

Corollary 5.1. *Let M be a positive integer which is congruent to 1 or 4 modulo 5. Let $N > M$ be an integer not congruent to 1 or 4 modulo 5. The number of partitions of n into parts congruent to 1 or 4 modulo 5, except M, or parts of* *size N, is equal to the number of partitions λ of n with difference at least 2, such that*

1. *λ has a single part, which is congruent to* $0, 1, \ldots,$ *or* $M-1 \mod N$,
2. *λ has at least two parts, and $\lambda_1 - \lambda_2 - 2$ is congruent to* $0, 1, \ldots,$ *or* $M-1 \mod N$.

Example 5.2. Let $N = 8$, $M = 4$, and $n = 9$. The equinumerous sets of partitions for Corollary 5.1 are

$$\{(9), (6,1,1,1), (8,1), (1^9)\} \leftrightarrow \{(9), (6,3), (7,2), (5,3,1)\}.$$

A related example occurs when two parts are shifted: 1 and 4 are replaced by 2 and 3. The appropriate identity is

$$\frac{1}{(1-q^2)(1-q^3)(q^6;q^5)_\infty (q^9;q^5)_\infty} = 1 + \frac{q^2(1+q)}{1-q^3} + \sum_{k=2}^{\infty} \frac{q^{k^2}}{(q^2;q)_{k-1}} \frac{1+q^2}{1-q^3}. \tag{5.1}$$

Theorem 5.3. *The number of partitions of n into parts from*

$$\{2, 3, 5k+1, 5k+4 : k \geq 1\}$$

is equal to the number of partitions λ of n with difference at least 2 and

1. *if λ has a single part, then $n \not\equiv 1 \mod 3$,*
2. *if λ has at least two parts, then $(\lambda_1 - \lambda_2 - 2) \not\equiv 1 \mod 3$.*

Example 5.4. Let $n = 13$. The two equinumerous sets of partitions in Theorem 5.3 are

$$\{(11,2), (9,2,2), (6,3,2,2), (3,2,2,2,2,2), (3,3,3,2,2)\}$$
$$\leftrightarrow \{(12,1), (10,3), (9,4), (8,4,1), (7,5,1)\}.$$

The possible partitions with difference at least 2

$$\{(13), (11,2), (8,5), (9,3,1), (7,4,2)\}$$

are disallowed.

Corollary 5.5. *Let M be an odd positive integer. Let $N > M$ be an even integer. The number of partitions of n into odd parts except M, or parts of size N, is equal to the number of partitions λ of n into distinct parts, such that*

1. *λ has a single part, which is congruent to* $0, 1, \ldots,$ *or* $M-1 \mod N$,
2. *λ has at least two parts, and $\lambda_1 - \lambda_2 - 1$ is congruent to* $0, 1, \ldots,$ *or* $M-1 \mod N$.

Example 5.6. If $N = 8$, $M = 3$ and $n = 9$, the equinumerous sets in Corollary 5.5 are

$$\{(9), (8,1), (7,1,1), (5,1^4), (1^9)\} \leftrightarrow \{(9), (5,4), (6,3), (5,3,1), (4,3,2)\}.$$

6. Marking a Sum of Parts

One may ask if Theorems 1.1 and 2.1 have combinatorial interpretations without the modular conditions on M. The sum sides retain the interpretations given by Theorems 1.2 and 2.2 and are positive as a power series in q and w. It remains to understand what the product side represents as a generating function of partitions. We give in Proposition 6.4 a general positive combinatorial expansion for the product side. We call this "marking a sum of parts".

As an example suppose that $M = A + B$ is a sum of two parts, where A and B are distinct integers congruent to 1 or 4 mod 5. The quotient in the product side of Theorem 1.1

$$\frac{1-q^{A+B}}{1-wq^{A+B}}\frac{1}{(1-q^A)(1-q^B)} = \frac{1}{(1-q^B)(1-wq^{A+B})} + \frac{q^A}{(1-q^A)(1-wq^{A+B})}$$

is a generating function for partitions with parts A or B. The first term allows the number of B's to be at least as many as the number of A's. The second term allows the number of A's to be greater than the number of B's. The exponent of w is the number of times a pair AB appears in a partition. For example, if $A = 6$, $B = 4$, the partition $(6, 6, 4, 4, 4, 4)$ contains 64 twice, along with two 4's. We have found a prototypical result.

Proposition 6.1. *Let $M = A + B$ for some $A, B \equiv 1, 4 \mod 5, A \neq B$. Then*

$$\frac{1-q^M}{1-wq^M}\frac{1}{(q;q^5)_\infty(q^4;q^5)_\infty}$$

is the generating function for all partitions μ with parts $\equiv 1, 4 \mod 5$ by the number of occurrences of the pair AB.

A more general statement holds for partitions other than $M = A + B$. To state this result, we need to define an analog of the multiplicity of a single part to a multiplicity of a partition. We again use the multiplicity notation for a partition, for example $(7^3, 4^1, 2^3)$ denotes the partition $(7, 7, 7, 4, 2, 2, 2)$.

Definition 6.2. Let $\lambda = (A_1^{m_1}, \ldots, A_k^{m_k})$ be a partition. We say λ is inside μ k times, $k = E_\lambda(\mu)$, if

$$k = \max\{j : j \geq 0, \mu \text{ contains at least } jm_s \text{ parts of size } A_s \text{ for all } s\}.$$

Example 6.3. Let $\lambda = (6^1, 4^2, 1^1)$, $\mu = (9^1, 6^7, 4^5, 1^8)$. Then $E_\lambda(\mu) = 2$ but not 3 because μ contains only five 4's.

With this definition, Proposition 6.1 holds for any partition. We let $||\lambda||$ denote the sum of the parts of λ.

Proposition 6.4. *Let $\lambda \vdash M$ be a fixed partition into parts congruent to* 1 *or* 4 mod 5. *Then*

$$\frac{1-q^M}{1-wq^M}\frac{1}{(q;q^5)_\infty(q^4;q^5)_\infty}$$

is the generating function for all partitions μ into parts congruent to 1 *or* 4 mod 5,

$$\sum_{\mu} q^{||\mu||} w^{E_\lambda(\mu)},$$

where $E_\lambda(\mu)$ is the number of times λ appears in μ.

The modular condition on the parts in Proposition 6.4 is irrelevant.

Proposition 6.5. *Let $\mathbb{A} = \{A_1, A_2, \ldots\}$ be any set of positive integers. Suppose that $\lambda = (B_1^{m_1}, \ldots, B_k^{m_k})$ is a partition whose parts come from $\mathbb{A}$ and $M = \sum_{i=1}^{k} m_i B_i$. Then*

$$\frac{1-q^M}{1-wq^M} \prod_{i=1}^{\infty} (1-q^{A_i})^{-1}$$

is the generating function for all partitions μ with parts from $\mathbb{A}$

$$\sum_{\mu} q^{||\mu||} w^{E_\lambda(\mu)}.$$

Proof. We start with the telescoping sum

$$1 - q^M = 1 - q^{m_1B_1} + q^{m_1B_1}(1-q^{m_2B_2}) + \cdots + q^{\sum_{i=1}^{k-1} m_iB_i}(1-q^{m_kB_k}),$$

which implies

$$\begin{aligned}(1-q^M)&\prod_{i=1}^{k}(1-q^{B_i})^{-1} \\ &= \sum_{i=1}^{k} q^{m_1B_1+\cdots+m_{i-1}B_{i-1}} \prod_{j=1}^{i-1}(1-q^{B_j})^{-1} \frac{1-q^{m_iB_i}}{1-q^{B_i}} \prod_{j=i+1}^{k}(1-q^{B_j})^{-1}.\end{aligned} \tag{6.1}$$

We see that (6.1) is the generating function for partitions μ with parts from $\{B_1, B_2, \ldots, B_k\}$ such that $E_\lambda(\mu) = 0$. The i-th term of the sum represents partitions $\mu = (B_1^{n_1}, B_2^{n_2}, \ldots, B_k^{n_k})$, where

$$n_1 \ge m_1,\ n_2 \ge m_2, \ldots, n_{i-1} \ge m_{i-1},\ n_i < m_i.$$

These disjoint sets cover all μ with $E_\lambda(\mu) = 0$.

Adding back the multiples of λ by multiplying by $(1-wq^M)^{-1}$, and also the unused parts from $\mathbb{A}$, gives the result. □

Definition 6.6. Let $\mathbb{A}$ be a set of parts. If λ has parts from $\mathbb{A}$, let $E_\lambda^{\mathbb{A}}(n,k)$ be the number of partitions μ of n with parts from $\mathbb{A}$ such that $E_\lambda(\mu) = k$.

Corollary 6.7. *For any set of part sizes $\mathbb{A}$, let λ_1 and λ_2 be two partitions of M into parts from $\mathbb{A}$. Then for all $n, k \ge 0$*

$$E_{\lambda_1}^{\mathbb{A}}(n,k) = E_{\lambda_2}^{\mathbb{A}}(n,k).$$

Here are the promised versions of Theorems 1.2 and 2.2 when M does not satisfy the mod 5 condition.

Corollary 6.8. *Suppose that λ is a partition of M into parts congruent to* 1 *or* 4 mod 5*. Then, Theorem 1.2 holds if the number of partitions having M of multiplicity k is replaced by $E^{\mathbb{A}}_{\lambda}(n,k)$, $\mathbb{A}=\{1,4,6,9,\ldots\}$. Also, if λ is a partition of M into parts congruent to* 2 *or* 3 mod 5*, then Theorem 2.2 holds if the number of partitions having M of multiplicity k is replaced by $E^{\mathbb{B}}_{\lambda}(n,k)$, $\mathbb{B}=\{2,3,7,8,\ldots\}$.*

Example 6.9. Let $\lambda=(6,1)$, $M=7$, and $n=17$. The equinumerous sets of partitions for Corollary 6.8 are

$$\{(9,6,1,1),(6,6,4,1),(6,4,4,1,1,1),(6,4,1^7),(6,1^{11})\}$$
$$\leftrightarrow \{(16,1),(15,2),(14,3),(13,4),(13,3,1)\}.$$

One corollary of the Rogers–Ramanujan identities is that there are more partitions of n into parts congruent to 1 or 4 mod 5 than into parts congruent to 2 or 3 mod 5. Kadell [8] gave an injection which proves this, and Berkovich–Garvan [3, Theorem 5.1] gave an injection for modulo 8. A general injection for finite products was given by Berkovich–Grizzell [4]. We can use Corollary 6.7, Theorems 1.1, and 2.1 to generalize this fact for the Rogers–Ramanujan identities.

Theorem 6.10. *Let*

$$\mathbb{A}=\{5k+1,5k+4:k\geq 0\},\quad \mathbb{B}=\{5k+2,5k+3:k\geq 0\}.$$

Fix partitions $\lambda\vdash M$ and $\theta\vdash M$, $M\geq 3$, with parts from $\mathbb{A}$ and $\mathbb{B}$, respectively. Then for all $n,k\geq 0$

$$E^{\mathbb{B}}_{\theta}(n,k)\leq E^{\mathbb{A}}_{\lambda}(n,k).$$

Proof. By Corollary 6.7, Theorems 1.1, and 2.1 we have

$$\sum_{n=0}^{\infty}\sum_{k=0}^{\infty}q^n w^k(E^{\mathbb{A}}_{\lambda}(n,k)-E^{\mathbb{B}}_{\theta}(n,k))=\frac{q-q^{M+1}}{1-wq^M}+\sum_{k=2}^{\infty}q^{k^2}\frac{[M]_q}{1-wq^M}\frac{1}{(q^2;q)_{k-2}}.$$

All terms are positive except for the first term. If we add the $k=2$ term to the first term, we have

$$\frac{q-q^{M+1}+q^4[M]_q}{1-wq^M},$$

whose numerator is positive for $M\geq 3$. □

One could also apply Theorems 1.2 and 2.2 to obtain this result combinatorially. The single part case would be considered separately.

7. Remarks

All of parts 1, 2, … may be simultaneously marked for all partitions by the largest part. The corresponding identity is

$$\frac{1}{\prod_{k=1}^{\infty}(1-x_kq^k)}=1+\sum_{j=1}^{\infty}\frac{x_jq^j}{\prod_{k=1}^{j}(1-x_kq^k)}.$$

A corresponding rational function identity for marking $1, 2, \ldots, n$ is

$$\frac{1}{\prod_{k=1}^{n}(1-x_k q^k)}\left(\sum_{j=0}^{n}\frac{q^j}{(q)_j}\right)=\frac{1}{(q)_n}\left(1+\sum_{j=1}^{n}\frac{x_j q^j}{\prod_{k=1}^{j}(1-x_k q^k)}\right).$$

We do not have such general marked versions for the Rogers–Ramanujan identities, which would be equivalent to bijections. There are partial results. In Ref. [9] marked versions of the second Rogers–Ramanujan identity are given for

1. a single part $\{M\}$,
2. two parts $\{2, M\}$,
3. four parts $\{2, 3, 7, 8\}$.

We do not have a general version of Proposition 3.1 which gives the last marked version.

A q-analog of Euler's odd=distinct theorem [10, Theorem 1] is the following. Let q be a positive integer. The number of partitions of N into q-odd parts $[2k+1]_q$ is equal to the the number of partitions of N into parts $[m]_q$ whose multiplicity is $\leq q^m$. A generating function identity equivalent to this result is

$$\prod_{n=0}^{\infty}\frac{1}{1-t^{[2n+1]_q}}=1+\sum_{m=1}^{\infty}t^{[m]_q}\frac{1-t^{q^m[m]_q}}{1-t^{[m]_q}}\prod_{k=1}^{m-1}\frac{1-t^{(q^k+1)[k]_q}}{1-t^{[k]_q}}.$$

We do not know how to perturb this identity to mark a part.

Given λ and μ, $E_\lambda(\mu)$ is an integer which counts the number of λ's in μ. One could imagine defining instead a rational value for this "multiplicity".

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Kathleen O'Hara
4356 Preston Forest Drive
Blacksburg
VA 24060
USA
e-mail: ohara.kathy1@gmail.com

Dennis Stanton
School of Mathematics
University of Minnesota
Minneapolis
MN 55455
USA
e-mail: stanton@math.umn.edu

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On Witten's Extremal Partition Functions

In celebration of George Andrews' 80th birthday

Ken Ono and Larry Rolen

Abstract. In his famous 2007 paper on three-dimensional quantum gravity, Witten defined candidates for the partition functions

$$Z_k(q) = \sum_{n=-k}^{\infty} w_k(n) q^n$$

of potential extremal conformal field theories (CFTs) with central charges of the form $c = 24k$. Although such CFTs remain elusive, he proved that these modular functions are well defined. In this note, we point out several explicit representations of these functions. These involve the partition function $p(n)$, Faber polynomials, traces of singular moduli, and Rademacher sums. Furthermore, for each prime $p \leq 11$, the p series $Z_k(q)$, where $k \in \{1, \ldots, p-1\} \cup \{p+1\}$, possess a Ramanujan congruence. More precisely, for every non-zero integer n we have that

$$w_k(pn) \equiv 0 \begin{cases} \pmod{2^{11}} & \text{if } p = 2, \\ \pmod{3^5} & \text{if } p = 3, \\ \pmod{5^2} & \text{if } p = 5, \\ \pmod{p} & \text{if } p = 7, 11. \end{cases}$$

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Keywords. Extremal partition function, Modular forms, Faber polynomials.

1. Introduction and Statement of Results

In Ref. [11], Witten defined a sequence of functions which he proposed encode quantum states of three-dimensional gravity. Namely, he purported the existence of an extremal conformal field theory (in the language of [9]) at any central charge $c = 24k$ with $k \geq 1$. They should have partition functions equal to the unique weakly holomorphic[1] modular functions on $\mathrm{SL}_2(\mathbb{Z})$ with principal part (i.e., the negative powers of q together with the constant term at $i\infty$) determined by:

$$Z_k(q) = \sum_{n=-k}^{\infty} w_k(n)q^n = q^{-k} \prod_{n \geq 2} \frac{1}{1-q^n} + O(q). \tag{1.1}$$

For positive integers k, these functions are Witten's candidates for the generating functions that count the quantum states of three-dimensional gravity in spacetime asymptotic to AdS_3 (see Sect. 3.1 of [11]).

It is well known that the Hauptmodul for $\mathrm{SL}_2(\mathbb{Z})$, given by (here $\sigma_k(n) = \sum_{d|n} d^k$ and $q = e^{2\pi i \tau}$)

$$J(\tau) = j(\tau) - 744 = \frac{\left(1 + 240 \sum_{n \geq 1} \sigma_3(n) q^n\right)^3}{q \prod_{n \geq 1} (1-q^n)^{24}} - 744 = q^{-1} + 196884q + \cdots,$$

generates the vector space of modular functions over $\mathbb{C}$. In particular, since there are no non-constant holomorphic modular functions, any polynomial in $J(\tau)$ which matches the principal part of $Z_k(q)$ is identically equal to $Z_k(q)$. For instance, we have

$$\begin{aligned} Z_1(q) &= J(\tau), \\ Z_2(q) &= J^2(\tau) - 393767, \\ Z_3(q) &= J^3(\tau) - 590651 J(\tau) - 64481279. \end{aligned}$$

Witten gave an elementary argument that proves that the $Z_k(q)$ are well defined. We offer several formulas for the modular functions $Z_k(q)$ in different guises. These formulas rely on expressions for the partition function $p(n)$, which counts the number of integer partitions of n, Faber polynomials, and Rademacher expansions, which are all standard in number theory. The hope is that these expressions might shed light on the search for these extremal CFTs.

Our first interpretation of the functions $Z_k(q)$ uses the following generating function, which encodes the classical Faber polynomials $F_d(X)$, and where each coefficient of q^d is a monic degree n polynomial in X (see [2,12]):

$$\begin{aligned} \Omega(X;\tau) &= \frac{1 - 24 \sum_{n \geq 1} \sigma_{13}(n) q^n}{q \prod_{n \geq 1} (1-q^n)^{24}} \cdot \frac{1}{J(\tau) - X} = \sum_{d \geq 0} F_d(X) q^d \\ &= 1 + (X - 744)q + (X^2 - 1488X + 159768)q^2 + \cdots. \end{aligned} \tag{1.2}$$

The Faber polynomials can be used to build the unique weakly holomorphic modular functions $J_d(\tau)$ satisfying $J_d(\tau) = q^{-d} + O(q)$ (see [2,12]). More precisely, they satisfy

$$J_d(\tau) = F_d(j(\tau)). \tag{1.3}$$

[1] A modular form on $\mathrm{SL}_2(\mathbb{Z})$ is said to be weakly holomorphic if its poles (if any) are supported at the cusp $i\infty$.

Our first result is then the following, where for any q-series

$$f(q) = a_{-m}q^{-m} + \cdots + a_{-1}q^{-1} + a_0 + a_1 q + \cdots ,$$

we define the following "principal part" operator by:

$$\mathrm{PP}(f)(q) = a_{-m}X_m + \cdots + a_{-1}X_1 + a_0 X_0.$$

Remark 1.1. The principal part operator PP can be thought of as the complement of MacMahon's $\Omega_{\geq}$ operator. For instance, the reader is also referred to [1].

Theorem 1.2. *If k is a positive integer, then the following are true.*

(i) *In terms of the partition function $p(n)$, we have*

$$Z_k(q) = p(k) + (J_k(q) - J_{k-1}(q)) + \sum_{n=1}^{k-1} p(n)\left(J_{k-n}(\tau) - J_{k-n-1}(\tau)\right).$$

(ii) *If we define $\Omega_k(X_0, X_1, \ldots)$ by*

$$\Omega_k(X_0, X_1, \ldots) = \mathrm{PP}\left(q^{-k}\prod_{n\geq 2}\frac{1}{(1-q^n)}\right),$$

then

$$Z_k(q) = \Omega_k(J_0(\tau), J_1(\tau), J_2(\tau), \ldots).$$

The first formula relies only on the elementary properties of the partition generating function, while, as we shall see, the second connects the function $Z_k(q)$ to the world of the Monster and its moonshine.

Our second main result writes $Z_k(q)$ as a blend of values of a canonical non-holomorphic modular function $\mathcal{P}(\tau)$ at CM points corresponding to elements of class groups (see Sect. 2.2 for the definition of $\mathcal{P}(\tau)$ and the sums of values of $\mathcal{P}(\tau)$ denoted by $\mathrm{Tr}(\mathcal{P}; n)$), and Rademacher sums $R_k(\tau)$ (see Sect. 2.3 for the precise definitions). Specifically, we will express the partition numbers as traces of singular moduli, that is, sums of values at CM points, of the special non-holomorphic modular function $\mathcal{P}(\tau)$. The definition of this function, the exact notion of traces of singular moduli we require here, and the Rademacher series in the following theorem will be given in Sect. 2.

Corollary 1.3. *We have the following identity:*

$$\begin{aligned} Z_k(q) &= \frac{1}{24k-1}\,\mathrm{Tr}(\mathcal{P}; k) + (R_k(\tau) - R_{k-1}(\tau)) \\ &\quad + \sum_{n=1}^{k-1}\frac{1}{24n-1}\,\mathrm{Tr}(\mathcal{P}; n)\left(R_{k-n}(\tau) - R_{k-n-1}(\tau)\right). \end{aligned}$$

Remark 1.4. The results in Ref. [6] indicate how to efficiently compute $p(n)$ as traces of singular moduli numerically, and may be useful to those wishing to implement the identities presented here.

Remark 1.5. Connections between class numbers (which count the number of terms in the traces of singular moduli discussed here), their algebraic structures, and black holes were also described recently in Ref. [3]. It would be interesting to see if the connections between the results discussed here and in that paper have a deeper connection.

Remark 1.6. Although the partition numbers $p(n)$ may also be written as Rademacher sums, here we have chosen to highlight their alternative algebraic representations. Specifically, a version of Rademacher's famous exact formula for $p(n)$ is

$$p(n) = \frac{2\pi}{(24n-1)^{\frac{3}{4}}} \sum_{k=1}^{\infty} \frac{A_k(n)}{k} I_{\frac{3}{2}}\left(\frac{\pi\sqrt{24n-1}}{6k}\right),$$

where

$$A_k(n) = \frac{1}{2}\sqrt{\frac{k}{12}} \sum_{\substack{d \pmod{24k} \\ d^2 \equiv -24n+1 \pmod{24k}}} \left(\frac{12}{d}\right) e^{\frac{\pi d i}{6k}}$$

is a Kloosterman sum $\left(\left(\frac{12}{d}\right)\right.$ denotes a Kronecker symbol$\left.\right)$ and $I_{\frac{3}{2}}$ is a modified I-Bessel function.

It turns out that the coefficients of some of the

$$Z_k(q) = \sum_{n=-k}^{\infty} w_k(n) q^n$$

possess striking systematic congruences which are analogous to the celebrated partition congruences of Ramanujan

$$\begin{aligned} p(5n+4) &\equiv 0 \pmod{5}, \\ p(7n+5) &\equiv 0 \pmod{7}, \\ p(11n+6) &\equiv 0 \pmod{11}. \end{aligned}$$

If $p \le 11$ is prime, then the p series $\{Z_1(q), \ldots, Z_{p-1}(q)\} \cup \{Z_{p+1}(q)\}$ all simultaneously satisfy Ramanujan congruences modulo fixed small powers of p. Namely, we prove the following theorem.

Theorem 1.7. *If $p \le 11$ is prime and $k \in \{1, \ldots, p-1\} \cup \{p+1\}$, then for every non-zero integer n we have that*

$$w_k(pn) \equiv 0 \begin{cases} \pmod{2^{11}} & \text{if } p = 2, \\ \pmod{3^5} & \text{if } p = 3, \\ \pmod{5^2} & \text{if } p = 5, \\ \pmod{p} & \text{if } p = 7, 11. \end{cases}$$

Remark 1.8. We have made no attempt to completely classify all of the Ramanujan-type congruences satisfied by the $Z_k(q)$. For each positive integer m and each $k \ge 1$, it is well known that there are arithmetic progressions $an + b$ for which

$$w_k(an+b) \equiv 0 \pmod{m}.$$

This follows easily from the theory of p-adic modular forms (for example, see Chapter 2 of [10]). The unexpected phenomenon here is the uniformity of these congruences among the low index q-series for the primes $p \leq 11$.

2. Nuts and Bolts and the Proofs

In this section, we review the basic definitions and results needed for the statements and proofs of the main theorems.

2.1. Faber Polynomials and the Connection to Monstrous Moonshine

Recall from above that for each $d \geq 0$, we have a function J_d, which is the unique weakly holomorphic modular function on $\mathrm{SL}_2(\mathbb{Z})$ with principal part

$$J_d(\tau) = q^{-d} + O(q). \tag{2.1}$$

In particular, $J_0 = 1$ and for $d \geq 1$, in terms of the normalized Hecke operators T_d (see [2,12]), we have

$$J_d(\tau) = d\left(J_1(\tau)|T_d\right).$$

These functions are, of course, monic degree n polynomials in $J(\tau)$. These polynomials are known as Faber polynomials and are closely related to the denominator formula for the Monster Lie algebra (for details on moonshine and related subjects, the reader is referred to the excellent exposition in Ref. [8]). Specifically, the denominator formula states that

$$J(z) - J(\tau) = e^{-2\pi iz} \prod_{\substack{m>0 \\ n\in\mathbb{Z}}} \left(1 - e^{2\pi imz} e^{2\pi in\tau}\right)^{c_{mn}},$$

where

$$J(\tau) = \sum_{n\geq -1} c_n q^n.$$

This formula plays a key role in the overall proof of moonshine, and in particular in connecting J with the natural infinite dimensional graded module of the monster which is appears in moonshine. Equivalently, Asai, Kaneko, and Ninomiya (cf. Theorem 3 of [2]) proved the following result for the logarithmic derivative (with respect to τ) of the preceding generating function [see (1.2)]:

$$\sum_{n=0}^{\infty} J_n(\tau) e^{2\pi inz} = \Omega(j(z);\tau). \tag{2.2}$$

2.2. An Algebraic Formula for $p(n)$

Here, we recall a finite, algebraic formula for the partition numbers obtained in Ref. [5]. To state this, we first require the quasimodular Eisenstein series

$$E_2(\tau) = 1 - 24 \sum_{n\geq 1} \sigma_1(n) q^n$$

and the Dedekind eta function

$$\eta(\tau) = q^{\frac{1}{24}} \prod_{n\geq 1} \left(1 - q^n\right).$$

Then we consider the weight -2, level 6 modular function

$$G(\tau) = \frac{1}{2}\frac{E_2(\tau) - 2E_2(2\tau) - 3E_2(3\tau) + 6E_2(6\tau)}{\eta(\tau)^2\eta(2\tau)^2\eta(3\tau)^2\eta(6\tau)^2}.$$

Our distinguished non-holomorphic modular function $\mathcal{P}$ is then obtained by applying a Maass raising operator to G:

$$\mathcal{P}(\tau) = \frac{i}{2\pi}\frac{\partial G}{\partial \tau} - \frac{G(\tau)}{2\pi\,\mathrm{Im}(\tau)}.$$

We then require the distinguished collection of binary quadratic forms given by

$$\mathcal{Q}_{D,6,1} = \{Q = [a,b,c] : a,b,c \in \mathbb{Z}, b^2 - 4ac = -24D + 1, \\ 6|a, a > 0,\ b \equiv 1 \pmod{12}\}.$$

For each quadratic form Q in this set, we define the corresponding CM point τ_Q to be the point in the upper half plane satisfying $a\tau_Q^2 + b\tau_Q + c = 0$. Finally, the trace of $\mathcal{P}$ at the relevant CM points is given by

$$\mathrm{Tr}(\mathcal{P}; n) = \sum_{Q \in \mathcal{Q}_{n,6,1}} \mathcal{P}(\tau_Q).$$

In terms of these notation, the main result of Ref. [5] is the following representation for $p(n)$ in terms of these traces.

Theorem 2.1. *For any $n \geq 1$, we have*

$$p(n) = \frac{1}{24n - 1}\,\mathrm{Tr}(\mathcal{P}; n).$$

Moreover, $(24n-1)\mathcal{P}(\tau_Q)$ is always an algebraic integer.

2.3. Rademacher Series

In this section, we recall the required expressions for our Rademacher series. These can be built out of well-known expressions for Poincaré series, for example, the reader is referred to Section 6.3 of [4]. However, these formulas here are very classical, and date back to the seminal work of Rademacher, Zuckerman, and others. From these classical results, we can write the following Rademacher series representation for $J_d(\tau)$.

Proposition 2.2. *If d is a positive integer, then*

$$J_d(\tau) = R_d(\tau) = q^{-d} + \sum_{n \geq 1} r_{d,n} q^n,$$

where

$$r_{d,n} = 2\pi\sqrt{\frac{d}{n}} \times \sum_{c>0} \frac{K(n;c)}{c} I_1\left(\frac{4\pi\sqrt{dn}}{c}\right),$$

and where

$$K(n;c) = \sum_{r \pmod{c}^\times} \exp\left(2\pi i\left(\frac{-d\overline{r} + nr}{c}\right)\right)$$

is a Kloosterman sum ($\bar{r}$ denotes the multiplicative inverse of r modulo c) and I_1 is a modified I-Bessel function.

2.4. Proofs of Theorem 1.2, Corollary 1.3 and Theorem 1.7

Here, we prove the main results of this paper.

Proof of Theorem 1.2. We begin with part (i). By (1.1) and (2.1), together with the fact that weakly holomorphic modular functions are determined by their principal parts, we have that

$$\begin{aligned} Z_k(q) &= (1-q)\cdot\sum_{n\geq 0} p(n)q^{n-k} + O\left(q\right) \\ &= \sum_{n=0}^{k}\left(p(n)-p(n-1)\right)q^{n-k} + O(q) \\ &= p(0)\left(q^{-k}-q^{1-k}\right)+p(1)\left(q^{1-k}-q^{2-k}\right) \\ &\quad + \cdots + p(k-1)\left(q^{-1}-q^{0}\right)+p(k)+O(q) \\ &= p(k)+\left(J_k(\tau)-J_{k-1}(\tau)\right)+\sum_{n=1}^{k-1}p(n)\left(J_{k-n}(\tau)-J_{k-n-1}(\tau)\right). \end{aligned}$$

The claim in part (ii) follows from part (i) and (2.2). $\square$

Proof of Corollary 1.3. Corollary 1.3 follows from Theorem 2.1 and Proposition 2.2. $\square$

Proof of Theorem 1.7. Recall that the Atkin $U(p)$-operator is defined by

$$\left(\sum_{n\gg-\infty} a(n)q^n\right) \mid U(p) = \sum_{n\gg-\infty} a(pn)q^n.$$

Suppose that $p\leq 11$ is prime. If $F(X)$ is a monic polynomial with integer coefficients, we let

$$F(j(\tau)) = \sum a(n)q^n.$$

If $p\leq 11$ is prime and $\deg(F(X))<p$, then Theorem 2.3 (2) of [7] implies that

$$F(j(\tau)) \mid U(p) \equiv a(0) \pmod{p}.$$

Theorem 1.2 then implies the result for $p=7$ and 11 and $1\leq k\leq p-1$. For the cases where $(p,k)\in\{(7,8),(11,12)\}$, one applies Theorem 2.3 (1) of [7].

For the remaining cases where $p\leq 5$ and the modulus of the congruence is a power of p, one may consider the weight $12kp$ holomorphic modular forms $Z_k(q)\cdot\Delta(p\tau)^{kp}$ on $\Gamma_0(p)$, where $\Delta(\tau)$ is the usual weight 12 normalized cusp form on $\mathrm{SL}_2(\mathbb{Z})$. It suffices to show that

$$(Z_k(q)\cdot\Delta(p\tau)^{kp}) \mid U(p) \equiv 0 \begin{cases} \pmod{2^{11}} & \text{if } p=2, \\ \pmod{3^5} & \text{if } p=3, \\ \pmod{5^2} & \text{if } p=5. \end{cases}$$

These congruences are easily confirmed using the well-known theorem of Sturm (for example, see p. 40 of [10]) which reduces each claim to a finite computation. In particular, one only needs to check the claimed congruences for the first $kp(p+1)$ terms. $\square$

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Ken Ono
Department of Mathematics and Computer Science
Emory University
Atlanta
GA 30322
USA
e-mail: ken.ono@emory.edu;
ken.ono.691@gmail.com

Larry Rolen
Department of Mathematics
Vanderbilt University
1326 Stevenson Center
Nashville
TN 37240
USA
e-mail: larry.rolen@vanderbilt.edu

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A Proof of the Weierstraß Gap Theorem not Using the Riemann–Roch Formula

Dedicated to our good friend George Andrews at the occasion of his 80th birthday

Peter Paule and Cristian-Silviu Radu

Abstract. Usually, the Weierstraß gap theorem is derived as a straightforward corollary of the Riemann–Roch theorem. Our main objective in this article is to prove the Weierstraß gap theorem by following an alternative approach based on "first principles", which does not use the Riemann–Roch formula. Having mostly applications in connection with modular functions in mind, we describe our approach for the case when the given compact Riemann surface is associated with the modular curve $X_0(N)$.

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Keywords. Weierstraß gap theorem, Modular functions.

1. Main Objective

Various topical areas in the theory of partitions, such as congruences for partition numbers, are connected to modular functions for congruence subgroups of $\mathrm{SL}_2(\mathbb{Z})$ as, for instance, $\Gamma_0(N)$; see Sect. 15 for definitions. Such functions live on compact Riemann surfaces, for instance, on $X_0(N)$ for $\Gamma_0(N)$. Number theoretic aspects then relate to properties of certain subalgebras formed by these functions. In cases where the genus of such surfaces is zero like, for instance, for $X_0(5)$ and $X_0(7)$, these algebras essentially have a relatively simple structure. For positive genus g, for example, in the case of $X_0(11)$, this changes. One explanation is this: when considering sets of meromorphic functions with poles only at one point p, the Weierstraß gap theorem says that one can obtain functions with all possible pole orders at p with exactly g exceptions.

Theorem 1.1 (Weierstraß gap theorem; e.g., Sect. III.5.3 in [6]). *Let X be a compact Riemann surface having genus $g \geq 1$. Then, for each $p \in X$, there* *are precisely g integers $n_j = n_j(p)$ with*

$$1 = n_1 < \cdots < n_g \leq 2g - 1, \tag{1.1}$$

such that there does not exist a meromorphic function on X which is holomorphic on $X \backslash \{p\}$ and which has a pole of pole order n_j at p.

We want to stress that "precisely" in the theorem means that for any positive integer n other than the g values n_j, a meromorphic function with a pole of order n at p exists.

Usually, as in [6, III. 5.3], this theorem is derived as a straightforward corollary of the Riemann–Roch theorem. Our main objective in this article is to prove the Weierstraß gap theorem by following an alternative approach based on "first principles" which does not use the Riemann–Roch formula. Having mostly applications in connection with modular functions in mind, we describe our approach for the case when the given compact Riemann surface X is associated with $X_0(N)$. Some ingredients of our setting are related to ideas from the celebrated paper [3] by Dedekind and Weber; see [2] for an English translation together with an excellent introduction by John Stillwell.

2. Introduction

To exemplify the usage of Weierstraß's gap theorem, we choose an example related to the classical Ramanujan congruences, which in further details are discussed in [14]. Following the definition given in Sect. 15, let

$$M(N) := \text{ field of meromorphic modular functions for } \Gamma_0(N).$$

To keep this article as much self-contained as possible, we list basic definitions and properties of modular functions in a separate Appendix Sect. 15.

One standard way to construct modular functions is by eta quotients, i.e., products of the form:

$$\prod_{d|m} \eta(\mathrm{d}\tau)^{r_d}, \tau \in \mathbb{H}. \tag{2.1}$$

Here, $\mathbb{H}$ denotes the upper half of the complex plane, $m \in \mathbb{Z}_{>0}$, r_d are chosen integers, and η denotes the Dedekind eta function defined as

$$\eta(\tau) = q(\tau/24) \prod_{n=1}^{\infty} (1 - q(\tau)^n) \quad \text{ where } q(\tau) = \exp(2\pi i \tau). \tag{2.2}$$

Usually one writes q instead of $q(\tau)$.

The case $m = \ell$, $\ell \geq 5$ a prime, gives rise to a simple but important class of eta quotients:

$$z_\ell(\tau) := \left(\frac{\eta(\ell\tau)}{\eta(\tau)} \right)^{\frac{24}{\gcd(\ell-1,24)}}, \tag{2.3}$$

which are modular functions in $M(\ell)$ with (e.g., [9, Chap. 7, Theorem 1])

$$\operatorname{ord}_{[\infty]_\ell} z_\ell^* = \frac{\ell - 1}{\gcd(\ell - 1, 12)}. \tag{2.4}$$

Here, the notation z_ℓ^* is explained by the fact that, in general, every modular function $f \in M(N)$ gives rise to an induced meromorphic function $f^*\colon X_0(N) \to \hat{\mathbb{C}} := \mathbb{C} \cup \{\infty\}$ which for $x = [\tau]_N$ is defined as

$$f^*(x) = f^*([\tau]_N) := f(\tau),\ \tau \in \hat{\mathbb{H}} := \mathbb{H} \cup \mathbb{Q} \cup \{\infty\};$$

see Sect. 15. There one also finds the definition of $[\tau]_N$ as the orbit of τ under $\Gamma_0(N)$, as well as definitions of basic notions like of $\mathrm{ord}_{[a/c]_N} f^*$, the order of f^* at a cusp $[a/c]_N$, $a/c \in \mathbb{Q} \cup \{\infty\}$. Note that $[\infty]_N = [1/0]_N$.

Example 2.1 [9, Chap. 7, Theorem 1]. Consider

$$z_5(\tau) = \left(\frac{\eta(5\tau)}{\eta(\tau)}\right)^6 = q \prod_{j=1}^{\infty} \left(\frac{1-q^{5j}}{1-q^j}\right)^6 = q + 6q^2 + 27q^3 + 98q^4 + \cdots. \tag{2.5}$$

We have $\mathrm{ord}_{[\infty]_N} f^* := \mathrm{ord}_q f$, confirming that

$$\mathrm{ord}_{[\infty]_5} z_5^* = \frac{5-1}{\gcd(5-1,12)} = 1 = \mathrm{ord}_q z_5. \tag{2.6}$$

Because of

$$z_\ell\left(-\frac{1}{\tau}\right) z_\ell\left(\frac{\tau}{\ell}\right) = \ell^{-\frac{12}{\gcd(\ell-1,12)}},\ \ell \ \text{ a prime} \geq 5, \tag{2.7}$$

$$z_5\left(-\frac{1}{\tau}\right) = \frac{5^{-3}}{z_5(\tau/5)} = \frac{1}{5^3}\left(\frac{1}{q^{1/5}} - 6 + 9q^{1/5} + 10q^{2/5} - \cdots\right), \tag{2.8}$$

which owing to $\mathrm{ord}_{[0]_N} f^* := \mathrm{ord}_{q^{1/N}} f(-1/\tau)$ confirms that

$$\mathrm{ord}_{[0]_5} z_5^* = -\frac{5-1}{\gcd(5-1,12)} = -1 = \mathrm{ord}_{q^{1/5}} z_5\left(-\frac{1}{\tau}\right). \tag{2.9}$$

In general, for ℓ a prime, $X_0(\ell)$ has exactly two cusps $[\infty]_\ell$ and $[0]_\ell$ with widths 1 and ℓ, respectively; see [9, Chap. 2, Sect. 2], resp. Sect. 15 for the definition of width. The q-series (2.5) and (2.8) are the local q-expansions of z_5^* at these cusps.

Being meromorphic, modular functions form fields. For example, a classical fact, e.g., [5, Proposition 7.5.1], is that $M(N) = \mathbb{C}(j(\tau), j(N\tau))$, where j is the modular invariant (the Klein j function). The subset

$$\begin{aligned} M^!(N) := \{f \in M(N)\colon & f^* \text{ has poles only at (finitely many) points } [\tau]_N \\ & \text{with } \tau \in \mathbb{Q} \cup \{\infty\}\} \end{aligned}$$

obviously is not a field but a $\mathbb{C}$-algebra.[1]

Example 2.2. By definition (2.5) together with (2.6) and (2.9), $z_5 \in M^!(5)$, because z_5^* has its only pole of pole order 1 at $[0]_5$.

[1] A $\mathbb{C}$-algebra is a commutative ring with 1 which is also a vector space over $\mathbb{C}$.

An important $\mathbb{C}$-subalgebra, in particular, with regard to algorithms, is

$$M^{\infty}(N) := \{f \in M^{!}(N)\colon f^{*} \text{ has poles only at } [\infty]_N\}.$$

By [15, Lemma 20], $M^{\infty}(N)$ for each $N \geq 1$ contains an eta quotient μ_N of the form as in (2.1), such that $\operatorname{ord}_{[a/c]_N} \mu_N^{*} > 0$ for all $a/c \in \mathbb{Q}$ with $[a/c]_N \neq [\infty]_N$. Hence, one can multiply with a suitable power of μ_N to turn any given $f \in M^{!}(N)$ into an element $\mu_N^{\alpha} f$ in $M^{\infty}(N)$.

Example 2.3. Choose $f(\tau) := j(\tau) \in M^{!}(N)$, the Klein j function, and α, such that $\mu_N^{\alpha} j \in M^{\infty}(N)$. Let $\beta := \operatorname{ord}_{[\infty]_N} \mu_N$, then $\operatorname{ord}_{[\infty]_N} \mu_N^{\alpha} j = \alpha\beta - 1$. In particular

$$\gcd(\operatorname{ord}_{[\infty]_N} \mu_N, \operatorname{ord}_{[\infty]_N} \mu_N^{\alpha} j) = 1,$$

which will be needed later. A description of how to construct such μ_N is given in [15].

Since we will prove the gap theorem in the version of Theorem 12.2, where $X = X_0(N)$, and with $p = [\infty]_N$, a key issue in our approach concerns the question of finding appropriate representations of $M^{\infty}(N)$.

Example 2.4. Owing to (2.6) and (2.9), $1/z_5 = \frac{1}{q} - 6 + 9q + \cdots \in M^{\infty}(5)$. Because of $\operatorname{ord}_{[\infty]_5}(1/z_5)^{*} = -1$, each $f \in M^{\infty}(5)$ can be written as a polynomial in $1/z_5$; in short:

$$M^{\infty}(5) = \mathbb{C}\Big[\frac{1}{z_5}\Big],$$

where $\mathbb{C}[x]$ denotes the ring of polynomials in x with complex coefficients. One also has

$$M^{\infty}(7) = \mathbb{C}\Big[\frac{1}{z_7}\Big],$$

but already for $\ell = 11$, the situation is quite different. For example, in [14], we proved (implicitly) that $M^{\infty}(11)$ can be represented as a $\mathbb{C}[1/z_{11}]$-module which is freely generated by modular functions F_2, F_3, F_4, and $F_6 \in M^{\infty}(11)$. More concretely

$$\begin{aligned} M^{\infty}(11) &= \langle 1, F_2, F_3, F_4, F_6\rangle_{\mathbb{C}[\frac{1}{z_{11}}]} \\ &:= \big\{p_0(z_{11}) + p_2(z_{11})F_2 + p_3(z_{11})F_3 \\ &\qquad + p_4(z_{11})F_4 + p_6(z_{11})F_6 : p_i(z_{11}) \in \mathbb{C}[1/z_{11}]\big\}, \end{aligned} \tag{2.10}$$

where the F_i are determined as follows [14, Sect. 9]: from the two functions $f_2, f_3 \in M^{!}(22)$:

$$f_2(\tau) := q^{-2} \prod_{n=1}^{\infty} \frac{(1-q^n)(1-q^{2n})^3}{(1-q^{11n})^3(1-q^{22n})}$$

and

$$f_3(\tau) := q^{-3} \prod_{n=1}^{\infty} \frac{(1-q^n)^3(1-q^{2n})}{(1-q^{11n})(1-q^{22n})^3},$$

one constructs the desired $F_i \in M^\infty(11)$ by

$$
\begin{aligned}
F_2(\tau) &:= f_2(\tau) - (U_2 f_3)(\tau) = q^{-2} + 2q^{-1} - 12 + 5q + 8q^2 + \cdots,\\
F_3(\tau) &:= f_3(\tau) - 4(U_2 f_2)(\tau) = q^{-3} - 3q^{-2} - 5q^{-1} + 24 - 13q - \cdots,\\
F_4(\tau) &:= f_2(\tau)^2 + \frac{1}{2}(U_2 f_3^2)(\tau) = q^{-4} - \frac{3}{2}q^{-3} - \frac{7}{2}q^{-2} - \frac{21}{2}q^{-1} + 48 - \cdots,\\
F_6(\tau) &:= f_3(\tau)^2 + 8(U_2 f_2^2)(\tau) = q^{-6} - 6q^{-5} + 7q^{-4} + 22q^{-3} - 41q^{-2} + \cdots,
\end{aligned}
$$

where U_2 is the special case $\ell = 2$ ("summing the even part") of the standard U-operator:

$$
U_\ell \sum_{k=N}^{\infty} a(k) q^k := \sum_{k=\lceil N/\ell \rceil}^{\infty} a(\ell k) q^k. \tag{2.11}
$$

In addition to $\operatorname{ord}_{[\infty]_{11}}(1/z_{11})^* = -5$, one has

$$
(\operatorname{ord}_{[\infty]_{11}} F_2^*, \operatorname{ord}_{[\infty]_{11}} F_3^*, \operatorname{ord}_{[\infty]_{11}} F_4^*, \operatorname{ord}_{[\infty]_{11}} F_6^*) = (-2, -3, -4, -6). \tag{2.12}
$$

Thus the minimal pole order of the functions which in the sense of (2.10) generate $M^\infty(11)$ is 2, not 1. Indeed, the gap at 1 is predicted by the Weierstraß gap theorem, Theorem 1.1, owing to the fact that the compact Riemann surface $X := X_0(11)$ has genus 1. A formula for the genus of $X_0(N)$, if $N = \ell$ is a prime, for instance, can be found in [5, Exercises 3.1.4(e)]; the genus for general N is determined in [5, Sect. 3.9].

In Sect. 12, we prove Theorem 12.2, a version of the gap Theorem 1.1 for the case $X = X_0(N)$ and with the only pole put at ∞, utilizing only first principles and avoiding the use of the Riemann–Roch formula. In particular, we avoid the use of any differentials. In addition, our approach provides new algebraic insight by consisting in a combination of module presentations of modular function algebras, integral bases, Puiseux series, and discriminants. For example, using our approach to prove the bound $\leq 2g - 1$ stated in the Weierstraß gap theorem is reduced to an elementary combinatorial argument, see Sect. 12. Another by-product of our proof of the Weierstraß gap Theorem 12.2 is a natural explanation of the genus $g = 0$ case as a consequence of the reduction to an integral basis.

In view of various constructive aspects involved, we are planning to exploit the algorithmic content of our approach for computer algebra applications, for instance, for the effective computation of suitable module bases for modular functions. As already mentioned, some ideas we used trace back to the celebrated work [3] by Dedekind and Weber; see [2] for an English translation together with an excellent introduction by John Stillwell.

Finally, we remark that the history of Weierstraß's gap theorem and related topics such as Weierstraß points somehow presents a challenge. The historical account [4] by Andrea Del Centina describes the scientific evolution of the gap theorem up to the 1970s. Concerning its beginnings Centina says, "The history of Weierstraß points is not marked by a precise starting date because it is not clear when Weierstraß stated and proved his *Lückensatz* (or "gap" theorem), but one can argue that probably it was in the early 1860s."

The rest of our article is structured as follows. In Sect. 3, we introduce order-complete bases of modules over a polynomial ring $\mathbb{C}[t]$ to describe modular function algebras. In Sect. 4, we describe how such bases can be stepwise modified to obtain an integral basis; i.e., an order-complete basis for the full algebra $M^\infty(N)$. Under particular circumstances, one can keep track of the total number of such steps, which then gives a proof of the Weierstraß gap Theorem 12.2. To do this bookkeeping, one can use "order-reduction" polynomials discussed in Sect. 5. In Sect. 6, we explain how to obtain order-reduction polynomials computationally; Sect. 7 deals with important special cases. In Sects. 8 and 9, we derive important ingredients of our proof of Theorem 12.2; for example, a factorization property of the discriminant polynomial in Proposition 9.3. In Sects. 10 and 11, we relate discriminant polynomials to order-reduction polynomials associated with integral bases. In Sect. 12, we use these results to prove the Weierstraß gap theorem in the version of Theorem 12.2. To prove the bound $2g-1$ for the size of the maximal gap, our approach allows a purely combinatorial argument (a gap property of monoids) which we describe in Sect. 13. At various places, we require functions to have the separation property, as defined in Sect. 9. In Sect. 14, we prove the existence of such functions by giving an explicit construction.

The first Appendix Sect. 15 gives a short account on basic modular function facts needed; the second Appendix Sect. 16 recollects some fundamental facts about meromorphic functions on Riemann surfaces.

3. Modular Function Algebras as $\mathbb{C}[t]$-Modules

We already used (implicitly) the convention that if a meromorphic function f has a pole, then the pole order is defined as the negative order at this point, that is

$$\operatorname{pord}_p f := -\operatorname{ord}_p f.$$

If $f \in M^\infty(N)$, we simplify notation using the convention for the pole order at infinity:

$$\operatorname{pord} f := -\operatorname{ord}_{[\infty]_N} f^*.$$

Definition 3.1. A tuple $(b_0, b_1, \ldots, b_{n-1})$, $n \geq 1$, of modular functions in $M^\infty(N)$ is called order-complete if

$$b_0 = 1 \text{ and } \operatorname{pord} b_i \equiv i \pmod{n} \text{ for } i = 1, \ldots, n-1.$$

Slightly more generally, any tuple $(1, \beta_1, \ldots, \beta_{n-1})$ which is a reordering of an order-complete tuple $(1, b_1, \ldots, b_{n-1})$, that is,

$$\{\beta_1, \ldots, \beta_{n-1}\} = \{b_1, \ldots, b_{n-1}\}, \tag{3.1}$$

is also called order-complete.

Example 3.2. The tuple $(1, F_6, F_2, F_3, F_4)$ with $F_j \in M^\infty(11)$ as in (2.10) is order-complete.

Example 3.3. Let

$$f(\tau) := q\,\frac{1}{z_{11}}\prod_{k=1}^{\infty}(1-q^{11k})\sum_{n=0}^{\infty}p(11n+6)q^n.$$

The tuple $(1, f, f^2, f^3, f^4)$ is order-complete. Notice that $\operatorname{pord} f = 4$. In [13], it is shown that the subalgebra $\mathbb{C}[1/z_{11}, f]$ of $M^\infty(11)$, which is generated by all bivariate polynomials in $1/z_{11}$ and f, has a representation as a $\mathbb{C}[1/z_{11}]$-module as follows:

$$\mathbb{C}\left[\frac{1}{z_{11}}, f\right] = \langle 1, f, f^2, f^3, f^4\rangle_{\mathbb{C}[\frac{1}{z_{11}}]}.$$

In view of these examples, we note that in contrast to (2.10), $\mathbb{C}[1/z_{11}, f] \neq M^\infty(11)$. For instance, it is obvious that this subalgebra does not contain any $g \in M^\infty(11)$ with $\operatorname{pord} g = 3$. Nevertheless, both function tuples

$$\langle 1, F_6, F_2, F_3, F_4\rangle_{\mathbb{C}[t]}, \text{ and } \langle 1, f, f^2, f^3, f^4\rangle_{\mathbb{C}[t]},$$

form a basis of the corresponding $\mathbb{C}[t]$-module they generate, where $t := 1/z_{11}$. Namely, since the generators have different pole-order modulo $\operatorname{pord} t = 5$, each element contained in these modules can be represented as a unique linear combination of the module generators with coefficients being polynomials in t. This motivates the following definition.

Definition 3.4. For $t \in M^\infty(N)$, let $n := \operatorname{pord} t \geq 1$. Let, M be the $\mathbb{C}[t]$-module generated by an order-complete tuple in $M^\infty(N)$, that is,

$$\begin{aligned} M &= \langle 1, b_1, b_2, \ldots, b_{n-1}\rangle_{\mathbb{C}[t]} \\ &:= \left\{p_0(t) + p_1(t)b_1 + \cdots + p_{n-1}(t)b_{n-1} : p_i(x) \in \mathbb{C}[x]\right\}. \end{aligned}$$

Then, we call $(1, b_1, \ldots, b_{n-1})$ an order-complete basis for M over $\mathbb{C}[t]$. Slightly more generally, any tuple $(1, \beta_1, \ldots, \beta_{n-1})$ which is a reordering, in the sense of (3.1), of an order-complete basis $(1, b_1, \ldots, b_{n-1})$ for M is also called an order-complete basis for M.

Proposition 3.5. *Let $t, f \in M^\infty(N)$ with $n := \operatorname{pord} t \geq 1$ and $\gcd(n, \operatorname{pord} f) = 1$. Then*

$$\mathbb{C}[t, f] = \langle 1, f, f^2, \ldots, f^{n-1}\rangle_{\mathbb{C}[t]},$$

where $(1, f, f^2, \ldots, f^{n-1})$ is an order-complete module basis.

Proof. If $\operatorname{pord} f^i \equiv \operatorname{pord} f^j \pmod{n}$, then $n \mid (i-j)\operatorname{pord} f$. This implies

$$\{1, 2, \ldots, n-1\} = \{\operatorname{pord} f \,(\mathrm{mod}\, n), \operatorname{pord} f^2 \,(\mathrm{mod}\, n), \ldots, \operatorname{pord} f^{n-1} \,(\mathrm{mod}\, n)\}.$$

In addition, as a consequence of Theorem 7.1 and Lemma 7.3 in [14], $f^n \in \langle 1, f, f^2, \ldots, f^{n-1}\rangle_{\mathbb{C}[t]}$. Hence, $\mathbb{C}[t, f] \subseteq \langle 1, f, f^2, \ldots, f^{n-1}\rangle_{\mathbb{C}[t]}$. The reverse direction of this inclusion is trivial, which completes the proof. ∎

4. Integral Bases

In Example 3.3, we saw that $(1, f, \ldots, f^4)$ is an order-complete basis of $\mathbb{C}[1/z_{11}, f]$ which is a proper subalgebra of $M^\infty(11)$.[2] In this section, we shall see how such an order-complete basis can be step-wise modified to obtain an order-complete basis for the full algebra $M^\infty(11)$.

Definition 4.1. Let $t \in M^\infty(N)$ with $n := \operatorname{pord} t \geq 1$. An order-complete tuple $(1, b_1, \ldots, b_{n-1})$, $b_j \in M^\infty(N)$ is called an integral basis for $M^\infty(N)$ over $\mathbb{C}[t]$ if

$$\langle 1, b_1, \ldots, b_{n-1} \rangle_{\mathbb{C}[t]} = M^\infty(N).$$

The motivation for this terminology comes from

Lemma 4.2. *Let $f \in M(N)$ and $t \in M^\infty(N)$ with $\operatorname{pord} t \geq 1$ and*

$$\gcd(\operatorname{pord} f, \operatorname{pord} t) = 1.$$

Then, f satisfies an algebraic relation

$$f^n + p_1(t) f^{n-1} + \cdots + p_n(t) = 0$$

with polynomials $p_j(x) \in \mathbb{C}[x]$ (i.e., f is integral over $\mathbb{C}[t]$) if and only if

$$f \in M^\infty(N).$$

Moreover, if $f \in M^\infty(N)$, then there exists an algebraic relation with $n = \operatorname{pord} t$.

Proof. The statement with the assumption $f \in M^\infty(N)$ follows immediately from Proposition 3.5. For the other direction, assume that $m := \operatorname{pord}_p f^* > 0$ for $p \neq [\infty]_N$. Then $\operatorname{pord}_p (f^n)^* = mn$, a contradiction to

$$\operatorname{pord}_p (p_1(t) f^{n-1} + \cdots + p_n(t))^* \leq (n-1)m.$$

$\square$

A crucial observation for the process to obtain an integral basis for $M^\infty(N)$ from an order-complete basis is stated in the following.

Proposition 4.3. *Let $t \in M^\infty(N)$ with $n := \operatorname{pord} t \geq 1$. Let $(1, b_1, \ldots, b_{n-1})$ with $b_j \in M^\infty(N)$ be an order-complete basis of the $\mathbb{C}[t]$-module:*

$$M := \langle 1, b_1, \ldots, b_{n-1} \rangle_{\mathbb{C}[t]} \subseteq M^\infty(N).$$

Then, for any $f \in M^\infty(N)$, there exist polynomials $q(x)$ and $p_j(x)$ in $\mathbb{C}[x]$, such that

$$f = \frac{p_0(t)}{q(t)} + \frac{p_1(t)}{q(t)} b_1 + \cdots + \frac{p_{n-1}(t)}{q(t)} b_{n-1}. \tag{4.1}$$

[2] Notice that $\operatorname{pord} 1/z_{11} = 5$.

Proof. For $j \in \mathbb{Z}_{\geq 0}$, consider the sets

$$G_j := \{t^j f - h : h \in M\}.$$

For each $j \geq 0$, choose a non-zero $g_j \in G_j$, such that $\operatorname{pord} g_j$ is minimal amongst all the elements in G_j. By construction, using the convention $n\mathbb{Z}_{\geq 0} := \{nk : k \in \mathbb{Z}_{\geq 0}\}$, we have for all $j \geq 0$:

$$\operatorname{pord} g_j \notin S := (0 + n\mathbb{Z}_{\geq 0}) \cup (\operatorname{pord} b_1 + n\mathbb{Z}_{\geq 0}) \cup \cdots \cup (\operatorname{pord} b_{n-1} + n\mathbb{Z}_{\geq 0}).$$

Obviously, S is an additive submonoid of $(\mathbb{Z}_{\geq 0}, +)$. Moreover, $\mathbb{Z}_{\geq 0} \backslash S$ has only finitely many elements; let k be the maximal element in this set. Then there exist $c_j \in \mathbb{C}$, not all zero, such that

$$c_0 g_0 + c_1 g_1 + \cdots + c_{k+1} g_{k+1} = 0. \tag{4.2}$$

This is owing to the fact that equating the coefficients of non-positive powers in the q-expansions of both sides (which are functions in $M^\infty(N)$) gives $k+1$ equations in $k+2$ variables c_j. Hence, the dimension of the $\mathbb{C}$-vector space G, which is generated by all the g_j, $j \geq 0$, is bounded by $k+1$. Using $g_j := t^j f - h_j$ with $h_j \in M$, (4.2) rewrites into the form:

$$\begin{aligned} &c_0(f - h_0) + c_1(tf - h_1) + \cdots + c_{k+1}(t^{k+1} f - h_{k+1}) \\ &\quad = (c_0 + c_1 t + \cdots + c_{k+1} t^{k+1}) f - (c_0 h_0 + c_1 h_1 + \cdots + c_{k+1} h_{k+1}) = 0. \end{aligned}$$

The linear combination of h_j is in M; hence, this gives the desired relation for f with $q(t) = c_0 + c_1 t + \cdots + c_{k+1} t^{k+1}$. □

Corollary 4.4. *Let $t \in M^\infty(N)$ with $n := \operatorname{pord} t \geq 1$. Let $(1, b_1, \ldots, b_{n-1})$ with $b_j \in M^\infty(N)$ be an order-complete basis of the $\mathbb{C}[t]$-module:*

$$M := \langle 1, b_1, \ldots, b_{n-1} \rangle_{\mathbb{C}[t]} \subseteq M^\infty(N).$$

If $M \neq M^\infty(N)$, then there exist $c_j \in \mathbb{C}$, not all zero, and α in $\mathbb{C}$, such that

$$h_\alpha := \frac{c_0 + c_1 b_1 + \cdots + c_{n-1} b_{n-1}}{t - \alpha} \in M^\infty(N) \backslash M. \tag{4.3}$$

In particular, there exists a uniquely determined $k \in \{1, \ldots, n-1\}$, such that

$$\operatorname{pord} h_\alpha = \operatorname{pord} b_k - n \geq k \;\text{ and }\; c_k \neq 0. \tag{4.4}$$

Proof. By Proposition 4.3, there exists an $f \in M^\infty(N) \backslash M$ of the form (4.1), such that $q(x) \nmid p_i(x)$ for some $i \in \{0, \ldots, n-1\}$. Hence, there exists $\alpha \in \mathbb{C}$, such that $x - \alpha \mid q(x)$, but $x - \alpha \nmid p_i(x)$. Consequently, $q(t)/(t - \alpha) \in M^\infty(N)$ and thus

$$g := f \frac{q(t)}{t - \alpha} = \frac{p_0(t) + p_1(t) b_1 + \cdots + p_{n-1}(t) b_{n-1}}{t - \alpha} \in M^\infty(N) \backslash M.$$

By division with remainder, there are polynomials $q_j(x) \in \mathbb{C}[x]$ and $c_j \in \mathbb{C}$, such that $p_j(x) = (x - \alpha) q_j(x) + c_j$, $j = 0, \ldots, n-1$. Rewriting the representation of g and noting that $c_i \neq 0$ proves the first part of the statement on h_α. To prove (4.4), consider

$$(t - \alpha) h_\alpha = c_0 + c_1 b_1 + \cdots + c_{n-1} b_{n-1},$$

which implies

$$\operatorname{pord}(th_\alpha) = n + \operatorname{pord} h_\alpha = \max_{\substack{1\le j\le n-1,\\ c_j\neq 0}} \{\operatorname{pord} b_j\}.$$

Let k be the index for which $\operatorname{pord} b_k$ becomes maximal with $c_k \neq 0$. Recalling $\operatorname{pord} b_j \equiv j \pmod n$, $j = 1, \ldots, n$, proves $\operatorname{pord} b_k \geq k + n$. Because otherwise $\operatorname{pord} b_k = k$ which owing to the choice of k would imply $\operatorname{pord} b_j = j$ for all $j = 1, \ldots, n$, and the given order-complete basis would be integral. This proves (4.4). □

Corollary 4.4 motivates the following.

Definition 4.5. Let $M = \langle 1, b_1, \ldots, b_{n-1}\rangle_{\mathbb{C}[t]}$ and $h_\alpha \in M^\infty(N)$ be as in Corollary 4.4; i.e., $M \neq M^\infty(N)$ and $\operatorname{pord} h_\alpha = \operatorname{pord} b_k - n \geq k$. The replacement

$$(1, \ldots, b_{k-1}, b_k, b_{k+1}, \ldots) \to (1, \ldots, b_{k-1}, h_\alpha, b_{k+1}, \ldots)$$

of b_k by h_α is called a pole-order-reduction step associated with $\alpha \in \mathbb{C}$.

We summarize in the form of

Proposition 4.6. *Let $t \in M^\infty(N)$ with $n := \operatorname{pord} t \geq 1$. Let $(1, b_1, \ldots, b_{n-1})$ with $b_j \in M^\infty(N)$ be an order-complete basis of the $\mathbb{C}[t]$-module:*

$$M := \langle 1, b_1, \ldots, b_{n-1}\rangle_{\mathbb{C}[t]} \subseteq M^\infty(N).$$

If $M \neq M^\infty(N)$, then:

(i) *By a finite sequence of pole-order-reduction steps the order-complete basis $(1, b_1, \ldots, b_{n-1})$ can be transformed into an integral basis $(1, \beta_1, \ldots, \beta_{n-1})$, such that*

$$\langle 1, \beta_1, \ldots, \beta_{n-1}\rangle_{\mathbb{C}[t]} = M^\infty(N).$$

(ii) *If $(1, \beta'_1, \ldots, \beta'_{n-1})$ is any another integral basis, that is,*

$$\langle 1, \beta'_1, \ldots, \beta'_{n-1}\rangle_{\mathbb{C}[t]} = M^\infty(N),$$

then

$$\{\operatorname{pord} \beta_1, \ldots, \operatorname{pord} \beta_{n-1}\} = \{\operatorname{pord} \beta'_1, \ldots, \operatorname{pord} \beta'_{n-1}\}. \tag{4.5}$$

Proof. The proof of part (i) is an immediate consequence of Corollary 4.4. Namely, owing to (4.4), each step reduces the pole order of one of the basis elements by n. This guarantees termination in finitely many steps. To prove (ii), without loss of generality, we can assume that $\operatorname{pord} \beta_j \equiv \operatorname{pord} \beta'_j \equiv j \pmod n$ for all j. Suppose $\operatorname{pord} \beta_j \neq \operatorname{pord} \beta'_j$ for some $j \in \{1, \ldots, n-1\}$, i.e., $\operatorname{pord} \beta'_j = \operatorname{pord} \beta_j + kn$ with $k \geq 1$. However, this implies that $\beta_j \notin \langle 1, \beta'_1, \ldots, \beta'_{n-1}\rangle_{\mathbb{C}[t]}$, because then, no element in this module can have the same pole order as β_j, a contradiction. □

5. Order-Reduction Polynomials

It was shown in the previous section that by applying a procedure using finitely many steps, any order-complete basis of a subalgebra of $M^\infty(N)$ can be extended to an integral basis of $M^\infty(N)$. Moreover, by (4.5), the pole orders of the integral basis functions are uniquely determined. It turns out that under particular circumstances, one can keep track of the number of order-reduction steps, which then gives a proof of the Weierstraß gap Theorem 12.2. To do this bookkeeping, one can use "order-reduction" polynomials. To our knowledge, for the first time such polynomials have been used by Dedekind and Weber [3], see [2] for Stillwell's translation into English.

Throughout this section, $t \in M^\infty(N)$ with $n := \operatorname{pord} t \geq 1$ and $(1, b_1, \ldots, b_{n-1})$ with $b_j \in M^\infty(N)$ is an order-complete basis of the $\mathbb{C}[t]$-module:

$$M := \langle 1, b_1, \ldots, b_{n-1} \rangle_{\mathbb{C}[t]} \subseteq M^\infty(N).$$

Owing to $t(\tau) = \infty$ if and only if $[\tau]_N = [\infty]_N$, t is a holomorphic function on $\mathbb{H}$. Moreover, the induced function t^*, which is meromorphic on the compact Riemann surface $X_0(N)$, has a pole only at $[\infty]_N$.

Remark 5.1 (A basic notational convention). In general, every modular function $f \in M(N)$ gives rise to an induced meromorphic function $f^*\colon X_0(N) \to \hat{\mathbb{C}}$ which for $x = [\tau]_N$ is defined as

$$f^*(x) = f^*([\tau]_N) := f(\tau), \ \tau \in \hat{\mathbb{H}}; \tag{5.1}$$

see Appendix Sect. 15. A central theme in what follows is to consider maps:

$$f^* \circ (t^* \mid U)^{-1}\colon V \to \hat{\mathbb{C}},$$

where $U \subseteq X_0(N)$ and $V \subseteq \mathbb{C}$ are open sets, such that

$$t^*\colon U \to V \quad \text{is bi-holomorphic.}$$

Hence, for $v \in V$, the evaluations

$$f^* \circ (t^* \mid U)^{-1}(v) = f^*((t^* \mid U)^{-1}(v))$$

have to be interpreted in the sense of (5.1), i.e., interpreting $x = (t^* \mid U)^{-1}(v)$ as $x = [\tau]_N$ for some $\tau \in \hat{\mathbb{H}}$.

Depending on the context, we will freely move between considering t as a function on $\mathbb{H}$, resp. $\hat{\mathbb{H}}$, and its induced version $t^*\colon X_0(N) \to \hat{\mathbb{C}}$.

Using the terminology explained in the Appendix Sect. 16, we assume that $v_0 \in \mathbb{C}$ is not a branch point of t^*; in short, $v_0 \notin \operatorname{BranchPts}(t^*)$. In this case, there are n pairwise distinct points $x_j = [\tau_j]_N \in X_0(N)$ with $\tau_j \in \mathbb{H}$, such that

$$t^{*-1}(v_0) = \{x_1, \ldots, x_n\}. \tag{5.2}$$

In addition, there exists a neighborhood V of v_0 and neighborhoods U_j of x_j, such that

$$t^{*-1}(V) = U_1 \cup \cdots \cup U_n,$$

as a disjoint union of open sets, and such that for $j = 1, \ldots, n$, the restricted functions

$$t^*|U_j \colon U_j \to V,$$

are bi-holomorphic.
Let

$$T_j := (t^*|U_j)^{-1} \colon V \to U_j,\ j = 1, \ldots, n.$$

Define

$$D_t(1, b_1, \ldots, b_{n-1}) \colon V \to \mathbb{C} \tag{5.3}$$

by

$$D_t(1, b_1, \ldots, b_{n-1})(v) := \begin{vmatrix} 1 & 1 & \cdots & 1 \\ (b_1^* \circ T_1)(v) & (b_1^* \circ T_2)(v) & \cdots & (b_1^* \circ T_n)(v) \\ \vdots & \vdots & \ddots & \vdots \\ (b_{n-1}^* \circ T_1)(v) & (b_{n-1}^* \circ T_2)(v) & \cdots & (b_{n-1}^* \circ T_n)(v) \end{vmatrix}^2 .$$

Taking the square of the determinant guarantees that the expression on the right side is symmetric with respect to any permutation of $T_1, \ldots, T_n$. Consequently, $D_t(1, b_1, \ldots, b_{n-1})$ is a holomorphic function on V. Carrying out the same construction on neighborhoods V for all $v_0 \in \mathbb{C}\backslash\mathrm{BranchPts}(t^*)$, and gluing the resulting functions $D_t(1, b_1, \ldots, b_{n-1}) : V \to \mathbb{C}$ together, gives a global holomorphic function:

$$D_t(1, b_1, \ldots, b_{n-1}) : \mathbb{C}\backslash\mathrm{BranchPts}(t^*) \to \mathbb{C}.$$

Using the same arguments as in the proof of Theorem 8.2 in [7], this function can be extended to a meromorphic function:

$$D_t(1, b_1, \ldots, b_{n-1}) : \hat{\mathbb{C}} := \mathbb{C} \cup \{\infty\} \to \hat{\mathbb{C}}$$

with ∞ as its only pole. Classical complex analysis tells that $\mathcal{M}(\hat{\mathbb{C}}) = \mathbb{C}(z)$, i.e., the field of meromorphic functions on $\hat{\mathbb{C}}$ are rational functions with coefficients in $\mathbb{C}$. Hence, we have the following.

Lemma 5.2. *The meromorphic function $D_t(1, b_1, \ldots, b_{n-1})(v)$ constructed above is a polynomial function in v.*

Definition 5.3. The polynomial $D_t(1, b_1, \ldots, b_{n-1})(x) \in \mathbb{C}[x]$ is called order-reduction polynomial for the order-complete basis $(1, b_1, \ldots, b_{n-1})$, $b_j \in M^\infty(N)$, of the $\mathbb{C}[t]$-module

$$\langle 1, b_1, \ldots, b_{n-1}\rangle_{\mathbb{C}[t]} \subseteq M^\infty(N),$$

where $t \in M^\infty(N)$ with $n := \operatorname{pord} t \geq 1$.

Example 5.4. Taking

$$t := \frac{1}{z_{11}} = \frac{1}{q^5} - \frac{12}{q^4} + \frac{54}{q^3} - \frac{88}{q^2} - \frac{99}{q} + 540 - 418q - \cdots \in M^\infty(11)$$

and

$$(1, b_1, \ldots, b_4) := (1, F_2, F_3, F_4, F_6),$$

where $F_j \in M^\infty(11)$ are as in Example 2.4, one obtains

$$D_{1/z_{11}}(1, F_2, F_3, F_4, F_6)(x) = x^4(5^5 11^6 - 2 \cdot 3^2 \cdot 439081x + 5^5 x^2). \quad (5.4)$$

Example 5.5. Taking t and the b_j as in Example 5.4, one obtains

$$D_{1/z_{11}}(1, F_2, F_4^2, F_4, F_6)(x) = (11^3 + x)^2 D_{1/z_{11}}(1, F_2, F_3, F_4, F_6)(x). \quad (5.5)$$

Remark 5.6. How such polynomials are computed is explained in Sect. 6.

In Corollary 4.4, we proved that if $M \neq M^\infty(N)$, then there exist $c_j \in \mathbb{C}$, not all zero, and v_0 in $\mathbb{C}$, such that

$$\frac{c_0 + c_1 b_1 + \cdots + c_{n-1} b_{n-1}}{t - v_0} \in M^\infty(N) \backslash M. \quad (5.6)$$

Recall that we denoted the n pairwise distinct preimages of v_0 as follows:

$$t^*(x_1) = t^*([\tau_1]_N) = t(\tau_1) = v_0, \ldots, t^*(x_n) = t^*([\tau_n]_N) = t(\tau_n) = v_0.$$

Relation (5.6) implies

$$c_0 + c_1 b_1(\tau_1) + \cdots + c_{n-1} b_{n-1}(\tau_1) = 0, \ldots, c_0 + c_1 b_1(\tau_n) + \cdots + c_{n-1} b_{n-1}(\tau_n) = 0.$$

As a necessary condition for the existence of $c_j \in \mathbb{C}$ not all zero, the determinant

$$\begin{vmatrix} 1 & b_1(\tau_1) & \cdots & b_{n-1}(\tau_1) \\ 1 & b_1(\tau_2) & \cdots & b_{n-1}(\tau_2) \\ \vdots & \vdots & \ddots & \vdots \\ 1 & b_1(\tau_n) & \cdots & b_{n-1}(\tau_n) \end{vmatrix}$$

of the corresponding linear system has to be zero. In view of

$$(b_i^* \circ T_j)(v_0) = b_i^*((t^* \mid U_j)^{-1}(v_0)) = b_i^*(x_j) = b_i(\tau_j),$$

the square of this determinant (taking the underlying matrix transposed) is $D_t(1, b_1, \ldots, b_{n-1})(v_0)$. Above, we used the fact that the definition for

$$D_t(1, b_1, \ldots, b_{n-1}) : \mathbb{C} \backslash \mathrm{BranchPts}(t^*) \to \mathbb{C}$$

extends to the polynomial function

$$D_t(1, b_1, \ldots, b_{n-1}) : \hat{\mathbb{C}} \to \hat{\mathbb{C}}.$$

This means, the case when $v_0 \in \mathbb{C}$ is a branch point of t^* is also covered by the same determinant condition:

$$D_t(1, b_1, \ldots, b_{n-1})(v_0) = 0.$$

However, if $v_0 \in \mathbb{C}$ is a branch point, this condition is automatically satisfied, because then at least the two rows

$$(1, b_1(\tau_i), \ldots, b_{n-1}(\tau_i)) \quad \text{and} \quad (1, b_1(\tau_j), \ldots, b_{n-1}(\tau_j)),$$

are equal for $i \neq j$. Summarizing, this gives

Lemma 5.7. *Let $t \in M^{\infty}(N)$ with $n := \operatorname{pord} t \geq 1$. Let $(1, b_1, \ldots, b_{n-1})$ with $b_j \in M^{\infty}(N)$ be an order-complete basis of the $\mathbb{C}[t]$-module*

$$M := \langle 1, b_1, \ldots, b_{n-1}\rangle_{\mathbb{C}[t]} \subseteq M^{\infty}(N) \text{ and } M \neq M^{\infty}(N).$$

Let $v_0 \in \mathbb{C}$ be such that[3]

$$\frac{c_0 + c_1 b_1 + \cdots + c_{n-1} b_{n-1}}{t - v_0} \in M^{\infty}(N) \backslash M.$$

for $c_j \in \mathbb{C}$, not all zero. Then

$$D_t(1, b_1, \ldots, b_{n-1})(v_0) = 0. \tag{5.7}$$

If v_0 is a branch point of t^*, the condition (5.7) is automatically satisfied.

6. How to Compute Order-Reduction Polynomials

Next, we explain how to compute the order-reduction polynomials in (5.4) and (5.5).

To this end, it will be convenient to introduce the following notation:

Definition 6.1. If

$$f(\tau) = \sum_{n=-K}^{\infty} f_n q^n$$

is the q-expansion at infinity for some $f \in M^{\infty}(N)$, we define

$$\tilde{f}(q) := \sum_{n=-K}^{\infty} f_n q^n,$$

that is,

$$f(\tau) = \tilde{f}(q(\tau)) = \tilde{f}(q) \text{ with } q = q(\tau) = e^{2\pi i \tau} \text{ for } \tau \in \mathbb{H}.$$

Returning to the setting (5.2), we again assume that $v_0 \in \mathbb{C}$ is not a branch point of t^*. This means that there exists pairwise distinct $x_j = [\tau_j]_N \in X_0(N)$ with $\tau_j \in \mathbb{H} \cup \mathbb{Q}$ such that $[\tau_j]_N \neq [\infty]_N$ and[4]

$$t^{*-1}(v_0) = \{x_1, \ldots, x_n\},$$

together with neighborhoods U_j of the x_j, such that for a suitable neighborhood V of v_0:

$$t^{*-1}(V) = U_1 \cup \cdots \cup U_n,$$

as a disjoint union of open sets, and such that the restricted functions

$$T_j = (t^*|U_j)^{-1} : V \to U_j$$

are bi-holomorphic.

[3]The existence of such a v_0 is owing to Corollary 4.4.
[4]Notice that, in particular, $\tau \neq \infty$.

For each $j = 1, \dots, n$ and $v \in V$ our goal, achieved in Lemma 6.2(ii), is to determine expressions for $q_j(v) := q^{2\pi i \tau(j)}$, where $\tau(j)$ is close to τ_j, such that

$$t^*([\tau(j)]_N) = t(\tau(j)) = \tilde{t}(q_j(v)) = v.$$

For $q = e^{2\pi i\tau}$ with $\tau \in \mathbb{H}$, we have

$$\tilde{t}(q) = \frac{1}{q^n}(1 + \varphi(q)) := \frac{1}{q^n}(1 + \varphi_1 q + \varphi_2 q^2 + \cdots).$$

Here, we assume that the first coefficient in this q-expansion of t is 1. Now, if

$$1 + \psi(q) := 1 + \psi_1 q + \psi_2 q^2 + \cdots := \frac{1}{1 + \varphi(q)},$$

and

$$(1 + \psi(q))^{1/n} := \sum_{l=0}^{\infty} \binom{1/n}{l} \psi(q)^l, \tag{6.1}$$

then

$$\tilde{t}(q) = \frac{1}{U(q)^n}, \quad \text{where}\, U(q) := q(1 + \psi(q))^{1/n}. \tag{6.2}$$

To fix a branch of the nth root, we choose the preimage τ_n and recall that

$$v_0 = t^*([\tau_n]_N) = t(\tau_n) = \tilde{t}(e^{2\pi i\tau_n}).$$

Now, for each $v \in \mathbb{C}$ close to v_0, there is for each $j \in \{1, \dots, n\}$ a uniquely determined $\tau(j) \in \mathbb{H}$ close to τ_j, such that

$$v = t^*([\tau(j)]_N) = t(\tau(j)) = \tilde{t}(e^{2\pi i\tau(j)}). \tag{6.3}$$

By choosing a neighborhood of τ_n, we fix a branch of the nth root of $v \in \mathbb{C}$ close to v_0:

$$\sqrt[n]{v} := \frac{1}{U(q)} \quad \text{with} \quad q = e^{2\pi i\tau(n)}, \tag{6.4}$$

where $\tau(n)$ is close to τ_n and determined as in (6.3).
In addition, let W be such that $U(W(q)) = W(U(q)) = q$, and define

$$\zeta_n := e^{\frac{2\pi i}{n}}.$$

After this preparation, in view of (6.4) we can put things together as follows.

Lemma 6.2. *In the given setting, for $j = 1, \dots, n$ and $v \in \mathbb{C}$ close to v_0, let*

$$q_j(v) := W\Big(\zeta_n^j \frac{1}{\sqrt[n]{v}}\Big),$$

where $\sqrt[n]{v}$ is defined as in (6.4).
Then, for $j = 1, \dots, n$ and $v \in \mathbb{C}$ close to v_0:

$$q_j(v) = e^{2\pi i\tau(j)}, \tag{i}$$

where $[\tau(j)]_N = T_j(v)$ with $\tau(j)$ as in (6.3), and

$$\tilde{t}(q_j(v)) = v, \tag{ii}$$

where the values $q_j(v)$ are pairwise distinct for $j = 1, \dots, n$.

Proof. The values $q_j(v)$, $j = 1, \ldots, n$, are defined by power series in $q = q(\tau(n))$:

$q_j(v){=}W(\zeta_n^j U(q)){=}\zeta_n^j q + O(q^2)$ with $q{=}e^{2\pi i\tau(n)}$ where $\tau(n)$ is close to τ_n.

For a fixed v close to v_0, these values are pairwise different for $j = 1, \ldots, n$, because

$$q_j(v) = W(\zeta_n^j U(q)) = W(\zeta_n^k U(q)) = q_k(v) \Rightarrow \zeta_n^j U(q) = \zeta_n^k U(q).$$

By (6.2)

$$\tilde{t}(q_j(v)) = \frac{1}{U(q_j(v))^n} = U\Big(W\Big(\zeta_n^j \frac{1}{\sqrt[n]{v}}\Big)\Big)^{-n} = v.$$

This implies (i) and (ii). $\square$

Lemma 6.2 enables us to compute the polynomial $D_t(1, b_1, \ldots, b_{n-1})(v)$, because by part (i) with $i = 1, \ldots, n-1$ and $j = 1, \ldots, n$:

$$(b_i^* \circ T_j)(v) = b_i^*(T_j(v)) = \widetilde{b_i}(e^{2\pi i\tau(j)}) = \sum_{\ell=-\operatorname{pord} b_i} \beta_\ell^{(i)} q_j(v)^\ell {=} \widetilde{b_i}\Big(W\Big(\zeta_n^j \frac{1}{\sqrt[n]{v}}\Big)\Big).$$

This means, each $(b_i^* \circ T_j)(v)$ can be represented as a Laurent series in powers of $1/v^{1/n}$:

$$(b_i^* \circ T_j)(v) = \zeta_n^{-j \operatorname{pord} b_i} v^{\frac{\operatorname{pord} b_i}{n}} + \alpha_{i,j} v^{\frac{\operatorname{pord} b_i - 1}{n}} + \cdots + \beta_{i,j} \frac{1}{v^{1/n}} + \cdots, \tag{6.5}$$

with coefficients $\alpha_{i,j}, \beta_{i,j}$, etc., in $\mathbb{C}$, and under the assumption that the first Laurent series coefficient $\beta^{(i)}_{-\operatorname{pord} b_i}$ of each $\widetilde{b_i}(q_j(v))$ is equal to 1. Owing to Lemma 5.2, $D_t(1, b_1, \ldots, b_{n-1})(v)$ as defined in (5.3) must be a polynomial in v. Consequently, we can compute it by taking suitable truncated versions of the expansions (6.5).

Remark 6.3. This is how we computed the order-reduction polynomials in (5.4) and (5.5).

7. Discriminant Polynomials

Important special cases of order-reduction polynomials are produced by order-complete module bases of $\mathbb{C}[t, f]$ of the form as in Proposition 3.5.

Definition 7.1. Let $t, f \in M^\infty(N)$ with $n := \operatorname{pord} t \geq 1$, and $\gcd(n, \operatorname{pord} f) = 1$. Then

$$D_t(f)(v) := D_t(1, f, f^2, \ldots, f^{n-1})(v)$$

is called the discriminant polynomial for the order-complete basis $(1, f, \ldots, f^{n-1})$ of the $\mathbb{C}[t]$-module:

$$\langle 1, f, f^2 \ldots, f^{n-1}\rangle_{\mathbb{C}[t]} = \mathbb{C}[t, f] \subseteq M^\infty(N).$$

The discriminant polynomial

$$D_t(f)(v) = \begin{vmatrix} 1 & 1 & \cdots & 1 \\ f^*(T_1(v)) & f^*(T_2(v)) & \cdots & f^*(T_n(v)) \\ \vdots & \vdots & \ddots & \vdots \\ f^*(T_1(v))^{n-1} & f^*(T_2(v))^{n-1} & \cdots & f(T_n(v))^{n-1} \end{vmatrix}^2$$
$$= \prod_{1 \le i < j \le n} \Big(f^*(T_i(v)) - f^*(T_j(v)) \Big)^2$$

factors as the square of a Vandermonde determinant. Now, invoking (6.5) with $b_i = f$, and thus, $\operatorname{pord} b_i = \operatorname{pord} f$, gives

$$f^*(T_j(v)) = \zeta_n^{-j \operatorname{pord} f} v^{\frac{\operatorname{pord} f}{n}} + \alpha_j v^{\frac{\operatorname{pord} f - 1}{n}} + \cdots + \beta_j \frac{1}{v^{1/n}} + \cdots .$$

Hence

$$f^*(T_i(v)) - f^*(T_j(v)) = (\zeta_n^{-i \operatorname{pord} f} - \zeta_n^{-j \operatorname{pord} f}) v^{\frac{\operatorname{pord} f}{n}} + \cdots$$

and thus

$$D_t(f)(v) = \text{constant} \cdot v^{2\binom{n}{2} \frac{\operatorname{pord} f}{n}} + \cdots .$$

Summarizing, we have the following.

Lemma 7.2. *Let $t, f \in M^\infty(N)$ with $n := \operatorname{pord} t \ge 1$, and $\gcd(n, \operatorname{pord} f) = 1$. Then, the degree of the discriminant polynomial $D_t(f)(x) \in \mathbb{C}[x]$ is*

$$\deg_x D_t(f)(x) = (n-1) \operatorname{pord} f. \tag{7.1}$$

8. Reduction Steps and Order-Reduction Polynomials

In Sect. 4, we described how order-complete bases can be transformed into integral bases of $M^\infty(N)$ by a finite sequence of pole-order-reduction steps. In this section, we establish a link between pole-order-reduction steps and order-reduction polynomials.

To this end, we consider again our standard situation: let $t \in M^\infty(N)$ with $n := \operatorname{pord} t \ge 1$, let $(1, b_1, \ldots, b_{n-1})$ with $b_j \in M^\infty(N)$ be an order-complete basis of the $\mathbb{C}[t]$-module:

$$M := \langle 1, b_1, \ldots, b_{n-1} \rangle_{\mathbb{C}[t]} \subseteq M^\infty(N).$$

By Corollary 4.4, when $M \ne M^\infty(N)$, there exist $c_j \in \mathbb{C}$, not all zero, and α in $\mathbb{C}$, such that

$$h_\alpha := \frac{c_0 + c_1 b_1 + \cdots + c_{n-1} b_{n-1}}{t - \alpha} \in M^\infty(N) \backslash M. \tag{8.1}$$

In particular, there exists a $k \in \{1, \ldots, n-1\}$, such that

$$\operatorname{pord} h_\alpha = \operatorname{pord} b_k - n \ge k \text{ and } c_k \ne 0. \tag{8.2}$$

Proposition 8.1. *With regard to order-reduction polynomials, this setting is reflected by*

$$D_t(1, b_1, \ldots, b_{k-1}, h_\alpha, b_{k+1}, \ldots, b_{n-1})(v)$$
$$= \frac{c_k^2}{(v-\alpha)^2} D_t(1, b_1, \ldots, b_{k-1}, b_k, b_{k+1}, \ldots, b_{n-1})(v). \tag{8.3}$$

Proof. After filling the right side of (8.1) into the determinant definition (5.3) of $D_t(1, b_1, \ldots, b_{k-1}, h_\alpha, b_{k+1}, \ldots, b_{n-1})(v)$ and noticing that $t^*(T_j(v)) = v$, $j = 1, \ldots, n$, the proof is a straightforward consequence of determinant calculus. $\square$

In other words, a pole-order-reduction step associated with $\alpha \in \mathbb{C}$:

$$(1, \ldots, b_{k-1}, b_k, b_{k+1}, \ldots) \to (1, \ldots, b_{k-1}, h_\alpha, b_{k+1}, \ldots),$$

from one order-complete basis to another corresponds to factoring the order-reduction polynomial as

$$D_t(1, b_1, \ldots, b_{k-1}, b_k, b_{k+1}, \ldots, b_{n-1})(x)$$
$$= \text{constant} \cdot (x-\alpha)^2 D_t(1, b_1, \ldots, b_{k-1}, h_\alpha, b_{k+1}, \ldots, b_{n-1})(x). \tag{8.4}$$

Example 8.2. In the situation of Example 5.5

$$F_3 + 12F_2 + 11^2 = -\frac{161{,}051 + 15{,}972F_2 + F_4^2 + 242F_4 - 121/4F_6}{1/z_{11} + 11^3}.$$

9. Local Puiseux Expansions

By considering local expansions at finitely many points $[\tau_j]_N \in X_0(N)$ for $\tau_j \in \hat{\mathbb{H}} = \mathbb{H} \cup \mathbb{Q} \cup \{\infty\}$, in this section, we derive important ingredients for our proof of Theorem 12.2. To this end, we consider charts $\varphi_{\tau_0} \colon U_0 \to \mathbb{C}$ with $\varphi_{\tau_0}([\tau]_N) := \phi_{\tau_0}(\tau)$ defined in a standard way either by

$$\phi_{\tau_0}(\tau) := \tau - \tau_0, \tag{9.1}$$

if $\tau_0 \in \mathbb{H}$ is not an elliptic point, or by

$$\phi_{\tau_0}(\tau) := \left(\frac{\tau - \tau_0}{\tau - \overline{\tau_0}}\right)^{h(\tau_0)}, \tag{9.2}$$

if $\tau_0 \in \mathbb{H}$ is an elliptic point (cf. (9.5)), or according to (15.3) by

$$\phi_{\tau_0}(\tau) := e^{2\pi i \gamma^{-1}\tau / w_N(c)}, \tag{9.3}$$

if $\tau_0 = \frac{a}{c} = \gamma\infty \in \mathbb{Q} \cup \{\infty\}$.
Here, $U_0 \subseteq X_0(N)$ is a neighborhood of $[\tau_0]_N$; furthermore, the periods $h(\tau_0)$ equal either 2 or 3. We note explicitly that all these charts are centered at 0, that is,

$$\phi_{\tau_0}(\tau_0) = 0. \tag{9.4}$$

Remark 9.1. The explanation why such charts have to be chosen can be found, for instance, in [5, Sect. 2.2 and Sect. 2.3]. Charts, being homeomorphisms between open subsets of Riemann surfaces and of $\mathbb{C}$, are used to set up local series expansions. Charts of the kind as in (9.2) have to be taken when $[\tau_0]_N$ is an *elliptic* point, i.e., if

$$\{\gamma \in \Gamma_0(N) : \gamma\tau_0 = \tau_0\} \neq \left\{ \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix} \right\}. \tag{9.5}$$

Throughout this section, again $t \in M^\infty(N)$ with $n := \operatorname{pord} t \geq 1$. Now, we reconsider the setting in Sect. 5 by dropping the assumption that $v_0 \in \mathbb{C}$ is not a branch point of t^*. This means, we allow $\ell \leq n$ pairwise distinct points $x_j = [\tau_j]_N \in X_0(N)$ with $\tau_j \in \mathbb{H} \cup \mathbb{Q}$, such that $[\tau_j]_N \neq [\infty]_N$ and[5]

$$t^{*-1}(v_0) = \{x_1, \ldots, x_\ell\}.$$

There exists a neighborhood V_0 of v_0 and neighborhoods U_j of the x_j, such that

$$t^{*-1}(V_0) = U_1 \cup \cdots \cup U_\ell,$$

as a disjoint union of open sets.
Now, if $\ell < n$, not all of the restricted functions

$$t^*|U_j : U_j \to V_0$$

are bi-holomorphic.
Summarizing this setting,

$$t^*(x) = v_0 \text{ has } \ell \leq n \text{ solutions } x_1 = [\tau_1]_N, \ldots, x_\ell = [\tau_\ell]_N$$

with multiplicities $k_1, \ldots, k_\ell$, respectively, i.e., $k_1 + \cdots + k_\ell = n$.

Hence, if V is an open subset of V_0 not containing v_0, then for each $j = 1, \ldots, \ell$, there exist pairwise disjoint open subsets $U_{j,k} \subseteq U_j$, $k = 1, \ldots, k_j$, such that

$$t^{*-1}(V) = \Big(U_{1,1} \cup \cdots \cup U_{1,k_1}\Big) \cup \cdots \cup \Big(U_{\ell,1} \cup \cdots \cup U_{\ell,k_\ell}\Big) \tag{9.6}$$

as a disjoint union, and for $k = 1, \ldots, k_j$, the restricted functions

$$t^*|U_{j,k} : U_{j,k} \to V$$

are bi-holomorphic.
For all $[\tau]_N \in U_j$, $j = 1, \ldots, \ell$, one has expansions

$$t^*([\tau]_N) = v_0 + a_{j,0}\phi_{\tau_j}(\tau)^{k_j} + a_{j,1}\phi_{\tau_j}(\tau)^{k_j+1} + \cdots \text{ with } a_{j,0} \neq 0. \tag{9.7}$$

Again, using (6.1), one has

$$t(\tau) - v_0 = B_j(\phi_{\tau_j}(\tau))^{k_j}, \tag{9.8}$$

where

$$B_j(z) := z\Big(a_{j,0} + a_{j,1}z + \cdots\Big)^{1/k_j}.$$

[5]Note that, in particular, $\tau_j \neq \infty$.

For $j = 1, \ldots, \ell$, let

$$A_j(z) = A_{j,1}z + A_{j,2}z^2 + \cdots \quad \text{such that } A_j(B_j(z)) = B_j(A_j(z)) = z.$$

Now, by inverting the relation (9.8) and using the Puiseux series, the situation of (9.6) is reflected as follows: for each $v \in V$, there is for fixed (j,k), $j = 1, \ldots, k_j$ and $k \in \{1, \ldots, k_j\}$, a uniquely determined $\tau = \tau(j,k) \in U_{j,k}$, such that

$$[\tau]_N = [\tau(j,k)]_N = (t^* \mid U_{j,k})^{-1}(v).$$

For such pairs $\tau = \tau(j,k)$ and v, one has

$$\begin{aligned} \phi_{\tau_j}(\tau) = \phi_{\tau_j}(\tau(j,k)) &= A_j\left(\zeta_{k_j}^k (v - v_0)^{1/k_j}\right) \\ &= A_{j,1}\zeta_{k_j}^k (v-v_0)^{1/k_j} + A_{j,2}\zeta_{k_j}^{2k}(v-v_0)^{2/k_j} + \cdots . \end{aligned} \tag{9.9}$$

As in Sect. 5, one works with a fixed branch of the k_jth root; moreover, we note that as a consequence of the definition of $A_j(z)$, $A_{j,1} \neq 0$ for all $j = 1, \ldots, \ell$.

To connect to discriminant polynomials, let $f \in M^\infty(N)$ be such that $\gcd(n, \operatorname{pord} f) = 1$. Moreover, without loss of generality, for $j = 1, \ldots, \ell$, we can assume that the neighborhoods U_j are chosen, such that the following expansions exist for all $[\tau]_N \in U_j$:

$$f^*([\tau]_N) = f(\tau) = f(\tau_j) + \sum_{m=1}^{\infty} b_{j,m}\phi_{\tau_j}(\tau)^m. \tag{9.10}$$

Invoking (9.9), one obtains

Lemma 9.2. *For $v_0 \in \mathbb{C}$ and $j = 1, \ldots, \ell$, suppose that open neighborhoods $U_{j,k}$, $k = 1, \ldots, k_j$ and V are chosen as above. Then, there exist series expansions with complex coefficients $c_{j,p}$, such that for all $v \in V$:*

$$f\Big((t^* \mid U_{j,k})^{-1}(v)\Big) = f(\tau_j) + \sum_{p=1}^{\infty} c_{j,p}\zeta_{k_j}^{pk}(v - v_0)^{p/k_j}. \tag{9.11}$$

Proof. Setting $[\tau]_N := (t^* \mid U_{j,k})^{-1}(v) \in U_{j,k}$, the statement follows from applying (9.9) to (9.10):

$$f(\tau) = f(\tau_j) + \sum_{m=1}^{\infty} b_{j,m}\left(A_{j,1}\zeta_{k_j}^k(v-v_0)^{1/k_j} + A_{j,2}\zeta_{k_j}^{2k}(v-v_0)^{2/k_j} + \cdots\right)^m.$$

□

To adapt to the refined setting (9.6), we extend our T_j-notation to the additional restricted functions:

$$T_{j,k} = (t^*|U_{j,k})^{-1} \colon V \to U_{j,k}.$$

Finally, we use the information we obtained in terms of the local holomorphic Puiseux series expansion to represent the discriminant polynomial at $v_0 \in \mathbb{C}$.

Namely, for all $v \in V$:

$$\begin{aligned}
D_t(f)(v) &= (-1)^{\binom{n}{2}} \prod_{(j,k)\neq(j',k')} (f(T_{j,k}(v)) - f(T_{j',k'}(v))) \\
&= (-1)^{\binom{n}{2}} \prod_{\substack{1\le j\le \ell \\ 1\le k,k'\le k_j, k\neq k'}} (f(T_{j,k}(v)) - f(T_{j,k'}(v))) \\
&\quad \prod_{\substack{1\le j,j'\le \ell, j\neq j' \\ 1\le k,k'\le k_j}} (f(T_{j,k}(v)) - f(T_{j',k'}(v))) \\
&= (-1)^{\binom{n}{2}} \prod_{\substack{1\le j\le \ell \\ 1\le k,k'\le k_j, k\neq k'}} \left(\sum_{p=1}^{\infty} c_{j,p}(\zeta_{k_j}^{pk} - \zeta_{k_j}^{pk'})(v-v_0)^{p/k_j}\right) \\
&\quad \prod_{\substack{1\le j,j'\le \ell, j\neq j' \\ 1\le k,k'\le k_j}} (f(\tau_j) - f(\tau_{j'}) + O((v-v_0)^{1/k_j}) - O((v-v_0)^{1/k_{j'}})).
\end{aligned}$$

The last equality is by (9.11); it gives rise to the following.

Proposition 9.3. *Let $t \in M^\infty(N)$ with $n = \operatorname{pord} t \geq 1$, let $f \in M^\infty(N)$ be such that* $\gcd(n, \operatorname{pord} f) = 1$. *For $v_0 \in \mathbb{C}$, suppose that*

$t^*(x) = v_0$ *has* $\ell \leq n$ *pairwise distinct solutions* $x_1 = [\tau_1]_N, \ldots, x_\ell = [\tau_\ell]_N$

with multiplicities $k_1, \ldots, k_\ell$, respectively, i.e., $k_1 + \cdots + k_\ell = n$.

If

$$\text{the values } f(\tau_1), \ldots, f(\tau_\ell) \text{ are pairwise distinct,} \tag{9.12}$$

and

$$f'(\tau_1) \neq 0, \ldots, f'(\tau_\ell) \neq 0, \tag{9.13}$$

then there exists a polynomial $p(x) \in \mathbb{C}[x]$, such that for all $v \in \mathbb{C}$:

$$D_t(f)(v) = (v - v_0)^{n-\ell} p(v) \quad \text{where} \quad p(v_0) \neq 0. \tag{9.14}$$

Proof. The statement follows from the last equality of the derivation preceding this proposition. Namely, under the condition (9.12), the second product on the right side of this equality is non-zero for $v = v_0$. Condition (9.13) means that $b_{j,1} \neq 0$ in (9.10), thus $c_{j,1} \neq 0$ for $j = 1, \ldots, \ell$ in (9.11). Consequently, from the first product in the expression under consideration, one can pull out $v - v_0$ as follows:

$$\begin{aligned}
&c_{1,1} \prod_{1\le k,k'\le k_1, k\neq k'} (\zeta_{k_1}^{k} - \zeta_{k_1}^{k'})(v-v_0)^{1/k_1} \cdots c_{\ell,1} \prod_{1\le k,k'\le k_\ell, k\neq k'} (\zeta_{k_\ell}^{k} - \zeta_{k_\ell}^{k'})(v-v_0)^{1/k_\ell} \\
&\quad = \text{constant} \cdot (v-v_0)^{2\binom{k_1}{2}\frac{1}{k_1} + \cdots + 2\binom{k_\ell}{2}\frac{1}{k_\ell}}.
\end{aligned}$$

Recalling that $k_1 + \cdots + k_l = n$ completes the proof for all $v \in V$, where V is an open subset of a neighborhood V_0 of v_0, such that V does not contain v_0. However, invoking the identity theorem from complex analysis, the statement extends to all $v \in \mathbb{C}$. $\square$

Properties (9.12) and (9.13) are sufficiently important to deserve a

Definition 9.4 (*separation property*). Let $t \in M^\infty(N)$ with $n := \operatorname{pord} t \geq 1$, let $f \in M^\infty(N)$ be such that $\gcd(n, \operatorname{pord} f) = 1$. For $v_0 \in \mathbb{C}$, suppose that

$$t^*(x) = v_0 \;\text{ has }\; \ell \leq n \;\text{ pairwise distinct solutions }\; x_1 = [\tau_1]_N, \ldots, x_\ell = [\tau_\ell]_N.$$

We say that f has the separation property for (t, v_0) if f satisfies (9.12) and (9.13).

Remark 9.5. In Sect. 14, we describe how to construct such an f having the separation property.

An immediate consequence of Proposition 9.3 is

Corollary 9.6. *Let f have the separation property for (t, β) with $\beta \in \mathbb{C}$. Then*

$$D_t(f)(\beta) = 0 \iff \beta \in \operatorname{BranchPts}(t^*). \tag{9.15}$$

Another consequence of our analysis above is

Proposition 9.7. *Let $t \in M^\infty(N)$ with $n := \operatorname{pord} t \geq 1$, and let $f \in M^\infty(N)$ be such that $\gcd(n, \operatorname{pord} f) = 1$. For $v_0 \in \mathbb{C}$, suppose that*

$$t^*(x) = v_0 \;\textit{ has }\; \ell \leq n \;\textit{ pairwise distinct solutions }\; x_1 = [\tau_1]_N, \ldots, x_\ell = [\tau_\ell]_N$$
$$\textit{with multiplicities }\; k_1, \ldots, k_\ell, \textit{respectively, i.e.,} k_1 + \cdots + k_\ell = n.$$

For complex numbers $a_0, \ldots, a_{n-1}$, not all zero, define a meromorphic function on $\mathbb{H}$ by

$$F(\tau) := \frac{a_0 + a_1 f(\tau) + \cdots + a_{n-1} f(\tau)^{n-1}}{t(\tau) - v_0}.$$

If f has the separation property for (t, v_0), then[6]

$$F(\tau_j) = \infty \;\; \textit{for some} \;\; j \in \{1, \ldots, \ell\}. \tag{9.16}$$

Proof. Suppose F^* is analytic on $X_0(N) \backslash \{[\infty]_N\}$. Then, assuming the setting as above, by (9.7), one has for all $\tau \in U_1$, a series expansion

$$\begin{aligned} &a_0 + a_1 f(\tau) + \cdots + a_{n-1} f(\tau)^{n-1} = (t(\tau) - v_0) F(\tau) \\ &\quad = (a_{1,0}\phi_{\tau_1}(\tau)^{k_1} + a_{1,1}\phi_{\tau_1}(\tau)^{k_1+1} + \cdots)(F_0 + F_1\phi_{\tau_1}(\tau) + \cdots) \end{aligned} \tag{9.17}$$

with $a_{1,0} \neq 0$. Hence, owing to (9.4),

$$a_0 + a_1 f(\tau_1) + \cdots + a_{n-1} f(\tau_1)^{n-1} = 0,$$

which implies a factorization

$$\begin{aligned} &a_0 + a_1 f(\tau) + \cdots + a_{n-1} f(\tau)^{n-1} \\ &\quad = (f(\tau) - f(\tau_1))(A_0 + A_1 f(\tau) + \cdots + A_{n-2} f(\tau)^{n-2}) \\ &\quad = (b_{1,0}\phi_{\tau_1}(\tau) + b_{1,1}\phi_{\tau_1}(\tau)^2 + \cdots)(A_0 + A_1 f(\tau) + \cdots + A_{n-2} f(\tau)^{n-2}), \end{aligned}$$

[6]Note that $\tau_j \in \mathbb{H} \cup \mathbb{Q}$ are such that $[\tau_j]_N \neq [\infty]_N$. Hence, (9.16) implies that $F \notin M^\infty(N)$.

where the last equality is by (9.10) with $b_{1,0} \neq 0$ owing to (9.13). As a consequence of (9.17), if $k_1 > 1$:

$$A_0 + A_1 f(\tau_1) + \cdots + A_{n-2} f(\tau_1)^{n-2} = 0,$$

and by iteration

$$\begin{aligned} &a_0 + a_1 f(\tau) + \cdots + a_{n-1} f(\tau)^{n-1} \\ &\quad = (f(\tau) - f(\tau_1))^{k_1} (B_0 + B_1 f(\tau) + \cdots + B_{n-1-k_1} f(\tau)^{n-1-k_1}). \end{aligned}$$

Notice that if

$$B(x) := B_0 + B_1 x + \ldots + B_{n-1-k_1} x^{n-1-k_1}$$

is the zero polynomial (e.g., if $k_1 = n$), the assumption that $(1, f, \ldots, f^{n-1})$ is an order-complete basis would imply that all $a_j = 0$, and the proof would stop with this contradiction.

Using the same argument, one derives

$$\begin{aligned} &a_0 + a_1 f(\tau) + \cdots + a_{n-1} f(\tau)^{n-1} \\ &\quad = (f(\tau) - f(\tau_2))^{k_2} (C_0 + C_1 f(\tau) + \ldots + C_{n-1-k_2} f(\tau)^{n-1-k_2}), \end{aligned}$$

etc., up to

$$\begin{aligned} &a_0 + a_1 f(\tau) + \ldots + a_{n-1} f(\tau)^{n-1} \\ &\quad = (f(\tau) - f(\tau_\ell))^{k_\ell} (D_0 + D_1 f(\tau) + \ldots + D_{n-1-k_\ell} f(\tau)^{n-1-k_\ell}). \end{aligned}$$

If one of the polynomial factors in the role of $B(x)$ above would be the zero polynomial, we are done. Otherwise, invoking condition (9.12) implies that

$$\begin{gathered} (f(\tau) - f(\tau_1))^{k_1} \cdots (f(\tau) - f(\tau_\ell))^{k_\ell} \ \text{ divides} \\ a_0 + a_1 f(\tau) + \cdots + a_{n-1} f(\tau)^{n-1}. \end{gathered}$$

Recalling that not all the a_j are zero and $k_1 + \cdots + k_\ell = n$, we obtain a contradiction to the assumption that F^* is analytic on $X_0(N) \backslash \{[\infty]_N\}$. □

Corollary 9.8. *Let* $t \in M^\infty(N)$ *with* $n := \operatorname{pord} t \geq 1$, *let* $f \in M^\infty(N)$ *be such that* $\gcd(n, \operatorname{pord} f) = 1$. *For* $v_0 \in \mathbb{C}$, *suppose that*

$$t^*(x) = v_0 \ \textit{ has } \ \ell \leq n \ \textit{ pairwise distinct solutions } \ x_1 = [\tau_1]_N, \ldots, x_\ell = [\tau_\ell]_N.$$

For complex numbers $a_0, \ldots, a_{n-1}$, *define a meromorphic function on* $\mathbb{H}$ *by*

$$F(\tau) := \frac{a_0 + a_1 f(\tau) + \cdots + a_{n-1} f(\tau)^{n-1}}{t(\tau) - v_0}.$$

If f *has the separation property for* (t, v_0), *then*

$$F \in M^\infty(N) \Longrightarrow F = 0.$$

Proof. If one of the a_j would be non-zero, Proposition 9.7 would imply a pole of F^* at some $[\tau_j]_N \neq [\infty]_N$, $\tau_j \in \mathbb{H} \cup \mathbb{Q}$. □

10. Order Reduction and Discriminant Polynomials

In this section, we relate discriminant polynomials to order-reduction polynomials associated with integral bases. Throughout this section, let $t \in M^{\infty}(N)$ with $n := \operatorname{pord} t \geq 1$, let $(1, b_1, \ldots, b_{n-1})$, $b_j \in M^{\infty}(N)$, be an order-complete tuple forming an integral basis for $M^{\infty}(N)$ over $\mathbb{C}[t]$, that is,

$$\langle 1, b_1, \ldots, b_{n-1} \rangle_{\mathbb{C}[t]} = M^{\infty}(N). \tag{10.1}$$

Moreover, let $f \in M^{\infty}(N)$ again be chosen, such that $\gcd(n, \operatorname{pord} f) = 1$. By Proposition 3.5, such an f gives rise to an order-complete basis $(1, f, \ldots, f^{n-1})$ of the $\mathbb{C}[t]$-module:

$$\langle 1, f, f^2, \ldots, f^{n-1} \rangle_{\mathbb{C}[t]} = \mathbb{C}[t, f] \subseteq M^{\infty}(N).$$

By exemplifying the case for $n = 3$, we shall see how the discriminant polynomial

$$D_t(f)(v) := D_t(1, f, f^2, \ldots, f^{n-1})(v)$$

is related to the order-reduction polynomial:

$$D_t(1, b_1, \ldots, b_{n-1})(v).$$

By the identity theorem from complex analysis, it is sufficient to consider the situation for v from a neighborhood V of $v_0 \in V$. With the setting as in (5.3), one has

$$\begin{pmatrix} 1 & 0 & 0 \\ r_0^{(1)}(v) & r_1^{(1)}(v) & r_2^{(1)}(v) \\ r_0^{(2)}(v) & r_1^{(2)}(v) & r_2^{(2)}(v) \end{pmatrix} \begin{pmatrix} 1 & 1 & 1 \\ (b_1 \circ T_1)(v) & (b_1 \circ T_2)(v) & (b_1 \circ T_3)(v) \\ (b_2 \circ T_1)(v) & (b_2 \circ T_2)(v) & (b_2 \circ T_3)(v) \end{pmatrix}$$
$$= \begin{pmatrix} 1 & 1 & 1 \\ f(T_1(v)) & f(T_2(v)) & f(T_3(v)) \\ f(T_1(v))^2 & f(T_2(v))^2 & f(T_3(v))^2 \end{pmatrix},$$

because owing to (10.1), there exist polynomials $r_j^{(i)}(x) \in \mathbb{C}[x]$, such that

$$f(\tau)^i = r_0^{(i)}(t(\tau)) + r_1^{(i)}(t(\tau))\, b_1(\tau) + r_2^{(i)}(t(\tau))\, b_2(\tau), \ \tau \in \mathbb{H}.$$

This implies

$$f(T_j(v))^i = r_0^{(i)}(v) + r_1^{(i)}(v)(b_1 \circ T_j)(v) + r_2^{(i)}(v)(b_2 \circ T_j)(v)$$

using

$$t(T_j(v)) = t((t \mid U_j)^{-1}(v)) = v.$$

Taking determinants of both sides of the matrix equation squared, this gives for the general case:

$$\text{as polynomials in } v\colon\ D_t(1, b_1, \ldots, b_{n-1})(v) \ \text{ divides } \ D_t(f)(v). \tag{10.2}$$

Next, we consider the other direction. By Proposition 4.3, there exist polynomials $q_j(x)$ and $p_i(x)$ in $\mathbb{C}[x]$, such that

$$b_j = \frac{p_0^{(j)}(t)}{q_j(t)} + \frac{p_1^{(j)}(t)}{q_j(t)} f + \cdots + \frac{p_{n-1}^{(j)}(t)}{q_j(t)} f^{n-1}, \ j = 1, \ldots, n-1, \tag{10.3}$$

where $q_j(t)$ is either a constant or such that

$$\gcd(q_j(x), p_0^{(j)}(x), \ldots, p_{n-1}^{(j)}(x)) = 1. \tag{10.4}$$

As before, this can be expressed as a matrix equation. We display the case for $n = 3$:

$$\begin{pmatrix} 1 & 0 & 0 \\ s_0^{(1)}(v) & s_1^{(1)}(v) & s_2^{(1)}(v) \\ s_0^{(2)}(v) & s_1^{(2)}(v) & s_2^{(2)}(v) \end{pmatrix} \begin{pmatrix} 1 & 1 & 1 \\ f(T_1(v)) & f(T_2(v)) & f(T_3(v)) \\ f(T_1(v))^2 & f(T_2(v))^2 & f(T_3(v))^2 \end{pmatrix}$$
$$= \begin{pmatrix} 1 & 1 & 1 \\ (b_1 \circ T_1)(v) & (b_1 \circ T_2)(v) & (b_1 \circ T_3)(v) \\ (b_2 \circ T_1)(v) & (b_2 \circ T_2)(v) & (b_2 \circ T_3)(v) \end{pmatrix},$$

where

$$s_i^{(j)}(v) := \frac{p_i^{(j)}(v)}{q_j(v)}.$$

Again, taking determinants of both sides of the matrix equation squared, for the general case, this gives another polynomial relation in v:

$$D_t(1, b_1, \ldots, b_{n-1})(v) = \frac{s(v)^2}{q_1(v)^2 \cdots q_{n-1}(v)^2} D_t(f)(v),$$

where $s(x), q_1(x), \ldots, q_{n-1}(x)$ are polynomials in $\mathbb{C}[x]$. It will be convenient to cancel out possible common factors and to write, as polynomials in $\mathbb{C}[x]$:

$$D_t(1, b_1, \ldots, b_{n-1})(x) = \frac{S(x)^2}{Q_1(x)^2 \cdots Q_{n-1}(x)^2}\, D_t(f)(x), \tag{10.5}$$

such that

$$S(x) \ \text{ and } \ Q_1(x) \cdots Q_{n-1}(x) \ \text{ are relatively prime polynomials,} \tag{10.6}$$

and

$$Q_j(x) \ \text{ divides } \ q_j(x), \ j = 1, \ldots, n-1, \tag{10.7}$$

where $q_j(x)$ are determined as in (10.3).

11. Order-Reduction Polynomials: Further Results

In this section, we continue the considerations made in the previous section. Again, $t \in M^\infty(N)$ with $n := \operatorname{pord} t \geq 1$, and $(1, b_1, \ldots, b_{n-1})$ with $b_j \in M^\infty(N)$ is assumed to be an integral basis for $M^\infty(N)$ over $\mathbb{C}[t]$.

Lemma 11.1. *Let $(1, \beta_1, \ldots, \beta_{n-1})$ with $\beta_j \in M^\infty(N)$ be an integral bases for $M^\infty(N)$ over $\mathbb{C}[t]$. Then, there exists a $c \in \mathbb{C}$ such that*

$$D_t(1, b_1, \ldots, b_{n-1})(x) = c \cdot D_t(1, \beta_1, \ldots, \beta_{n-1})(x).$$

Proof. Applying the same kind of argument as used to derive (10.2), we obtain the polynomial relations:

$$D_t(1, b_1, \dots, b_{n-1})(x) \;\text{ divides }\; D_t(1, \beta_1, \dots, \beta_{n-1})(x) \tag{11.1}$$

and

$$D_t(1, \beta_1, \dots, \beta_{n-1})(x) \;\text{ divides }\; D_t(1, b_1, \dots, b_{n-1})(x). \tag{11.2}$$

This proves the statement. ◻

Another application of the argument we used to derive (10.2) is the existence of some polynomial $R(x) \in \mathbb{C}[x]$, such that

$$R(x)^2 \; D_t(1, b_1, \dots, b_{n-1})(x) = D_t(f)(x).$$

This, using (10.5), implies

$$\frac{R(x)S(x)}{Q_1(x) \cdots Q_{n-1}(x)} = 1 \;\text{ or }\; -1.$$

Finally, as a consequence of (10.6), we obtain

$$R(x) = \frac{1}{c} \cdot Q_1(x) \cdots Q_{n-1}(x) \;\text{ and }\; S(x) = c \;\text{ for some non-zero }\; c \in \mathbb{C}. \tag{11.3}$$

We summarize the following.

Lemma 11.2. *There is $c \in \mathbb{C}$ such that*

$$D_t(f)(x) = c \cdot Q_1(x)^2 \cdots Q_{n-1}(x)^2 \; D_t(1, b_1, \dots, b_{n-1})(x), \tag{11.4}$$

where for $j = 1, \dots, n-1$, the polynomials $Q_j(x)$ divide the polynomials $q_j(x)$ which are determined as in (10.3).

Lemma 11.3. *Let $Q_j(x)$ be the polynomials as in Lemma 11.2. Suppose f has the separation property for (t, β) for some $\beta \in \mathbb{C}$. Then*

$$Q_j(\beta) \neq 0 \;\text{ for all }\; j = 1, \dots, n-1.$$

Proof. Suppose $x - \beta \mid Q_j(x)$ for some $j \in \{1, \dots, n-1\}$. By Lemma 11.2, $Q_j(x) \mid q_j(x)$ with $q_j(x)$ as in relation (10.3). Hence, $x - \beta$ divides $q_j(x)$, and (10.3) can be rewritten as

$$\frac{q_j(t)}{t - \beta}\, b_j = \frac{p_0^{(j)}(t)}{t - \beta} + \frac{p_1^{(j)}(t)}{t - \beta} f + \cdots + \frac{p_{n-1}^{(j)}(t)}{t - \beta} f^{n-1}.$$

As in the proof of Corollary 4.4, by division with remainder, there are polynomials $p_l(x) \in \mathbb{C}[x]$ and $a_l \in \mathbb{C}$, such that $p_l^{(j)}(x) = (x - \beta)p_l(x) + a_l$, $l = 0, \dots, n-1$. This means

$$\begin{aligned} \frac{q_j(t)}{t - \beta}\, b_j = {} & \frac{a_0 + a_1 f + \cdots + a_{n-1} f^{n-1}}{t - \beta} \\ & + p_0(t) + p_1(t) f + \cdots + p_{n-1}(t) f^{n-1} \in M^{\infty}(N). \end{aligned}$$

Owing to the fact that f has the separation property for (t, β), one has by Corollary 9.8:

$$\frac{a_0 + a_1 f + \cdots + a_{n-1} f^{n-1}}{t - \beta} = 0.$$

Iterating this argument cancels out all powers of $t - \beta$ and one arrives at a representation of b_j of the form:

$$Q(t)\ b_j = P_0(t) + P_1(t) f + \cdots + P_{n-1}(t) f^{n-1}$$

with polynomials $P_l(x)$ and $Q(x)$, such that

$$x - \beta \nmid Q(x). \tag{11.5}$$

Comparing this to the representation (10.3), which rewrites as

$$q_j(t)\ b_j = p_0^{(j)}(t) + p_1^{(j)}(t) f + \cdots + p_{n-1}^{(j)}(t) f^{n-1},$$

produces a contradiction to the uniqueness of the basis representation since in contrast to (11.5), $x - \beta$ divides the denominator polynomial $q_j(x)$. □

Proposition 11.4. *For any $\beta \in \mathbb{C}$:*

$$D_t(1, b_1, \ldots, b_{n-1})(\beta) = 0 \iff \beta \in \mathrm{BranchPts}(t^*). \tag{11.6}$$

Proof. For the proof we choose f having the separation property for (t, β).[7] For the "$\Rightarrow$" direction of the statement, suppose $D_t(1, b_1, \ldots, b_{n-1})(\beta) = 0$. Then (11.4) implies $D_t(f)(\beta) = 0$ which, owing to Corollary 9.6, is true if and only if $\beta \in \mathrm{BranchPts}(t)$. For the other direction, we use the reverse direction of this "if and only if" relation: $\beta \in \mathrm{BranchPts}(t^*)$ implies $x - \beta \mid D_t(f)(x)$. Next, we apply Lemma 11.3 to the equation (10.5) and obtain

$$x - \beta \mid D_t(1, b_1, \ldots, b_{n-1})(x),$$

which completes the proof. □

From all this we obtain the complete factorization of order-reduction polynomials of integral bases. To state it, it is convenient to define

$$\mathrm{BranchPts}_{\mathbb{C}}(t^*) := \mathrm{BranchPts}(t^*) \cap \mathbb{C},$$

in order to keep the point ∞ out, as the image of the only pole at $[\infty]_N$.

Proposition 11.5. *Let $t \in M^\infty(N)$ with $n := \operatorname{pord} t \geq 1$ and $(1, b_1, \ldots, b_{n-1})$ with $b_j \in M^\infty(N)$ be an integral basis for $M^\infty(N)$ over $\mathbb{C}[t]$. Then, there exists a $c \in \mathbb{C}$ and positive integers m_β, such that*

$$D_t(1, b_1, \ldots, b_{n-1})(x) = c \cdot \prod_{\beta \in \mathrm{BranchPts}_{\mathbb{C}}(t^*)} (x - \beta)^{m_\beta}. \tag{11.7}$$

Moreover, for any $\beta \in \mathrm{BranchPts}_{\mathbb{C}}(t^*)$, suppose that $t^*(x) = \beta$ has

$$\ell(\beta) < n \ \textit{pairwise distinct solutions} \ x_1^{(\beta)} = [\tau_1^{(\beta)}]_N, \ldots, x_{\ell(\beta)}^{(\beta)} = [\tau_{\ell(\beta)}^{(\beta)}]_N$$

with multiplicities $k_1^{(\beta)}, \ldots, k_{\ell(\beta)}^{(\beta)}$, respectively, i.e., $k_1^{(\beta)} + \cdots + k_{\ell_\beta}^{(\beta)} = n$.

[7] How to construct such f is described in Sect. 14.

Then

$$m_\beta = n - \ell(\beta). \tag{11.8}$$

Proof. The factorization (11.7) is immediate from Proposition 11.4. To prove (11.8), let $\beta \in \mathrm{BranchPts}_{\mathbb{C}}(t^*)$ be a branch point of the kind as stated. Choose f to have the separation property for (t,β). Then (9.14) implies the existence of a polynomial $p(x) \in \mathbb{C}[x]$, such that

$$D_t(f)(x) = (x-\beta)^{n-\ell(\beta)}p(x), \quad \text{where} \ \ p(\beta) \neq 0.$$

According to (11.4), there exist polynomials $Q_j(x)$, such that

$$D_t(f)(x) = c \cdot Q_1(x)^2 \cdots Q_{n-1}(x)^2 \ D_t(1, b_1, \dots, b_{n-1})(x)$$

and owing to Lemma 11.3,

$$Q_j(\beta) \neq 0 \ \text{ for all } \ j = 1, \dots, n-1.$$

Hence, $(x-\beta)^{n-\ell(\beta)}$ divides $D_t(1, b_1, \dots, b_{n-1})(x)$ with the maximal power, which proves (11.8). □

For the next consideration, we again have to use the charts as in (9.1), (9.2), and (9.3). In the setting of Proposition 11.5, one has

$$\begin{aligned}
D_t(1, b_1, \dots, b_{n-1})(x) &= c \cdot \prod_{\beta \in \mathrm{BranchPts}_{\mathbb{C}}(t^*)} (x-\beta)^{k_1^{(\beta)}+\cdots+k_{\ell(\beta)}^{(\beta)}-\ell(\beta)} \\
&= c \cdot \prod_{\beta \in \mathrm{BranchPts}_{\mathbb{C}}(t^*)} (x-\beta)^{k_1^{(\beta)}-1} \cdots (x-\beta)^{k_{\ell(\beta)}^{(\beta)}-1} \\
&= c \cdot \prod_{\beta \in \mathrm{BranchPts}_{\mathbb{C}}(t^*)} (x-t(\tau_1^{(\beta)}))^{k_1^{(\beta)}-1} \cdots (x-t(\tau_{\ell(\beta)}^{(\beta)}))^{k_{\ell(\beta)}^{(\beta)}-1} \\
&= c \cdot \prod_{\beta \in \mathrm{BranchPts}_{\mathbb{C}}(t^*)} \prod_{j=1}^{\ell(\beta)} (x-t(\tau_j^{(\beta)}))^{-1+\mathrm{mult}_{[\tau_j^{(\beta)}]_N}(\tau^*)} \\
&= c \cdot \prod_{\substack{\text{all orbits } [\tau_0]_N \in X_0(N), \\ [\tau_0]_N \neq [\infty]_N}} (x-t(\tau_0))^{-1+\mathrm{mult}_{[\tau_0]_N}(\tau^*)},
\end{aligned}$$

where the last line is by the fact that if $t(\tau_0) \notin \mathrm{BranchPts}(t^*)$, then

$$-1 + \mathrm{mult}_{[\tau_0]_N}(\tau^*) = 0.$$

Here, we use the notion of multiplicity $\mathrm{mult}_x(f)$, also explained in Sect. 16, which stands for the multiplicity at the point $x \in X$ of a meromorphic function f on a (compact) Riemann surface X. For $x_0 = [\tau_0]_N \in X_0(N)$, one has (e.g., [11, Lemma 4.7] and [5, Sect. 2.4]) with respect to our charts $\phi_{\tau_0}(\tau)$ centered at 0:[8]

$$\mathrm{mult}_{x_0}(t^*) = \begin{cases} \mathrm{ord}_{\phi_{\tau_0}(\tau)}(t(\tau) - t(\tau_0)), & \text{if } \ [\tau_0]_N \ \text{ is no pole of } \ t^*, \\ -\,\mathrm{ord}_{\phi_{\tau_0}(\tau)}\, t(\tau), & \text{if } \ [\tau_0]_N \ \text{ is a pole of } \ t^*. \end{cases} \tag{11.9}$$

Hence, we obtain Proposition 11.5.

[8] I.e., $\phi_{\tau_0}(\tau_0) = 0$.

Corollary 11.6. *Let $t \in M^\infty(N)$ with $n := \operatorname{pord} t \geq 1$ and $(1, b_1, \ldots, b_{n-1})$ with $b_j \in M^\infty(N)$ be an integral basis for $M^\infty(N)$ over $\mathbb{C}[t]$. Then*

$$\deg_x D_t(1, b_1, \ldots, b_{n-1})(x) = \sum_{\substack{x_0 \in X_0(N), \\ x_0 \neq [\infty]_N}} (-1 + \operatorname{mult}_{x_0}(t^*)).$$

Next, recall from Sect. 16 the definition of $\operatorname{Deg}(f)$, the degree of a meromorphic function f on a compact Riemann surface X:

$$\operatorname{Deg}(f) := \sum_{x \in f^{-1}(v)} \operatorname{mult}_x(f) \text{ where } v \text{ is any element in } \hat{\mathbb{C}}.$$

Choosing $v := \infty$, we have $\operatorname{Deg}(t^*) = n$. Let

$$g(X) := \text{ genus of a compact Riemann surface } X.$$

Recall the Riemann–Hurwitz formula [11, Thmorem 4.16][9] for a non-constant holomorphic map $F\colon X \to Y$ between compact Riemann surfaces:

$$2\, g(X) - 2 = \operatorname{Deg}(F)(2\, g(Y) - 2) + \sum_{x \in X} (\operatorname{mult}_x(F) - 1). \tag{11.10}$$

Now, we apply this to our setting where $X := X_0(N)$ and $F := t^*\colon X_0(N) \to \hat{\mathbb{C}}$. Owing to $g(\hat{\mathbb{C}}) = 0$, together with (11.9) and Corollary 11.6, this gives

$$\begin{aligned} 2\, g(X_0(N)) - 2 &= -2n + \sum_{x \in X_0(N)} (\operatorname{mult}_x(F) - 1) \\ &= -2n + \sum_{\substack{x_0 \in X_0(N), \\ x_0 \neq [\infty]_N}} (\operatorname{mult}_{x_0}(t^*) - 1) + \operatorname{pord} t - 1 \\ &= -n - 1 + \deg_x D_t(1, b_1, \ldots, b_{n-1})(x). \end{aligned}$$

We summarize in

Corollary 11.7. *Let $t \in M^\infty(N)$ with $n := \operatorname{pord} t \geq 1$ and $(1, b_1, \ldots, b_{n-1})$ with $b_j \in M^\infty(N)$ be an integral basis for $M^\infty(N)$ over $\mathbb{C}[t]$. Then*

$$\deg_x D_t(1, b_1, \ldots, b_{n-1})(x) = 2\, g(X_0(N)) + n - 1. \tag{11.11}$$

12. Proof of the Weierstraß Gap Theorem

In this section, we prove the gap theorem for modular functions in $M^\infty(N)$.

Definition 12.1. (*Gaps in modular function algebras*) Let M be a subalgebra of $M^\infty(N)$, the modular functions for $\Gamma_0(N)$ which are holomorphic in $\mathbb{H}$ and with a pole at ∞. A positive integer n is called a gap in M if there is no $f \in M$ with $\operatorname{pord} f = n$. We also define the gap number g_M as the total number of gaps in M, that is,

$$g_M := \#\{n \in \mathbb{Z}_{>0} : n \text{ is a gap of } M\}.$$

In this section, we prove the gap theorem in the following version.

[9] Actually the special case we need, $Y = \hat{\mathbb{C}}$, was given by Riemann; e.g., [2].

Theorem 12.2. (Weierstraß gap theorem for $X_0(N)$) *Let $g := g(X_0(N))$ be the genus of $X_0(N)$. If $g \geq 1$, then $M^\infty(N)$ has exactly g gaps n_j with*

$$1 = n_1 < \cdots < n_g \leq 2g - 1. \tag{12.1}$$

If $g = 0$, then $M^\infty(N)$ has no gaps i.e., there exists an $h \in M^\infty(N)$, such that $\operatorname{pord} h = 1$.[10]

To prepare for the proof, we determine the gap number $g_{\mathbb{C}[t,f]}$, where $t, f \in M^\infty(N)$ with $n := \operatorname{pord} t \geq 2$, $l := \operatorname{pord} f \geq 2$, and $\gcd(l, n) = 1$. To construct such functions with relatively prime pole orders is straightforward; see, for instance, Example 2.3. By Proposition 3.5, we know that

$$\mathbb{C}[t, f] = \langle 1, f, f^2, \ldots, f^{n-1} \rangle_{\mathbb{C}[t]},$$

where $(1, f, f^2, \ldots, f^{n-1})$ is an order-complete module basis. Hence, there are $l_j \in \mathbb{Z}_{\geq 0}$, $j = 1, 2, \ldots, n-1$, such that

$$\begin{aligned}\{\operatorname{pord} f, \operatorname{pord} f^2, \ldots, \operatorname{pord} f^{n-1}\} &= \{l, 2l, \ldots, (n-1)l\} \\ &= \{l_1 n + 1, l_2 n + 2, \ldots, l_{n-1} n + n - 1\}.\end{aligned}$$

Thus, inspecting each of the residue classes modulo n for $j \in \{1, \ldots, n-1\}$ makes clear that one cannot find any function of pole order

$$j, n + j, \ldots, (l_j - 1)n + j \ \text{ if } \ l_j > 0$$

in $\mathbb{C}[t, f]$. Hence, for fixed j, l_j pole orders are missing; summing j from 1 to $n - 1$ gives the total number of missing pole orders of functions in $\mathbb{C}[t, f]$:

$$\begin{aligned}l_1 + l_2 + \cdots + l_{n-1} &= \left\lfloor \frac{l_1 n + 1}{n} \right\rfloor + \left\lfloor \frac{l_2 n + 2}{n} \right\rfloor + \cdots + \left\lfloor \frac{l_{n-1} n + n - 1}{n} \right\rfloor \\ &= \sum_{j=1}^{n-1} \left\lfloor \frac{jl}{n} \right\rfloor = \frac{(l-1)(n-1)}{2},\end{aligned}$$

where the last equality is by [8, (3.32)]. We summarize in

Lemma 12.3. *Let $t, f \in M^\infty(N)$ with $n := \operatorname{pord} t \geq 1$, $l := \operatorname{pord} f \geq 1$, and $\gcd(l, n) = 1$. Then, the total number of missing pole orders of functions in $\mathbb{C}[t, f]$ is*

$$\frac{(l-1)(n-1)}{2}.$$

Proof. If $n = 1$ or $\ell = 1$ then $\mathbb{C}[t, f] = M^\infty(N)$; i.e., there is no gap. The case both n and ℓ greater or equal to 2 was treated above. □

Proof of Theorem 12.2.. Recalling Definition 4.5, each pole-order-reduction step associated to some $\alpha \in \mathbb{C}$,

$$(1, \ldots, \beta_{k-1}, \beta_k, \beta_{k+1}, \ldots) \to (1, \ldots, \beta_{k-1}, h_\alpha, \beta_{k+1}, \ldots)$$

between order-complete bases

(i) by Corollary 4.4 (4.4) reduces the total number of gaps by exactly one.

[10] This means, in this case, one has $M^\infty(N) = \mathbb{C}[h]$.

(ii) by Proposition 8.1 (8.3) reduces the degree of the order-reduction polynomial by exactly two.

The gap theorem now will be proved by successively applying pole-order-reduction steps to the order-complete basis $(1, f, \ldots, f^{n-1})$ of $\mathbb{C}[t, f]$, where t and f are chosen from $M^\infty(N)$, such that $n := \operatorname{pord} t \geq 2$, $l := \operatorname{pord} f \geq 2$, and $\gcd(l, n) = 1$. Such a pair (t, f) can be easily constructed, see, for instance, Example 2.3.

Suppose that after r reduction steps, we arrive at the integral basis $(1, b_1, \ldots, b_{n-1})$ of $M^\infty(N)$. Defining

$$d_f := \deg_x D_t(f)(x) \ \text{ and } \ d_b := \deg_x D_t(1, b_1, \ldots, b_{n-1})(x),$$

the reduction observation (ii) made above gives

$$d_b = d_f - 2r, \tag{12.2}$$

furthermore, reduction observation (i) implies for the gap numbers:

$$g_{M^\infty(N)} = g_{\mathbb{C}[t,f]} - r. \tag{12.3}$$

Combining (12.2) and (12.3) gives the desired Weierstraß estimate for the total number of gaps in $M^\infty(N)$ in terms of the genus g:

$$\begin{aligned}
g_{M^\infty(N)} &= g_{\mathbb{C}[t,f]} - \frac{1}{2}(d_f - d_b) \quad \text{(by (12.2), (12.3))}\\
&= \frac{1}{2}((n-1)(l-1) - d_f + d_b) \quad \text{(by Lemma 12.3)}\\
&= \frac{1}{2}(-(n-1) + d_b) \quad \text{(by (7.1))}\\
&= \frac{1}{2}(-(n-1) + n - 1 + 2g(X_0(N))) \quad \text{(by (11.11))}\\
&= g(X_0(N)) = g.
\end{aligned}$$

Hence, we proved that $M^\infty(N)$ has exactly g gaps. If $g = 0$ there are no gaps; i.e., in this case, after relabelling indices,

$$M^\infty(N) = \langle 1, b_1, \ldots, b_{n-1} \rangle_{\mathbb{C}[t]} \ \text{ with } \ \operatorname{pord} b_j = j, j = 1, \ldots, n-1.$$

Hence, $M^\infty(N) = \mathbb{C}[h]$ for $h := b_1$.

To prove the remaining part of the gap theorem, namely, the bound (12.1) for the gaps $\{n_1 = 1, n_2, \ldots, n_g\}$ where $g \geq 1$, we will use a general combinatorial argument. Notice that $n_1 = 1$, because otherwise there would be no gap, which, as we proved, is only possible if $g = 0$.

To prepare for the combinatorial argument, recall that after choosing t and f from $M^\infty(N)$ as above, by applying pole-order-reduction steps, we arrived, after relabelling indices, at an integral basis $(1, b_1, \ldots, b_{n-1})$ for $M^\infty(N)$ where $\operatorname{pord} b_j \equiv j \pmod{n}$, $j = 1, \ldots, n-1$. Defining

$$r_1 := \operatorname{pord} b_1, \ldots, r_{n-1} := \operatorname{pord} b_{n-1},$$

this basis gives rise to the additive submonoid

$$S := (0 + n\mathbb{Z}_{\geq 0}) \cup (r_1 + n\mathbb{Z}_{\geq 0}) \cup \cdots \cup (r_{n-1} + n\mathbb{Z}_{\geq 0})$$

of $(\mathbb{Z}_{\geq 0}, +)$ which describes the gap set of $M^\infty(N)$:

$$\mathbb{Z}_{\geq 0} \backslash S = \{n_1 = 1, n_2, \ldots, n_g\}.$$

Let $m+1$ be the smallest non-gap of $M^\infty(N)$; $m \geq 1$ owing to $n_1 = 1$.

To prove the desired bound (12.1) for the gap sizes n_j, we change the representation of the monoid S with respect to $m+1$. Namely, it is easy to see that there exist positive integers $s_1, \ldots, s_m \in \mathbb{Z}_{>0}$, such that $s_j \equiv j \pmod{m+1}$ for all $j \in \{1, \ldots, m\}$ and

$$S = (0 + (m+1)\mathbb{Z}_{\geq 0}) \cup (s_1 + (m+1)\mathbb{Z}_{\geq 0}) \cup \cdots \cup (s_m + (m+1)\mathbb{Z}_{\geq 0}).$$

In Sect. 13, we denote the number of gaps in a monoid S by $\gamma(S)$. Hence, in the given context, $g = \gamma(S)$. Recall that $m+1$ is chosen to be the smallest non-gap of $M^\infty(N)$. Therefore, we choose a monoid representation with respect to $m+1$. Concretely, in this case, there are $k_j \in \mathbb{Z}_{>0}$, such that

$$s_j = j + (m+1)k_j \text{ for } j = 1, \ldots, m.$$

Now, Lemma 13.1 implies

$$2\gamma(S) - 1 \geq j + (m+1)(k_j - 1), \; j = 1, \ldots, m.$$

Since $j + (m+1)(k_j - 1)$ are the largest non-gaps in each residue class modulo $m+1$, this proves the bound given in (12.1), and the proof of the Weierstraß gap Theorem 12.2 is completed. □

13. A Gap Property of Monoids

Let $m \in \mathbb{Z}_{>0}$ and $s_1, \ldots, s_m \in \mathbb{Z}_{>0}$, such that $s_j \equiv j \pmod{m+1}$ for all $j \in \{1, \ldots, m\}$. We consider the additive submonoid:

$$S := (0 + (m+1)\mathbb{Z}_{\geq 0}) \cup (s_1 + (m+1)\mathbb{Z}_{\geq 0}) \cup \cdots \cup (s_m + (m+1)\mathbb{Z}_{\geq 0})$$

of $(\mathbb{Z}_{\geq 0}, +)$. A positive integer $\ell \notin S$ is called a gap of S.[11] Let $\gamma(S)$ be the total number of gaps of S. Relating to our proof setting in Sect. 12, we choose this representation of S under the assumption that $m+1$ is the smallest non-gap of S. By the definition of S, there are *positive* integers k_j, such that

$$s_j = j + (m+1)k_j \quad \text{ for } j = 1, \ldots, m.$$

An easy count gives

$$\gamma(S) = k_1 + \cdots + k_m. \tag{13.1}$$

Lemma 13.1 (Monoid gap lemma). *Under these assumptions, one has for all* $j = 1, \ldots, m$:

$$2\gamma(S) - 1 \geq j + (m+1)(k_j - 1). \tag{13.2}$$

In other words, the largest possible gap is bounded by $2\gamma(S) - 1$. Before proving this statement, we prove two elementary observations.

[11] The largest gap is called the Frobenius number of S.

Lemma 13.2. *If i and ℓ in $\mathbb{Z}_{>0}$ are such that $i+\ell=j$ for $j\in\{1,\dots,m\}$, then*

$$k_i + k_\ell \geq k_j.$$

Proof.

$$\begin{aligned} s_i + s_\ell &= i + (m+1)k_i + \ell + (m+1)k_\ell \\ &= j + (m+1)(k_i + k_\ell) \geq j + (m+1)k_j. \end{aligned}$$

The inequality is by $s_i + s_\ell \in S$ with $s_i + s_\ell \equiv j \pmod{m+1}$, and $s_j \in S$ is minimal with this property. □

Lemma 13.3. *If i and ℓ in $\mathbb{Z}_{>0}$ are such that $i+\ell=j+m+1$ for $j\in\{1,\dots,m\}$, then $k_i+k_\ell+1\geq k_j$.*

Proof.

$$s_i + s_\ell = i + (m+1)k_i + \ell + (m+1)k_\ell = j + (m+1)(k_i + k_\ell + 1) \geq s_j.$$

The inequality is by $s_i + s_\ell \in S$ with $s_i + s_\ell \equiv j \pmod{m+1}$, and $s_j = j + (m+1)k_j \in S$ is minimal with this property. □

Proof of Lemma 13.1. By (13.1), the statement to prove is equivalent to

$$2(k_1 + \cdots + k_m) \geq j + (m+1)k_j - m. \tag{13.3}$$

By Lemma 13.2,

$$k_j \leq k_1 + k_{j-1}, k_j \leq k_2 + k_{j-2}, \dots, k_j \leq k_{j-1} + k_1.$$

Summing the left and right sides, respectively; of these, $j-1$ inequalities gives

$$(j-1)k_j \leq 2(k_1 + \cdots + k_{j-1}).$$

By Lemma 13.3,

$$k_j \leq k_{j+1} + k_{m+1-1} + 1, k_j \leq k_{j+2} + k_{m+1-2} + 1, \dots, k_j \leq k_m + k_{j+1} + 1.$$

Summing the left and right sides, respectively; of these, $m-j$ inequalities gives

$$(m-j)k_j \leq 2(k_{j+1} + \cdots + k_m) + m - j.$$

Combining the two inequalities, we obtain that

$$(m-1)k_j \leq 2(k_1 + \cdots + k_m) - 2k_j + m - j,$$

which is (13.3). □

14. Functions with Separation Property

The setting which we use throughout this section is: $t \in M^\infty(N)$ with $n := \operatorname{pord} t \geq 2$ and $(1, b_1, \ldots, b_{n-1})$ with $b_j \in M^\infty(N)$ is an integral basis for $M^\infty(N)$ over $\mathbb{C}[t]$. Because of $\operatorname{pord} t = n$, for any fixed $\alpha \in \mathbb{C}$, we have that[12]

$$\begin{aligned} &t^*(x) = \alpha \text{ has } \ell \leq n \text{ pairwise distinct solutions } x_1 = [\tau_1]_N, \ldots, x_\ell = [\tau_\ell]_N, \\ &\text{with multiplicities } k_1, \ldots, k_\ell, \text{ respectively. (I.e., } k_1 + \cdots + k_\ell = n.) \end{aligned} \tag{14.1}$$

We note that, as above, owing to $t \in M^\infty(N)$, $x_j = [\tau_j]_N \in \Gamma_0(N)$ with $\tau_j \in \mathbb{H} \cup \mathbb{Q}$ are such that $[\tau_j]_N \neq [\infty]_N$.

In Definition 9.4, we defined the separation property of f for (t, v_0) with $v_0 = \alpha$ as in (14.1). At various places, we required f to have this property, for instance, in Proposition 11.5. In this section, we prove the existence of such f. In addition, here, we have to use the charts as in (9.1), (9.2), and (9.3).

Lemma 14.1. *Given the setting of this section with $\ell \geq 2$, let $\tau, \tau' \in \{\tau_1, \ldots, \tau_\ell\}$ be such that $\tau \neq \tau'$. Then*

$$b_i(\tau) \neq b_i(\tau') \quad \textit{for some} \quad i \in \{1, \ldots, n-1\}.$$

Proof. We are free to relabel the indices of the preimages of α. Hence, it is sufficient to prove the statement for $\tau_1 := \tau$ and $\tau_2 := \tau'$. Suppose

$$b_j(\tau_1) = b_j(\tau_2) \quad \text{for all} \quad j \in \{1, \ldots, n-1\}. \tag{$\star$}$$

As in (9.7), for $j = 1, \ldots, \ell$ and suitable neighborhoods U_j, one has local expansions for $\tau \in U_j$:

$$t(\tau) - \alpha = a_{j,0}\phi_{\tau_j}(\tau)^{k_j} + a_{j,1}\phi_{\tau_j}(\tau)^{k_j+1} + \cdots \quad \text{with} \quad a_{j,0} \neq 0.$$

Moreover, as in (9.10), for $j = 1, \ldots, \ell$, we can assume that the neighborhoods U_j are chosen, such that the following expansions exist for all $\tau \in U_j$:

$$\begin{aligned} b_1(\tau) &= b_1(\tau_j) + \sum_{l=1}^\infty d_l^{(1,j)} \phi_{\tau_j}(\tau)^l, \\ &\vdots \\ b_{n-1}(\tau) &= b_{n-1}(\tau_j) + \sum_{l=1}^\infty d_l^{(n-1,j)} \phi_{\tau_j}(\tau)^l. \end{aligned}$$

Taking $a_j \in \mathbb{C}$ the quotient

$$g := \frac{a_0 + a_1 b_1 + \cdots + a_{n-1} b_{n-1}}{t - \alpha} \tag{14.2}$$

defines a modular function $g \in M(N)$. Now, $g \in M^\infty(N)$ if and only if all the zeros τ_j of $t(\tau_j) - \alpha = 0$ cancel out. Indeed, assuming ($\star$), one can determine $a_j \in \mathbb{C}$, not all zero, such that this cancellation happens. Namely, using the

[12] Recall that $t^*([\tau]_n) = t(\tau)$.

local expansions the cancellation condition translates into a system of linear equations, where j runs form 1 to ℓ:[13]

$$\begin{aligned} a_0 + a_1 b_1(\tau_j) + \cdots + a_{n-1} b_{n-1}(\tau_j) &= 0, \\ a_1 d_1^{(1,j)} + \cdots + a_{n-1} d_1^{(n-1,j)} &= 0, \\ &\vdots \\ a_1 d_{k_j-1}^{(1,j)} + \cdots + a_{n-1} d_{k_j-1}^{(n-1,j)} &= 0. \end{aligned}$$

This gives in total $k_1 + \cdots + k_\ell = n$ equations. However, owing to ($\star$), two of these equations are the same. This means, we are left with $n-1$ equations in n unknowns $a_0, \ldots, a_{n-1}$. This implies that there exists a solution to the system with the a_j not all 0 . This produces a contradiction: since $g \in M^\infty(N)$, there are polynomials $p_j(x) \in \mathbb{C}[x]$, such that

$$g = p_0(t) + p_1(t) b_1 + \cdots + p_{n-1}(t) b_{n-1}.$$

Combining this with (14.2), the uniqueness of the basis representation gives

$$a_0 = (t-\alpha) p_0(t), \ldots, a_{n-1} = (t-\alpha) p_{n-1}(t).$$

Consequently, all $p_j(x)$ and all a_j must be zero. Hence, ($\star$) leads to a contradiction and the lemma is proved. □

Lemma 14.2. *Given the setting of this section with $\ell \geq 2$, let*

$$S_m := \{[\tau_{i_1}]_N, \ldots, [\tau_{i_m}]_N\}$$

be a subset of $m \in \{2, \ldots, \ell\}$ pairwise distinct preimages of α. Then, there exist $\alpha_i \in \mathbb{C}$, such that for $f = \alpha_1 b_1 + \cdots + \alpha_{n-1} b_{n-1}$:

$$f(\tau_{i_1}), \ldots, f(\tau_{i_m}) \quad \textit{are pairwise distinct.} \tag{14.3}$$

Proof. We proceed by induction on m. If $m = 2$, then by the previous lemma, there exists an $i \in \{1, \ldots, n-1\}$, such that $b_i(\tau_{i_1}) \neq b_i(\tau_{i_2})$, and we choose $f := b_i$. Suppose $m \geq 2$. Because of index relabeling, we can choose $S_m := \{\tau_1, \ldots, \tau_m\}$, and the induction hypothesis gives an $F \in M^\infty(N)$ as a $\mathbb{C}$-linear combination of b_j, such that the values $F(\tau_1), \ldots, F(\tau_m)$ are pairwise distinct. If $F(\tau_{m+1}) \neq F(\tau_i)$ for all $i = 1, \ldots, m$, the induction step is done. Otherwise, $F(\tau_{m+1}) = F(\tau_r)$ for some $r \in \{1, \ldots, m\}$. By the previous lemma, there is some $k \in \{1, \ldots, n-1\}$, such that $b_k(\tau_{m+1}) \neq b_k(\tau_r)$, and we can choose a non-zero $c \in \mathbb{C}$, such that

$$c \neq \frac{F(\tau_j) - F(\tau_i)}{b_k(\tau_i) - b_k(\tau_j)} \quad \text{for all } \ 1 \leq i < j \leq m+1 \ \text{ with } \ b_k(\tau_i) \neq b_k(\tau_j).$$

By inspection, one verifies for $F_r := F + c b_k$ that $F_r(\tau_{m+1}) \neq F_r(\tau_r)$ and also that the values

$$F_r(\tau_1), \ldots, F_r(\tau_m) \quad \text{are pairwise distinct.}$$

[13]Notice that the charts $\phi_{\tau_j}(\tau)$ are centered at 0; i.e., $\phi_{\tau_j}(\tau_j) = 0$. Consequently, for fixed j, the numerator in (14.2) has to be of the form: $\phi_{\tau_j}(\tau)^{k_j}(c_0 + c_1 \phi_{\tau_j}(\tau) + \cdots)$.

Suppose $F_r(\tau_{m+1}) = F_r(\tau_s)$ for some $s \in \{1,\dots,m\}\backslash\{r\}$. If there is no such s, we are done with $f := F_r$. Otherwise, by the previous lemma, there is some $l \in \{1,\dots,n-1\}$, such that $b_l(\tau_{m+1}) \neq b_l(\tau_s)$, and we can choose a non-zero $d \in \mathbb{C}$, such that

$$d \neq \frac{F_r(\tau_j) - F_r(\tau_i)}{b_l(\tau_i) - b_l(\tau_j)} \quad \text{for all} \quad 1 \le i < j \le m+1 \quad \text{with} \quad b_l(\tau_i) \neq b_l(\tau_j).$$

Now, we set $F_{r,s} := F_r + d b_l$, and see that,

$$F_{r,s}(\tau_{m+1}) \neq F_{r,s}(\tau_r) \quad \text{and} \quad F_{r,s}(\tau_{m+1}) \neq F_{r,s}(\tau_s)$$

together with pairwise distinct values $F_{r,s}(\tau_1),\dots,F_{r,s}(\tau_m)$. Iterating this argument exhausts all possibilities and the induction step is proved. $\square$

Under the assumptions as in (14.1), to have the separation property for (t,α), f additionally has to satisfy the conditions (9.13) which rewritten as order conditions are

$$\mathrm{ord}_{\phi_{\tau_j}(\tau)}(f(\tau) - \alpha) = 1 \quad \text{for all} \quad j = 1,\dots,\ell. \tag{14.4}$$

Remark 14.3. The order in (14.4) has to be interpreted in view of (9.10) and in the sense of the ϕ-order defined in Definition 15.3. This deviates slightly from the standard notation used in the theory of the Riemann surfaces, where (14.4) would be stated in the format:

$$\mathrm{ord}_{\tau_j}(f - \alpha) = 1 \quad \text{for all} \quad j = 1,\dots,\ell. \tag{14.5}$$

This notation suppresses the explicit mentioning of the chart. We also use this notation, for instance, in cases like (2.4), where the chart is clear from the context, or in Lemma 16.1 when citing from Riemann surface theory.

Lemma 14.4. *Given the setting of this section with $\ell \ge 1$, assume that $k_i = \mathrm{ord}_{\phi_{\tau_i}(\tau)}(t(\tau) - \alpha) > 1$ for some $i \in \{1,\dots,n-1\}$. Then*

$$\mathrm{ord}_{\phi_{\tau_i}(\tau)}(b_j(\tau) - b_j(\tau_i)) = 1 \quad \textit{for some} \quad j \in \{1,\dots,n-1\}.$$

Proof. Let us assume that

$$\mathrm{ord}_{\phi_{\tau_i}(\tau)}(b_k(\tau) - b_k(\tau_i)) > 1 \quad \text{for all} \quad k \in \{1,\dots,n-1\}. \tag{$\star\star$}$$

As in (9.7), for $j = 1,\dots,\ell$ and suitable neighborhoods U_j, one has local expansions for $\tau \in U_j$:

$$t(\tau) - \alpha = a_{j,0}\phi_{\tau_j}(\tau)^{k_j} + a_{j,1}\phi_{\tau_j}(\tau)^{k_j+1} + \cdots \quad \text{with} \quad a_{j,0} \neq 0.$$

Now, we proceed with the proof exactly as above. Namely, as in (9.10), for $j = 1,\dots,\ell$, we can assume that the neighborhoods U_j are chosen such that the following expansions exist for all $\tau \in U_j$ and $k \in \{1,\dots,n-1\}$:

$$b_k(\tau) = b_k(\tau_j) + \sum_{l=1}^{\infty} d_l^{(k,j)} \phi_{\tau_j}(\tau)^l.$$

Now, we apply the same strategy as in the proof of Lemma 14.1 and determine $a_j \in \mathbb{C}$, not all zero, such that

$$g := \frac{a_0 + a_1 b_1 + \cdots + a_{n-1} b_{n-1}}{t - \alpha} \in M^\infty(N).$$

This leads us to consider the same system of $k_1 + \cdots + k_\ell = n$ linear equations. This time, owing to ($\star\star$), we have $d_1^{(k,i)} = 0$ for all $k = 1, \ldots, n-1$, and the equation containing the $d_1^{(k,i)}$ is always satisfied and can be removed. This means, we are left with $n-1$ equations in n unknowns $a_0, \ldots, a_{n-1}$.[14] This implies that there exists a solution to the system with the a_j not all 0, which produces a contradiction as in the proof of Lemma 14.2. $\square$

Lemma 14.5. *Again, we assume the setting of this section with $\ell \geq 1$. Then, there exist $\alpha_i \in \mathbb{C}$, such that for $f := \alpha_0 t + \alpha_1 b_1 + \cdots + \alpha_{n-1} b_{n-1}$:*

$$f(\tau_1), \ldots, f(\tau_\ell) \quad \textit{are pairwise distinct, and} \tag{14.6}$$

$$\mathrm{ord}_{\phi_{\tau_j}(\tau)}(f(\tau) - f(\tau_j)) = 1, \quad j = 1, \ldots, \ell. \tag{14.7}$$

Proof. For the proof, it is convenient to introduce an auxiliary function:

$$g_i(\tau) := \begin{cases} t(\tau), & \text{if } \ \mathrm{ord}_{\phi_{\tau_i}(\tau)}(t(\tau) - \alpha) = 1, \\ b_j(\tau), & \text{if } \ \mathrm{ord}_{\phi_{\tau_i}(\tau)}(t(\tau) - \alpha) > 1, \end{cases} \quad i = 1, \ldots, \ell,$$

where j is chosen according to Lemma 14.4, namely, such that $\mathrm{ord}_{\phi_{\tau_i}(\tau)}(b_j(\tau) - b_j(\tau_i)) = 1$. We will show: given $F = a_0 t + a_1 b_1 + \cdots + a_{n-1} b_{n-1}$ with $a_j \in \mathbb{C}$, such that

$$F(\tau_1), \ldots, F(\tau_\ell) \ \text{ are pairwise distinct, and} \tag{14.8}$$

$$\mathrm{ord}_{\phi_{\tau_j}(\tau)}(F(\tau) - F(\tau_j)) = 1 \ \text{ for } \ j = 1, \ldots, k, \tag{14.9}$$

then there is an $f = \alpha_0 t + \alpha_1 b_1 + \cdots + \alpha_{n-1} b_{n-1}$ with $\alpha_j \in \mathbb{C}$ which satisfies (14.6) and

$$\mathrm{ord}_{\phi_{\tau_j}(\tau)}(f(\tau) - f(\tau_j)) = 1 \ \text{ for } \ j = 1, \ldots, k+1. \tag{14.10}$$

In other words, we prove Lemma 14.5 by induction on k.

The base case $k = 1$ corresponds to the induction step from $k = 0$ to $k = 1$. The existence of F such that (14.8) holds is by Lemma 14.2. If, in addition,

$$\mathrm{ord}_{\phi_{\tau_1}(\tau)}(F(\tau) - F(\tau_1)) = 1,$$

we take $f := F$, and the base case $k = 1$ is done. If $\mathrm{ord}_{\phi_{\tau_1}(\tau)}(F(\tau) - F(\tau_1)) > 1$, define

$$f := F + c g_1 \ \text{ with } \ c \in \mathbb{C}\backslash\{0\} \text{ such that } \ c \neq \frac{F(\tau_j) - F(\tau_i)}{g_1(\tau_i) - g_1(\tau_j)},$$

[14] Notice that for this argument to work, we invoke $k_i > 1$.

where the quotient is taken for all $i, j \in \{1, \ldots, \ell\}$ for which the denominator is non-zero. Now, it is a straightforward verification that for this f the condition (14.6) holds and also that

$$\mathrm{ord}_{\phi_{\tau_1}(\tau)}(f(\tau) - f(\tau_1)) = 1.$$

This settles the base case $k = 1$.

For the induction step $k \to k+1$, we assume that we have F of required form, such that (14.8) and (14.9) hold. If, in addition,

$$\mathrm{ord}_{\phi_{\tau_{k+1}}(\tau)}(F(\tau) - F(\tau_{k+1})) = 1,$$

we are done. Otherwise, define

$$f := F + cg_{k+1} \quad \text{with } c \in \mathbb{C}\backslash\{0\} \text{ such that } \; c \neq \frac{F(\tau_j) - F(\tau_i)}{g_{k+1}(\tau_i) - g_{k+1}(\tau_j)},$$

where the quotient is taken for all $i, j \in \{1, \ldots, \ell\}$ for which the denominator is non-zero. Now, we additionally require that

$$c \neq \frac{a_{1,j}}{b_{1,j}} \text{ for all } \; j \in \{1, \ldots, \ell\} \; \text{ when } \; b_{1,j} \neq 0, \tag{14.11}$$

for $a_{1,j}$ and $b_{1,j}$ coming from the expansions:

$$F(\tau) = a_{0,j} + a_{1,j}\phi_j(\tau) + \cdots \quad \text{and} \quad g_{k+1}(\tau) := b_{0,j} + b_{1,j}\phi_j(\tau) + \cdots.$$

Again, it is straightforward to verify that for such a choice f has the properties (14.6) and (14.10). The extra requirement (14.11) is needed to guarantee the first k instances of the latter condition. This completes the proof of the induction step and also the proof of Lemma 14.5. □

Summarizing, using an integral basis $(1, b_1, \ldots, b_{n-1})$ for $M^\infty(N)$ over $\mathbb{C}[t]$, we constructed an f which proves

Corollary 14.6. *For every $\alpha \in \mathbb{C}$, there is an $f \in M^\infty(N)$, such that f has the separation property for (t, α).*

15. Appendix: Modular Functions–Basic Notions

To make this article as much self-contained as possible, in this section, we recall most of the facts we need about modular functions.

The modular group $\mathrm{SL}_2(\mathbb{Z}) = \{\left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \in \mathbb{Z}^{2\times 2} \colon ad - bc = 1\}$ acts on the upper half $\mathbb{H}$ of the complex plane by $\left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right)\tau := \frac{a\tau+b}{c\tau+d}$; this action is inherited by the congruence subgroups:

$$\Gamma_0(N) := \left\{\left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \in \mathrm{SL}_2(\mathbb{Z}) \colon N \mid c\right\},$$

where throughout this paper, N is a fixed positive integer. Note that $\Gamma_0(1) = \mathrm{SL}_2(\mathbb{Z})$. These subgroups have a finite index in $\mathrm{SL}_2(\mathbb{Z})$:

$$[\mathrm{SL}_2(\mathbb{Z}) : \Gamma_0(N)] = N \prod_{\text{prime } p|N} \left(1 + \frac{1}{p}\right), \quad N \geq 2; \tag{15.1}$$

see the standard literature on modular forms like [1] or [5]. Particularly related to our context are [9] and [12].

The action of $\mathrm{SL}_2(\mathbb{Z})$ on $\mathbb{H}$ extends to an action on meromorphic functions $f\colon \mathbb{H} \to \hat{\mathbb{C}} := \mathbb{C} \cup \{\infty\}$. A meromorphic function $f\colon \mathbb{H} \to \hat{\mathbb{C}}$ is called a *meromorphic modular function for* $\Gamma_0(N)$, if (i) for all $\left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \in \Gamma_0(N)$:

$$f\Big(\frac{a\tau + b}{c\tau + d}\Big) = f(\tau), \quad \tau \in \mathbb{H},$$

and (ii) for any $\gamma = \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \in \mathrm{SL}_2(\mathbb{Z})$, there exists an $M = M(\gamma) \in \mathbb{Z}$ together with a Fourier expansion:

$$f(\gamma\tau) = f\left(\frac{a\tau + b}{c\tau + d}\right) = \sum_{n=-M}^{\infty} f_n(\gamma) q^{n/w_N(c)},$$

where $q = q(\tau) := e^{2\pi i \tau}$ and $w_N(c) := N/\gcd(c^2, N)$. By $M(N)$, we denote the set of meromorphic modular functions for $\Gamma_0(N)$.

By (ii) with $\left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) = \left(\begin{smallmatrix} 1 & 0 \\ 0 & 1 \end{smallmatrix}\right)$, any $f \in M(N)$ admits a Laurent series expansion in powers of q with finite principal part, that is

$$f(\tau) = \sum_{n=-M}^{\infty} f_n q^n. \tag{15.2}$$

Hence, in view of $\lim_{\mathrm{Im}(\tau)\to\infty} q(\tau) = 0$, one can extend f to $\mathbb{H} \cup \{\infty\}$ by defining $f(\infty) := \infty$, if $M > 0$, and $f(\infty) := f_0$, otherwise. Subsequently, a Laurent expansion of f as in (15.2) will be also called *q-expansion of f at infinity*.[15]

Given $\gamma = \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \in \mathrm{SL}_2(\mathbb{Z})$ and $f \in M(N)$, consider the Laurent series expansion of $f(\gamma\tau)$ in powers of $q^{1/w_N(c)}$,

$$f(\gamma\tau) = \sum_{n=-M}^{\infty} g_n q^{n/w_N(c)}. \tag{15.3}$$

In view of $\gamma\infty = \lim_{\mathrm{Im}(\tau)\to\infty} \gamma\tau = a/c$, we say that (15.3) is a *q-expansion of f at a/c*. Understanding that $a/0 = \infty$, this also covers the definition of q-expansions at ∞. Concerning the uniqueness of such expansions, let $\gamma' \in \mathrm{SL}_2(\mathbb{Z})$ be such that $\gamma'\infty = \gamma\infty = a/c$, then the q-expansion of $f(\gamma'\tau)$ differs from that of $f(\gamma\tau)$ only by a root-of-unity factor in the coefficients, namely, we have then $\gamma' = \gamma \left(\begin{smallmatrix} \pm 1 & m \\ 0 & \pm 1 \end{smallmatrix}\right)$ for some $m \in \mathbb{Z}$, which implies

$$f(\gamma'\tau) = \sum_{n=-M}^{\infty} g_n \left(e^{\pm 2\pi i m/w_N(c)}\right)^n q^{n/w_N(c)}.$$

As a consequence, one can extend f from $\mathbb{H}$ to $\hat{\mathbb{H}} := \mathbb{H} \cup \{\infty\} \cup \mathbb{Q}$ by defining $f(a/c) := \lim_{\mathrm{Im}(\tau)\to\infty} f(\gamma\tau)$, where $\gamma \in \mathrm{SL}_2(\mathbb{Z})$ is chosen, such that $\gamma\infty = a/c$. Another consequence is that the q-expansions of f at ∞ are uniquely determined owing to

$$\gamma\infty = \infty \Leftrightarrow \gamma = \left(\begin{smallmatrix} \pm 1 & m \\ 0 & \pm 1 \end{smallmatrix}\right) \text{ and } w_\ell(0) = 1. \tag{15.4}$$

[15] This expansion and also those for $f(\gamma\tau)$ are required to converge for all $\tau \in \mathbb{H}$ with $\mathrm{Im}(\tau)$ sufficiently large.

Next, notice that the action of $\mathrm{SL}_2(\mathbb{Z})$, and thus of $\Gamma_0(N)$, extends in an obvious way to an action on $\hat{\mathbb{H}}$. The orbits of the $\Gamma_0(N)$ action are denoted by

$$[\tau]_N := \{\gamma\tau\colon \gamma \in \Gamma_0(N)\}, \quad \tau \in \hat{\mathbb{H}}.$$

In cases where N is clear from the context, one also writes $[\tau]$ instead of $[\tau]_N$. The set of all such orbits is denoted by

$$X_0(N) := \{[\tau]_N\colon \tau \in \hat{\mathbb{H}}\}.$$

The $\Gamma_0(N)$ action maps $\mathbb{Q} \cup \{\infty\}$ to itself, and owing to (15.1), each $\Gamma_0(N)$ produces only finitely many orbits $[\tau]_N$ with $\tau \in \mathbb{Q} \cup \{\infty\}$; such orbits are called *cusps* of $X_0(N)$. One has, for example,

Lemma 15.1. *For any prime* ℓ

(1) $X_0(\ell)$ *has two cusps*: $[\infty]_\ell$ *and* $[0]_\ell$*;*
(2) $X_0(\ell^2)$ *has* $\ell + 1$ *cusps*: $[\infty]_{\ell^2}, [0]_{\ell^2},$ *and* $[k/\ell]_{\ell^2},\ k = 1, \ldots, \ell - 1.$

Proof. This fact can be found in many sources; a detailed description of how to construct a set of representatives for the cusps of $\Gamma_0(N)$, for instance, is given in [16, Lemma 5.3]. $\square$

Suppose that the domain of $f \in M(N)$ is extended from $\mathbb{H}$ to $\hat{\mathbb{H}}$ as described above, i.e., $f\colon \mathbb{H} \to \hat{\mathbb{C}}$ is extended to $f\colon \hat{\mathbb{H}} \to \hat{\mathbb{C}}$, where we keep the same name for the extended function. Then, using this extension gives rise to a function $f^*\colon X_0(N) \to \hat{\mathbb{C}}$, which is defined as follows:

$$f^*([\tau]_N) := f(\tau), \quad \tau \in \hat{\mathbb{H}}.$$

The fact that f^* is well-defined follows from our previous discussion. We say that f^* is induced by f.

As described in detail in [5], $X_0(N)$ can be equipped with the structure of a compact Riemann surface. This analytic structure turns the induced functions f^* into meromorphic functions on $X_0(N)$. The following classical lemma [11, Theorem 1.37], a Riemann surface version of Liouville's theorem, is crucial for zero recognition of modular functions.

Lemma 15.2. *Let* X *be a compact Riemann surface. Suppose that* $g\colon X \to \mathbb{C}$ *is a holomorphic function on all of* X*. Then,* g *is a constant function.*

Being meromorphic, modular functions form fields. A classic example is that $M(N) = \mathbb{C}(j(\tau), j(N\tau))$, e.g., [5, Proposition 7.5.1], where j is the modular invariant (the Klein j function). The subset

$$M^!(N) := \{f \in M(N)\colon f^* \text{ has poles only at } [\tau]_N \text{ with } \tau \in \mathbb{Q} \cup \{\infty\}\},$$

which is important for our context and, obviously, is not a field but a $\mathbb{C}$-algebra. In this case, owing to the definition of induced functions f^*, all possible poles of f^* can be spotted by checking whether $f^*([a/c]) = f(a/c) = \infty$ for $a/c \in \mathbb{Q} \cup \{\infty\}$. Because of (15.1), $\mathbb{Q} \cup \{\infty\}$ splits only into a *finite* number of cusps:

$$\mathbb{Q} \cup \{\infty\} = [a_1/c_1]_N \cup \cdots \cup [a_k/c_k]_N.$$

Hence, knowing all the cusps $[a_j/c_j]$ reduces the task of finding all possible poles to the inspection of q-expansions of f at a_j/c_j; i.e., of q-expansions of $f(\gamma_j\tau)$ as in (15.3) with $\gamma_j \in \mathrm{SL}_2(\mathbb{Z})$ such that $\gamma_j\infty = a_j/c_j$. We call these expansions also local q-expansions of f^* at the cusps $[a_j/c_j]_N$; $w_N(c_j)$ is called the width of the cusp $[a_j/c_j]_N$. It is straightforward to show that it is independent of the choice of the representative a_j/c_j of the cusp $[a_j/c_j]_N$, and that $w_N(c_j) = N/\gcd(c_j^2, N)$ for relatively prime a_j and c_j. Note that $[\infty]_N = [1/0]_N$.

Definition 15.3 (*Order and ϕ-order*). Let $f = \sum_{n=m}^{\infty} a_n q^n$ with $m \in \mathbb{Z}$, such that $a_m \neq 0$. Then we define the order of f as

$$\operatorname{ord} f := m.$$

More generally, if $\phi = \sum_{n=1}^{\infty} b_n q^{n/w}$ for some fixed $w \in \mathbb{Z}_{>0}$, and $F = f \circ \phi := \sum_{n=m}^{\infty} a_n \phi^n$, then we define the ϕ-order of f as

$$\operatorname{ord}_\phi f := m.$$

(e.g., if $m = \operatorname{ord} f = -1$ and $\phi = q^2$, then $\operatorname{ord}_\phi F = -1$, but $\operatorname{ord} F = -2$; if $m = \operatorname{ord} f = -2$ and $\phi = q^{1/2}$, then $\operatorname{ord}_\phi F = -2$, but $\operatorname{ord} F = -1$.) In addition, more generally, we extend this definition of ϕ-order to the case, where $\phi := \phi_{\tau_0}(\tau)$ is one of the charts as in (9.1), (9.2), and (9.3).

The order $\operatorname{ord}_{[a/c]_\ell} f^*$ of f^* at a cusp $[a/c]_\ell$ is defined to be the $q^{1/w_\ell(c)}$-order of a local q-expansion of f^* at $[a/c]_N$, that is,

$$\operatorname{ord}_{[a/c]_N} f^* := \operatorname{ord}_{q^{1/w_N(c)}} f(\gamma\tau) \text{ where } \gamma = \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \in \mathrm{SL}_2(\mathbb{Z}).$$

It is straightforward to verify that $\operatorname{ord}_{[a/c]} f^*$ is well-defined. For a concrete example, see Example 2.1.

16. Appendix: Meromorphic Functions on Riemann Surfaces–Basic Notions

To make this article as much self-contained as possible, in this second appendix section, we recall most of the facts that we need about meromorphic functions on Riemann surfaces. For the terminology, we basically follow [7]; other classic texts are [6] and [11].

Lemma 15.2 states the fundamental fact that any analytic function on a compact Riemann surface is constant. In Example 2.1, we have seen that z_5^* has its only zero of order 1 at $[\infty]_5$ and its only pole at $[0]_5$ with multiplicity 1, i.e., z_5^* has order -1 at $[\infty]_5$.[16] This is also in accordance with Lemma 16.1, a corollary of another fundamental fact which says that meromorphic functions on compact Riemann surfaces have exactly as many zeroes as poles (counting multiplicities); see, for instance, [11, Proposition 4.12]:

[16] Notice that we also say that z_5^* has pole order 1 at $[\infty]_5$.

Lemma 16.1. *Let g be a non-constant meromorphic function on a compact Riemann surface X. Then*

$$\sum_{x \in X} \operatorname{ord}_x g = 0.$$

Here, $\operatorname{ord}_{x_0} g$ is defined as follows. Suppose $g(x) = \sum_{n=m}^{\infty} c_n(\varphi(x) - \varphi(x_0))^n$, $c_m \neq 0$, is the local Laurent expansion of g at x_0 using the local coordinate chart $\varphi\colon U_0 \to \mathbb{C}$ which homeomorphically maps a neighborhood U_0 of $x_0 \in X$ to an open set $V_0 \subseteq \mathbb{C}$. Then, $\operatorname{ord}_{x_0} g := m$.

Let $\mathcal{M}(S)$ denote the field of meromorphic functions $f\colon S \to \hat{\mathbb{C}}$ on a Riemann surface S.[17] Let $f \in \mathcal{M}(S)$ be non-constant: then for every neighborhood U of $x \in S$, there exist neighborhoods $U_x \subseteq U$ of x and V of $f(x)$, such that the set $f^{-1}(v) \cap U_x$ contains exactly k elements for every $v \in V \backslash \{f(x)\}$. This number k is called the multiplicity of f at x; notation: $k = \operatorname{mult}_x(f)$.[18] If S is compact, $f \in \mathcal{M}(S)$ is surjective and each $v \in \hat{\mathbb{C}}$ has the same number of preimages, say n, counting multiplicities; i.e., $n = \sum_{x \in f^{-1}(v)} \operatorname{mult}_x(f)$, see, e.g., [7, Theorem 4.24]. This number n is called the degree of f; notation: $n = \operatorname{Deg}(f)$. One of the consequences is that non-constant functions on compact Riemann surfaces have as many (finitely many) zeros as poles counting multiplicities; this is Lemma 16.1.

$\operatorname{RamiPts}(f) := \{x \in S : \operatorname{mult}_x(f) \geq 2\}$ denotes the set of ramification points of f; $\operatorname{BranchPts}(f) := f(\operatorname{RamiPts}(f)) \subseteq \hat{\mathbb{C}}$ denotes the set of branch points of f. Ramification points, and also branch points, of a function f form sets having no accumulation point. Hence, for functions on compact Riemann surfaces, these sets have finitely many elements.

17. Conclusion

In this article, we present the first proof of the Weierstraß gap theorem (for modular functions) without using the Riemann–Roch theorem. The main ingredient in our proof is the concept of order-reduction polynomials which corresponds to the discriminant of a field extension of $\mathbb{Q}$ in the setting of algebraic number theory, see, for instance, [10, III, §3]. In the field case, the structure of this discriminant is related to the ramification index [10, III, §2, Proposition 8, and III, §3, Proposition 14]. Analogously, in Proposition 11.5, we give a factorization of the order-reduction polynomial which in direct fashion relates to the branch points of the modular function t. This relation allows us to connect the degree of this polynomial to the genus of $X_0(N)$. This observation is crucial for our proof of the Weierstraß gap theorem.

In addition, our approach gives new algebraic and algorithmic insight based on module presentations of modular function algebras, in particular, the usage of integral bases. For example, our proof also gives a method to

[17]In this context, $\hat{\mathbb{C}} := \mathbb{C} \cup \{\infty\}$ is understood to be a compact Riemann surface isomorphic to the Riemann sphere.
[18]If x is a pole of f: $\operatorname{mult}_x f = -\operatorname{ord}_x f$; otherwise, $\operatorname{mult}_x f = \operatorname{ord}_x(f - f(x))$.

compute the order-reduction polynomial by using the Puiseux series expansions at infinity. Another new feature concerns the gap bound: the main task of our proof is to show that there are exactly g gaps for any modular function algebra. The proof that the corresponding pole orders are bounded by $2g-1$, with the help of an elementary combinatorial argument turns out to be an immediate consequence of our approach. Another by-product of our framework is a natural explanation of the genus $g = 0$ case as a consequence of the reduction to an integral basis.

Summarizing, our setting generalizes ideas from algebraic number theory, but still stays close to "first principles." Hence, we feel that our approach has potential for further extensions and applications. For example, we are planning to exploit the algorithmic content of our approach for computer algebra applications, for instance, for the effective computation of suitable module bases for modular function algebras.

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Peter Paule and Cristian-Silviu Radu
Research Institute for Symbolic Computation (RISC)
Johannes Kepler University
4040 Linz
Austria

Peter Paule
e-mail: Peter.Paule@risc.uni-linz.ac.at

Cristian-Silviu Radu
e-mail: Silviu.Radu@risc.uni-linz.ac.at

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Richaud–Degert Real Quadratic Fields and Maass Waveforms

Dedicated to Professor George E. Andrews on the occasion of his 80th birthday

Larry Rolen and Karen Taylor

Abstract. In this paper, we place the work of Andrews et al. (Invent Math 91(3):391–407, 1988) and Cohen (Invent Math 91(3):409–422, 1988), relating arithmetic in $\mathbb{Q}(\sqrt{6})$ to modularity of Ramanujan's function $\sigma(q)$, in the context of the general family of Richaud–Degert real quadratic fields $\mathbb{Q}(\sqrt{2p})$. Moreover, we give the resulting generalizations of the function σ as indefinite theta functions and invoke Zwegers' work, (Q J Math 63(3):753–770, 2012), to prove the modular properties of the completed functions.

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1. Introduction

In [3], Andrews, Dyson, and Hickerson (ADH) studied the Fourier coefficients of the function:

$$\sigma(q) = 1 + \sum_{n=1}^{\infty} \frac{q^{\frac{n(n+1)}{2}}}{(1+q)(1+q^2)\cdots(1+q^n)}.$$

The function σ originally appeared in the "lost" notebook of Ramanujan. (See [2] for a discussion of Ramanujan's entries involving σ.) Using Bailey pairs, ADH proved the q-identity:

$$\sigma(q) = \sum_{\substack{n\geq 0\\ |j|\leq n}} (-1)^{n+j} q^{\frac{n(3n+1)}{2}-j^2}(1-q^{2n+1}). \tag{1.1}$$

The Karen Taylor would like to thank the department of mathematics at Bryn Mawr College for hosting her sabbatical year.

In particular, they used such identities to prove Andrews' earlier speculation [1] that the coefficients of $\sigma(q)$ satisfy the unique property that the lim sup of their absolute values is infinity, but they also vanish infinitely often.

Specifically, from this identity, they deduced that the *kth* Fourier coefficient of σ is determined by congruence conditions on solutions to the generalized Pell's equation:

$$u^2 - 6v^2 = 24k + 1.$$

To describe these coefficients in a manner convenient for our purposes, we let $D = 2p$, where p is a prime, $m = 8pk + \delta_l^2$, $\delta_l = 2l - 1$ with $1 \leq l \leq \frac{p-1}{2}$, and

$$X_D(m) = \{(u, v) \in \mathbb{Z} \times \mathbb{Z} \colon u^2 - Dv^2 = m\}.$$

As usual, we denote $(u, v) \sim (U, V)$ if $u + \sqrt{D}v$ and $U + \sqrt{D}V$ are associates in $\mathbb{Z}[\sqrt{D}]$. Let

$$T_{2p}(m) = \sum_{[(u,v)] \in (X_{2p}(m)/\sim)} \left(\frac{4p}{u + pv} \right),$$

where, as above, $m \equiv \delta_l^2 \pmod{8p}$. In the case studied by ADH, we write $T(m) = T_6(m)$. Their identity (1.1) shows that $\sigma(q)$ is the generating function of $T(m)$ for $m > 0$. The "companion" generating function, $\sigma^*(q)$, of $T(m)$, for $m < 0$ is (see [3, Theorem 5]):

$$\sigma^*(q) = \sum_{n=1}^{\infty} \frac{(-1)^n q^{n^2}}{(1-q)(1-q^3)\cdots(1-q^{2n+1})}.$$

In the conclusion of their paper, ADH define a counting function $V_3(m)$ for $\mathbb{Q}(\sqrt{3})$, which can be expressed as the character sum:

$$V_3(m) = \sum_{[(u,v)] \in (X_3(m)/\sim)} \left(\frac{24}{u + 3v} \right).$$

There also is a counting function $U_2(m)$ for $\mathbb{Q}(\sqrt{2})$. They state, without proof, that

$$T(m) = V_3(m) = U_2(m).$$

(Hickerson gave an algebraic proof of these identities in a personal correspondence to Andrews [7]).

Cohen [5] further shed light on the observations of ADH by placing them in the context of Maass waveforms. To be more specific, he considered the completed function:

$$\phi(q) = q^{\frac{1}{24}} \sigma(q) + q^{-\frac{1}{24}} \sigma^*(q) = \sum_{m \equiv 1 \pmod{24}} T(m) q^{\frac{|m|}{24}}.$$

He employed the following dihedral lattice to derive connections between $k_1 = \mathbb{Q}(\sqrt{6}), k_2 = \mathbb{Q}(\sqrt{2})$, and $k_3 = \mathbb{Q}(\sqrt{3})$.

$$\begin{tikzcd}
 & L = \mathbb{Q}(\sqrt{2}, \sqrt{3+\sqrt{3}}) \arrow[d, dash] & \\
 & B = \mathbb{Q}(\sqrt{2}, \sqrt{3}) \arrow[dl, dash] \arrow[d, dash] \arrow[dr, dash] & \\
k_3 = \mathbb{Q}(\sqrt{3}) \arrow[dr, dash] & k_1 = \mathbb{Q}(\sqrt{6}) \arrow[d, dash] & k_2 = \mathbb{Q}(\sqrt{2}) \arrow[dl, dash] \\
 & \mathbb{Q} &
\end{tikzcd}$$

Here, $\mathrm{Gal}(L/\mathbb{Q}) \simeq D_4$ and $\mathrm{Gal}(L/k_1) \simeq V_4$. Since k_1 has class number one, χ_1 is a ray class character on the ray class group $CL((12)) = I^{(12)}/P_1^{(12)}$. The functional equation of the Hecke–Weber function, $L(s, \chi_1)$, was then used, via the Mellin transform, to prove that

$$\phi_0(\tau) = y^{\frac{1}{2}} \sum_{m \equiv 1 \pmod{24}} T(m) e^{\frac{2\pi i m x}{24}} K_0\left(\frac{2\pi |m| y}{24}\right)$$

is a Maass waveform. The Artin map, $\mathrm{Art}^{(12)}$, gives the isomorphism

$$\mathrm{Art}^{(12)} \colon CL((12))/\ker \mathrm{Art}^{(12)}/P_1^{(12)} \simeq \mathrm{Gal}(L/k_1).$$

The conductor of χ_1 is $\mathfrak{m} = (4(3+\sqrt{6}))$; it is the smallest modulus (in this case ideal), $\mathfrak{m}$, such that $\mathfrak{m}|(12)$ and

$$\mathrm{Art}^{\mathfrak{m}} \colon \mathrm{CL}(\mathfrak{m}) \simeq \mathrm{Gal}(L/k_1).$$

In this case, the Hecke–Weber L-function $L(s, \chi_1)$ for the ray class character χ_1 is, by the Artin map isomorphism, identical to the Artin L function $L(s, \tilde{\chi}_1)$ for the one-dimensional representation $\tilde{\chi}_1 = \chi_1 \circ (\mathrm{Art}^{\mathfrak{m}})^{-1}$ of $\mathrm{Gal}(L/k_1)$. The representation $\tilde{\chi}_1$ induces a two-dimensional representation, ρ, of $\mathrm{Gal}(L/\mathbb{Q})$. Artin's induction theorem gives $L(s, \tilde{\chi}_1) = L(s, \rho)$.

Cohen obtained the quadratic identities $T(m) = V_3(m) = U_2(m)$, for $m \equiv 1 \pmod{24}$, by showing that the characters χ_2 and χ_3 which define $U_2(m)$ and $V_3(m)$, respectively, also induce the unique two-dimensional representation ρ of $\mathrm{Gal}(L/\mathbb{Q})$.

Later, Zwegers [10] placed Cohen's construction into a more general context, generalizing the theory of mock modular forms to what he calls "mock Maass theta functions". The starting point for Zwegers is the fact that the generating functions $\sigma(q)$ and $\sigma^*(q)$ can also be expressed as the following indefinite theta functions:

$$q^{\frac{1}{24}}\sigma(q) = \left(\sum_{\substack{n+j\geq 0\\ n-j\geq 0}} + \sum_{\substack{n+j<0\\ n-j<0}}\right)(-1)^{n+j}q^{\frac{3}{2}(n+\frac{1}{6})^2-j^2},$$

$$q^{-\frac{1}{24}}\sigma^*(q) = \left(\sum_{\substack{2j+3n\geq 0\\ 2j-3n>0}} + \sum_{\substack{2j+3n<0\\ 2j-3n\leq 0}}\right)(-1)^{n+j}q^{-\frac{3}{2}(n+\frac{1}{6})^2+j^2}.$$

Given the special nature of these functions and the interest which they have generated in the literature, it is natural to consider whether the functions of Andrews, Dyson, Hickerson, and Cohen fit into a broader framework. One such generalization was explored by Bringmann, Lovejoy, and the first author in [4]. This was developed in the context of Bailey pairs and indefinite theta functions, with an eye towards quantum modular properties. However, these examples did not explore the connection to real quadratic characters; indeed, the authors of that paper were unable to identify explicit Hecke characters, in general. Here, we search for a framework in the context of Cohen's original Hecke character observation. Specifically, in this paper, we consider the generating functions $\sigma_{p,l}$, $\sigma^*_{p,l}$ for $T_{2p}(m)$ with $m \equiv \delta_l^2$, $m > 0$, and $m < 0$. Using Zwegers' formalism for the modularity properties of indefinite theta functions of this shape, we find the following general picture which naturally extends the original example of σ and σ^*.

Theorem 1.1. *Let*

$$\phi_{0,l}(\tau;p) = y^{\frac{1}{2}} \sum_{m\equiv\delta_l^2 \pmod{8p}} T_{2p}(m)e^{\frac{2\pi i m x}{8p}}K_0\left(\frac{2\pi|m|y}{8p}\right).$$

For primes p of the form $p = 2M^2+1$, where M is odd, the function $\phi_{0,l}(\tau;p)$ is a Maass waveform, with multiplier, on a congruence subgroup of $\mathrm{SL}(2,\mathbb{Z})$.

Remark 1.2. The primes p of the form $p = 2M^2+1$ have the property that the quadratic field $Q(\sqrt{2p})$ are of Richaud–Degert type and their fundamental units, ϵ_{2p}, are given explicitly by $\epsilon_{2p} = 2p-1+2M\sqrt{2p}$ (see [6, p. 50]).

Example 1.3. We list the first four primes of the form $2M^2+1$ with M odd, along with the corresponding primes p and class numbers h of $Q(\sqrt{2p})$:

M	p	h
1	3	1
3	19	1
9	163	3
21	883	5

The paper is organized as follows. In Sect. 2 we develop the multiplicative properties of the function $\chi_{19,1}$ which is used to define a (finite) Hecke character and to define $T_{2p}(m)$ as a character sum. In Sect. 3, we give an example for $p = 19$ where, since $\mathbb{Q}(\sqrt{38})$ has class number 1 and, thus, Cohen's argument applies. In Sect. 4, we prove our main theorem using Zwegers' machinery. In the final section, we gather questions for future study.

2. Arithmetic in $\mathbb{Z}[\sqrt{2p}]$ and Indefinite Theta Functions

In this section, we introduce the multiplicative function and character sums that generalize the case $p = 3$ which occurs in ADH. We then determine their generating functions as indefinite theta functions.

2.1. Character Sums

Let $R_{2p} = \{u+v\sqrt{2p}\colon (u^2-2pv^2, 2p) = 1\}$. We define $\chi_{p;1}$, on R_{2p}, as follows:

$$\chi_{p;1}(\alpha) = \begin{cases} \left(\frac{4p}{u+pv}\right), & \text{if } v \text{ is even,} \\ \left(\frac{4p}{(p-1)u+pv}\right), & \text{if } v \text{ is odd.} \end{cases}$$

Let $\alpha = u+v\sqrt{2p}$ and $N(\alpha) = \alpha\alpha'$, and then, $(N(\alpha), 2p) = 1$ implies $u \equiv 2l-1 \pmod{2p}$, with $1 \le l \le p$ and $l \ne \frac{p+1}{2}$. Since $u^2 \equiv (2p-u)^2 \pmod{8p}$, it is enough to consider $u \equiv \pm(2l-1) \pmod{8p}$, for $1 \le l \le M^2$. Thus, $N(\alpha) \equiv (2l-1)^2 \pmod{8p}$ or $N(\alpha) \equiv (2l-1)^2 + 6p \pmod{8p}$.

The function $\chi_{p;1}$ enjoys the following properties.

Proposition 2.1. *The following are true.*

1. *The function $\chi_{p;1}$ is multiplicative on R_{2p}.*
2. *We have the identity:*

$$\chi_{p;1}(\alpha') = \begin{cases} \chi_{p;1}(\alpha), & \text{if } N(\alpha) = (2l-1)^2 \pmod{8p}, 1 \le l \le M^2, \\ -\chi_{p;1}(\alpha), & \text{if } N(\alpha) = (2l-1)^2 + 6p \pmod{8p}, 1 \le l \le M^2. \end{cases}$$

Proof. Let $\alpha = u + v\sqrt{2p}$ and $\beta = U + V\sqrt{2p}$. Then

$$\alpha\beta = uU + 2pVv + (uV + Uv)\sqrt{2p} = (\alpha\beta)_1 + (\alpha\beta)_2\sqrt{2p}.$$

Since α and β are in R_{2p}, u and U are both odd. There are three cases to consider depending on the parities of v and V.
Case (i): If v and V are even, then

$$\chi_{p;1}(\alpha)\chi_{p;1}(\beta) = \chi_{p;1}(\alpha\beta),$$

since $(\alpha\beta)_1 + p(\alpha\beta)_2 \equiv (u+pv)(U+pV) \pmod{4p}$.
Case (ii): When v and V are odd, we compute

$$\begin{aligned}(\alpha\beta)_1 + p(\alpha\beta)_2 - ((p-1)u+pv)((p-1)U+pV) \\ = (p-2)p(u+v)(U+V) = 0 \pmod{4p}.\end{aligned}$$

Case (iii): Finally, if v even and V is odd, then we have

$$\begin{aligned}(p-1)(\alpha\beta)_1 + p(\alpha\beta)_2 - (u+pv)((p-1)U+pV)\\ = p(p-2)v(V-U) = 0 \pmod{4p}.\end{aligned}$$

$\square$

2.2. Indefinite Theta Functions

To determine convenient formulas for the generating function of $T_{2p}(m)$, we make repeated use of the following lemma, given in [3].

Lemma 2.2. *Let (u_0, v_0) be the fundamental solution to $U^2 - DV^2 = 1$. Each equivalence class of solutions to $U^2 - DV^2 = m$ has a representative (u, v) satisfying:*
Case (i) $m > 0$, *then* $u > 0$ *and*

$$-\frac{v_0}{u_0+1}u < v \leq \frac{v_0}{u_0+1}u;$$

Case (ii) $m < 0$, *then* $v > 0$ *and*

$$-\frac{Dv_0}{u_0+1}v < u \leq \frac{Dv_0}{u_0+1}v.$$

In our case, we have $D = 2p$ and, since $\epsilon_{2p} = 2p-1+2M\sqrt{2p}$ satisfies $\mathrm{N}_{k_1/\mathbb{Q}}(\epsilon_{2p}) = 1$, $u_0 = 2p-1$ and $v_0 = 2M$. We also need

$$\frac{v_0}{u_0+1} = \frac{M}{p}.$$

Set $\delta_l = 2l-1$. Then, $u^2 - 2pv^2 = 8pk + \delta_l^2$ implies that

$$u \equiv \pm\delta_l \pmod{2p}, \text{ and } v \text{ is even.}$$

Set $u = 2pn + \delta$, $\delta \in \{\delta_l, 2p-\delta_l\}$, $n \geq 0$ and $v = 2j$. Then

$$u + pv = 2p(n+j) + \delta \equiv \pm\delta_l \pmod{4p}$$

implies that $n + j$ is even and $\delta = \delta_l$ and, by the lemma:

$$-\frac{M}{p}(2pn+\delta_l) < 2j \leq \frac{M}{p}(2pn+\delta_l).$$

Thus

$$Mn + j > -\frac{\delta_l M}{2p} \quad \text{and} \quad Mn - j \geq -\frac{\delta_l M}{2p}.$$

Indeed, $n + j \equiv 1 \pmod 2$, $\delta = 2p - \delta_l$, and

$$-\frac{M}{p}(2p(n+1)-\delta_l) < 2j \leq \frac{M}{p}(2p(n+1)-\delta_l)$$

implies

$$M(n+1) + j > \frac{\delta_l M}{2p} \quad \text{and} \quad M(n+1) - j \geq \frac{\delta_l M}{2p}.$$

Similarly, for

$$u + pv = 2p(n+j) + \delta \equiv \pm(2p-\delta_l) \pmod{4p},$$

we have $n + j \equiv 0 \pmod 2$, $\delta = 2p - \delta_l$, and

$$M(n+1) + j > \frac{\delta_l M}{2p} \quad \text{and} \quad M(n+1) - j \geq \frac{\delta_l M}{2p},$$

or $n + j \equiv 1 \pmod 2$, $\delta = \delta_l$, and

$$Mn + j > -\frac{\delta_l M}{2p} \quad \text{and} \quad Mn - j \geq -\frac{\delta_l M}{2p}.$$

Recall that we have assumed that M is odd. Thus, $p \equiv 3 \pmod 4$ and the parity of $n \pm j$ is equal to the parity of $Mn+j$. Therefore, for $m = 8pk + \delta_l^2, k \geq 0$, we have:

$$\begin{aligned}
T_{2p}(m) &= \sum_{\substack{Mn+j>-\frac{\delta_l M}{2p} \\ Mn-j\geq-\frac{\delta_l M}{2p} \\ (2pn+\delta_l)^2-2p(2j)^2=8pk+\delta_l^2}} (-1)^{Mn+j} - \sum_{\substack{M(n+1)+j>\frac{\delta_l M}{2p} \\ M(n+1)-j\geq\frac{\delta_l M}{2p} \\ (2pn+2p-\delta_l)^2-2p(2j)^2=8pk+\delta_l^2}} (-1)^{Mn+j} \\
&= \sum_{\substack{Mn+j>-\frac{\delta_l M}{2p} \\ Mn-j\geq-\frac{\delta_l M}{2p} \\ (2pn+\delta_l)^2-2p(2j)^2=8pk+\delta_l^2}} (-1)^{Mn+j} + \sum_{\substack{Mn+j>\frac{\delta_l M}{2p} \\ Mn-j\geq\frac{\delta_l M}{2p} \\ (2pn-\delta_l)^2-2p(2j)^2=8pk+\delta_l^2}} (-1)^{Mn+j} \\
&= \sum_{\substack{Mn+j>-\frac{\delta_l M}{2p} \\ Mn-j\geq-\frac{\delta_l M}{2p} \\ (2pn+\delta_l)^2-2p(2j)^2=8pk+\delta_l^2}} (-1)^{Mn+j} + \sum_{\substack{Mn+j\leq-\frac{\delta_l M}{2p} \\ Mn-j<-\frac{\delta_l M}{2p} \\ (2pn+\delta_l)^2-2p(2j)^2=8pk+\delta_l^2}} (-1)^{Mn+j}.
\end{aligned}$$

In the second term of the last line, we have made the substitution $n \mapsto -n$. For $m < 0$, the lemma gives $-2Mv < u \leq 2Mv$. Now, if $n + j$ is even and $\delta = \delta_l$, then we find

$$2Mj + pn > -\frac{\delta_l}{2} \quad \text{and} \quad 2Mj - pn \geq \frac{\delta_l}{2};$$

whereas if $n + j$ odd and $\delta = 2p - \delta_l$, then

$$2Mj + p(n+1) > \frac{\delta_l}{2} \quad \text{and} \quad 2Mj - p(n+1) \geq -\frac{\delta_l}{2}.$$

Thus, we have

$$\begin{aligned}
T_{2p}(m) &= \sum_{\substack{2Mj+pn>-\frac{\delta_l}{2} \\ 2Mj-pn\geq\frac{\delta_l}{2} \\ (2pn+\delta_l)^2-2p(2j)^2=8pk+\delta_l^2}} (-1)^{Mn+j} - \sum_{\substack{2Mj+p(n+1)>\frac{\delta_l}{2} \\ 2Mj-p(n+1)\geq-\frac{\delta_l}{2} \\ (2p(n+1)-\delta_l)^2-2p(2j)^2=8pk+\delta_l^2}} (-1)^{Mn+j} \\
&= \sum_{\substack{2Mj+pn>-\frac{\delta_l}{2} \\ 2Mj-pn\geq\frac{\delta_l}{2} \\ (2pn+\delta_l)^2-2p(2j)^2=8pk+\delta_l^2}} (-1)^{Mn+j} + \sum_{\substack{2Mj+pn>\frac{\delta_l}{2} \\ 2Mj-pn\geq-\frac{\delta_l}{2} \\ (2pn-\delta_l)^2-2p(2j)^2=8pk+\delta_l^2}} (-1)^{Mn+j}.
\end{aligned}$$

Now, in the second term, make the changes of variables $n \mapsto -n$ and $j \mapsto -j$, to arrive at

$$T_{2p}(m) = \sum_{\substack{2Mj+pn>-\frac{\delta_l}{2} \\ 2Mj-pn\geq\frac{\delta_l}{2} \\ (2pn+\delta_l)^2-2p(2j)^2=8pk+\delta_l^2}} (-1)^{Mn+j} + \sum_{\substack{2Mj+pn<-\frac{\delta_l}{2} \\ 2Mj-pn\leq\frac{\delta_l}{2} \\ (2pn+\delta_l)^2-2p(2j)^2=8pk+\delta_l^2}} (-1)^{Mn+j}.$$

For $m = 8pk + \delta_l^2$, the generating functions, $\sigma_{2p,l}$ for $m > 0$ and $\sigma_{2p,l}^*$ for $m < 0$, for $T_{2p}(m)$, are

$$q^{\frac{\delta_l^2}{8p}}\sigma_{2p,l}(q) = \left(\sum_{\substack{Mn+j>-\frac{\delta_l M}{2p} \\ Mn-j\geq-\frac{\delta_l M}{2p}}} + \sum_{\substack{Mn+j\leq-\frac{\delta_l M}{2p} \\ Mn-j<-\frac{\delta_l M}{2p}}} \right) (-1)^{Mn+j} q^{\frac{p}{2}(n+\frac{\delta_l}{2p})^2-j^2},$$

$$q^{-\frac{\delta_l^2}{8p}}\sigma_{2p,l}^*(q) = \left(\sum_{\substack{2Mj+pn>-\frac{\delta_l}{2} \\ 2Mj-pn\geq\frac{\delta_l}{2}}} + \sum_{\substack{2Mj+pn<-\frac{\delta_l}{2} \\ 2Mj-pn\leq\frac{\delta_l}{2}}} \right) (-1)^{Mn+j} q^{-\frac{p}{2}(n+\frac{\delta_l}{2p})^2+j^2}.$$

Finally, for each l, we define

$$\phi_l(q;p) = q^{\frac{\delta_l^2}{8p}}\sigma_{2p,l}(q) + q^{-\frac{\delta_l^2}{8p}}\sigma_{2p,l}^*(q) = \sum_{m\equiv\delta_l^2 \pmod{8p}} T_{2p}(m) q^{\frac{|m|}{8p}}.$$

3. Example: $p = 19$

In the case of $p = 19$, the class number of $\mathbb{Q}(\sqrt{38})$ is 1. In this case, the proof of modularity applies exactly as in Cohen's argument. Specifically, we identify $\chi_{19,1}$ with a character on $I^{(76)}$; it gives a ray class character on $CL(\mathfrak{m})$ with conductor $\mathfrak{m} = (4(19 + 3\sqrt{38}))$. However, the ray class field is a degree 36 extension, L, of k_1. To complete the Artin theory argument for the quadratic identities, we need to identify L and show that the one-dimensional representations $\chi_{19,1} \circ (\mathrm{Art}^{\mathfrak{m}})^{-1}$, $\chi_{19,2} \circ (\mathrm{Art}^{\mathfrak{m}})^{-1}$, $\chi_{19,3} \circ (\mathrm{Art}^{\mathfrak{m}})^{-1}$ all induce the same representation of $\mathrm{Gal}(L/\mathbb{Q})$.

We then have the theta series:

$$\phi(q) = \sum_{\mathfrak{a}\subset\mathbb{Z}[\sqrt{38}]} \chi_{19,1}(\mathfrak{a}) q^{\frac{N\mathfrak{a}}{152}},$$

and the Hecke L-series:

$$L(s, \chi_{19,1}) = \sum_{\mathfrak{a}\subset\mathbb{Z}[\sqrt{38}]} \frac{\chi_{19,1}(\mathfrak{a})}{N(\mathfrak{a})^s}.$$

The completed L-function $\Lambda_{19}(s) = (2^7 \cdot 19^2)^{\frac{s}{2}} \pi^{-s} \Gamma(\frac{s}{2})^2 L(s, \chi_{19,1})$ satisfies the functional equation:

$$\Lambda_{19}(1-s) = \Lambda_{19}(s).$$

Following Cohen, set

$$\phi_0(\tau; 19) = y^{\frac{1}{2}} \sum_{\substack{n \equiv \delta_l^2 \\ 1 \le l \le 9}} T_{38}(n) e^{\frac{2\pi i n x}{152}} K_0\left(\frac{2\pi |n| y}{152}\right).$$

This construction yields the following result.

Proposition 3.1. *$\phi_0(\tau; 19)$ is a Maass waveform on $\Gamma(2)$ with eigenvalue $\lambda = \frac{1}{4}$.*

Proof. By construction, $\phi_0(\tau; 19)$ satisfies:

$$\left(\Delta - \frac{1}{4}\right) \phi_0(\tau; 19) = 0.$$

Thus, it is determined by the values $\phi_0(\tau; 19)|_{x=0}$ and $\frac{\partial \phi_0(\tau;19)}{\partial x}|_{x=0}$. Equivalently, by the Maass Converse Theorem [8], it is determined by the functional equations of the two Dirichlet series:

$$\sum_{\substack{n \equiv \delta_l^2 \\ 1 \le l \le 9}} \frac{T_{38}(n)}{|n|^s} \quad \text{and} \quad \sum_{\substack{n \equiv \delta_l^2 \\ 1 \le l \le 9}} \frac{\operatorname{sgn}(n) T_{38}(n)}{|n|^s}.$$

Consider the completed functions:

$$\Lambda_{19}(s) = 2^{\frac{s}{2}} \left(\frac{152}{\pi}\right)^s \Gamma\left(\frac{s}{2}\right)^2 L(s, \chi_{19,1}),$$
$$\tilde{\Lambda}_{19}(s) = 2^{\frac{s+1}{2}} \left(\frac{152}{\pi}\right)^{s+1} \Gamma\left(\frac{s+1}{2}\right)^2 \sum_{\substack{n \equiv \delta_l^2 \\ 1 \le l \le 9}} \frac{\operatorname{sgn}(n) T_{38}(n)}{|n|^s},$$

where $\frac{N(\mathfrak{m})}{2} = 152$. Their functional equations are $\Lambda_{19}(1-s) = \Lambda_{19}(s)$ and $\tilde{\Lambda}_{19}(1-s) = -\tilde{\Lambda}_{19}(s)$, [9]. We start with (see [8]):

$$K_0(y) = \frac{1}{8\pi i} \int_{\sigma - i\infty}^{\sigma + i\infty} \Gamma\left(\frac{s}{2}\right)^2 2^s y^{-s} \mathrm{d}s,$$

where $y > 0$ and $\sigma = \operatorname{Re}(s) > 0$. It follows that

$$y^{\frac{1}{2}} K_0\left(\frac{2\pi |n|}{152} y\right) = \frac{1}{8\pi i} \int_{\sigma - i\infty}^{\sigma + i\infty} \frac{\left(\frac{152}{\pi}\right)^{s+\frac{1}{2}} \Gamma\left(\frac{s+\frac{1}{2}}{2}\right)^2 y^{-s} \mathrm{d}s}{|n|^{s+\frac{1}{2}}},$$

for $y > 0, \sigma > \frac{1}{2}$. Thus:

$$\phi_0(iy; 19) = \frac{2^{-\frac{9}{4}}}{2\pi i} \int_{\sigma - i\infty}^{\sigma + i\infty} y^{-s} 2^{-\frac{s}{2}} \Lambda_{19}\left(s + \frac{1}{2}, \chi_{19,1}\right) \mathrm{d}s.$$

Moving the line of integration to the vertical line $-\sigma+it$ and letting $t\mapsto -t$, it gives:

$$\begin{aligned}\phi_0(iy;19) &= \frac{2^{-\frac{9}{4}}}{2\pi i}\int_{\sigma-i\infty}^{\sigma+i\infty} y^s 2^{\frac{s}{2}}\Lambda_{19}\left(-s+\frac{1}{2},\chi_{19,1}\right)\mathrm{d}s\\ &= \frac{2^{-\frac{9}{4}}}{2\pi i}\int_{\sigma-i\infty}^{\sigma+i\infty}\left(\frac{1}{2y}\right)^{-s} 2^{-\frac{s}{2}}\Lambda_{19}\left(s+\frac{1}{2},\chi_{19,1}\right)\mathrm{d}s\\ &= \phi_0\left(\frac{-1}{2iy};19\right).\end{aligned}$$

Set $\phi_1(\tau)=\phi_0(\frac{-1}{2\tau};19)$ and note that

$$\left.\frac{\partial\phi_1}{\partial x}\right|_{x=0} = -\frac{1}{2y^2}\frac{\partial\phi_0}{\partial x}\left(\frac{-1}{2\tau};19\right)\Bigg|_{x=0}.$$

To evaluate $\frac{\partial\phi_0}{\partial x}(iy;19)$, we consider, for $y>0,\sigma>\frac{1}{2}$:

$$ny^{\frac{1}{2}}K_0\left(\frac{2\pi|n|}{152}y\right) = \frac{1}{8\pi i}\int_{\sigma-i\infty}^{\sigma+i\infty}\frac{\left(\frac{152}{\pi}\right)^{s+\frac{1}{2}}\Gamma\left(\frac{s+\frac{1}{2}}{2}\right)^2\operatorname{sgn}(n)y^{-s}\mathrm{d}s}{|n|^{s-\frac{1}{2}}}.$$

Hence

$$\frac{\partial\phi_0}{\partial x}(iy;19) = \left(\frac{2\pi i}{152}\right)\frac{1}{8\pi i}\int_{\sigma-i\infty}^{\sigma+i\infty}2^{-(\frac{s+\frac{1}{2}}{2})}\tilde{\Lambda}_{19}\left(s-\frac{1}{2}\right)y^{-s}\mathrm{d}s.$$

Finally, shift the line of integration to the vertical line $2-\sigma+it$ and let $t\mapsto -t$, to obtain

$$\begin{aligned}\frac{\partial\phi_0}{\partial x}(iy;19) &= \left(\frac{2\pi i}{152}\right)\frac{1}{8\pi i}\int_{\sigma-i\infty}^{\sigma+i\infty}2^{-(\frac{2-s+\frac{1}{2}}{2})}\tilde{\Lambda}_{19}\left(\frac{3}{2}-s\right)y^{s-2}\mathrm{d}s\\ &= \frac{1}{2y^2}\left(\frac{2\pi i}{152}\right)\frac{1}{8\pi i}\int_{\sigma-i\infty}^{\sigma+i\infty}2^{-(\frac{s+\frac{1}{2}}{2})}\tilde{\Lambda}_{19}\left(\frac{3}{2}-s\right)\left(\frac{1}{2y}\right)^{-s}\mathrm{d}s\\ &= -\frac{1}{2y^2}\left(\frac{2\pi i}{152}\right)\frac{1}{8\pi i}\int_{\sigma-i\infty}^{\sigma+i\infty}2^{-(\frac{s+\frac{1}{2}}{2})}\tilde{\Lambda}_{19}\left(s-\frac{1}{2}\right)\left(\frac{1}{2y}\right)^{-s}\mathrm{d}s\\ &= -\frac{1}{2y^2}\frac{\partial\phi_0}{\partial x}\left(\frac{-1}{2yi};19\right)\\ &= \left.\frac{\partial\phi_1}{\partial x}\right|_{x=0}.\end{aligned}$$

□

4. Modularity of Indefinite Theta Series a la Zwegers

For each δ_l, our generalized σ functions fit into Zwegers' formalism. In this section, we briefly summarize Zwegers' results. Let $Q(\nu_1,\nu_2)=\frac{1}{2}\nu^tA\nu$ be an indefinite binary quadratic form and $B(\nu,\mu)=\nu^tA\mu$ the associated bilinear form. We shall also denote the components of ν by $\left(\begin{smallmatrix}\nu_1\\ \nu_2\end{smallmatrix}\right)$. It is assumed that the binary quadratic form $Q=[a,b,c]$ has $a,c\in\frac{1}{2}\mathbb{Z}$ and $b\in\mathbb{Z}$. The hyperbola $\{\left(\begin{smallmatrix}\nu_1\\ \nu_2\end{smallmatrix}\right):Q(\nu)=-1\}$ is the disjoint union $C_Q^+\cup C_Q^-$, where $C_Q^+=\{\left(\begin{smallmatrix}\nu_1\\ \nu_2\end{smallmatrix}\right):Q(\nu)=$

-1 and $\nu_2 > 0\}$. It then turns out that there is a change of basis matrix P, such that the following three properties hold.

1. We have $Q(x,y) = (Q_0 \circ P)(x,y)$ for $Q_0(\nu_1,\nu_2) = \nu_1\nu_2$.
2. The vector $P^{-1}\begin{pmatrix}1\\-1\end{pmatrix}$ lies in the component C_Q^+.
3. All of C_Q^+ is parameterized by $c\colon \mathbb{R} \longrightarrow C_Q^+$ given by $c(t) = P^{-1}\left(\begin{smallmatrix}e^t\\-e^{-t}\end{smallmatrix}\right)$.

Remark 4.1. In Zwegers' normalization, $2Q = [2a, 2b, 2c]$ is a primitive indefinite binary quadratic form with matrix $A = \begin{pmatrix}2a & b\\ b & 2c\end{pmatrix}$. We assume $a > 0$. We have

$$Q(x,y) = a(x-\eta y)(x-\eta' y),$$

with $\eta = \frac{-b+\sqrt{D}}{2a}$, η' its conjugate, and $D = b^2 - 4ac \in \mathbb{Z}^+$. If

$$c_0 = \sqrt{\frac{a}{D}}\begin{pmatrix}-\frac{b}{a}\\ 2\end{pmatrix},$$

then $Q(c_0) = -1$, and $B(\nu, c_0) < 0$ if and only if $\nu_0 > 0$. We have

$$P = \sqrt{a}\begin{pmatrix}1 & -\eta'\\ 1 & -\eta\end{pmatrix}$$

is the change of basis matrix, such that

$$c_0 = P^{-1}\begin{pmatrix}1\\-1\end{pmatrix}.$$

The parameterization of C_Q^+ is then given by:

$$c(t) = \begin{pmatrix}\frac{-b\cosh(t)+\sqrt{D}\sinh(t)}{2a}\\ \cosh(t)\end{pmatrix}.$$

We will restrict the automorphisms of $2Q$ to $\mathrm{SL}(2,\mathbb{Z})$, and thus, we define

$$\begin{aligned}\mathrm{Aut}^+(2Q) &= \{\gamma \in \mathrm{SL}(2,\mathbb{Z}) : \gamma^t A\gamma = A\}\\ &= \{\gamma \in \mathrm{SL}(2,\mathbb{Z}) : \gamma\eta = \eta, \gamma\eta' = \eta'\} = \langle\gamma_\eta\rangle,\end{aligned}$$

where

$$\gamma_\eta = \begin{pmatrix}u_0 - bv_0 & -2cv_0\\ 2av_0 & u_0 + bv_0\end{pmatrix}.$$

Here, (u_0, v_0) is a fundamental solution to Pell's equation $u^2 - Dv^2 = 1$, and D is the discriminant of Q.

Zwegers defined the following family of lattice sums [10], which are constructed to be eigenfunctions of the hyperbolic Laplacian:

$$\Delta = -y^2\left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}\right),$$

but are only legitimate Maass waveforms in special cases.

Definition 4.2. For $c_1, c_2 \in C_Q^+$ and $a, b \in \mathbb{R}^2$, we consider the indefinite theta function of Zwegers:

$$\Phi_{a,b}^{c_1,c_2}(\tau) = \mathrm{sgn}(t_2 - t_1) y^{\frac{1}{2}} \frac{1}{2}$$
$$\times \left(\sum_{\nu = a+\mathbb{Z}^2} (1 - \mathrm{sgn} B(\nu, c_1) B(\nu, c_2)) e^{2\pi i Q(\nu) x} e^{2\pi i B(\nu, b)} K_0(2\pi Q(\nu) y) \right.$$
$$\left. + \sum_{\nu = a+\mathbb{Z}^2} (1 - \mathrm{sgn} B(\nu, c_1^\perp) B(\nu, c_2^\perp)) e^{2\pi i Q(\nu) x} e^{2\pi i B(\nu, b)} K_0(2\pi Q(\nu) y) \right).$$

Here, t_j, for $j = 1, 2$, is defined by $c_j = c(t_j)$ and $c_j^\perp$ is defined by $c_j^\perp = c^\perp(t_j)$, with

$$c^\perp(t) = P^{-1} \begin{pmatrix} e^t \\ e^{-t} \end{pmatrix}.$$

Note that the inclusion of the exponential and K-Bessel functions implies that $\Phi_{a,b}^{c_1,c_2}(\tau)$ satisfies

$$\Delta \Phi_{a,b}^{c_1,c_2}(\tau) = \frac{1}{4} \Phi_{a,b}^{c_1,c_2}(\tau),$$

so that it turns the q-series indefinite theta functions into functions which have a hope of (and sometimes are) Maass waveforms. In Zwegers' language, we can express the functions from our new family as follows.

Proposition 4.3. *For* $A = \begin{pmatrix} p & 0 \\ 0 & -2 \end{pmatrix}, c_1 = \frac{1}{\sqrt{p}} \begin{pmatrix} -2M \\ p \end{pmatrix}, c_2 = \frac{1}{\sqrt{p}} \begin{pmatrix} 2M \\ p \end{pmatrix}, a_l = \begin{pmatrix} \frac{\delta_l}{2p} \\ 0 \end{pmatrix}$ *and* $b = \begin{pmatrix} \frac{M}{2p} \\ \frac{1}{4} \end{pmatrix}$, *we have*

$$\Phi_{a_l,b}^{c_1,c_2}(\tau) = \zeta_{4p,l}^M \phi_{l,0}(\tau),$$

where $\zeta_{4p,l} = e^{\frac{2\pi i \delta_l}{4p}}$.

Proof. From

$$(2pn + \delta_l)^2 - 2p(2j)^2 = 8pk + \delta_l^2,$$

we have the following setup:

$$Q(\nu_1, \nu_2) = \frac{p}{2}\nu_1^2 - \nu_2^2, \ A = \begin{pmatrix} p & 0 \\ 0 & -2 \end{pmatrix}, \ B(\mu, \nu) = p\mu_1\nu_1 - 2\mu_2\nu_2, \ a_l = \begin{pmatrix} \frac{\delta_l}{2p} \\ 0 \end{pmatrix}.$$

To find $b = \begin{pmatrix} b_1 \\ b_2 \end{pmatrix}$, we want

$$e^{2\pi i B(\nu, b)} = \lambda (-1)^{Mn+j},$$

when $\nu = \begin{pmatrix} n + \frac{\delta_l}{2p} \\ j \end{pmatrix}$. Thus, we have the condition:

$$2B\left(\begin{pmatrix} n + \frac{\delta_l}{2p} \\ j \end{pmatrix}, \begin{pmatrix} b_1 \\ b_2 \end{pmatrix}\right) = (Mn \pm j) + C.$$

This gives $b = \begin{pmatrix} \frac{M}{2p} \\ \frac{1}{4} \end{pmatrix}$ and

$$\exp\left(2\pi i B\left(\begin{pmatrix} n + \frac{1}{2p} \\ j \end{pmatrix}, \begin{pmatrix} \frac{M}{2p} \\ \frac{1}{4} \end{pmatrix}\right)\right) = \zeta_{4p,l}^{M}(-1)^{Mn+j}.$$

Next, we take $c \in C_Q^+$ of the form $c = \alpha b$. Since

$$\alpha^2 \left(\frac{p}{2}\left(\frac{M}{2p}\right)^2 - \left(\frac{1}{4}\right)^2\right) = -1,$$

this gives $\alpha = 4\sqrt{p}$, and we set

$$c_1 = \frac{1}{\sqrt{p}}\begin{pmatrix} -2M \\ p \end{pmatrix}, \quad \text{and} \quad c_2 = \frac{1}{\sqrt{p}}\begin{pmatrix} 2M \\ p \end{pmatrix}.$$

Since

$$Q(x,y) = \frac{p}{2}x^2 - y^2 = \frac{p}{2}\left(x - \sqrt{\frac{2}{p}}y\right)\left(x + \sqrt{\frac{2}{p}}y\right),$$

by the remark at the start of this section, the change of basis matrix is:

$$P = \sqrt{\frac{p}{2}}\begin{pmatrix} 1 & \sqrt{\frac{2}{p}} \\ 1 & -\sqrt{\frac{2}{p}} \end{pmatrix}.$$

The generator of $\mathrm{Aut}^+(2Q)$ is then given by:

$$\gamma_{\sqrt{\frac{2}{p}}} = \begin{pmatrix} 2p-1 & 4M \\ 2Mp & 2p-1 \end{pmatrix},$$

since $N(\epsilon_{2p}) = 1$ and the fundamental solution to

$$x^2 - 2py^2 = 1$$

is $(2p-1, 2M)$. Note that c_1 is the unique point in C_Q^+ satisfying

$$\gamma_{\sqrt{\frac{2}{p}}} \cdot \begin{pmatrix} a \\ b \end{pmatrix} = \begin{pmatrix} -a \\ b \end{pmatrix}.$$

Employing the parameterization

$$c(t) = \begin{pmatrix} \sqrt{\frac{2}{p}}\sinh(t) \\ \cosh(t) \end{pmatrix},$$

we find

$$t_1 = \log(\sqrt{p} - \sqrt{2}\cdot M) \quad \text{and} \quad t_2 = \log(\sqrt{p} + \sqrt{2}\cdot M).$$

It follows that

$$c_1{}^{\perp} = P^{-1}\begin{pmatrix}\sqrt{p}-\sqrt{2}\cdot M\\ \sqrt{p}+\sqrt{2}\cdot M\end{pmatrix} = \begin{pmatrix}\sqrt{2}\\ -\sqrt{2}\cdot M\end{pmatrix} \quad \text{and} \quad c_2{}^{\perp} = \begin{pmatrix}\sqrt{2}\\ \sqrt{2}\cdot M\end{pmatrix}.$$

Next, we determine the required cone conditions to obtain

$$\begin{aligned}
\operatorname{sgn}(B(\nu,c_1)) &= -\operatorname{sgn}\left(Mn+j+\frac{\delta_l M}{2p}\right),\\
\operatorname{sgn}(B(\nu,c_2)) &= \operatorname{sgn}\left(Mn-j+\frac{\delta_l M}{2p}\right),\\
\operatorname{sgn}(B(\nu,c_1{}^{\perp})) &= \operatorname{sgn}\left(pn-2Mj+\frac{\delta_l}{2}\right),\\
\operatorname{sgn}(B(\nu,c_2{}^{\perp})) &= \operatorname{sgn}\left(pn+2Mj+\frac{\delta_l}{2}\right),
\end{aligned}$$

and

$$1-\operatorname{sgn}(B(\nu,c_1)B(\nu,c_2)) = \begin{cases} 2, & \text{if } Mn+j > -\frac{\delta_l M}{2p} \text{ and } Mn-j > -\frac{\delta_l M}{2p},\\ 2, & \text{if } Mn+j < -\frac{\delta_l M}{2p} \text{ and } Mn-j < -\frac{\delta_l M}{2p},\\ 0, & \text{otherwise,}\end{cases}$$

$$1-\operatorname{sgn}(B(\nu,c_1{}^{\perp})B(\nu,c_2{}^{\perp})) = \begin{cases} 2, & \text{if } 2Mj+pn < -\frac{\delta_l}{2} \text{ and } 2Mj-pn < \frac{\delta_l}{2},\\ 2, & \text{if } 2Mj+pn > -\frac{\delta_l}{2} \text{ and } 2Mj-pn > \frac{\delta_l}{2},\\ 0, & \text{otherwise.}\end{cases}$$

Plugging our data into Definition 4.2, we have

$$\begin{aligned}
\Phi_{a_l,b}^{c_1,c_2}(\tau) &= \zeta_{4p,l}^{M} y^{\frac{1}{2}} \sum_{n\geq 0} \left\{ \sum_{\substack{Mn+j>-\frac{\delta_l M}{2p}\\ Mn-j>-\frac{\delta_l M}{2p}\\ Q\begin{pmatrix} n+\frac{\delta_l}{2p}\\ j\end{pmatrix}=k+\frac{\delta_l^2}{8p}}} + \sum_{\substack{Mn+j<-\frac{\delta_l M}{2p}\\ Mn-j<-\frac{\delta_l M}{2p}\\ Q\begin{pmatrix} n+\frac{\delta_l}{2p}\\ j\end{pmatrix}=k+\frac{\delta_l^2}{8p}}} \right\}\\
&\quad \times (-1)^{Mn+j} e^{2\pi i\left(k+\frac{\delta_l^2}{8p}\right)x} K_0\left(2\pi\left(k+\frac{\delta_l^2}{8p}\right)y\right)\\
&\quad + \zeta_{4p,l}^{M} y^{\frac{1}{2}} \sum_{n<0} \left\{ \sum_{\substack{2Mj+pn>-\frac{\delta_l}{2}\\ 2Mj-pn>\frac{\delta_l}{2}\\ Q\begin{pmatrix} n+\frac{\delta_l}{2p}\\ j\end{pmatrix}=k+\frac{\delta_l^2}{8p}}} + \sum_{\substack{2Mj+pn<-\frac{\delta_l}{2}\\ 2Mj-pn<\frac{\delta_l}{2}\\ Q\begin{pmatrix} n+\frac{\delta_l}{2p}\\ j\end{pmatrix}=k+\frac{\delta_l^2}{8p}}} \right\}\\
&\quad \times (-1)^{Mn+j} e^{2\pi i\left(k+\frac{\delta_l^2}{8p}\right)x} K_0\left(2\pi\left|k+\frac{\delta_l^2}{8p}\right| y\right),
\end{aligned}$$

as claimed. $\square$

Zwegers further constructed the "completed" function:

$$\widehat{\Phi}_{a,b}^{c_1,c_2}(\tau) = y^{\frac{1}{2}} \sum_{\nu \in a+\mathbb{Z}^2} q^{Q(\nu)} e^{2\pi i B(\nu,b)} \int_{t_1}^{t_2} e^{-\pi y B(\nu,c(t))^2} \mathrm{d}t,$$

which naturally contains $\Phi_{a,b}^{c_1,c_2}(\tau)$ as a piece and transforms as a modular form, but may no longer be an eigenfunction of the hyperbolic Laplacian. However, its image under $\Delta - 1/4$ is a "simpler" function, and so, in analogy with the theory of mock modular forms and harmonic Maass forms, Zwegers called $\Phi_{a,b}^{c_1,c_2}$ a mock Maass theta function. In particular, he proved that $\widehat{\Phi}_{a,b}^{c_1,c_2}(\tau)$ has the following properties which will be useful for us here:

1. We have the following relationships under transformations of the parameters of $\widehat{\Phi}_{a,b}^{c_1,c_2}(\tau)$:
   1.1. $\widehat{\Phi}_{a+\lambda,b+\mu}^{c_1,c_2}(\tau) = e^{2\pi i B(a,\mu)} \widehat{\Phi}_{a,b}^{c}(\tau), \quad \lambda \in \mathbb{Z} \times \mathbb{Z}, \quad \mu \in \frac{1}{p}\mathbb{Z} \times \frac{1}{2}\mathbb{Z},$
   1.2. $\widehat{\Phi}_{-a,-b}^{c_1,c_2}(\tau) = \widehat{\Phi}_{a,b}^{c_1,c_2}(\tau),$
   1.3. $\widehat{\Phi}_{\left(\begin{smallmatrix} -a_1 \\ a_2 \end{smallmatrix}\right),\left(\begin{smallmatrix} -b_1 \\ b_2 \end{smallmatrix}\right)}^{c_1,c_2}(\tau) = \widehat{\Phi}_{\left(\begin{smallmatrix} a_1 \\ a_2 \end{smallmatrix}\right),\left(\begin{smallmatrix} b_1 \\ b_2 \end{smallmatrix}\right)}^{c_1,c_2}(\tau).$
2. The modularity transformations of $\widehat{\Phi}_{a,b}^{c_1,c_2}(\tau)$ are given as follows. Under translation, we have

$$\widehat{\Phi}_{a,b}^{c_1,c_2}(\tau+1) = e^{-2\pi i Q(a) - \pi i B(AA^*,a)} \widehat{\Phi}_{a,a+b+\frac{1}{2}AA^*}^{c_1,c_2}(\tau),$$

   where A^* denotes the vector of diagonal entries of A. Under inversion, we have

$$\widehat{\Phi}_{a,b}^{c_1,c_2}\left(-\frac{1}{\tau}\right) = e^{2\pi i B(a,\mu)} \frac{e^{2\pi i B(a,b)}}{\sqrt{-\det A}} \sum_{\mu \in A^{-1}\mathbb{Z}^2 \pmod{\mathbb{Z}^2}} \widehat{\Phi}_{-b+\mu,a}(\tau).$$

3. We have that $\widehat{\Phi}_{a,b}^{c_1,c_2}(\tau)$ is related to $\Phi_{a,b}^{c_1,c_2}(\tau)$ by

$$\widehat{\Phi}_{a,b}^{c_1,c_2}(\tau) = \Phi_{a,b}^{c_1,c_2}(\tau) + \phi_{a,b}^{c_1}(\tau) - \phi_{a,b}^{c_2}(\tau),$$

   where

$$\phi_{a,b}^{c_0}(\tau) = y^{\frac{1}{2}} \sum_{\nu \in a+\mathbb{Z}^2} \alpha_{t_0}(\nu y^{\frac{1}{2}}) q^{Q(\nu)} e^{2\pi i B(\nu,b)},$$

   and where

$$\alpha_{t_0}(\nu) = \begin{cases} \int_{t_0}^{\infty} e^{-\pi y B(\nu,c(t))^2} \mathrm{d}t, & \text{if } B(\nu,c_0)B(\nu,{c_0}^{\perp}) > 0, \\ -\int_{-\infty}^{t_0} e^{-\pi y B(\nu,c(t))^2} \mathrm{d}t, & \text{if } B(\nu,c_0)B(\nu,{c_0}^{\perp}) < 0, \\ 0, & \text{if } B(\nu,c_0)B(\nu,{c_0}^{\perp}) = 0. \end{cases}$$

   Moreover, the functions $\phi_{a,b}^{c_1}$ satisfy the parameter identities 1.1–1.3 and the functional equation

$$\phi_{\gamma a,\gamma b}^{\gamma c}(\tau) = \phi_{a,b}^{c}(\tau) \quad \text{for any } \gamma \in \mathrm{Aut}^+(2Q).$$

4. Instead of being in the kernel of $\Delta - 1/4$ as a typical Maass waveform would be, in general:
$$(\Delta - 1/4)(\widehat{\Phi}_{a,b}^{c_1,c_2})$$
has an explicit representation in terms of "simpler" functions.

Property 3 is especially convenient for showing that special examples of $\Phi_{a,b}^{c_1,c_2}$ are actually Maass waveforms, which happens precisely when $\phi_{a,b}^{c_1} = \phi_{a,b}^{c_2}$, and hence, $\widehat{\Phi}_{a,b}^{c_1,c_2} = \Phi_{a,b}^{c_1,c_2}$. In this situation, the function inherits both the eigenfunction under the Laplacian property of $\Phi_{a,b}^{c_1,c_2}$ as well as the full modularity transformation properties of $\widehat{\Phi}_{a,b}^{c_1,c_2}$. We conclude this section by noting that in our case, the modular transformations of the completed functions take the form:
$$\begin{aligned}\widehat{\Phi}_{a,b}(\tau+1) &= e^{-2\pi i Q(a) - \pi i (pa_1 - 2a_2)} \widehat{\Phi}_{a,a+b+\begin{pmatrix}\frac{1}{2}\\ \frac{1}{2}\end{pmatrix}}(\tau),\\ \widehat{\Phi}_{a,b}\left(-\frac{1}{\tau}\right) &= \frac{e^{2\pi i B(a,b)}}{\sqrt{2p}} \sum_{\substack{m_1 \pmod p\\ m_2 \pmod 2}} \widehat{\Phi}_{-b+\begin{pmatrix}\frac{m_1}{p}\\ \frac{m_2}{2}\end{pmatrix},a}(\tau).\end{aligned} \tag{4.1}$$

4.1. Proof of Theorem 1

By Proposition 4.3:
$$\zeta_{4p,l}^{M}\phi_{0,l}(\tau;p) = \Phi^{c_1,c_2}_{\begin{pmatrix}\frac{\delta_l}{2p}\\ 0\end{pmatrix},\begin{pmatrix}\frac{M}{2p}\\ \frac{1}{4}\end{pmatrix}}(\tau).$$
Zwegers' machinery gives that $\Phi^{c_1,c_2}_{\begin{pmatrix}\frac{\delta_c}{2p}\\ 0\end{pmatrix},\begin{pmatrix}\frac{M}{2p}\\ \frac{1}{4}\end{pmatrix}}(\tau)$ is a component of a vector-valued Maass waveform on $\mathrm{SL}(2,\mathbb{Z})$ whenever
$$\phi^{c_1}_{\begin{pmatrix}\frac{\delta_l}{2p}\\ 0\end{pmatrix},\begin{pmatrix}\frac{M}{2p}\\ \frac{1}{4}\end{pmatrix}}(\tau) = \phi^{c_2}_{\begin{pmatrix}\frac{\delta_l}{2p}\\ 0\end{pmatrix},\begin{pmatrix}\frac{M}{2p}\\ \frac{1}{4}\end{pmatrix}}(\tau).$$
Since $\gamma_{\sqrt{\frac{2}{p}}}c_1 = c_2$, we have
$$\begin{aligned}\phi^{c_1}_{\begin{pmatrix}\frac{\delta_l}{2p}\\ 0\end{pmatrix},\begin{pmatrix}\frac{M}{2p}\\ \frac{1}{4}\end{pmatrix}}(\tau) &= \phi^{\gamma_{\sqrt{\frac{2}{p}}}c_1}_{\gamma_{\sqrt{\frac{2}{p}}}\begin{pmatrix}\frac{\delta_l}{2p}\\ 0\end{pmatrix},\gamma_{\sqrt{\frac{2}{p}}}\begin{pmatrix}\frac{M}{2p}\\ \frac{1}{4}\end{pmatrix}}(\tau)\\ &= \phi^{c_2}_{\begin{pmatrix}\frac{-\delta_l}{2p}\\ 0\end{pmatrix}+\begin{pmatrix}\delta_l\\ M\delta_l\end{pmatrix},\begin{pmatrix}-\frac{M}{2p}\\ -\frac{1}{4}\end{pmatrix}+\begin{pmatrix}2M\\ M^2+\frac{p}{2}\end{pmatrix}}(\tau)\\ &= e^{-2\pi i \delta_l M}\phi^{c_2}_{\begin{pmatrix}-\frac{\delta_l}{2p}\\ 0\end{pmatrix},\begin{pmatrix}-\frac{M}{2p}\\ -\frac{1}{4}\end{pmatrix}}(\tau) = \phi^{c_2}_{\begin{pmatrix}\frac{\delta_l}{2p}\\ 0\end{pmatrix},\begin{pmatrix}\frac{M}{2p}\\ \frac{1}{4}\end{pmatrix}}(\tau).\end{aligned}$$
Thus, $\phi_{0,l}(\tau;p)$ is a Maass waveform, with multiplier, on a congruence subgroup of $\mathrm{SL}(2,\mathbb{Z})$.

4.2. Vector-Valued Transformations

For $1 \leq \nu \leq \frac{p+M}{2} - 1$, let

$$\widehat{\Phi} = \begin{pmatrix} \widehat{\Phi}_{\begin{pmatrix} -\frac{M}{2p} + \frac{\nu}{p} \\ 0 \end{pmatrix}, \begin{pmatrix} \frac{M}{2p} \\ \frac{1}{4} \end{pmatrix}} \\ \widehat{\Phi}_{\begin{pmatrix} -\frac{M}{2p} + \frac{\nu}{p} \\ \frac{1}{4} \end{pmatrix}, \begin{pmatrix} \frac{M}{2p} \\ 0 \end{pmatrix}} \\ \widehat{\Phi}_{\begin{pmatrix} -\frac{M}{2p} + \frac{\nu}{p} \\ \frac{1}{4} \end{pmatrix}, \begin{pmatrix} \frac{M}{2p} \\ \frac{1}{4} \end{pmatrix}} \end{pmatrix}.$$

Employing the modular transformations (4.1) and the parametric identities 1.1–1.3, we can determine matrices A, B, C, A_1, B_1, C_1, each of size $(\frac{p+M}{2} - 1) \times (\frac{p+M}{2} - 1)$, such that

$$\widehat{\Phi}\left(\frac{-1}{\tau}\right) = \begin{pmatrix} 0 & A & 0 \\ B & 0 & 0 \\ 0 & 0 & C \end{pmatrix} \widehat{\Phi}(\tau)$$

and

$$\widehat{\Phi}(\tau + 1) = \begin{pmatrix} A_1 & 0 & 0 \\ 0 & 0 & B_1 \\ 0 & C_1 & 0 \end{pmatrix} \widehat{\Phi}(\tau).$$

5. Questions

We conclude with several questions.

1. The construction of a Maass waveform attached to a real quadratic field using Zwegers' formalism depends only on knowing the fundamental unit explicitly and the lemma from ADH. It would be interesting to construct the forms attached to other families of real quadratic fields where the fundamental unit is known.
2. Are there an infinite number of primes of the form $p = qM^2 + 1$?
3. Can Cohen's argument be extended when the class number of $\mathbb{Q}(\sqrt{2p})$ is greater than one?
4. Can the generating functions for $V_3(m)$ and $U_2(m)$ be used to prove the identities $T(m) = V_3(m) = U_2(m)$ using Bailey pairs?

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Larry Rolen
Department of Mathematics
Vanderbilt University
Nashville
TN 37240
USA
e-mail: larry.rolen@vanderbilt.edu

Karen Taylor
Department of Mathematics and Computer Science
Bronx Community College, City University of New York
Bronx
NY 10453
USA
e-mail: karen.taylor@bcc.cuny.edu

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Sequentially Congruent Partitions and Related Bijections

In honor of George E. Andrews on his 80th birthday

Maxwell Schneider and Robert Schneider

Abstract. We study a curious class of partitions, the parts of which obey an exceedingly strict congruence condition we refer to as "sequential congruence": the mth part is congruent to the $(m+1)$th part modulo m, with the smallest part congruent to zero modulo the length of the partition. It turns out these obscure-seeming objects are embedded in a natural way in the theory of partitions. We show that sequentially congruent partitions with the largest part n are in bijection with the partitions of n. Moreover, we show sequentially congruent partitions induce a bijection between partitions of n and partitions of length n whose parts obey a strict "frequency congruence" condition—the frequency (or multiplicity) of each part is divisible by that part—and prove families of similar bijections, connecting with G. E. Andrews' theory of partition ideals.

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1. Introduction

Here we consider a somewhat exotic subset of integer partitions, which turns out to be naturally embedded in the theory of partitions.

Let $\mathcal{P}$ denote the set of *partitions*, with elements $\lambda = (\lambda_1, \lambda_2, \dots, \lambda_r)$, $\lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_r \geq 1$, including the *empty partition* $\emptyset$. Alternatively, following Andrews [4], Fine [8] and other authors, one sometimes writes $\lambda = (1^{m_1}\ 2^{m_2}\ 3^{m_3}\ \dots)$ with $m_i = m_i(\lambda)$ the *frequency* (or *multiplicity*) of $i \in \mathbb{N}$ as a part of λ, setting $m_i(\emptyset) = 0$ for all i. Furthermore, for a given partition λ, let $|\lambda|$ denote its *size* (sum of the parts) and $\ell(\lambda) := r$ denote its *length* (number of parts), with the conventions $|\emptyset| := 0,\ \ell(\emptyset) := 0$.

We define the set $\mathcal{S} \subset \mathcal{P}$ of *sequentially congruent partitions* as follows.

Definition 1.1. We define a partition λ to be *sequentially congruent* if the following congruences between the parts are all satisfied:

$$\lambda_1 \equiv \lambda_2 \pmod{1},\ \lambda_2 \equiv \lambda_3 \pmod{2},\ \lambda_3 \equiv \lambda_4 \pmod{3},\ \dots,$$
$$\lambda_{r-1} \equiv \lambda_r \pmod{r-1},$$

and for the smallest part, $\lambda_r \equiv 0 \pmod{r}$.

For example, the partition $(20, 17, 15, 9, 5)$ is sequentially congruent, because $20 \equiv 17 \pmod{1}$ trivially, $17 \equiv 15 \pmod{2}$, $15 \equiv 9 \pmod{3}$, $9 \equiv 5 \pmod{4}$, and finally $5 \equiv 0 \pmod{5}$. On the other hand, $(21, 18, 16, 10, 6)$ is *not* sequentially congruent, for while the first four congruences still hold, clearly $6 \not\equiv 0 \pmod{5}$. Note that increasing the largest part λ_1 of any $\lambda \in \mathcal{S}$ yields another partition in $\mathcal{S}$, as does adding or subtracting a fixed integer multiple of the length r to all its parts, so long as the resulting parts are still positive.

No doubt, this strict congruence restriction on the parts hardly appears natural. However, it turns out sequentially congruent partitions are in one-to-one correspondence with the entire set $\mathcal{P}$.

2. Bijections Between $\mathcal{S}$ and $\mathcal{P}$

Let $\mathcal{P}_n$ denote the set of partitions of n, as usual let $p(n) = \#\mathcal{P}_n$, with $\#Q$ standing for the cardinality of a set Q, and let $\mathcal{S}_{\text{lg}=n}$ denote sequentially congruent partitions λ' whose the largest part λ'_1 equals n.

Theorem 2.1. *There exists a bijection π between the set $\mathcal{P}$ and the set $\mathcal{S}$ such that*

$$\pi(\mathcal{P}_n) = \mathcal{S}_{\text{lg}=n}.$$

Moreover, we have

$$\#\mathcal{S}_{\text{lg}=n} = p(n).$$

Proof. We prove the theorem directly by construction.

For a partition $\lambda = (\lambda_1, \lambda_2, \dots, \lambda_i, \dots, \lambda_r)$, one constructs a sequentially congruent dual

$$\lambda' = (\lambda'_1, \lambda'_2, \dots, \lambda'_i, \dots, \lambda'_r)$$

by taking the parts equal to

$$\lambda'_i = i\lambda_i + \sum_{j=i+1}^{r} \lambda_j. \tag{2.1}$$

Note that $\lambda'_r \equiv 0 \pmod{r}$ as $\sum_{j=r+1}^{r}$ is empty; the other congruences between successive parts of λ' are also immediate from Eq. (2.1).

Let us take

$$\pi \colon \mathcal{P} \to \mathcal{S}$$

to be the map defined by this construction, with $\lambda' = \pi(\lambda)$. In fact, the above argument establishes that $\pi \colon \mathcal{P}_n \to \mathcal{S}_{\text{lg}=n}$.

Conversely, given a sequentially congruent partition λ', one can recover the dual partition λ by working from right-to-left. Begin by computing the smallest part

$$\lambda_r = \frac{\lambda'_r}{r}, \tag{2.2}$$

then compute $\lambda_{r-1}, \lambda_{r-2}, \ldots, \lambda_1$ in this order by taking

$$\lambda_i = \frac{1}{i}\left(\lambda'_i - \sum_{j=i+1}^{r} \lambda_j\right). \tag{2.3}$$

We define the inverse map π^{-1} from the algorithm in (2.2) and (2.3), i.e., $\pi^{-1}(\lambda') = \lambda$:

$$\pi^{-1} : \mathcal{S} \to \mathcal{P}. \tag{2.4}$$

Noting that the uniqueness of λ implies the uniqueness of λ', and vice versa, the bijection between $\mathcal{S}$ and $\mathcal{P}$ follows from this two-way construction.

Furthermore, since $\lambda'_1 = |\lambda|$, then every partition λ of n corresponds to a sequentially congruent partition λ' with the largest part n, and vice versa. □

The sets $\mathcal{P}$ and $\mathcal{S}$ enjoy another interrelation that can be used to compute the coefficients of infinite products. Now, it is a rewriting of Equation 22.16 in Fine [8] that for a function $f : \mathbb{N} \to \mathbb{C}$ and $q \in \mathbb{C}$ with f, q chosen such that the product converges absolutely, we have

$$\prod_{n=1}^{\infty}(1 - f(n)q^n)^{-1} = \sum_{\lambda \in \mathcal{P}} q^{|\lambda|} \prod_{i \geq 1} f(i)^{m_i}, \tag{2.5}$$

where $m_i = m_i(\lambda)$ is the frequency of i as a part of λ, and the sum on the right is taken over all partitions λ. Of course the canonical case would be, for $|q| < 1$, the identity

$$\prod_{n=1}^{\infty}(1 - xq^n)^{-1} = \sum_{\lambda \in \mathcal{P}} x^{\ell(\lambda)} q^{|\lambda|}, \tag{2.6}$$

which enjoys many beautiful q-series representations (see [4,6,8]).

It follows from an extension of (2.5) in [10] that the product on the left side of (2.5) can also be expressed as a sum over *sequentially congruent* partitions.

Let $\lg(\lambda) = \lambda_1$ denote the *largest part* of partition λ, and set $\lambda_k = 0$ if $k > \ell(\lambda)$.

Theorem 2.2. *For $f : \mathbb{N} \to \mathbb{C}, q \in \mathbb{C}$ such that the product converges absolutely, we have*

$$\prod_{n=1}^{\infty}(1 - f(n)q^n)^{-1} = \sum_{\lambda \in \mathcal{S}} q^{\lg(\lambda)} \prod_{i \geq 1} f(i)^{(\lambda_i - \lambda_{i+1})/i}.$$

Proof. For $j = 1, 2, 3, \ldots$, let $\mathcal{P}_{T_j}$ denote partitions whose parts are all in some subset $T_j \subseteq \mathbb{N}$, with $\emptyset \in \mathcal{P}_{T_j}$ for all j, and define $f_j : T_j \to \mathbb{C}$. To prove Theorem 2.2, we begin by recalling Corollary 2.9 of [10] in the case that "±" signs are set to minus:

$$\prod_{j=1}^{n} \prod_{k_j \in T_j} \left(1 - f_j(k_j) q^{k_j}\right)^{-1} = \sum_{k=0}^{\infty} c_k q^k,$$

with the coefficients c_k given by the somewhat unwieldy $(n-1)$-tuple sum

$$c_k = \sum_{k_2=0}^{k} \sum_{k_3=0}^{k_2} \cdots \sum_{k_n=0}^{k_{n-1}} \left(\sum_{\substack{\lambda \vdash k_n \\ \lambda \in \mathcal{P}_{T_n}}} \prod_{\lambda_i \in \lambda} f_n(\lambda_i) \right) \left(\sum_{\substack{\lambda \vdash (k_{n-1} - k_n) \\ \lambda \in \mathcal{P}_{T_{n-1}}}} \prod_{\lambda_i \in \lambda} f_{n-1}(\lambda_i) \right)$$
$$\times \left(\sum_{\substack{\lambda \vdash (k_{n-2} - k_{n-1}) \\ \lambda \in \mathcal{P}_{T_{n-2}}}} \prod_{\lambda_i \in \lambda} f_{n-2}(\lambda_i) \right) \cdots \left(\sum_{\substack{\lambda \vdash (k - k_2) \\ \lambda \in \mathcal{P}_{T_1}}} \prod_{\lambda_i \in \lambda} f_1(\lambda_i) \right),$$

where "$\lambda \vdash r$" indicates λ is a partition of r and the interior products are taken over the parts λ_i of each λ, which can be proved from (2.5) by repeated application of the Cauchy product formula.

Now, for every $j \in \mathbb{N}$ take $T_j = \{j\}$ and fix $f_j = f$. In this case, $\lambda \in \mathcal{P}_{T_j}$ means if $\lambda \neq \emptyset$ that $\lambda = (j, j, \ldots, j)$, so we must have $j | (k_j - k_{j+1})$ in any nonempty partition sum on the right side above. Then every summand comprising c_k vanishes unless all the $k_i \leq k$ are parts of a sequentially congruent partition having length $\leq n$: each sum over partitions is empty (i.e., equal to zero) if j does not divide $k_j - k_{j+1}$; is equal to 1 if $k_j - k_{j+1} = 0$ as then $\lambda = \emptyset$ and $\prod_{\lambda_i \in \emptyset}$ is an empty product; or else has one term $f(j)^{m_j} = f(j)^{(k_j - k_{j+1})/j}$ as there is exactly one $\lambda = (j, j, \ldots, j)$ with $|\lambda| = m_j j = k_j - k_{j+1} > 0$. Finally, let $n \to \infty$ so this argument encompasses partitions in $\mathcal{S}$ of unrestricted length. □

Remark 2.3. We note that setting $f = 1$, then comparing Eq. (2.5) to Theorem 2.2, gives another proof of Theorem 2.1: the sets $\mathcal{S}_{\mathrm{lg}=n}$ and $\mathcal{P}_n$ (and thus, the sets $\mathcal{S}$ and $\mathcal{P}$) have the same product generating function.

Remark 2.4. If we instead take every ± equal to plus in Corollary 2.9 of [10], similar arguments reveal there is also a bijection between partitions into *distinct* parts and the subset of $\mathcal{S}$ containing partitions into parts with differences $\lambda_i - \lambda_{i+1} = i$ exactly.

3. Cyclic Sequentially Congruent Maps

Comparing Theorem 2.2 with (2.5) above, we have two formally different-looking decompositions of the coefficients of $\prod_{n \geq 1}(1 - f(n)q^n)^{-1}$ as sums over partitions of the form $\sum_{\lambda \in \mathcal{P}_n}$ and $\sum_{\lambda \in \mathcal{S}_{\mathrm{lg}=n}}$, yet one observes the summands in each case consist of the same terms in different orders. Then one wonders: precisely *which* partition $\gamma \in \mathcal{P}_n$ is such that

$$\prod_{i\geq 1} f(i)^{(\phi_i-\phi_{i+1})/i} = \prod_{j\geq 1} f(j)^{m_j(\gamma)} \tag{3.1}$$

for a given $\phi \in \mathcal{S}_{\mathrm{lg}=n}$? One observes that γ is generally not the same partition $\lambda = \pi^{-1}(\phi)$ as in (2.4).

Evidently, the set $\mathcal{S}$ enjoys a second map to $\mathcal{P}$ (apart from π^{-1}). Let

$$\sigma\colon \mathcal{S} \to \mathcal{P}$$

denote this map. We can write σ down by comparing the forms of the products in (3.1):

$$\sigma(\phi) := (1^{\phi_1-\phi_2}\ 2^{(\phi_2-\phi_3)/2}\ 3^{(\phi_3-\phi_4)/3}\ldots) = \gamma \in \mathcal{P}_n,$$

where $\phi \in \mathcal{S}_{\mathrm{lg}=n}$ as above. For example, $\sigma(5,3,3) = (1^{5-3}\ 2^{(3-3)/2}\ 3^{(3-0)/3}) = (3,1,1)$.

Under this map, we have $\sigma(\mathcal{S}_{\mathrm{lg}=n}) = \mathcal{P}_n$, thus the composite map is

$$\sigma\circ\pi\colon \mathcal{P}_n \to \mathcal{P}_n,$$

and, similarly, we have the map $\pi\circ\sigma\colon \mathcal{S}_{\mathrm{lg}=n} \to \mathcal{S}_{\mathrm{lg}=n}$.

A natural question to ask is: what kind of permutation structure arises as we alternately compose π, σ, that is, what if we apply $\sigma\circ\pi\circ\sigma\circ\pi\circ\cdots\circ\sigma\circ\pi$ to a partition of n? For a concrete example, let us check by repeatedly applying $\sigma\circ\pi\circ\cdots\circ\sigma\circ\pi$ to the partitions of $n = 4$:

$$\begin{aligned}
&(4) \xmapsto{\pi} (4) \xmapsto{\sigma} (1,1,1,1) \xmapsto{\pi} (4,4,4,4) \xmapsto{\sigma} (4),\\
&(3,1) \xmapsto{\pi} (4,2) \xmapsto{\sigma} (2,1,1) \xmapsto{\pi} (4,3,3) \xmapsto{\sigma} (3,1),\\
&(2,2) \xmapsto{\pi} (4,4) \xmapsto{\sigma} (2,2),\\
&(2,1,1) \xmapsto{\pi} (4,3,3) \xmapsto{\sigma} (3,1) \xmapsto{\pi} (4,2) \xmapsto{\sigma} (2,1,1),\\
&(1,1,1,1) \xmapsto{\pi} (4,4,4,4) \xmapsto{\sigma} (4) \xmapsto{\pi} (4) \xmapsto{\sigma} (1,1,1,1).
\end{aligned}$$

There appears to be a cyclic behavior of order 1 or 2; also evident is the following fact.

Theorem 3.1. *The composite map $\sigma\circ\pi\colon \mathcal{P}_n \to \mathcal{P}_n$ takes partitions to their conjugates.*

Proof. If we write

$$\lambda = (a_1^{m_{a_1}} a_2^{m_{a_2}} a_3^{m_{a_3}} \cdots\ a_r^{m_{a_r}}),\ a_1 > a_2 > \cdots > a_r \geq 1,$$

then we can compute the parts and frequencies of the conjugate partition

$$\lambda^* = (b_1^{m_{b_1}} b_2^{m_{b_2}} b_3^{m_{b_3}} \cdots\ b_s^{m_{b_s}}),\ b_1 > b_2 > \cdots > b_s \geq 1$$

directly from the parts and frequencies of λ by comparing the Ferrers–Young diagrams of λ, λ^*. The conjugate partition λ^* has the largest part b_1 given by

$$b_1 = \ell(\lambda) = m_{a_1} + m_{a_2} + \cdots + m_{a_r}, \quad \text{with}\quad m_{b_1}(\lambda^*) = a_r, \tag{3.2}$$

and for $1 < i \leq s$, the parts and their frequencies are given by

$$b_i = m_{a_1} + m_{a_2} + \cdots + m_{a_{r-i+1}}, \qquad m_{b_i}(\lambda^*) = a_{r-i+1} - a_{r-i+2}. \tag{3.3}$$

Moreover, we have that $s = r$. The theorem results from using the definitions of the maps π and σ, keeping track of the parts in the transformation $\lambda \mapsto (\sigma \circ \pi)(\lambda)$, then comparing the parts of $(\sigma \circ \pi)(\lambda)$ with the parts of λ^* in (3.2) and (3.3) above to see they are the same. □

The preceding considerations also make explicit our observation above about cyclic orders.

Corollary 3.2. *We have that $(\sigma \circ \pi)(\lambda) = \lambda$ when λ is self-conjugate, and $(\sigma \circ \pi)^2(\lambda) = \lambda$ holds for all $\lambda \in \mathcal{P}$. Likewise, for ϕ sequentially congruent it is the case that $(\pi \circ \sigma)(\phi) = \phi$ when $\sigma(\phi)$ is self-conjugate, and $(\pi \circ \sigma)^2(\phi) = \phi$ holds for all $\phi \in \mathcal{S}$.*

Remark 3.3. Interestingly, the map $\pi \circ \sigma \colon \mathcal{S}_{\mathrm{lg}=n} \to \mathcal{S}_{\mathrm{lg}=n}$ defines a duality analogous to conjugation in $\mathcal{P}_n$ that instead connects partitions ϕ and $(\pi \circ \sigma)(\phi)$ in $\mathcal{S}_{\mathrm{lg}=n}$. For instance, from the above examples, it is the case in $\mathcal{P}_4$ that $(2,1,1)$ and $(3,1) = (\sigma \circ \pi)(2,1,1)$ are conjugates, while on the same row, $(4,3,3)$ and $(4,2) = (\pi \circ \sigma)(4,3,3)$ are paired under this new, analogous duality in $\mathcal{S}_{\mathrm{lg}=4}$.

4. Frequency Congruent Partitions and Infinite Families of Bijections

The conjugates of sequentially congruent partitions are themselves interesting combinatorial objects.

Theorem 4.1. *A sequentially congruent partition ϕ is mapped by conjugation to a partition ϕ^* whose frequencies $m_i = m_i(\phi^*)$ obey the congruence condition*

$$m_i \equiv 0 \pmod{i}.$$

Conversely, any partition with parts obeying this congruence condition has a sequentially congruent partition as its conjugate.

Proof. The theorem is immediate by conjugation of the relevant Young diagrams. □

Let us codify the objects highlighted in the preceding theorem.

Definition 4.2. We define a partition to be *frequency congruent* if it has the property that each part divides its frequency.[1]

Then Theorem 4.1 implies the following result.

Corollary 4.3. *Frequency congruent partitions of length n are in bijection with the partitions of n, viz.*

$$\#\{\lambda \in \mathcal{P} \colon \ell(\lambda) = n,\ i | m_i(\lambda)\} \ = \ p(n).$$

[1] As in Theorem 4.1.

Proof. This statement follows from Theorem 4.1 together with Theorem 2.1. For a combinatorial proof, take any partition $\lambda = (1^{m_1}2^{m_2}3^{m_3}\cdots i^{m_i}\ldots)$ of n, and multiply each m_i by i to yield a frequency congruent partition $(1^{m_1}2^{2m_2}3^{3m_3}\cdots i^{im_i}\ldots)$ with length $m_1+2m_2+3m_3+\cdots = |\lambda| = n$. Conversely, by the same principle, divide the frequency of each part of a length-n frequency congruent partition by the part itself for a partition of n.

Alternatively, we can prove the bijection using generating functions. For $|x|<1, |q|<1$, consider the following identities in light of (2.5) and (2.6):

$$\begin{aligned}
\prod_{n=1}^{\infty}\frac{1}{1-x^nq^{n^2}}
&= (1+x^1q^1+x^2q^{1+1}+x^3q^{1+1+1}+\cdots)\\
&\quad\times(1+x^2q^{2+2}+x^4q^{2+2+2+2}+x^6q^{2+2+2+2+2+2}+\cdots)\\
&\quad\times(1+x^3q^{3+3+3}+x^6q^{3+3+3+3+3+3}+x^9q^{3+3+3+3+3+3+3+3+3}+\cdots)\times\cdots\\
&= \sum_{\substack{\lambda\in\mathcal{P}\\ i|m_i(\lambda)}} x^{\ell(\lambda)}q^{|\lambda|} = \sum_{n=0}^{\infty}x^n\sum_{\substack{\ell(\lambda)=n\\ i|m_i(\lambda)}} q^{|\lambda|},
\end{aligned}$$

where the final two (absolutely convergent) sums are taken over frequency congruent partitions.

To count the number of frequency congruent partitions of length n, let $q\to 1$ from within the unit circle in the right-most series above, noting in the limit we still have convergence since $|x|<1$. Then by comparison with the product side of the generating function, the resulting coefficient of x^n is equal to $p(n)$ by Euler's identity (see [4]). $\square$

Remark 4.4. We note that the generating function proof above provides (by conjugation) another proof that $\#\mathcal{S}_{\mathrm{lg}=n} = p(n)$.

Indeed, the steps of the preceding proof suggest a highly general frequency congruence phenomenon yielding infinite families of partition bijections.

As before, let $\mathcal{P}_T \subseteq \mathcal{P}$ be the set of partitions (including $\emptyset$) with parts from $T=\{t_1,t_2,t_3,\ldots\}\subseteq\mathbb{N}$; we allow $\mathcal{P}_T$ to also denote partitions with parts from a sequence T of natural numbers if they are distinct. Let $p_T(n)$ denote the number of partitions of $n\geq 0$ in $\mathcal{P}_T$. Moreover, for a sequence $S=(s_1,s_2,s_3,\ldots)$ of natural numbers, define

$$\mathcal{P}_T(S) := \{\lambda\in\mathcal{P}_T\colon s_i|m_{t_i}\},$$

and let $\mathcal{P}_T(S,n)$ denote partitions in $\mathcal{P}_T(S)$ of length n. Thus, $\mathcal{P}_{\mathbb{N}}((1,1,1,\ldots)) = \mathcal{P}$ and $\#\mathcal{P}_{\mathbb{N}}((1,1,1,\ldots),n) = p(n)$. Then we have the following.

Theorem 4.5. *Let $|x|<1, |q|<1$. For a sequence $A=(a_1,a_2,a_3,\ldots)$ of natural numbers and subset $B=\{b_1,b_2,b_3,\ldots\}\subseteq\mathbb{N}$, we have*

$$\prod_{n=1}^{\infty}\frac{1}{1-x^{a_n}q^{a_nb_n}} = \sum_{\lambda\in\mathcal{P}_B(A)} x^{\ell(\lambda)}q^{|\lambda|} = \sum_{n=0}^{\infty}x^n\sum_{\lambda\in\mathcal{P}_B(A,n)} q^{|\lambda|}.$$

If the $a_i \in A$ are distinct then the sets $\mathcal{P}_A$ and $\mathcal{P}_B(A)$ are in bijection, and

$$\#\mathcal{P}_B(A, n) \;=\; p_A(n).$$

We note that Eq. (2.6) represents the case $a_i = 1, b_i = i$, and the generating function in the proof of Corollary 4.3 is the case $a_i = b_i = i$.

Proof. For the first identity, much as in the proof of Corollary 4.3, for $|x| < 1, |q| < 1$, rewrite the infinite product on the left side of Theorem 4.5 as a product of geometric series:

$$\prod_{n=1}^{\infty} \left(1 + x^{a_n} q^{b_n+b_n+\cdots+b_n} + x^{2a_n} q^{b_n+\cdots+b_n} + x^{3a_n} q^{b_n+\cdots+b_n} + \cdots\right),$$

where in each term $x^{ia_n} q^{b_n+\cdots+b_n}$ there are ia_n repetitions of b_n in the exponent of q. Expanding the product immediately gives the first equality, and collecting coefficients of x^n gives the right-most equality.

To prove the second identity in the theorem, just as in the proof of Corollary 4.3, let $q \to 1$ from within the unit circle in the right-most summation of the first identity. But if the a_i are distinct the infinite product becomes

$$\prod_{n=1}^{\infty} \frac{1}{1 - x^{a_n}} = \prod_{n \in A} \frac{1}{1 - x^n} = \sum_{n=0}^{\infty} p_A(n) x^n.$$

Equating coefficients of x^n completes the proof.

One can also prove the second identity by mapping every partition $(a_1^{m_{a_1}} a_2^{m_{a_2}} a_3^{m_{a_3}} \ldots) \in \mathcal{P}_A$ of size n (noting these a_i are not necessarily in increasing order) to partition $(b_1^{a_1 m_{a_1}} b_2^{a_2 m_{a_2}} b_3^{a_3 m_{a_3}} \ldots) \in \mathcal{P}_B(A, n)$ and, conversely, mapping each $(b_1^{a_1 n_1} b_2^{a_2 n_2} b_3^{a_3 n_3} \ldots) \in \mathcal{P}_B(A, n)$ to $(a_1^{n_1} a_2^{n_2} a_3^{n_3} \ldots)$.[2] $\square$

Observe that in the above notation, frequency congruent partitions represent the set $\mathcal{P}_{\mathbb{N}}((1, 2, 3, 4, \ldots))$. Recalling that the conjugates of frequency congruent partitions are sequentially congruent, then the set $\mathcal{S}_B(A)$ of *conjugates of partitions in* $\mathcal{P}_B(A)$ is evidently an analog of the set $\mathcal{S}$. For example, for $B = \mathbb{N}$ and a sequence A, the conjugates of the set of partitions $\mathcal{P}_{\mathbb{N}}(A)$ such that a_i divides m_i have a nice sequential congruence property:

$$\mathcal{S}_{\mathbb{N}}(A) = \{\lambda \in \mathcal{P} \colon \lambda_i \equiv \lambda_{i+1} \pmod{a_i}\}. \tag{4.1}$$

We conjecture that there are bijective maps in this extended regime analogous to those in Sects. 2 and 3 above; however, they alternate between $\mathcal{P}_A$ and $\mathcal{S}_B(A)$ under composition instead of between $\mathcal{P}$ and $\mathcal{S}$.

Remark 4.6. For $T \subseteq \mathbb{N}$ and $f, g \colon T \to \mathbb{C}$, the two-variable generating functions of the general form $\prod_{n \in T}(1 - x^{f(n)} q^{g(n)})^{-1}$ used in this section are analytic and combinatorial objects (see [4,8]). We note if $0 < x < e^{-1}, |q| <$

[2] There is a resemblance here to maps generating other classes of partitions with nontrivial weightings on the frequencies, e.g. see [1,5,7] regarding identities of Capparelli and Primc.

$1, 1 \notin T$, taking $f(n) = \log\ n$ and letting $q \to 1$ as we did above yields a class of "partition zeta functions" studied in [9,10]:

$$\lim_{q\to 1} \prod_{n\in T} (1 - x^{\log n} q^{g(n)})^{-1} = \prod_{n\in T} (1 - n^{\log x})^{-1} = \sum_{\lambda\in\mathcal{P}_T} N(\lambda)^{-s},$$

where $s := -\log x$, thus $s > 1$ for convergence, and $N(\lambda) := \prod_{\lambda_i\in\lambda} \lambda_i$. (By the same token, one may rewrite the Riemann zeta function as $\zeta(s) = \zeta(-\log x) = \sum_{n=1}^{\infty} x^{\log n}$.)

5. Further Thoughts: Partition Ideals

In a series of papers in the 1970s (e.g. see [2,3]), Andrews developed a *theory of partition ideals* which uses ideas from lattice theory to unify and extend many classical results on generating functions and partition bijections, summarized in Chapter 8 of [4].

Definition 5.1. A partition ideal is a subset $\mathcal{C} \subseteq \mathcal{P}$ with the property that if any parts are deleted from a partition in $\mathcal{C}$, the resulting partition is an element of $\mathcal{C}$ as well.

Remark 5.2. We note Andrews' definition is stated in terms of frequencies.

For example, partitions into distinct parts form a partition ideal. Andrews identifies relations between partition ideals which break the set $\mathcal{P}$ into algebraic subclasses.

Definition 5.3. We say two partition ideals $\mathcal{C}, \mathcal{C}'$ are equivalent and write $\mathcal{C} \sim \mathcal{C}'$ if $\#\{\lambda \in \mathcal{C} \colon |\lambda| = n\} = \#\{\lambda \in \mathcal{C}' \colon |\lambda| = n\}$ for all $n \geq 1$.

Andrews carries out the study of equivalences where one subset $\mathcal{C}$ is a partition ideal of "order one" in great detail (see [4] for specifics). These are "nice" subsets of $\mathcal{P}$ including many of interest classically, e.g., partitions into distinct parts form a partition ideal of order one. Sets $\mathcal{P}_A, \mathcal{P}_B$ as in Theorem 4.5 are also partition ideals of order one. Naturally, then, one wonders if Andrews' theory extends in some way to sets like $\mathcal{P}_B(A)$.

A moment's thought convinces one that such sets are not generally partition ideals. However, they do enjoy a tantalizing "quasi-ideal" property: *If a_i copies (or a multiple thereof) of any part b_i are deleted from a partition in $\mathcal{P}_B(A)$, the resulting partition is an element of $\mathcal{P}_B(A)$ as well.*

This feels like a refinement of Definition 5.1. Furthermore, if the a_i in the sequence A of distinct terms are rearranged to form a new sequence A' (the same terms in a different order), clearly $p_{A'}(n) = p_A(n)$ even though $\mathcal{P}_B(A') \neq \mathcal{P}_B(A)$; thus, Theorem 4.5 gives

$$\#\mathcal{P}_B(A, n) = \#\mathcal{P}_B(A', n). \tag{5.1}$$

Similarly, noting B is arbitrary in Theorem 4.5 and could be replaced by another subset $B' \subseteq \mathbb{N}$ without changing the right side of the second identity, then

$$\#\mathcal{P}_B(A, n) = \#\mathcal{P}_{B'}(A, n). \tag{5.2}$$

In light of the correspondence between length-n partitions in $\mathcal{P}_B(A)$ and size-n partitions in $\mathcal{P}_A$, Eqs. (5.1) and (5.2) feel similar to partition ideal equivalence in Definition 5.3.

Moreover, the two-variable generating functions in Sect. 4 are of a similar shape to Andrews' formulas for "linked partition ideals" in Chapter 8.4 of [4]. Are there maps between these schemes? If subsets of partitions such as $\mathcal{P}_B(A)$ are analogous to partition ideals, do there exist closely-related subsets analogous to equivalent ideals in Andrews' theory? Conversely, might cyclic maps like those in Sect. 3 exist between equivalent partition ideals?

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Maxwell Schneider
Honors Program
University of Georgia
Athens GA30602
USA
e-mail: maxwell.schneider@uga.edu

Robert Schneider
Department of Mathematics
University of Georgia
Athens GA30602
USA
e-mail: robert.schneider@uga.edu

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Singular Overpartitions and Partitions with Prescribed Hook Differences

Dedicated to George Andrews for his 80th birthday

Seunghyun Seo and Ae Ja Yee

Abstract. Singular overpartitions, which are Frobenius symbols with at most one overlined entry in each row, were first introduced by Andrews in 2015. In his paper, Andrews investigated an interesting subclass of singular overpartitions, namely, (K, i)-singular overpartitions for integers K, i with $1 \leq i < K/2$. The definition of such singular overpartitions requires successive ranks, parity blocks and anchors. The concept of successive ranks was extensively generalized to hook differences by Andrews, Baxter, Bressoud, Burge, Forrester and Viennot in 1987. In this paper, employing hook differences, we generalize parity blocks. Using this combinatorial concept, we define (K, i, α, β)-singular overpartitions for positive integers α, β with $\alpha+\beta < K$, and then we show some connections between such singular overpartitions and ordinary partitions.

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1. Introduction

A partition of a positive integer n is a weakly decreasing sequence of positive integers whose sum equals n [3]. The integers in the sequence are called parts. An overpartition of n is a partition in which the first occurrence of a part may be overlined [12].

A Frobenius symbol is a two-rowed array of nonnegative integers such that entries in each row are strictly decreasing and the numbers of entries in

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the top and bottom rows are equal [4]. There is a natural one-to-one correspondence between partitions and Frobenius symbols [4,15,18]. For self-containedness, the definition of the Frobenius symbol of a partition is given later in Definition 2.8. For an overpartition, one can define the corresponding Frobenius symbol by allowing overlined entries in a similar way.

In [5], Andrews introduced singular overpartitions, which are Frobenius symbols with at most one overlined entry in each row. For integers K, i with $1 \leq i < K/2$, Andrews defined a subclass of singular overpartitions with some restrictions subject to K and i, namely (K,i)-singular overpartitions. He then showed interesting combinatorial and arithmetic properties of (K,i)-singular overpartitions. As seen in [5], (K,i)-singular overpartitions are closely related to partitions counted by partition sieves, which were first employed by Andrews [1,2] to discover Rogers–Ramanujan type partitions and later generalized further by Bressoud [7].

Successive ranks are the differences between the top and bottom entries of the columns in a Frobenius symbol. In the partition sieves, they are vital combinatorial statistics and have led to a number of discoveries of Rogers–Ramanujan type partitions [1,2,7,9–11]. The concept of successive ranks was extensively generalized to hook differences by Andrews, Baxter, Bressoud, Burge, Forrester and Viennot in [6], which concerns partitions with prescribed hook differences. The work in [6] was further extended by Gessel and Krattenthaler [14].

The main purpose of this paper is to generalize (K,i)-singular overpartitions by utilizing the concept of hook differences. Throughout this paper, we assume that K, i, α and β are positive integers with $i < K/2$ and $\alpha + \beta < K$. For a positive integer n, let $\overline{Q}_{K,i,\alpha,\beta}(n)$ be the number of singular overpartitions of n with prescribed overlining constraints subject to K, i, α and β. Such singular overpartitions will be called (K,i,α,β)-singular overpartitions. Because of the complexity of the constraints, we defer the exact definition of (K,i,α,β)-singular overpartitions to Sect. 2.

One of our results is given in the following theorem.

Theorem 1.1. *We have*

$$\sum_{n=0}^{\infty} \overline{Q}_{K,i,\alpha,\alpha}(n) q^n = \frac{(-q^{i\alpha}; q^{K\alpha})_\infty (-q^{(K-i)\alpha}; q^{K\alpha})_\infty (q^{K\alpha}; q^{K\alpha})_\infty}{(q;q)_\infty},$$

where $\overline{Q}_{K,i,\alpha,\alpha}(0) = 1$ *and* $(a;q)_\infty = \lim_{n\to\infty} \prod_{j=0}^{n} (1 - aq^j)$.

We note that $\overline{Q}_{K,i,1,1}(n)$ becomes the number of (K,i)-singular overpartitions of n given by Andrews in [5].

For any positive integers m and n, let us define a refined partition function $\overline{Q}_{K,i,\alpha,\beta}(m,n)$ by the number of (K,i,α,β)-singular overpartitions of n with an overlined entry in its mth anchor. Again, because of the complexity, anchors are defined in Sect. 2. Then we have the following theorem.

Theorem 1.2. *For* $m \geq 1$ *and* $n \geq 0$,

$$\overline{Q}_{K,i,\alpha,\alpha}(m,n) = p\left(n - \alpha K \binom{m+1}{2} + \alpha i m\right) + p\left(n - \alpha K \binom{m}{2} - \alpha i m\right),$$

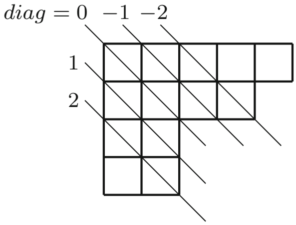


FIGURE 1. $\lambda = (5, 4, 2, 2)$ with diagonals

where $p(N)$ denotes the number of ordinary partitions of N with $p(0) = 1$ and $p(N) = 0$ for $N < 0$.

For arbitrary positive integers α, β, more general and refined results than Theorems 1.1 and 1.2 are presented in Sect. 6. Our proofs are combinatorial and bijective generalizing the proof methods used in [15]. One of the main ingredients of the methods in [15] was Dyson's map [13]. We will generalize this map for our purpose.

The rest of this paper is organized as follows. In Sect. 2, some basic definitions and notions are recollected followed by the definition of (K, i, α, β)-singular overpartitions. In Sect. 3, Dyson's map and its generalization are presented along with the shift map from [15]. Necessary lemmas for later use are given in Sect. 4. In Sect. 5, another representation of (K, i, α, β)-singular overpartitions is given and it is shown bijectively that (K, i, α, β)-singular overpartitions are related to ordinary partitions. In Sect. 6, our theorem on (K, i, α, β)-singular overpartitions is proved along with Theorems 1.1 and 1.2. Some remarks are given in Sect. 7.

2. (K, i, α, β)-Singular Overpartitions

For a partition λ of n, we denote it by $\lambda \vdash n$, the sum of parts by $|\lambda|$, and the number of parts by $\ell(\lambda)$. The Ferrers diagram of λ is a left-justified graphical representation whose jth row has as many boxes as the jth part λ_j. The box in row x and column y of the Ferrers diagram is called node (x, y). If a node (x, y) is inside the Ferrers diagram, i.e., $1 \leq x \leq \ell(\lambda)$ and $1 \leq y \leq \lambda_x$, then we denote it by $(x, y) \in \lambda$. For an integer k, the diagonal diag $= k$ is the line passing through nodes (x, y) with $x = y + k$ [6]. Figure 1 shows some diagonals on the partition $(5, 4, 2, 2)$.

The conjugate of λ is the partition resulting from reflecting the Ferrers diagram of λ about the main diagonal, and we denote the conjugate partition by λ'. For instance, $\lambda' = (4, 4, 2, 2, 1)$ in Fig. 1.

Definition 2.1. For a partition λ, we define the hook difference at a node $(x, y) \in \lambda$ as

$$h_{(x,y)} = h_{(x,y)}(\lambda) = (\lambda_x - y) - (\lambda'_y - x) = \lambda_x - \lambda'_y + (x - y). \tag{2.1}$$

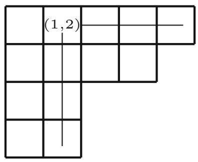


FIGURE 2. $\lambda = (5, 4, 2, 2)$ and its hook at $(1, 2)$

For convenience, we also define the hook difference at a node $(x, y) \notin \lambda$ as

$$h_{(x,y)} = h_{(x,y)}(\lambda) = \begin{cases} -\infty, & \text{if } x > y, \\ 0, & \text{if } x = y, \\ +\infty, & \text{if } x < y. \end{cases} \tag{2.2}$$

Here we note that the hook difference at a node $(x, y) \in \lambda$ is defined as $\lambda_x - \lambda'_y$ in [6].

Figure 2 shows the hook at node $(1, 2)$ in the partition $(5, 4, 2, 2)$ and its hook difference equals 0. For a node not in λ, (2.2) says that the hook difference at that node is defined to be 0 if the node is on the main diagonal, $-\infty$ if it is below the main diagonal, and ∞ if it is above the main diagonal.

In the next two lemmas, we will show how the hook difference at a node on the diagonal $1 - \beta$ affects the hook differences at nodes on the diagonal $\alpha - 1$, and vice versa.

Lemma 2.2. *For a partition λ, suppose that $h_{(j,j+\beta-1)}(\lambda) \le 1 - i$. Then, for any nonnegative integer x with $0 \le x \le \beta - 1$,*

$$h_{(j+x+\alpha-1,j+x)} \le K - i - 2.$$

Proof. Let us consider two cases: $(j, j + \beta - 1) \in \lambda$ and $(j, j + \beta - 1) \notin \lambda$.

Case 1: $(j, j + \beta - 1) \in \lambda$. If $(j + x + \alpha - 1, j + x) \in \lambda$, then

$$\begin{aligned} h_{(j+x+\alpha-1,j+x)} &= \lambda_{j+x+\alpha-1} - \lambda'_{j+x} + \alpha - 1 \\ &\le \lambda_j - \lambda'_{j+x} + \alpha - 1 \\ &\le \lambda_j - \lambda'_{j+\beta-1} + \alpha - 1 \\ &= h_{(j,j+\beta-1)} + (\alpha + \beta - 2) \\ &\le 1 - i + (\alpha + \beta) - 2 \\ &\le K - i - 2, \end{aligned}$$

where the second to last inequality follows from $h_{(j,j+\beta-1)} \le 1 - i$ and the last inequality follows from $K > \alpha + \beta$.

If $(j + x + \alpha - 1, j + x) \notin \lambda$, then

$$h_{(j+x+\alpha-1,j+x)} \le 0 \le K - i - 2,$$

where the left inequality follows from (2.2) and the right inequality follows from $1 \le i < K/2$.

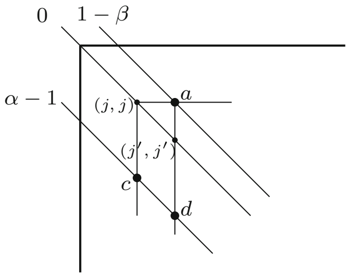


FIGURE 3. Lemma 2.2

Case 2: $(j, j+\beta-1) \notin \lambda$. Then, by (2.2), we know that

$$h_{(j,j+\beta-1)} = 0 \text{ or } +\infty.$$

However, since $h_{(j,j+\beta-1)} \le 1-i$, it has to be

$$h_{(j,j+\beta-1)} = 0,$$

and then, by (2.2), it has to be $\beta = 1$, i.e., $(j,j) \notin \lambda$. Then, for any nonnegative integer x, $(j+x+\alpha-1, j+x) \notin \lambda$. As seen above in the second case in Case 1, we get the desired inequality.

□

Lemma 2.2 can be summarized as follows. In Fig. 3, if $h_a \le 1-i$, then $h_z \le K-i-2$ for any nodes z between c and d. Further explanations on nodes (j,j) and (j',j') will be given in Remark 2.7.

Lemma 2.3. *For a partition λ, suppose that $h_{(j+\alpha-1,j)}(\lambda) \ge K-i-1$. Then, for any nonnegative integer x with $0 \le x \le \alpha-1$,*

$$h_{(j+x,j+x+\beta-1)}(\lambda) \ge 2-i.$$

Proof. By conjugation, it is clear that

$$(j+\alpha-1, j) \in \lambda \quad \text{if and only if} \quad (j, j+\alpha-1) \in \lambda'$$

and

$$h_{(j+\alpha-1,j)}(\lambda) = -h_{(j,j+\alpha-1)}(\lambda').$$

Thus, by the assumption that $h_{(j+\alpha-1,j)}(\lambda) \ge K-i-1$, we have

$$h_{(j,j+\alpha-1)}(\lambda') \le 1-(K-i),$$

which implies by Lemma 2.2 that

$$h_{(j+x+\beta-1,j+x)}(\lambda') \le K-(K-i)-2.$$

This is equivalent to

$$h_{(j+x,j+x+\beta-1)}(\lambda) \ge 2-i,$$

as desired. □

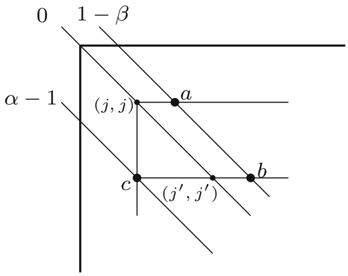


FIGURE 4. Lemma 2.3

Lemma 2.3 can be summarized as follows. In Fig. 4, if $h_c \geq K-i-1$, then $h_z \geq 2-i$ for any nodes z between a and b. Further explanations on nodes (j,j) and (j',j') will be given in Remark 2.7.

Remark 2.4. By the cases when $x=0$ in Lemmas 2.2 and 2.3, we see that it does not happen simultaneously that $h_{(j,j+\beta-1)} \leq 1-i$ and $h_{(j+\alpha-1,j)} \geq K-i-1$ for any j. That is, in Fig. 3 or 4, it is impossible that $h_a \leq 1-i$ and $h_c \geq K-i-1$ hold at the same time.

We now define the sign of a node on the main diagonal. For a node (j,j), its sign will be determined by the hook differences at the nodes that are in the hook of (j,j) and on the diagonals $1-\beta$ or $\alpha-1$.

Definition 2.5. Let λ be a partition. For a positive integer j, suppose that a node $(j,j)\in\lambda$. Then the node (j,j) is said to be

- (K,i,α,β)-negative if
$$(j,j+\beta-1)\in\lambda \quad \text{and} \quad h_{(j,j+\beta-1)} \leq 1-i; \tag{2.3}$$
- (K,i,α,β)-positive if
$$(j+\alpha-1,j)\in\lambda \quad \text{and} \quad h_{(j+\alpha-1,j)} \geq K-i-1; \tag{2.4}$$
- (K,i,α,β)-neutral otherwise.

By Remark 2.4, we see that the node (j,j) cannot be (K,i,α,β)-negative and positive at the same time. Also, we see that the node (j,j) is (K,i,α,β)-neutral if and only if

$$h_{(j,j+\beta-1)} \geq 2-i \quad \text{and} \quad h_{(j+\alpha-1,j)} \leq K-i-2,$$

which works even if $(j,j+\beta-1)\notin\lambda$ or $(j+\beta-1,j)\notin\lambda$ due to (2.2).

In Fig. 5, we set $(K,i,\alpha,\beta)=(7,2,3,2)$. The number in each box on the diagonals -1 and 2 is the hook difference at the node. Then the sign of node $(1,1)$ is positive since $h_{(1,2)}=3>1-i=-1$, but $h_{(3,1)}=5\geq K-i-1=4$. The signs of the other nodes on the main diagonal can be determined in the same way. Here we note that the sign of node $(5,5)$ is neutral since there are no nodes in the hook of $(5,5)$ that are on the diagonals -1 or 2. The letters p, n and e in the boxes stand for positive, negative and neutral, respectively.

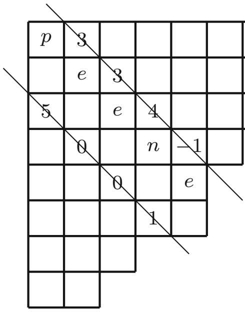


FIGURE 5. $(7,2,3,2)$-positive, negative and neutral

Remark 2.6. By the definition, if a node (j,j) is (K,i,α,β)-negative, then $(j,j+\beta-1)\in\lambda$, so $\lambda_j \geq j+\beta-1$. Similarly, if a node (j,j) is (K,i,α,β)-positive, then $(j+\alpha-1,j)\in\lambda$, so $\lambda'_j \geq j+\alpha-1$.

Remark 2.7. By Lemmas 2.2 and 2.3, we also see that the following statements hold true:

(i) Suppose that a node (j,j) is (K,i,α,β)-negative. Then for any x with $1\leq x\leq \beta-1$ and $(j+x,j+x)\in\lambda$, a node $(j+x,j+x)$ cannot be (K,i,α,β)-positive. That is, in Fig. 3, if (j,j) is negative, then any nodes between (j,j) and (j',j') are negative or neutral.

(ii) Suppose that a node (j,j) is (K,i,α,β)-positive. Then for any x with $1\leq x\leq \alpha-1$ and $(j+x,j+x)\in\lambda$, a node $(j+x,j+x)$ cannot be (K,i,α,β)-negative. That is, in Fig. 4, if (j,j) is positive, then any nodes between (j,j) and (j',j') are positive or neutral.

Next, we give the definition of Frobenius symbols and then rephrase our definition of the sign of nodes in terms of Frobenius symbol.

Definition 2.8. For a partition λ, let δ be the largest x such that $\lambda_x \geq x$. Then the Frobenius symbol of λ is defined as

$$\lambda = \begin{pmatrix} a_1 & \cdots & a_\delta \\ b_1 & \cdots & b_\delta \end{pmatrix},$$

where $a_x = \lambda_x - x$ and $b_y = \lambda'_y - y$ for $1\leq x,y\leq\delta$.

For $1\leq j\leq\delta$, the jth column of the Frobenius symbol of λ is said to be

- (K,i,α,β)-negative if the node $(j,j)\in\lambda$ is (K,i,α,β)-negative;
- (K,i,α,β)-positive if the node $(j,j)\in\lambda$ is (K,i,α,β)-positive;
- (K,i,α,β)-neutral if the node $(j,j)\in\lambda$ is (K,i,α,β)-neutral.

For instance, the partition in Fig. 5 can be written as follows:

$$\begin{pmatrix} 11 & 9 & 8 & 2 & 0 \\ 7 & 6 & 4 & 2 & 1 \end{pmatrix},$$

and the first column is positive, the fourth column is negative, and the others are neutral.

We note that from the definitions of hook differences and Frobenius symbols, it is clear that for $1 \le x, y \le \delta$,

$$h_{(x,y)} = a_x - b_y + 2(x - y). \tag{2.5}$$

Remark 2.9. When we compute hook differences from a Frobenius symbol, perhaps it is more convenient to write the Frobenius symbol in a slightly different form by shifting the second row to the right or left by $\alpha - 1$ units or $\beta - 1$ units.

In particular, let δ be the number of columns in the Frobenius symbol and $\delta \ge \max(\alpha, \beta)$. Then we can explicitly write the conditions for a node being (K, i, α, β)-negative, positive, or neutral in terms of the entries in the Frobenius symbol. If $\delta < \min(\alpha, \beta)$, then it is better to use Definitions 2.5 and 2.8 with (2.1).

In what follows, assuming that $\delta \ge \max(\alpha, \beta)$, we explain how to determine if a node is (K, i, α, β)-negative or positive.

(i) To check more easily if a node is (K, i, α, β)-negative, we write the Frobenius symbol as follows:

$$\begin{array}{ccccccc} & & a_1 & \cdots & a_{\delta-\beta+1} & \cdots & a_\delta \\ b_1 & \cdots & b_\beta & \cdots & b_\delta. & & \end{array}$$

Then, the hook differences to be needed to compute are the difference of the entries in each column with two entries. Namely, for $1 \le j \le \delta - \beta + 1$, by (2.3) and (2.5), if

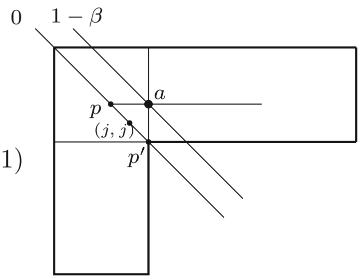
$$a_j - b_{\beta+j-1} \le 2\beta - 1 - i, \tag{2.6}$$

then the node (j, j) is negative.

Let us turn to nodes (j, j) for $\delta - \beta + 2 \le j \le \delta$. See the figure below. If the node p was negative, i.e., (2.6) holds true for $j = \delta - \beta + 1$, then next $\beta - 1$ nodes on the main diagonal after p could not be positive by Remark 2.7 (i). Thus, the nodes (j, j) between p and p' would be negative or neutral. However, if the node p was not negative, then the next node might be positive.

To compute the hook difference, the Ferrers diagram is more convenient to use than the Frobenius symbol.

$a = (\delta - \beta + 1, \delta)$
$p = (\delta - \beta + 1, \delta - \beta + 1)$
$p' = (\delta, \delta)$



(ii) Similarly, to check if a node is (K, i, α, β)-positive, we write the Frobenius symbol as follows.

$$\begin{array}{ccccccccc} a_1 & \cdots & a_\alpha & \cdots & a_\delta & & & & \\ & & b_1 & \cdots & b_{\delta-\alpha+1} & \cdots & b_\delta \end{array}$$

Then, the hook differences to be needed to compute are the difference of the entries in each column with two entries. For $1 \leq j \leq \delta - \alpha + 1$, by (2.4) and (2.5), if

$$a_{\alpha+j-1} - b_j \geq K - i + 1 - 2\alpha, \tag{2.7}$$

then the node (j, j) is positive. Similarly, we can show that nodes (j, j) for $\delta - \alpha + 2 \leq j \leq \delta$ cannot be positive if the node $(\delta - \alpha + 1, \delta - \alpha + 1)$ is positive, i.e., (2.7) holds true for $j = \delta - \alpha + 1$.

(iii) Adopting the notation for cylindric partitions [14], we may combine the expressions in (i) and (ii) above as follows:

$$\begin{array}{cccccccccc} & & & & a_1 & \cdots & \cdots & \cdots & a_\delta \\ & & b_1 & \cdots & b_\beta & \cdots & \cdots & b_\delta & \\ a_1 & \cdots & a_\alpha & \cdots & \cdots & a_\delta. & & & \end{array}$$

We note that it should be checked if the difference between the first and second rows satisfies (2.6) and the difference between the second and third rows satisfies (2.7).

We give an example to demonstrate how to find positive or negative columns in a Frobenius symbol.

Example 2.10. Consider the following Frobenius symbol:

$$\begin{pmatrix} 31 & 28 & 27 & 22 & 18 & 9 & 8 & 7 & 1 & 0 \\ 29 & 26 & 25 & 23 & 22 & 8 & 5 & 4 & 1 & 0 \end{pmatrix}.$$

Let $(K, i, \alpha, \beta) = (5, 2, 2, 2)$.

First, for negative nodes, by (2.6), we compute and see if

$$a_j - b_{j+1} \leq 1.$$

So, we shift the second row to the left by 1 unit and compute the difference of the entries in each column with two entries. Again, the upper entries with difference at most 1 are circled:

$$\begin{array}{ccccccccccc} & 31 & 28 & 27 & 22 & 18 & 9 & 8 & 7 & 1 & 0 \\ 29 & 26 & 25 & 23 & 22 & 8 & 5 & 4 & 1 & 0 & \end{array}. \tag{2.8}$$

These circled entries indicate the negative columns in the original Frobenius symbol. Since $\beta = 2$ and the second to last column is negative, the last column cannot be positive. Actually, from the Ferrers diagram of the Frobenius symbol, we see that the node $(10, 11)$ does not exist in the partition, so the sign of the last column is neutral.

Similarly, for positive nodes, by (2.7), we compute and see if

$$a_{j+1} - b_j \geq 0.$$

So, we shift the first row of the Frobenius symbol to the left by 1 unit and compute the difference of the entries in each column with two entries. For convenience, if the difference is at least 0, then the lower entry will be circled:

$$\begin{array}{ccccccccccc} 31 & 28 & 27 & 22 & 18 & 9 & 8 & 7 & 1 & 0 & \\ & 29 & 26 & 25 & 23 & 22 & 8 & 5 & 4 & 1 & 0 \end{array}. \tag{2.9}$$

These circled entries indicate the positive columns in the original Frobenius symbol. For the last column, we cannot say if it is not positive because the second to last column is not positive. However from the earlier calculations for negative nodes, we already know that it is neutral.

We now take the first row in (2.8) and the second row in (2.9) to form the following Frobenius symbol:

$$\begin{pmatrix} 31 & 28 & 27 & 22 & 18 & 9 & 8 & 7 & 1 & 0 \\ 29 & 26 & 25 & 23 & 22 & 8 & 5 & 4 & 1 & 0 \end{pmatrix},$$

where columns with circled entries in the first row are negative and columns with circled entries in the second row are positive. Therefore, we have

$$\begin{array}{c} \begin{array}{cccccccccc} & p & & n & & p & p & & n & \end{array} \\ \begin{pmatrix} 31 & 28 & 27 & 22 & 18 & 9 & 8 & 7 & 1 & 0 \\ 29 & 26 & 25 & 23 & 22 & 8 & 5 & 4 & 1 & 0 \end{pmatrix} \end{array},$$

where p and n indicate that the corresponding columns are positive and negative, respectively.

2.1. (K, i, α, β)-Parity Blocks and Anchors

Following [5], we generalize the notion of parity blocks of Frobenius symbols. Unlike in [5], for convenience, we introduce neutral blocks. For a Frobenius symbol, if there are consecutive neutral columns starting from the first column, then separate them to form a neutral block. We shall say that the block is neutral and we denote it by E. For the remaining columns, we take sets of contiguous columns maximally extended to the right, where all the columns have either the same parity or neutral. We shall say that a block is positive (or negative) if it contains no negative (or no positive, resp.) nodes, and we denote it by P (or N, resp.).

Example 2.11. We consider the same Frobenius symbol as in Example 2.10. Then, the $(5, 2, 2, 2)$-parity blocks are

$$\left(\begin{array}{c|cc|cc|ccc|cc} \overset{E}{31} & 28 & 27 & 22 & 18 & 9 & 8 & 7 & 1 & 0 \\ 29 & 26 & 25 & 23 & 22 & 8 & 5 & 4 & 1 & 0 \end{array}\right),$$

where the superscripts E, P, and N denote that the block is neutral, positive, and negative, respectively.

Definition 2.12. For a positive (or negative) block, we define its anchor as the first column in the block.

Lemma 2.13. *Let λ be a partition. For a non-last block of λ, if it is positive or negative, then there are at least α or β columns, respectively.*

This lemma follows from Remark 2.7 on the sign of $\alpha-1$ (or $\beta-1$) nodes next to a positive (or negative, resp.) node. So we omit the proof.

Example 2.14. Let $(K,i,\alpha,\beta)=(7,3,3,2)$. We consider the following Frobenius symbol:

$$\begin{pmatrix} 31\ 28\ 27\ 22\ 18\ 10\ 9\ 8\ 7\ 1 \\ 29\ 26\ 25\ 23\ 22\ 20\ 8\ 5\ 4\ 3 \end{pmatrix}.$$

Following Example 2.10, we will decompose this Frobenius symbol into $(7,3,3,2)$-parity blocks and identify their anchors.

First, by (2.6), for negative nodes, we compute and see if

$$a_j - b_{j+1} \le 0.$$

So, we shift the second row to the left by 1 unit and compute the difference of the entries in each column with two entries to see if the difference is at most 0.

$$\begin{array}{ccccccccccc} & 31 & 28 & 27 & 22 & 18 & 10 & 9 & 8 & 7 & 1 \\ 29 & 26 & 25 & 23 & 22 & 20 & 8 & 5 & 4 & 3 & \end{array},$$

where the circled $22, 18$ and 1 indicate the negative columns in the original Frobenius symbol. Here, we note that since the second to last column is not negative, we cannot determine the sign of the last column from the sign of the second to last column, so we used the Ferrers diagram to compute the hook difference at the corresponding node $(10,11)$, which is 0.

Similarly, by (2.7), for positive nodes, we compute and see if

$$a_{j+2} - b_j \ge -1.$$

We shift the first row of the Frobenius symbol to the left by 2 units and compute the difference of the entries in each column with two entries.

$$\begin{array}{cccccccccccc} 31 & 28 & 27 & 22 & 18 & 10 & 9 & 8 & 7 & 1 & & \\ & & 29 & 26 & 25 & 23 & 22 & 20 & 8 & 5 & 4 & 3, \end{array}$$

where the circled 8 indicates the positive column in the original Frobenius symbol. Here, we used the Ferrers diagram to determine the sign of each of the last two columns.

Therefore, we see that the $(7,3,3,2)$-parity blocks with anchors are as follows:

$$\begin{array}{c} \begin{array}{cccccccccc} & E & & & N & & & P & & N \end{array} \\ \left(\begin{array}{ccc|ccc|ccc|c} 31 & 28 & 27 & 22 & 18 & 10 & \boxed{9} & 8 & 7 & 1 \\ 29 & 26 & 25 & \boxed{23} & 22 & 20 & 8 & 5 & 4 & \boxed{3} \end{array}\right) \end{array},$$

where the bottom entries in the anchors of the negative blocks are boxed and the top entry in the anchor of the positive block is boxed.

2.2. (K, i, α, β)-Singular Overpartitions

We are now ready to define (K, i, α, β)-singular overpartitions. An overpartition is (K, i, α, β)-singular if its Frobenius symbol satisfies one of the following conditions:

- there are no overlined entries;
- if there is one overlined entry on the top row, then it occurs in the anchor of a positive block;
- if there is one overlined entry on the bottom row, then it occurs in the anchor of a negative block;
- if there are two overlined entries, then they occur in adjacent anchors with one on the top row of the positive block and the other on the bottom row of the negative block.

Example 2.15. We consider the same Frobenius symbol as in Example 2.14. Then, all $(7, 3, 3, 2)$-singular overpartitions are:

$$\left(\begin{array}{ccc|ccc|ccc|c} 31 & 28 & 27 & 22 & 18 & 10 & 9 & 8 & 7 & 1 \\ 29 & 26 & 25 & 23 & 22 & 20 & 8 & 5 & 4 & 3 \end{array}\right), \left(\begin{array}{ccc|ccc|ccc|c} 31 & 28 & 27 & 22 & 18 & 10 & \overline{9} & 8 & 7 & 1 \\ 29 & 26 & 25 & 23 & 22 & 20 & 8 & 5 & 4 & 3 \end{array}\right),$$
$$\left(\begin{array}{ccc|ccc|ccc|c} 31 & 28 & 27 & 22 & 18 & 10 & 9 & 8 & 7 & 1 \\ 29 & 26 & 25 & \overline{23} & 22 & 20 & 8 & 5 & 4 & 3 \end{array}\right), \left(\begin{array}{ccc|ccc|ccc|c} 31 & 28 & 27 & 22 & 18 & 10 & 9 & 8 & 7 & 1 \\ 29 & 26 & 25 & 23 & 22 & 20 & 8 & 5 & 4 & \overline{3} \end{array}\right),$$
$$\left(\begin{array}{ccc|ccc|ccc|c} 31 & 28 & 27 & 22 & 18 & 10 & \overline{9} & 8 & 7 & 1 \\ 29 & 26 & 25 & \overline{23} & 22 & 20 & 8 & 5 & 4 & 3 \end{array}\right), \left(\begin{array}{ccc|ccc|ccc|c} 31 & 28 & 27 & 22 & 18 & 10 & \overline{9} & 8 & 7 & 1 \\ 29 & 26 & 25 & 23 & 22 & 20 & 8 & 5 & 4 & \overline{3} \end{array}\right).$$

3. A Generalization of the Dyson Map and the Shift Map

In this section, we modify a map of Dyson, which was used to prove a symmetry of the partition function $p(n)$ [13]. We also recall a map, the so-called shift map, from [15].

3.1. A Generalization of the Dyson Map

Let t be a positive integer and m be an integer. We define a generalized Dyson map d_m^t as follows:

- Case 1: $m \leq 0$. Let π be a partition such that $(1, t) \in \pi$ and
$$h_{(1,t)} \leq m \quad \text{(if and only if} \quad \pi_1 - \pi_t' \leq t - 1 + m). \tag{3.1}$$
  – Delete the first t columns in the Ferrers diagram of π;
  – Add rows of size $\pi_j' + m - 1$ for $1 \leq j \leq t$ on the top of the resulting Ferrers diagram from the previous step.
  – Define $d_m^t(\pi)$ to be the partition whose Ferrers diagram is the one obtained in the previous step.

  Here we note that d_m^t can be given explicitly as follows:
$$d_m^t(\pi) = (\pi_1' + m - 1, \ldots, \pi_t' + m - 1, \pi_1 - t, \pi_2 - t, \ldots),$$
  which is indeed a partition since $\pi_j \geq \pi_{j+1}$, $\pi_j' \geq \pi_{j+1}'$ for any $j \geq 1$, and
$$\pi_t' + m - 1 \geq \pi_1 - t$$

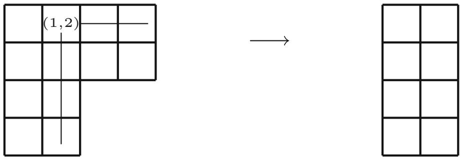


FIGURE 6. $d^2_{-1}(4,4,2,2) = (2,2,2,2)$

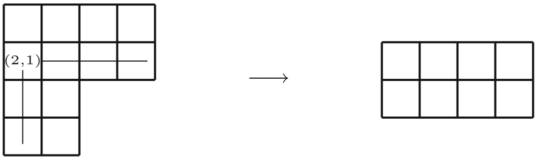


FIGURE 7. $d^2_1(4,4,2,2) = (4,4)$

by (3.1). More details are given in Lemma 3.2 (Figs. 6, 7).

For example, let $t = 2, m = -1$ and $\pi = (4,4,2,2)$. We have

$$h_{(1,2)} = -1 \leq m = -1.$$

So,

$$d^2_{-1}(\pi) = (2,2,2,2).$$

- Case 2: $m > 0$. Let π be a partition such that $(t,1) \in \pi$ and

$$h_{(t,1)} \geq m \quad (\text{if and only if} \quad \pi_t - \pi'_1 \geq 1 - t + m). \tag{3.2}$$

Then,

$$d^t_m(\pi) = \left(d^t_{-m}(\pi')\right)'.$$

For example, let $t = 2, m = 1$ and $\pi = (4,4,2,2)$. We have

$$h_{(2,1)} = 1 \geq m = 1.$$

By the definition,

$$d^2_1(\pi) = (4,4).$$

Remark 3.1. Some facts about the generalized Dyson map are noted below.

(i) $|\pi| - |d^t_m(\pi)| = t(|m|+1)$.
(ii) For $m \leq 0$, d^1_m becomes the Dyson map d_m appeared in [13,15].
(iii) For $m \leq 0$, if (3.1) is not satisfied, then d^t_m is not defined. Also, for $m > 0$, if (3.2) is not satisfied, then d^t_m is not defined.

Lemma 3.2. *For $m \leq 0$, let λ be a partition with $(1,t) \in \lambda$ and $h_{(1,t)} \leq m$. If $d^t_m(\lambda) = \mu \neq \emptyset$, then μ is a partition with $\mu_j = \lambda'_j + m - 1$ for $1 \leq j \leq t$ and $\mu'_1 = \lambda'_{t+1} + t$.*

Proof. Note that μ is a partition if and only if the parts are weakly decreasing. By the definition of d^t_m, we know that

$$\mu_j = \begin{cases} \lambda'_j + m - 1, & \text{for } 1 \le j \le t, \\ \lambda_{j-t} - t, & \text{for } j > t. \end{cases}$$

Thus, clearly $\mu_j \ge \mu_{j+1}$ for $1 \le j < t$ and $j > t$. Also,

$$\begin{aligned} \mu_t - \mu_{t+1} &= (\lambda'_t + m - 1) - (\lambda_1 - t) \\ &= \lambda'_t - \lambda_1 + (t-1) + m \\ &\ge 0. \end{aligned}$$

In addition, it is clear that $\mu_j = \lambda'_j + m - 1$ for $1 \le j \le t$ and $\mu'_1 = \lambda'_{t+1} + t$. □

For $m \le 0$, we first note that if $\lambda = (t^{1-m})$, that is, λ is a partition with part t occurring $1-m$ times, then $d^t_m(\lambda) = \emptyset$. We now give a close look at the entries in the first column of the Frobenius symbol of $d^t_m(\lambda)$ when $\lambda \ne (t^{1-m})$. Suppose that

$$\lambda = \begin{pmatrix} a_1 & \cdots & a_\delta \\ b_1 & \cdots & b_\delta \end{pmatrix}.$$

By Lemma 3.2,

$$\mu = \begin{pmatrix} \lambda'_1 + m - 2 & \cdots \\ \lambda'_{t+1} + t - 1 & \cdots \end{pmatrix} = \begin{pmatrix} b_1 + m - 1 & \cdots \\ \gamma & \cdots \end{pmatrix}, \tag{3.3}$$

where $\gamma = \lambda'_{t+1} + t - 1$. If $\delta > t$, then $\lambda'_{t+1} = b_{t+1} + t + 1$, so

$$\gamma = b_{t+1} + 2t;$$

if $\delta \le t$, then $\lambda'_{t+1} \le t$, so

$$\gamma = \lambda'_{t+1} + t - 1 \le 2t - 1.$$

Lemma 3.3. *For $m > 0$, let λ be a partition with $(t,1) \in \lambda$ and $h_{(t,1)} \ge m$. If $d^t_m(\lambda) = \mu$, then μ is a partition with $\mu_1 = \lambda_{t+1} + t$ and $\mu'_1 = \lambda_1 - m - 1$ for $1 \le j \le t$.*

The proof of this lemma is similar to that of Lemma 3.2, so we omit it.

For $m > 0$, we first note that if $\lambda = (m+1)^t$, then $d^t_m(\lambda) = \emptyset$. Suppose that $\lambda \ne (m+1)^t$ and

$$\lambda = \begin{pmatrix} a_1 & \cdots & a_\delta \\ b_1 & \cdots & b_\delta \end{pmatrix}.$$

By Lemma 3.3,

$$\mu = \begin{pmatrix} \gamma & \cdots \\ a_1 - m - 1 & \cdots \end{pmatrix}, \tag{3.4}$$

where $\gamma = \lambda_{t+1} + t - 1$. If $\delta > t$, then $\lambda_{t+1} = a_{t+1} + t + 1$, so

$$\gamma = a_{t+1} + 2t;$$

if $\delta \le t$, then $\lambda_{t+1} \le t$, so

$$\gamma = \lambda_{t+1} + t - 1 \le 2t - 1.$$

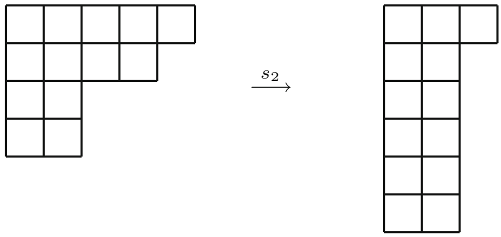


Figure 8. Shift map

3.2. The Shift Map

Given an integer u, a shift map s_u is defined as follows [15]:

$$\begin{pmatrix} a_1 & a_2 & \cdots & a_\delta \\ b_1 & b_2 & \cdots & b_\delta \end{pmatrix} \xrightarrow{s_u} \begin{pmatrix} a_1-u & a_2-u & \cdots & a_\delta-u \\ b_1+u & b_2+u & \cdots & b_\delta+u \end{pmatrix}. \tag{3.5}$$

Let $\lambda = \begin{pmatrix} a_1 & a_2 & \cdots & a_\delta \\ b_1 & b_2 & \cdots & b_\delta \end{pmatrix}$ with $a_\delta \geq u$, and $\mu = s_u(\lambda)$. Then, it is clear that μ is still a partition. Also, for $1 \leq x, y \leq \delta$,

$$h_{(x,y)}(\mu) = h_{(x,y)}(\lambda) - 2u. \tag{3.6}$$

Figure 8 shows that $s_2(5,4,2,2) = (3,2,2,2,2,2)$.

4. Lemmas

The main purpose of this section is to prove lemmas that will be used in Sect. 5.

We first introduce a necessary definition and then present lemmas.

Definition 4.1. Let ν and σ be two partitions whose Frobenius symbols are

$$\nu = \begin{pmatrix} a_1 & \cdots & a_t \\ b_1 & \cdots & b_t \end{pmatrix} \quad \text{and} \quad \sigma = \begin{pmatrix} a_{t+1} & \cdots \\ b_{t+1} & \cdots \end{pmatrix}.$$

We define the concatenation of ν and σ as

$$\nu\sigma = \begin{pmatrix} a_1 & \cdots & a_t & a_{t+1} & \cdots \\ b_1 & \cdots & b_t & b_{t+1} & \cdots \end{pmatrix}.$$

Let $\lambda = \nu\sigma$. First, note that if $a_t > a_{t+1}$ and $b_t > b_{t+1}$, then λ is indeed a partition. Even if λ is not a partition, we relax the definition of parts of a partition and define the xth part of λ as

$$\lambda_x = \nu_x + \sigma_{x-t},$$

where $\nu_x = 0$ if $x > \ell(\nu)$, and $\sigma_{x-t} = 0$ if $x - t > \ell(\sigma)$ or $x - t \leq 0$. In other words, λ_x counts the number of boxes in row x in the concatenated diagram of λ (see Fig. 9). We now define the hook difference at a node (x, y) of λ as

$$h_{(x,y)}(\lambda) = \lambda_x - \lambda'_y + (x - y)$$

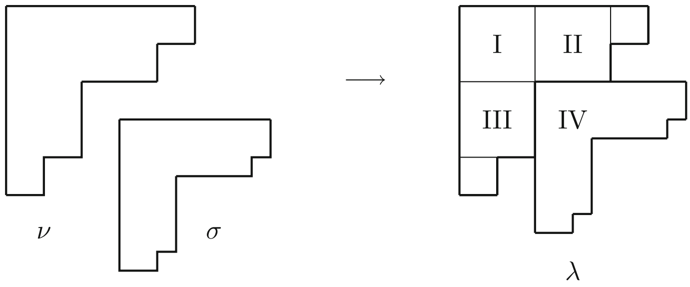


FIGURE 9. Concatenation of two partitions

provided λ_x and λ'_y exist. We also define the weight of λ by the sum of the weights of ν and σ, i.e.,

$$|\lambda| = |\nu| + |\sigma|.$$

For instance, let

$$\nu = \begin{pmatrix} 5 & 4 \\ 4 & 2 \end{pmatrix}, \sigma = \begin{pmatrix} 3 & 1 \\ 1 & 0 \end{pmatrix}.$$

Then

$$\nu\sigma = \begin{pmatrix} 5 & 4 & 3 & 1 \\ 4 & 2 & 1 & 0 \end{pmatrix},$$

which is clearly a partition. However, if we consider

$$\nu = \begin{pmatrix} 5 & 4 \\ 4 & 2 \end{pmatrix}, \sigma = \begin{pmatrix} 4 & 1 \\ 3 & 0 \end{pmatrix},$$

then

$$\nu\sigma = \begin{pmatrix} 5 & 4 & 4 & 1 \\ 4 & 2 & 3 & 0 \end{pmatrix},$$

which is not a partition. In either case, $|\nu\sigma| = |\nu| + |\sigma|$.

Although we can evaluate hook difference at any node, in this paper, we are interested in the hook differences at nodes only in the regions I–IV in Fig. 9, where

$$\begin{cases} \mathrm{I} = \{(x, y) | 1 \le x, y \le t\}, \\ \mathrm{II} = \{(x, y) | 1 \le x \le t < y \le t + a_t\}, \\ \mathrm{III} = \{(x, y) | 1 \le y \le t < x \le t + b_t\}, \\ \mathrm{IV} = \{(x, y) | t < x, y\}. \end{cases} \tag{4.1}$$

In the following lemma, we evaluate those hook differences at nodes in each of the regions I–IV in terms of ν and σ for later use.

Lemma 4.2. *Given partitions*

$$\nu = \begin{pmatrix} a_1 & \cdots & a_t \\ b_1 & \cdots & b_t \end{pmatrix} \neq \emptyset \quad \text{and} \quad \sigma = \begin{pmatrix} a_{t+1} & \cdots \\ b_{t+1} & \cdots \end{pmatrix} \neq \emptyset,$$

let $\lambda = \nu\sigma$. *Then*

$$h_{(x,y)}(\lambda) = \begin{cases} h_{(x,y)}(\nu) = a_x - b_y + 2(x-y), & \text{if } (x,y) \in \mathrm{I}, \\ h_{(x,y)}(\nu) - \sigma'_{y-t} = (a_x + x) - t + (x-y) - \sigma'_{y-t}, & \text{if } (x,y) \in \mathrm{II}, \\ h_{(x,y)}(\nu) + \sigma_{x-t} = t - (b_y + y) + (x-y) + \sigma_{x-t}, & \text{if } (x,y) \in \mathrm{III}, \\ h_{(x-t,y-t)}(\sigma) = a_x - b_y + 2(x-y), & \text{if } (x,y) \in \mathrm{IV}. \end{cases} \tag{4.2}$$

Proof. Note that

$$\lambda_x = \begin{cases} \nu_x = a_x + x, & \text{for } 1 \le x \le t, \\ \nu_x + \sigma_{x-t} = t + \sigma_{x-t}, & \text{for } t < x \le t + b_t, \end{cases}$$

$$\lambda'_y = \begin{cases} \nu'_y = b_y + y, & \text{for } 1 \le y \le t, \\ \nu'_y + \sigma'_{y-t} = t + \sigma'_{y-t}, & \text{for } t < y \le t + a_t. \end{cases}$$

It is easy to prove the statement and we omit the details. □

In the next two lemmas, we will deal with pairs of Frobenius symbols with certain conditions, and we will show that after the shift map and the Dyson map are applied to such a pair, the concatenation of the resulting pair becomes a partition.

Lemma 4.3. *Given integers* f, g, h *with* $g \ge 1$, $f \le 2g - 1$, *and* $h \ge f$, *let* $\lambda = \nu\sigma$ *with*

$$\nu = \begin{pmatrix} a_1 & \cdots & a_t \\ b_1 & \cdots & b_t \end{pmatrix} \quad \textit{and} \quad \sigma = \begin{pmatrix} a_{t+1} & \cdots \\ b_{t+1} & \cdots \end{pmatrix} \neq \emptyset,$$

such that

(i) $h_{(t+1,t+\beta)}(\lambda) \le f - 2g + 1$;

and if $\nu \neq \emptyset$, *then*

(ii) $h_{(j,j+\beta-1)}(\lambda) \ge \begin{cases} f, & \textit{for } 1 \le j \le t - \beta + 1, \\ f - g + 1, & \textit{for } t - \beta + 2 \le j \le t, \end{cases}$

(iii) $a_t > a_{t+1} + g - 1$, $a_{t+1} \ge \beta - 1$,

(iv) $b_t > b_{t+1} - g + 1$, $b_t \ge 1$.

If $\mu = s_{f-g-1}(\nu)(d^{\beta}_{f-2g+1}(\sigma))'$, *then the following are true.*

(a) μ *is a partition.*

(b) $h_{(j,j+\beta-1)}(\mu) \ge -f + 2g + 2$ *for* $1 \le j \le t$, *and if* $(t+1, t+\beta) \in \mu$, *then* $h_{(t+1,t+\beta)}(\mu) \le -f + 2g + 1$.

(c) *For an integer* α *with* $1 \le \alpha \le t$, *if* $\nu \neq \emptyset$ *and* $h_{(\alpha,1)}(\nu) \ge h$, *then* $h_{(\alpha,1)}(\mu) \ge h - 2f + 2g + 2$.

(d) *The map from* λ *to* μ *is reversible.*

(e) $|\lambda| - |\mu| = (2g - f)\beta$.

Figure 10 sketches how to get μ from ν and σ.

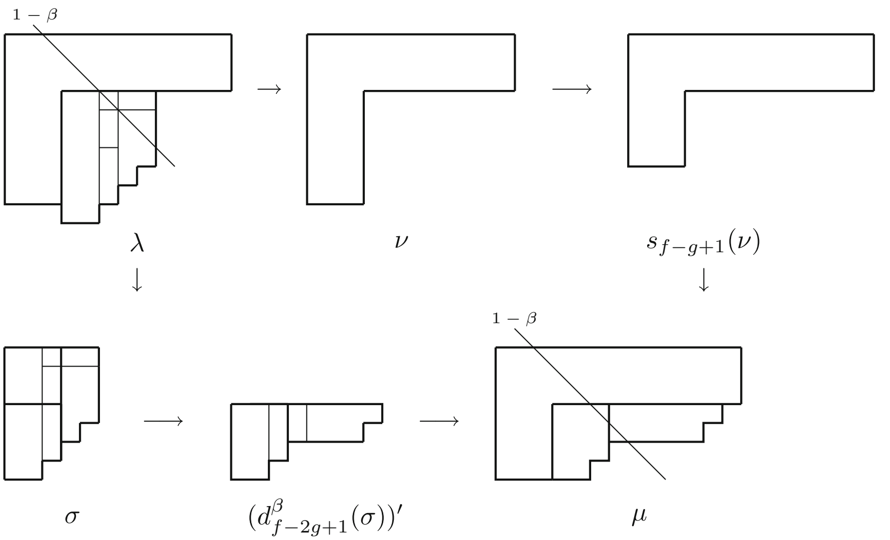


FIGURE 10. Lemma 4.3

Proof. First, if $\nu = \emptyset$, then $t = 0$ and $\lambda = \sigma$, so $h_{(t+1,t+\beta)}(\lambda) = h_{(1,\beta)}(\sigma)$. Even if $\nu \neq \emptyset$, we see that by (4.2), $h_{(t+1,t+\beta)}(\lambda) = h_{(1,\beta)}(\sigma)$. Thus in both cases, by Condition i) with $f \leq 2g-1$,

$$h_{(1,\beta)}(\sigma) = h_{(t+1,t+\beta)}(\lambda) \leq f - 2g + 1 \leq 0. \tag{4.3}$$

Hence, $d^{\beta}_{f-2g+1}(\sigma)$ is well defined.

We now check if

$$s_{f-g-1}(\nu) = \begin{pmatrix} a_1 - f + g + 1 & \cdots & a_t - f + g + 1 \\ b_1 + f - g - 1 & \cdots & b_t + f - g - 1 \end{pmatrix}$$

is well defined. Namely, all the entries in each row of $s_{f-g-1}(\nu)$ are nonnegative and strictly decreasing. Since a_j and b_j are strictly decreasing, the resulting sequences $a_j - f + g + 1$ and $b_j + f - g - 1$ are strictly decreasing. For non-negativity, it suffices to check $a_t - f + g + 1 \geq 0$ and $b_t + f - g - 1 \geq 0$. Since

$$h_{1,\beta}(\sigma) = \sigma_1 - \sigma'_\beta + (1-\beta) \geq \sigma_1 - \sigma'_1 + (1-\beta) = (a_{t+1} - b_{t+1}) + (1-\beta),$$

(4.3) with $a_{t+1} \geq \beta - 1$ from Condition (iii) guarantees that

$$b_{t+1} \geq (a_{t+1} + 1 - \beta) + 2g - f - 1 \geq 2g - f - 1.$$

So, if $\nu \neq \emptyset$, by Condition (iv)

$$b_t + f - g - 1 \geq 0.$$

Also, if $\nu \neq \emptyset$, by Condition (iii)

$$a_t \geq \beta + g - 1.$$

Thus

$$a_t - f + g + 1 \geq \beta - f + 2g \geq \beta + 1 \geq 2,$$

where the second inequality follows from the condition that $f \leq 2g - 1$. So we showed that $s_{f-g-1}(\nu)$ is well defined.

We now prove each of the five statements.

(a) If $\nu = \emptyset$, then it follows from Lemma 3.2 that $\mu = (d^{\beta}_{f-2g+1}(\sigma))'$ is a partition. Also, we note that if $d^{\beta}_{f-2g+1}(\sigma) = \emptyset$, then it is clear that $\mu = s_{f-g-1}(\nu)$, which is a partition as shown above.

Now assume that $\nu \neq \emptyset$ and $d^{\beta}_{f-2g+1}(\sigma) \neq \emptyset$. Let $\tilde{\nu} = s_{f-g-1}(\nu)$ and $\tilde{\sigma} = (d^{\beta}_{f-2g+1}(\sigma))'$. Note that by (3.5),

$$\tilde{\nu} = \begin{pmatrix} a_1 - f + g + 1 & \cdots & a_t - f + g + 1 \\ b_1 + f - g - 1 & \cdots & b_t + f - g - 1 \end{pmatrix}. \tag{4.4}$$

Also, by Lemma 3.2,

$$\tilde{\sigma}_1 = \sigma'_{\beta+1} + \beta, \tag{4.5}$$

and

$$\tilde{\sigma}'_j = \sigma'_j + (f - 2g) \quad \text{for } 1 \leq j \leq \beta. \tag{4.6}$$

So the Frobenius symbol of $\tilde{\sigma}$ is as follows:

$$\tilde{\sigma} = \begin{pmatrix} \sigma'_{\beta+1} + \beta - 1 & \cdots \\ \sigma'_1 + f - 2g - 1 & \cdots \end{pmatrix}.$$

For μ to be a partition, it has to hold that

$$a_t - f + g + 1 > \sigma'_{\beta+1} + \beta - 1 \tag{4.7}$$

and

$$b_t + f - g - 1 > \sigma'_1 + (f - 2g) - 1. \tag{4.8}$$

We first prove (4.7). If $\beta = 1$, then $(t, t + \beta - 1) \in \mathrm{I}$ in (4.1), so by (4.2) and Condition (ii),

$$h_{(t,t+\beta-1)}(\lambda) = a_t - b_t \geq f,$$

so

$$(a_t - f + g + 1) - \sigma'_2 \geq b_t + g + 1 - \sigma'_2 \geq b_t + g + 1 - b_{t+1} - 1 > 1,$$

where the second inequality follows from $\sigma'_2 \leq \sigma'_1 = b_{t+1} + 1$ and the last inequality follows from Condition (iv). If $\beta > 1$, then $(t, t+\beta-1) \in \mathrm{II}$ in (4.1). Thus, by (4.2) and Condition (ii),

$$h_{(t,t+\beta-1)}(\lambda) = (a_t + t) - t - (\beta - 1) - \sigma'_{\beta-1} = a_t - (\beta - 1) - \sigma'_{\beta-1} \geq f - g + 1,$$

so

$$\begin{aligned} &(a_t - f + g + 1) - (\sigma'_{\beta+1} + \beta - 1) \\ &\quad \geq \sigma'_{\beta-1} + (\beta - 1) + 2 - (\sigma'_{\beta+1} + \beta - 1) \\ &\quad = \sigma'_{\beta-1} - \sigma'_{\beta+1} + 2 > 1, \end{aligned}$$

since $\sigma'_{\beta-1} \geq \sigma'_{\beta+1}$. This proves (4.7).

For (4.8), note that $\sigma'_1 = b_{t+1} + 1$, so

$$(b_t + f - g - 1) - (\sigma'_1 + f - 2g - 1) = (b_t - b_{t+1}) + g - 1 > 0,$$

where the inequality follows from Condition (iv). Therefore, μ is a partition.

(b) For $1 \le j \le t - \beta + 1$, i.e., $(j, j+\beta-1) \in \mathrm{I}$ in (4.1), by (4.2) and (4.4),

$$\begin{aligned} h_{(j,j+\beta-1)}(\mu) &= h_{(j,j+\beta-1)}(\tilde{\nu}) \\ &= (a_j - f + g + 1) - (b_{j+\beta-1} + f - g - 1) - 2(\beta - 1) \\ &= (a_j - b_{j+\beta-1}) - 2(\beta-1) - 2f + 2g + 2 \\ &= h_{(j,j+\beta-1)}(\nu) - 2f + 2g + 2 \\ &= h_{(j,j+\beta-1)}(\lambda) - 2f + 2g + 2 \\ &\ge -f + 2g + 2, \end{aligned}$$

where the last inequality follows from Condition (ii).

For $t - \beta + 2 \le j \le t$, i.e., $(j, j+\beta-1) \in \mathrm{II}$ in (4.1), by (4.2), (4.4) and (4.6),

$$\begin{aligned} h_{(j,j+\beta-1)}(\mu) &= h_{(j,j+\beta-1)}(\tilde{\nu}) - \tilde{\sigma}'_{j+\beta-1-t} \\ &= (a_j - f + g + 1 + j) - t - (\beta - 1) - (\sigma'_{j+\beta-1-t} + f - 2g) \\ &= \left(a_j + j - t - (\beta-1) - \sigma'_{j+\beta-1-t}\right) - 2f + 3g + 1 \\ &= \left(h_{(j,j+\beta-1)}(\nu) - \sigma'_{j+\beta-1-t}\right) - 2f + 3g + 1 \\ &= h_{(j,j+\beta-1)}(\lambda) - 2f + 3g + 1 \\ &\ge -f + 2g + 2, \end{aligned}$$

where the last inequality follows from Condition (ii).

Also, $(t+1, t+\beta) \in \mathrm{IV}$ in (4.1). So, by (4.2), (4.5) and (4.6),

$$\begin{aligned} h_{(t+1,t+\beta)}(\mu) &= h_{(1,\beta)}(\tilde{\sigma}) \\ &= \tilde{\sigma}_1 - \tilde{\sigma}'_\beta + (1 - \beta) \\ &= \sigma'_{\beta+1} + \beta - \sigma'_\beta - (f - 2g) + (1 - \beta) \\ &\le -f + 2g + 1, \end{aligned}$$

where the last inequality follows from $\sigma'_{\beta+1} \le \sigma'_\beta$.

(c) Suppose $h_{(\alpha,1)}(\nu) \ge h$ for $1 \le \alpha \le t$. Then

$$h_{(\alpha,1)}(\mu) = h_{(\alpha,1)}(\tilde{\nu}) = h_{(\alpha,1)}(\nu) - 2f + 2g + 2 \ge h - 2f + 2g + 2,$$

where (3.6) is used for the second equality.

(d) By (b), t is uniquely determined, so we can decompose μ into $\tilde{\nu}$ and $\tilde{\sigma}$. Since the shift map and the Dyson map are reversible, we can recover ν and σ from μ.

(e) Finally, since $|\tilde{\nu}| = |s_{f-g-1}(\nu)| = |\nu|$, $|\tilde{\sigma}| = |d^{\beta}_{f-2g+1}(\sigma)| = |\sigma| - (2g - f)\beta$, $|\lambda| = |\nu| + |\sigma|$, $|\mu| = |\tilde{\nu}| + |\tilde{\sigma}|$, we have

$$|\lambda| - |\mu| = (2g - f)\beta,$$

as desired. $\square$

Lemma 4.4. *Given integers f, g, h with $g - f \geq 1$, $f \leq 2g-1$, $h \leq f$, let $\lambda = \nu\sigma$ with*

$$\nu = \begin{pmatrix} a_1 & \cdots & a_t \\ b_1 & \cdots & b_t \end{pmatrix} \quad \textit{and} \quad \sigma = \begin{pmatrix} a_{t+1} & \cdots \\ b_{t+1} & \cdots \end{pmatrix} \neq \emptyset,$$

such that

(i) $h_{(t+\alpha,t+1)}(\lambda) \geq -f + 2g - 1$;

and if $\nu \neq \emptyset$, then

(ii) $h_{(j+\alpha-1,j)}(\lambda) \leq \begin{cases} f, & \textit{for } 1 \leq j \leq t-\alpha+1, \\ g-1, & \textit{for } t-\alpha+2 \leq j \leq t, \end{cases}$

(iii) $a_t > a_{t+1} + f - g + 1$, $a_t \geq 1$,

(iv) $b_t > b_{t+1} - f + g - 1$, $b_{t+1} \geq \alpha - 1$.

If $\mu = s_{g+1}(\nu)\,(d^{\alpha}_{2g-f-1}(\sigma))'$, then the following are true.

(a) μ *is a partition.*

(b) $h_{(j+\alpha-1,j)}(\mu) \leq f - 2g - 2$ *for all* $1 \leq j \leq t$, *and if* $(t+\alpha, t+1) \in \mu$, *then* $h_{(t+\alpha,t+1)}(\mu) \geq f - 2g - 1$.

(c) For an integer β with $1 \leq \beta \leq t$, if $\nu \neq \emptyset$ and $h_{(1,\beta)}(\nu) \leq h$, then $h_{(1,\beta)}(\mu) \leq h - 2g - 2$.

(d) The map from λ to μ is reversible.

(e) $|\lambda| - |\mu| = (2g-f)\alpha$.

Proof. Note that $h_{(x,y)}(\lambda) = -h_{(y,x)}(\lambda')$. We substitute ν', σ', α, $-f$, $g-f$, and $-h$ for ν, σ, β, f, g, and h, respectively, in Lemma 4.3. Then we can easily check that ν', σ', $\alpha, -f, g-f, -h$ satisfy the conditions in Lemma 4.3. Thus all the statements (a)–(e) hold true. □

In the next two lemmas, we will discuss a lower bound of the largest part of a partition and a lower bound of the number of parts.

Lemma 4.5. *Suppose that λ is a partition with the following non-neutral block decomposition:*

$$\lambda = \left(D_{2w} \middle| D_{2w-1} \middle| \cdots \middle| D_2 \middle| D_1 \right).$$

(a) If D_1 is negative, then $\lambda_1 \geq Kw$;

(b) If D_1 is positive, then $\lambda'_1 \geq Kw$.

Proof. (a) For $1 \leq j \leq 2w$, let x_j and y_j be the first and last entries in the top row of D_j, and $\tilde{x}_j$ and $\tilde{y}_j$ be the first and last entries in the bottom row of D_j, i.e.,

$$D_j = \begin{matrix} x_j & \cdots & y_j \\ \tilde{x}_j & \cdots & \tilde{y}_j \end{matrix}.$$

First, we note that since D_1 is negative, $x_1 \geq \beta - 1$ by Remark 2.6.

For $j > 1$, we note that

$$y_j > x_{j-1}, \quad \tilde{y}_j > \tilde{x}_{j-1}, \tag{4.9}$$

since λ is a partition. Also, we note from Lemma 2.13 that for $j > 1$, D_j has at least β columns if j is odd (i.e., D_j is negative) and at least α columns if j is even (i.e., D_j is positive). Thus

$$x_j \geq \begin{cases} y_j + \beta - 1, & \text{if } j \text{ is odd,} \\ y_j + \alpha - 1, & \text{if } j \text{ is even,} \end{cases} \tag{4.10}$$

$$\tilde{x}_j \geq \begin{cases} \tilde{y}_j + \beta - 1, & \text{if } j\text{is odd,} \\ \tilde{y}_j + \alpha - 1, & \text{if } j \text{ is even.} \end{cases} \tag{4.11}$$

For convenience, we let $\nu = (D_{2w})$ and $\sigma = (D_{2w-1})$. We use induction on w.

Suppose $w = 1$. By (4.11), (4.9), we have

$$\nu_1' = \tilde{x}_2 + 1 \geq \tilde{y}_2 + \alpha \geq \tilde{x}_1 + 1 + \alpha = \sigma_1' + \alpha. \tag{4.12}$$

On the other hand, since D_1 is negative,

$$h_{(1,\beta)}(\sigma) = \sigma_1 - \sigma_\beta' + (1 - \beta) \leq 1 - i,$$

from which we have

$$\sigma_1' \geq \sigma_\beta' \geq i - 1 + \sigma_1 + (1 - \beta) \geq i, \tag{4.13}$$

where the last inequality follows since $\sigma_1 \geq \beta$ by Remark 2.6. Also, since D_2 is positive,

$$h_{(\alpha,1)}(\nu) = \nu_\alpha - \nu_1' + (\alpha - 1) \geq K - i - 1. \tag{4.14}$$

Hence,

$$\begin{aligned} \lambda_1 = \nu_1 \geq \nu_\alpha &\geq \nu_1' + K - i - 1 + (1 - \alpha) \\ &\geq \sigma_1' + \alpha + K - i - 1 + (1 - \alpha) \\ &\geq i + \alpha + K - i - 1 + (1 - \alpha) = K, \end{aligned}$$

where the second, third, and last inequalities follow from (4.14), (4.12) and (4.13), respectively.

Suppose $w > 1$, and let $\mu = (D_{2w-2}|\cdots|D_1)$. Then by the induction hypothesis, we know that

$$\mu_1 \geq K(w - 1). \tag{4.15}$$

Note that $\sigma_1 = x_{2w-1} + 1$ and $\mu_1 = x_{2w-2} + 1$. Then, by (4.10), (4.9), we have that

$$\sigma_1 = x_{2w-1} + 1 \geq y_{2w-1} + \beta \geq x_{2w-2} + 1 + \beta = \mu_1 + \beta \geq K(w - 1) + \beta, \tag{4.16}$$

where the last inequality follows from (4.15).

We now apply the same analysis as the $w = 1$ case to $\nu\sigma = (D_{2w}|D_{2w-1})$. Then, the only difference happens in (4.13), which becomes

$$\sigma_1' \geq \sigma_\beta' \geq i - 1 + \sigma_1 + (1 - \beta) \geq K(w - 1) + i, \tag{4.17}$$

where the last inequality follows from (4.16). This change leads to

$$\lambda_1 = \nu_1 \geq \nu_\alpha \geq \nu_1' + K - i - 1 + (1 - \alpha)$$

$$\geq \sigma_1' + \alpha + K - i - 1 + (1 - \alpha)$$
$$\geq K(w-1) + i + \alpha + K - i - 1 + (1-\alpha) = Kw,$$

where the second, third, and last inequalities follow from (4.14), (4.12), and (4.17), respectively.

(b) We take the conjugate of λ, switch α and β, and then replace i by $K - i$ in (a). Then we see that $\lambda' \geq Kw$. We omit the details. □

Lemma 4.6. *Suppose that λ is a partition with the following non-neutral block decomposition:*

$$\lambda = \left(D_{2w+1} \middle| D_{2w} \middle| \cdots \middle| D_2 \middle| D_1 \right).$$

(a) If D_1 is negative, then $\lambda_1 \geq Kw + \beta$;
(b) If D_1 is positive, then $\lambda_1' \geq Kw + \alpha$.

Proof. (a) First, if $w = 0$, then $\lambda = D_1$. Since D_1 is negative, $\lambda_1 \geq \beta$ by Remark 2.6.

Suppose $w > 0$, and let $\mu = (D_{2w}|\cdots|D_2|D_1)$. By (a) in Lemma 4.5, we know that $\mu_1 \geq Kw$. Since $\lambda = (D_{2w+1}|D_{2w}|\cdots|D_2|D_1)$ is a Frobenius symbol, the entries in each row are strictly decreasing, so the first entry in the top row of D_{2w+1} is at least the first entry in the top row of D_{2w} plus the number of columns in D_{2w+1}. Since D_{2w+1} is negative, by Lemma 2.13, there must be at least β columns in D_{2w+1}. Thus

$$\lambda_1 \geq \mu_1 + \beta \geq Kw + \beta.$$

(b) We can prove this in a way similar to the proof of (a), so we omit the details. □

5. Bijections

5.1. (K, i, α, β)-Singular Overpartitions and Dotted Parity Blocks

For convenience, we introduce another representation of singular overpartitions, namely partitions with dotted parity blocks. Let λ be a (K, i, α, β)-singular overpartition. If there is exactly one overlined entry in λ, we put a dot on the top of each of the blocks between the first non-neutral block and the block of the overlined entry. If there are two overlined entries in λ, then we put a dot on the top of each block between the second non-neutral block and the block of the last overlined entry. In both cases, we remove the overlines from the entries. It is clear that

S1. there are no dotted blocks, or
S2. there are consecutive dotted blocks starting from the first non-neutral block, or
S3. there are consecutive dotted blocks starting from the second non-neutral block.

For instance, if a sequence of parity blocks is $EPNPN$, then the following are all the dotted blocks:

$EPNPN$,
$E\dot{P}NPN$, $E\dot{P}\dot{N}PN$, $E\dot{P}\dot{N}\dot{P}N$, $E\dot{P}\dot{N}\dot{P}\dot{N}$,
$EP\dot{N}PN$, $EP\dot{N}\dot{P}N$, $EP\dot{N}\dot{P}\dot{N}$.

Since there is a one-to-one correspondence between (K, i, α, β)-singular overpartitions and Frobenius symbols with a sequence of parity blocks satisfying S1, or S2, or S3, we will use the latter form in this section.

For a positive integer m, let $\dot{p}^{-}_{K,i,\alpha,\beta}(m, n)$ (or $\dot{p}^{+}_{K,i,\alpha,\beta}(m, n)$) be the number of partitions of n with exactly m dotted parity blocks and the last block negative (or positive, resp.).

Theorem 5.1. *For $m \geq 1$ and $n \geq 0$,*

$$\dot{p}^{-}_{K,i,\alpha,\beta}(m,n) = p\left(n - \left(K\left\lfloor\frac{m}{2}\right\rfloor^2 + i\left\lfloor\frac{m}{2}\right\rfloor\right)\alpha - \left(K\left\lceil\frac{m}{2}\right\rceil^2 - (K-i)\left\lceil\frac{m}{2}\right\rceil\right)\beta\right), \quad (5.1)$$

$$\dot{p}^{+}_{K,i,\alpha,\beta}(m,n) = p\left(n - \left(K\left\lceil\frac{m}{2}\right\rceil^2 - i\left\lceil\frac{m}{2}\right\rceil\right)\alpha - \left(K\left\lfloor\frac{m}{2}\right\rfloor^2 + (K-i)\left\lfloor\frac{m}{2}\right\rfloor\right)\beta\right). \quad (5.2)$$

The proof of Theorem 5.1 will be given in Sect. 5.2. When $\alpha = \beta$, Theorem 5.1 yields the following theorem.

Theorem 5.2. *For $m \geq 1$ and $n \geq 0$,*

$$\dot{p}^{-}_{K,i,\alpha,\alpha}(m,n) = p\left(n - \alpha K\binom{m}{2} - \alpha i m\right),$$

$$\dot{p}^{+}_{K,i,\alpha,\alpha}(m,n) = p\left(n - \alpha K\binom{m+1}{2} + \alpha i m\right).$$

We note that when $\alpha = 1$, Theorem 5.2 yields Theorem 3.1 in [15].

5.2. The Bijection $\psi_m^{\alpha,\beta}$

In this section, we will prove Theorem 5.1 by constructing a bijection between partitions with dotted parity blocks and ordinary partitions. We will prove only (5.1). The proof of (5.2) will be similar, so it will be omitted.

Let us denote the set of partitions of n by $\mathcal{P}(n)$. Also, let $\dot{\mathcal{P}}^{-}_{K,i,\alpha,\beta}(m, n)$ be the set of partitions of n with exactly m dotted parity blocks with the last block negative. We will construct a bijection $\psi_m^{\alpha,\beta}$ from $\dot{\mathcal{P}}^{-}_{K,i,\alpha,\beta}(m, n)$ to $\mathcal{P}(N)$, where

$$N = \begin{cases} n - (Ku^2 + iu)\alpha - \left(Ku^2 - (K-i)u\right)\beta, & \text{if } m = 2u, \\ n - (Ku^2 + iu)\alpha - \left(K(u+1)^2 - (K-i)(u+1)\right)\beta, & \text{if } m = 2u+1. \end{cases} \quad (5.3)$$

Let λ be a partition in $\dot{\mathcal{P}}^{-}_{K,i,\alpha,\beta}(m, n)$. First let D_1 be the union of the last dotted block and the blocks on the right of the last dotted block if any. From right to left, denote each of the unchosen dotted blocks by D_v for $1 < v \leq m$. Let D_{m+1} be the union of the blocks on the left of D_m if any.

Let us recall the $(5,2,2,2)$-singular overpartition from Example 2.11:

$$\lambda = \left(\begin{array}{c|cc|cc|ccc|cc} 31 & 28 & 27 & 22 & 18 & 9 & 8 & 7 & 1 & 0 \\ 29 & 26 & 25 & 23 & 22 & 8 & 5 & 4 & 1 & 0 \end{array} \right),$$

with its sequence of dotted blocks $EP\dot{N}\dot{P}\dot{N}$. Then we have

$$D_4 = \left(\begin{array}{c|cc} 31 & 28 & 27 \\ 29 & 26 & 25 \end{array} \right),\ D_3 = \begin{pmatrix} 22 & 18 \\ 23 & 22 \end{pmatrix},\ D_2 = \begin{pmatrix} 9 & 8 & 7 \\ 8 & 5 & 4 \end{pmatrix},\ D_1 = \begin{pmatrix} 1 & 0 \\ 1 & 0 \end{pmatrix}.$$

We then define $\Gamma_1, \ldots, \Gamma_{m+1}$ and $\psi_m^{\alpha,\beta}(\lambda)$ as follows:

- Set $\Gamma_1 = D_1$.
- For $1 \le v \le m$, set

$$\Gamma_{v+1} = \begin{cases} s_{-i-wK}(D_{v+1})\,(d^{\beta}_{1-i-(v-1)K}(\Gamma_v))', & \text{if } v = 2w+1 \text{ for some } w \ge 0, \\ s_{wK}(D_{v+1})\,(d^{\alpha}_{-1+i+(v-1)K}(\Gamma_v))', & \text{if } v = 2w \text{ for some } w > 0. \end{cases}$$

- Define $\psi_m^{\alpha,\beta}(\lambda) = \Gamma_{m+1}$.

Now we will inductively show that for each $1 \le v \le m$, Γ_v is a partition satisfying

$$h_{(1,\beta)}(\Gamma_v) \le 1 - i - (v-1)K, \quad \text{if } v = 2w+1, \tag{5.4}$$

$$h_{(\alpha,1)}(\Gamma_v) \ge -1 + i + (v-1)K, \quad \text{if } v = 2w. \tag{5.5}$$

First, since $\Gamma_1 = D_1$ and its first column is (K,i,α,β)-negative, Γ_1 is a partition satisfying (5.4).

Assume that for $1 \le v < m$, Γ_v is well defined and satisfies (5.4) or (5.5). We now prove that Γ_{v+1} is a partition satisfying (5.4) or (5.5). Let t be the number of columns in D_{v+1}.

Case 1: Suppose $v = 2w+1$ for some $w \ge 0$. Then we can write Γ_{v+1} as

$$\Gamma_{v+1} = s_{-i-wK}(D_{v+1})\,(d^{\beta}_{1-i-(v-1)K}(\Gamma_v))'.$$

In Lemma 4.3, set $f = 2-i$, $g = wK+1$, $h = K-i-1$, and $\nu = D_{v+1}$, $\sigma = \Gamma_v$. Clearly f,g,h satisfy $g \ge 1$, $f \le 2g-1$, $h \ge f$, since $1 \le i < K/2$.

Let us check the four conditions of Lemma 4.3. First, $(t+1, t+\beta) \in$ IV in (4.1). So, by (4.2),

$$h_{(t+1,t+\beta)}(D_{v+1}\Gamma_v) = h_{(1,\beta)}(\Gamma_v).$$

Thus, by the induction hypothesis (5.4),

$$h_{(t+1,t+\beta)}(D_{v+1}\Gamma_v) = h_{(1,\beta)}(\Gamma_v) \le 1 - i - 2wK = f - 2g + 1,$$

so Condition (i) holds true.

Let us verify Condition (ii) of the lemma. To that end, we have to consider two different regions where a node $(j, j+\beta-1)$ is placed, namely $j \le t-\beta+1$ and $j \ge t-\beta+2$, i.e., $(j, j+\beta-1) \in$ I and $(j, j+\beta-1) \in$ II in (4.1). In Fig. 11, the node a falls in the first case and the node b falls in the second case. The nodes c and d will be discussed later in Case 2.

First note that the hook difference at the node a is unchanged after D_v and Γ_{v-1} are merged to become Γ_v, namely,

$$h_{(j,j+\beta-1)}(D_{v+1}\Gamma_v) = h_{(j,j+\beta-1)}(D_{v+1}D_v\Gamma_{v-1}). \tag{5.6}$$

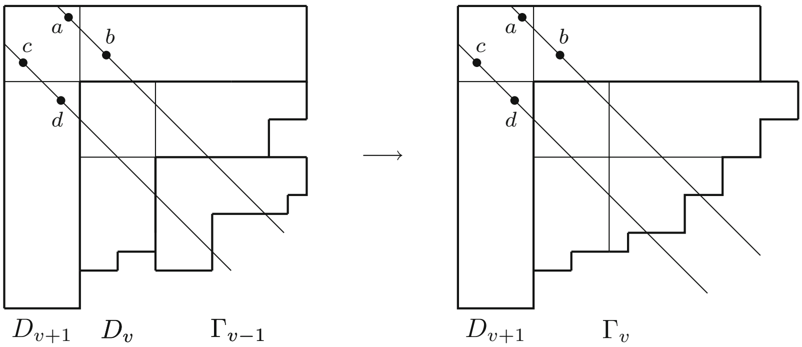


FIGURE 11. $D_{v+1}D_v\Gamma_{v-1} \to D_{v+1}\Gamma_v$

On the other hand, since

$$\Gamma_v = s_{wK}(D_v)\left(d^{\alpha}_{-1+i+(v-2)K}(\Gamma_{v-1})\right)',$$

and D_v is negative, i.e., there are at least β columns, the hook difference at the node b is affected only by the shift map s_{wK} that is applied to D_v, namely,

$$\begin{aligned} h_{(j,j+\beta-1)}(D_{v+1}\Gamma_v) &= h_{(j,j+\beta-1)}(D_{v+1}D_v\Gamma_{v-1}) - wK \\ &= h_{(j,j+\beta-1)}(D_{v+1}D_v\Gamma_{v-1}) - (g-1). \end{aligned} \tag{5.7}$$

Also, we note that D_{v+1} cannot be (K,i,α,β)-negative because the last dotted block is negative and the signs of blocks are alternating. Thus,

$$h_{(j,j+\beta-1)}(D_{v+1}D_v\Gamma_{v-1}) \geq 2-i = f. \tag{5.8}$$

Therefore, by (5.6), (5.7), and (5.8),

$$h_{(j,j+\beta-1)}(D_{v+1}\Gamma_v) \geq \begin{cases} f, & \text{for } 1 \leq j \leq t-\beta+1, \\ f-g+1, & \text{for } t-\beta+2 \leq j \leq t, \end{cases}$$

which verifies that Condition (ii) holds true.

Lastly, let $\begin{matrix} x_1 \\ x_2 \end{matrix}$ and $\begin{matrix} z_1 \\ z_2 \end{matrix}$ be the last column of D_{v+1} and the first column of D_v, respectively, i.e.,

$$(D_{v+1}|D_v) = \left(\begin{array}{cc|cc} \cdots & x_1 & z_1 & \cdots \\ \cdots & x_2 & z_2 & \cdots \end{array}\right).$$

Since $D_{v+1}D_v$ forms a Frobenius symbol, we have

$$x_1 > z_1, \quad x_2 > z_2.$$

By Lemma 4.6 (a), we know that $z_1 \geq wK + \beta - 1$.

Since

$$\Gamma_v = \begin{cases} D_1, & \text{for } v = 1, \\ s_{wK}(D_v)(d^{\alpha}_{-1+i+(v-2)K}(\Gamma_{v-1}))', & \text{for } v > 2, \end{cases}$$

the first column of Γ_v is $\begin{matrix} z_1 - wK \\ z_2 + wK \end{matrix}$. Thus, we have

$$\begin{aligned} &x_1 > (z_1 - wK) + (wK + 1) - 1 \text{ and } z_1 - wK \geq \beta - 1, \\ &x_2 > (z_2 + wK) - (wK + 1) + 1 \text{ and } x_2 > z_2 \geq 0, \end{aligned}$$

which verify Conditions (iii) and (iv). Since all the four conditions in Lemma 4.3 are satisfied, by Statement (a) of Lemma 4.3, Γ_{v+1} is a partition. Also, since $v < m$, $D_{v+1} \neq \emptyset$ is indeed a positive block, so

$$h_{(\alpha,1)}(D_{v+1}) \geq K - i - 1 = h.$$

Thus, by Statement (c) of Lemma 4.3,

$$h_{(\alpha,1)}(\Gamma_{v+1}) \geq h - 2f + 2g + 2 = -1 + i + (2w + 1)K,$$

which verifies (5.5).
Case 2: Suppose $v = 2w$ for some $w \geq 1$. Then we can write Γ_{v+1} as

$$\Gamma_{v+1} = s_{wK}(D_{v+1})\,(d^{\alpha}_{-1+i+(v-1)K}(\Gamma_v))'.$$

In Lemma 4.4, set $f = K - i - 2$, $g = wK - 1$, $h = 1 - i$, and $\nu = D_{v+1}$, $\sigma = \Gamma_v$. Clearly, f, g and h satisfy $g - f \geq 1$, $f \leq 2g - 1$, $h \leq f$.

Next, let us verify the four conditions of Lemma 4.4. First, note that $(t + \alpha, t + 1) \in \mathrm{IV}$ in (4.1). So, by (4.2), we know that

$$h_{(t+\alpha,t+1)}(D_{v+1}\Gamma_v) = h_{(\alpha,1)}(\Gamma_v).$$

Thus, by the induction hypothesis (5.5),

$$\begin{aligned} h_{(t+\alpha,t+1)}(D_{v+1}\Gamma_v) = h_{(\alpha,1)}(\Gamma_v) &\geq -1 + i + (2w - 1)K \\ &= -f + 2g - 1, \end{aligned}$$

so Condition (i) holds true.

For Condition (ii), we have to consider two different regions where a node $(j + \alpha - 1, j)$ is placed, namely $j \leq t - \alpha + 1$ and $j \geq t - \alpha + 2$, i.e., $(j + \alpha - 1, j) \in \mathrm{I}$ and $(j + \alpha - 1, j) \in \mathrm{III}$ in (4.1). In Fig. 11, the node c falls in the first case and the node d falls in the second case. First note that the hook difference at the node c is unchanged after D_v and Γ_{v-1} are merged to become Γ_v, namely,

$$h_{(j+\alpha-1,j)}(D_{v+1}\Gamma_v) = h_{(j+\alpha-1,j)}(D_{v+1}D_v\Gamma_{v-1}). \tag{5.9}$$

On the other hand, since

$$\Gamma_v = s_{-i-(w-1)K}(D_v)\,(d^{\beta}_{1-i-(v-2)K}(\Gamma_{v-1}))',$$

and D_v is positive, i.e., there are at least α columns, the hook difference at the node d is affected only by the shift map $s_{-i-(w-1)K}$ that is applied to D_v, namely,

$$\begin{aligned} h_{(j+\alpha-1,j)}(D_{v+1}\Gamma_v) &= h_{(j+\alpha-1,j)}(D_{v+1}D_v\Gamma_{v-1}) + i + (w - 1)K \\ &= h_{(j+\alpha-1,j)}(D_{v+1}D_v\Gamma_{v-1}) - f + g - 1. \end{aligned} \tag{5.10}$$

Also, we note that D_{v+1} cannot be (K, i, α, β)-positive because the last dotted block is negative and the signs of blocks are alternating. Thus,

$$h_{(j+\alpha-1,j)}(D_{v+1}D_v\Gamma_{v-1}) \le K - i - 2 = f. \tag{5.11}$$

Therefore, by (5.9), (5.10), and (5.11),

$$h_{(j+\alpha-1,j)}(D_{v+1}\Gamma_v) \le \begin{cases} f, & \text{for } 1 \le j \le t-\alpha+1, \\ g-1, & \text{for } t-\alpha+2 \le j \le t, \end{cases}$$

which verifies that Condition (ii) holds true.

Lastly, $D_{v+1}D_v$ forms a Frobenius symbol. Thus, in the same way as in Case 1, Conditions (iii) and (iv) in Lemma 4.4 can be verified. We omit the details. Therefore, by Statements (a) and (c) of Lemma 4.4, Γ_{v+1} is a partition satisfying (5.4).

We now have that Γ_m is a partition satisfying (5.4) or (5.5) from the induction. Also, D_{m+1} is a partition. We can easily check that D_{m+1} and Γ_m satisfy the conditions for Lemmas 4.3 or 4.4. Therefore, Γ_{m+1} is a partition by Statement (a) of each lemma. Here we note that all the arguments for $v < m$ hold for $v = m$ except that if the first column of D_{m+1} is neutral, then Statement (c) does not hold. However, Statement (c) is not needed to complete our proof, for what we need to prove is that Γ_{m+1} is a partition.

Let us then check the weight difference. By Statement (e) of each of Lemmas 4.3 and 4.4, we have

$$|D_{v+1}\Gamma_v| - |\Gamma_{v+1}| = \begin{cases} (i+(v-1)K)\,\beta, & \text{if } v = 2w+1, \\ (i+(v-1)K)\,\alpha, & \text{if } v = 2w, \end{cases} \tag{5.12}$$

for $v = 1, \dots, m$. By (5.12), we have

$$\begin{aligned}
|\lambda| &= |D_{m+1}D_m \cdots D_5D_4D_3D_2D_1| \\
&= |D_{m+1}D_m \cdots D_5D_4D_3D_2\Gamma_1| \\
&= |D_{m+1}D_m \cdots D_5D_4D_3\Gamma_2| + i\beta \\
&= |D_{m+1}D_m \cdots D_5D_4\Gamma_3| + (i+K)\alpha + i\beta \\
&= |D_{m+1}D_m \cdots D_5\Gamma_4| + (i+2K)\beta + (i+K)\alpha + i\beta \\
&\ \ \vdots \\
&= |\Gamma_{m+1}| \\
&\quad + \begin{cases} \sum_{v=1}^{u} \Big(\big(i+(2v-1)K\big)\alpha + \big(i+(2v-2)K\big)\beta \Big), & \text{if } m = 2u, \\ (i+2uK)\beta + \sum_{v=1}^{u} \Big(\big(i+(2v-1)K\big)\alpha + \big(i+(2v-2)K\big)\beta \Big), & \text{if } m = 2u+1. \end{cases}
\end{aligned}$$

Thus

$$|\Gamma_{m+1}| = \begin{cases} |\lambda| - (Ku^2+iu)\alpha - (Ku^2-Ku+iu)\beta, & \text{if } m = 2u, \\ |\lambda| - (Ku^2+iu)\alpha - (Ku^2+Ku+iu+i)\beta, & \text{if } m = 2u+1. \end{cases}$$

This shows that $\psi_m^{\alpha,\beta}$ is a map from $\dot{\mathcal{P}}^-_{K,i,\alpha,\beta}(m,n)$ to $\mathcal{P}(N)$, where N is given in (5.3) as desired.

In addition, by Statement (d) of Lemma 4.3 and Lemma 4.4, each process of producing Γ_{v+1} is reversible. Therefore, $\psi_m^{\alpha,\beta}$ is indeed a bijection.

Example 5.3. Consider a $(5,2,2,2)$-singular overpartition

$$\lambda = \left(\begin{array}{c|cc|cc|ccc|cc} 31 & 28 & 27 & 22 & 18 & 9 & 8 & 7 & 1 & 0 \\ 29 & 26 & 25 & 23 & 22 & 8 & 5 & 4 & 1 & 0 \end{array}\right),$$

with its sequence of dotted blocks $EP\dot{N}\dot{P}\dot{N}$. Note that $K=5$, $i=\alpha=\beta=2$, and $m=3$. We have the following Γ_v for $v=1,2,3,4$:

- $\Gamma_1 = D_1 = \begin{pmatrix} 1 & 0 \\ 1 & 0 \end{pmatrix}$,
- $\Gamma_2 = s_{-2}(D_2)\ \left(d_{-1}^2(\Gamma_1)\right)' = \left(\begin{array}{ccc|} 11 & 10 & 9 \\ 6 & 3 & 2 \end{array}\right)$,
- $\Gamma_3 = s_5(D_3)\ \left(d_6^2(\Gamma_2)\right)' = \left(\begin{array}{cc|ccc} 17 & 13 & 4 & 3 & 2 \\ 28 & 27 & 13 & 3 & 2 \end{array}\right)$,
- $\Gamma_4 = s_{-7}(D_4)\ \left(d_{-11}^2(\Gamma_3)\right)' = \left(\begin{array}{ccc|ccccc} 38 & 35 & 34 & 17 & 7 & 6 & 3 & 2 \\ 22 & 19 & 18 & 16 & 15 & 13 & 9 & 0 \end{array}\right)$,

where the dashed line is put to indicate the concatenated two arrays in each Γ_v. Here Γ_4 is the ordinary partition corresponding to the $(5,2,2,2)$-singular overpartition λ. Lastly, we check their weight difference

$$|\lambda| - |\Gamma_4| = 304 - 262 = 42,$$

which matches $(Ku^2+iu)\alpha + (Ku^2+Ku+iu+i)\beta$ as desired.

6. Results

In this section, we will relate (K,i,α,β)-singular overpartitions with ordinary partitions.

For a (K,i,α,β)-singular overpartition with overlined entries in anchors, if the first overlined entry occurs in the mth block, then the mth block can be negative or positive, and the next anchor can have an overlined entry if exists. In all these four cases, i.e., only one overlined entry in either a negative or a positive block, or two overlined entries in two consecutive and opposite parity blocks, we see from the definition of dotted parity blocks given in the beginning of Sect. 5.1 that such singular overpartitions are partitions with exactly m dotted parity blocks. Thus,

$$\overline{Q}_{K,i,\alpha,\beta}(m,n) = \dot{p}^{-}_{K,i,\alpha,\beta}(m,n) + \dot{p}^{+}_{K,i,\alpha,\beta}(m,n), \tag{6.1}$$

where $\overline{Q}_{K,i,\alpha,\beta}(m,n)$ is the number of (K,i,α,β)-singular overpartitions of n with an overlined entry in its mth anchor, which is defined before Theorem 1.2 in Introduction.

Theorem 6.1. *For $m \geq 1$ and $n \geq 0$,*

$$\begin{aligned}&\overline{Q}_{K,i,\alpha,\beta}(m,n)\\&\quad = p\left(n - \left(K\left\lceil\frac{m}{2}\right\rceil^2 - i\left\lceil\frac{m}{2}\right\rceil\right)\alpha - \left(K\left\lfloor\frac{m}{2}\right\rfloor^2 + (K-i)\left\lfloor\frac{m}{2}\right\rfloor\right)\beta\right)\\&\qquad + p\left(n - \left(K\left\lfloor\frac{m}{2}\right\rfloor^2 + i\left\lfloor\frac{m}{2}\right\rfloor\right)\alpha - \left(K\left\lceil\frac{m}{2}\right\rceil^2 - (K-i)\left\lceil\frac{m}{2}\right\rceil\right)\beta\right),\end{aligned}$$

where $p(N)$ denotes the number of ordinary partitions of N with $p(0) = 1$ and $p(N) = 0$ for $N < 0$.

Proof. This theorem follows from (6.1) and Theorem 5.1. □

Remark 6.2. Some facts about $\overline{Q}_{K,i,\alpha,\beta}(m,n)$ are noted below.

(i) Since $\left\lfloor\frac{m}{2}\right\rfloor = -\left\lceil\frac{-m}{2}\right\rceil$ and $\left\lceil\frac{m}{2}\right\rceil = -\left\lfloor\frac{-m}{2}\right\rfloor$, we have

$$\begin{aligned}&\overline{Q}_{K,i,\alpha,\beta}(m,n)\\&\quad = p\left(n - \left(K\left\lceil\frac{m}{2}\right\rceil^2 - i\left\lceil\frac{m}{2}\right\rceil\right)\alpha - \left(K\left\lfloor\frac{m}{2}\right\rfloor^2 + (K-i)\left\lfloor\frac{m}{2}\right\rfloor\right)\beta\right)\\&\qquad + p\left(n - \left(K\left\lceil\frac{-m}{2}\right\rceil^2 - i\left\lceil\frac{-m}{2}\right\rceil\right)\alpha - \left(K\left\lfloor\frac{-m}{2}\right\rfloor^2 + (K-i)\left\lfloor\frac{-m}{2}\right\rfloor\right)\beta\right)\end{aligned}$$

for $m \geq 1$ and $n \geq 0$.

(ii) Each ordinary partition of n can be regarded as a (K,i,α,β)-singular overpartition without any overlined entries. Thus

$$\overline{Q}_{K,i,\alpha,\beta}(0,n) = p(n).$$

Let us recall $\overline{Q}_{K,i,\alpha,\beta}(n)$:

$$\overline{Q}_{K,i,\alpha,\beta}(n) = \sum_{m=0}^{\infty} \overline{Q}_{K,i,\alpha,\beta}(m,n). \tag{6.2}$$

Theorem 6.3. *We have*

$$\begin{aligned}&\sum_{n=0}^{\infty} \overline{Q}_{K,i,\alpha,\beta}(n)q^n\\&\quad = \frac{(q^{2K(\alpha+\beta)}, -q^{(K+i)\alpha+i\beta}, -q^{(K-i)\alpha+(2K-i)\beta}; q^{2K(\alpha+\beta)})_\infty}{(q;q)_\infty}\\&\qquad + q^{i\beta}\frac{(q^{2K(\alpha+\beta)}, -q^{(K-i)\alpha-i\beta}, -q^{(K+i)\alpha+(2K+i)\beta}; q^{2K(\alpha+\beta)})_\infty}{(q;q)_\infty},\end{aligned}$$

where $(a_1, a_2, \ldots, a_M; q)_\infty = (a_1;q)_\infty(a_2;q)_\infty \cdots (a_M;q)_\infty$.

Proof. By (6.2), Theorem 6.1, and Remark 6.2,

$$\begin{aligned}&\overline{Q}_{K,i,\alpha,\beta}(n)\\&\quad = \sum_{m=-\infty}^{\infty} p\left(n - \left(K\left\lfloor\frac{m}{2}\right\rfloor^2 + i\left\lfloor\frac{m}{2}\right\rfloor\right)\alpha - \left(K\left\lceil\frac{m}{2}\right\rceil^2 - (K-i)\left\lceil\frac{m}{2}\right\rceil\right)\beta\right)\end{aligned}$$

$$= \sum_{u=-\infty}^{\infty} p\left(n-(Ku^2+iu)\alpha-(Ku^2-Ku+iu)\beta\right) \qquad (m=2u)$$

$$+ \sum_{u=-\infty}^{\infty} p\left(n-(Ku^2+iu)\alpha-(Ku^2+Ku+iu+i)\beta\right) \quad (m=2u+1).$$

Thus,

$$\begin{aligned}
&\sum_{n=0}^{\infty}\overline{Q}_{K,i,\alpha,\beta}(n)q^n \\
&\quad=\sum_{n=0}^{\infty}\sum_{u=-\infty}^{\infty} p\left(n-(Ku^2+iu)\alpha-(Ku^2-Ku+iu)\beta\right)q^n \\
&\qquad+\sum_{n=0}^{\infty}\sum_{u=-\infty}^{\infty} p\left(n-(Ku^2+iu)\alpha-(Ku^2+Ku+iu+i)\beta\right)q^n \\
&\quad=\frac{1}{(q;q)_\infty}\sum_{u=-\infty}^{\infty} q^{(Ku^2+iu)\alpha+(Ku^2-Ku+iu)\beta} \\
&\qquad+\frac{1}{(q;q)_\infty}\sum_{u=-\infty}^{\infty} q^{(Ku^2+iu)\alpha+(Ku^2+Ku+iu+i)\beta} \\
&\quad=\frac{(q^{2K(\alpha+\beta)},-q^{(K+i)\alpha+i\beta},-q^{(K-i)\alpha+(2K-i)\beta};q^{2K(\alpha+\beta)})_\infty}{(q;q)_\infty} \\
&\qquad+q^{i\beta}\frac{(q^{2K(\alpha+\beta)},-q^{(K+i)\alpha+(2K+i)\beta},-q^{(K-i)\alpha-i\beta};q^{2K(\alpha+\beta)})_\infty}{(q;q)_\infty},
\end{aligned}$$

where the last equality follows from Jacobi's triple product identity [3,17]. □

6.1. Proof of Theorem 1.1

When $\alpha=\beta$, Theorem 6.3 can be simplified further. By Theorem 5.2, we have

$$\overline{Q}_{K,i,\alpha,\alpha}(n)=\sum_{m=-\infty}^{\infty} p\left(n-\alpha K\binom{m}{2}-\alpha i m\right),$$

where we use the fact $\binom{m+1}{2}=\binom{-m}{2}$. Thus,

$$\begin{aligned}
&\sum_{n=0}^{\infty}\overline{Q}_{K,i,\alpha,\alpha}(n)q^n \\
&\quad=\sum_{n=0}^{\infty}\sum_{m=-\infty}^{\infty} p\left(n-\alpha K\binom{m}{2}-\alpha i m\right)q^n \\
&\quad=\sum_{m=-\infty}^{\infty} q^{\alpha K\binom{m}{2}+\alpha i m}\sum_{n=0}^{\infty} p\left(n-\alpha K\binom{m}{2}-\alpha i m\right)q^{n-\alpha K\binom{m}{2}-\alpha i m} \\
&\quad=\sum_{m=-\infty}^{\infty} q^{\alpha K\binom{m}{2}+\alpha i m}\sum_{n=0}^{\infty} p(n)q^n
\end{aligned}$$

$$= \frac{1}{(q;q)_\infty} \sum_{m=-\infty}^{\infty} q^{\alpha K\binom{m}{2}+\alpha i m}$$
$$= \frac{(-q^{i\alpha}, -q^{(K-i)\alpha}, q^{K\alpha}; q^{K\alpha})_\infty}{(q;q)_\infty}, \tag{6.3}$$

where the last equality follows from Jacobi's triple product identity [3,17]. This proves Theorem 1.1.

We easily see that the right hand side of (6.3) is the generating function of overpartitions in which parts $\not\equiv 0 \bmod K\alpha$ and only parts $\equiv \pm i\alpha \bmod K\alpha$ may be overlined.

6.2. Proof of Theorem 1.2

Theorem 1.2 follows immediately from (6.1) and Theorem 5.2.

7. Remarks

We provide a few remarks. First, for a positive integer $k > 1$, the case when $i = k$ and $K = 2k$ is investigated by Bressoud in [8]. In [16], when $K = 3, i = \alpha = \beta = 1$, further refined cases were studied.

Finally, let $p^E_{K,i,\alpha,\beta}(n)$ be the number of partitions of n without any signed blocks. Since $\overline{Q}_{K,i,\alpha,\beta}(m,n)$ counts the number of singular overpartitions of n with an overlined entry in the mth anchor, it is the same as the number of partitions of n with at least m signed blocks. By the sieving method, we have

$$p^E_{K,i,\alpha,\beta}(n) = p(n) + \sum_{m\geq 1} (-1)^m \overline{Q}_{K,i,\alpha,\beta}(m,n),$$

from which we can deduce Theorem 2 in [6].

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Seunghyun Seo
Department of Mathematics Education
Kangwon National University
Chuncheon 24341 Gangwon
Republic of Korea
e-mail: shyunseo@kangwon.ac.kr

Ae Ja Yee
Department of Mathematics
The Pennsylvania State University
University Park
PA 16802
USA
e-mail: auy2@psu.edu;
yee@psu.edu

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Annals of Combinatorics

The Combinatorics of MacMahon's Partial Fractions

Dedicated to my teacher, George E. Andrews, on the occasion of his 80th birthday

Andrew V. Sills

Abstract. MacMahon showed that the generating function for partitions into at most k parts can be decomposed into a partial fraction-type sum indexed by the partitions of k. In the present work, a generalization of MacMahon's result is given, which in turn provides a full combinatorial explanation.

Mathematics Subject Classification. Primary 05A17.

Keywords. Partitions, Partition function, Compositions, Symmetric group.

1. Introduction

1.1. An Excerpt from MacMahon's *Combinatory Analysis*

In *Combinatory Analysis*, vol. 2 [7, p. 61ff], P.A. MacMahon writes:

> We commence by observing the two identities
>
> $$\frac{1}{(1-x)(1-x^2)} = \frac{1}{2}\frac{1}{(1-x)^2} + \frac{1}{2}\frac{1}{(1-x^2)},$$
> $$\frac{1}{(1-x)(1-x^2)(1-x^3)} = \frac{1}{6}\frac{1}{(1-x)^3} + \frac{1}{2}\frac{1}{(1-x)(1-x^2)} + \frac{1}{3}\frac{1}{1-x^3},$$
>
> which we will also write in the illuminating notation so often employed:
>
> $$\frac{1}{(\mathbf{1})(\mathbf{2})} = \frac{1}{2}\frac{1}{(\mathbf{1})^{\mathbf{2}}} + \frac{1}{2}\frac{1}{(\mathbf{2})},$$

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$$\frac{1}{(\mathbf{1})(\mathbf{2})(\mathbf{3})} = \frac{1}{6}\frac{1}{(\mathbf{1})^{\mathbf{3}}} + \frac{1}{2}\frac{1}{(\mathbf{1})(\mathbf{2})} + \frac{1}{3}\frac{1}{(\mathbf{3})}.$$

[Elsewhere [7, p. 5] the "very convenient notation" $(\mathbf{j}) = 1 - x^j$ is defined and attributed to Cayley.]

The observation leads to the conjecture that we are in the presence of partial fractions of a new and special kind. We note that in the first identity we have a fraction corresponding to each of the partitions (1^2), (2) of the number 2 and in the second fractions corresponding to and derived from each of the partitions (1^3), (21), (3) of the number 3. ... [In general] we find

$$\frac{1}{(\mathbf{1})(\mathbf{2})\cdots(\mathbf{i})} = \sum \frac{1}{1^{p_1}.2^{p_2}.3^{p_3}\dots p_1!p_2!p_3!\dots} \frac{1}{(\mathbf{1})^{p_1}(\mathbf{2})^{p_2}(\mathbf{3})^{p_3}\dots}, \tag{1.1}$$

where $(1^{p_1}2^{p_2}3^{p_3}\dots)$ is a partition of i and the summation is in regard to all partitions of i. This remarkable result shows the decomposition of the generating function into as many fractions as the number i possesses partitions. The denominator of each fraction is directly derived from one of the partitions and is of degree i in x. The numerator does not involve x and the coefficient is the easily calculable number

$$\frac{1}{1^{p_1}.2^{p_2}.3^{p_3}\dots p_1!p_2!p_3!\dots}.$$

Remark 1.1. In [3, p. 209, Ex. 1] Andrews attributes (1.1) to Cayley, and this attribution has been repeated by other authors in the literature. However, the author has been unable to find (1.1) anywhere in Cayley's works [5], and indeed MacMahon, in the chapter where he presents his "partial fractions of a new and special kind" [7, Sect. VII, Chapter V] contrasts his results with those of Cayley several times.

1.2. Some Definitions and Notation

1.2.1. Partitions and Related Objects. It will be necessary to employ partitions and compositions of positive integers, sometimes allowing 0's as parts, and sometimes not. Accordingly, we will formalize terminology via the following definitions.

Definition 1.2. A *weak k-composition* γ is a k-tuple of nonnegative integers $(\gamma_1, \gamma_2, \dots, \gamma_k)$. Each γ_i (even if $\gamma_i = 0$) is called a *part* of γ. The *weight* of γ, denoted $|\gamma|$, is $\sum_{i=1}^m \gamma_i$. The *length* of γ, denoted $l(\gamma)$, is the number of parts in γ. The *frequency* (or *multiplicity*) of part j in γ, denoted $f_j(\gamma)$ or simply f_j when γ is clear from context, is the number of times that j appears as a part in γ:

$$f_j(\gamma) := \#\{i\colon \gamma_i = j\}.$$

The *frequency sequence* associated with γ is

$$f(\gamma) := (f_0(\gamma), f_1(\gamma), f_2(\gamma), f_3(\gamma), \dots).$$

The set of all weak k-compositions will be denoted $\mathscr{C}_k$.

Remark 1.3. The author prefers to use the term *frequency* and the corresponding notation f_j over the term *multiplicity* (with the notation m_j) to be consistent with the works of Andrews [1–3].

Definition 1.4. A weak k-composition $w = (w_1, w_2, \dots, w_k)$ is called a *weak k-partition* if its parts occur in nonincreasing order

$$w_1 \geq w_2 \geq \cdots \geq w_k.$$

The set of weak k-partitions of weight n will be denoted $\mathscr{W}_k(n)$ and the cardinality of this set by $p_k(n)$.

Definition 1.5. If w is a weak k-partition and γ is a weak k-composition, we shall say that γ *is of type* w if γ is a permutation of w.

Definition 1.6. A *partition* λ is any nonincreasing finite or infinite sequence $(\lambda_1, \lambda_2, \lambda_3, \dots)$ of nonnegative integers. However, in contrast to Definition 1.2, only positive integers are considered parts, thus for a partition λ, $l(\lambda) = \#\{i : \lambda_i > 0\}$. Analogous to the frequency sequence of a weak composition, the *frequency sequence* $f(\lambda)$ of λ is

$$f(\lambda) = (f_1(\lambda), f_2(\lambda), f_3(\lambda), \dots).$$

Remark 1.7. In fact, no distinction will be drawn between, e.g., $\lambda = (5, 2, 1, 1)$ and $\lambda = (5, 2, 1, 1, 0, 0, 0, 0, 0, 0, 0, 0, \dots)$; both will be considered the same partition of length 4 and weight 9. Also, $f\big((5, 2, 1, 1)\big) = (2, 1, 0, 0, 1, 0, 0, \dots)$.

It will be convenient to consider $\lambda_j = 0$ for any $j > l(\lambda)$, even when λ is not explicitly constructed with a tail of zeros.

Definition 1.8. The set of all partitions of weight n is denoted by $\mathscr{P}(n)$ and the cardinality of $\mathscr{P}(n)$ by $p(n)$. The notation $\lambda \vdash n$ means "λ is a partition of weight n", i.e., $\lambda \in \mathscr{P}(n)$.

For example,

$$\mathscr{P}(4) = \{(4), (3, 1), (2, 2), (2, 1, 1), (1, 1, 1, 1)\},$$

so $p(4) = 5$.

Definition 1.9. For a partition λ, the partition $\lambda - \mathbf{1}$ is the partition obtained from λ by decreasing each of its parts by 1:

$$\lambda - \mathbf{1} := (\lambda_1 - 1, \lambda_2 - 1, \dots, \lambda_{l(\lambda)} - 1).$$

Notice that $l(\lambda - \mathbf{1}) = l(\lambda) - f_1(\lambda)$.

It is often convenient to denote a partition (respectively, weak k-partition) by the *superscript frequency notation* $\langle 1^{f_1} 2^{f_2} 3^{f_3} \cdots \rangle$ (respectively, $\langle 0^{f_0} 1^{f_1} 2^{f_2} 3^{f_3} \cdots \rangle$) where it is permissible to omit f_j if $f_j = 1$ and to omit j^{f_j} if $f_j = 0$. Thus,

$$(5, 5, 5, 5, 3, 2, 2, 1, 1, 1) = \langle 1^3 2^2 3\ 5^4 \rangle$$

are two ways of expressing that particular (weak 10-)partition of 30.

A variant on this notation for weak compositions (to emphasize runs of adjacent equal parts) will also be useful. For example, let us allow ourselves to write the weak 9-composition $(3, 3, 2, 0, 0, 3, 3, 3, 3)$ of 20 as $[3^2 2^1 0^2 3^4]$.

The following quantities will arise often enough to warrant these definitions:

Definition 1.10. Following Schneider [11], for a partition $\lambda = (\lambda_1, \lambda_2, \dots, \lambda_k)$, we define its *norm* to be the product of its parts,

$$N(\lambda) := \lambda_1 \lambda_2 \cdots \lambda_k.$$

Further, the *factorial* of a partition λ is $\lambda! = (\lambda_1!, \lambda_2!, \dots, \lambda_k!)$, so that $N(\lambda!) = \lambda_1!\lambda_2! \cdots \lambda_k!$. Analogously for a weak k-composition $\gamma = (\gamma_1, \dots, \gamma_k)$,

$$N(\gamma!) := \gamma_1!\gamma_2! \dots \gamma_k!$$

Observe that f effectively maps a partition $\lambda = (\lambda_1, \lambda_2, \dots)$ to a weak λ_1-composition of weight $l(\lambda)$ if we ignore the infinite tail of zeros in $f(\lambda)$. For example, $f((5, 5, 5, 5, 3, 2, 2, 1, 1, 1)) = (3, 2, 1, 0, 4)$, a weak 5-composition of weight 10. Likewise, f maps a weak k-composition γ to a weak L-composition of weight k, where L is the largest part of γ. Thus, the *frequency factorial product* of a weak k-composition γ may be consistently notated as

$$N\big(f(\gamma)!\big) = f_0(\gamma)!f_1(\gamma)!f_2(\gamma)! \cdots ,$$

and that of a partition λ as

$$N\big(f(\lambda)!\big) = f_1(\lambda)!f_2(\lambda)!f_3(\lambda)! \cdots .$$

Observation 1.11. *The number of weak k-compositions of type $w = (w_1, \dots, w_k)$, where w is a weak k-partition, is $k!/N\big(f(w)!\big)$.*

Definition 1.12. A *multipartition* is a t-tuple of partitions for some t.

For example, $((1, 1, 1), (4, 1), (3, 2, 2))$ is a multipartition simply because $(1, 1, 1)$, $(4, 1)$, and $(3, 2, 2)$ are all partitions.

Definition 1.13. The *multipartition dissection $MD(\lambda)$* of the partition λ is the following Cartesian product:

$$MD(\lambda) := \mathscr{P}(\lambda_1) \times \mathscr{P}(\lambda_2) \times \cdots \times \mathscr{P}(\lambda_{l(\lambda)}).$$

We will require the result given by Fine [6, p. 38, Eq. (22.2)],

$$\sum_{\lambda \vdash n} \frac{1}{1^{f_1} 2^{f_2} 3^{f_3} \cdots f_1!f_2!f_3! \cdots} = 1, \tag{1.2}$$

which may be expressed in the present notation as

$$\sum_{\lambda \vdash n} \frac{1}{N(\lambda)\, N\big(f(\lambda)!\big)} = 1, \tag{1.3}$$

in the iterated form

$$\sum_{i=1}^{p(\lambda_1)p(\lambda_2)\cdots p(\lambda_{l(\lambda)})} \prod_{j=1}^{l(\lambda)} \frac{1}{N\big(f(\mu^{ij})!\big) N(\mu^{ij})} = 1. \tag{1.4}$$

The superscript notation on μ is to be understood as follows: if $\lambda = (\lambda_1, \ldots, \lambda_l)$ is a partition, then μ^{ij} is the ith partition of λ_j where the $p(\lambda_1)p(\lambda_2)\cdots p(\lambda_l)$ multipartitions of $MD\,(\lambda)$ have been placed in *some* order; any order is fine. See also (3.3) below for an explicit illustration.

Notice that (1.2) states that the sum of the coefficients that appear in the MacMahon partial fraction decomposition

$$\frac{1}{(1-x)(1-x^2)\cdots(1-x^k)} = \sum_{\lambda\vdash k} \frac{1}{1^{f_1}2^{f_2}3^{f_3}\cdots f_1!f_2!f_3!\cdots} g(\lambda; x, x, \ldots, x),$$

where g is defined below in Eq. (1.5), must be 1.

1.2.2. Combinatorial Generating Functions. As part of the combinatorial construction to be undertaken, we will need to associate with each partition λ a certain rational generating function in the indeterminates $x_1, x_2, \ldots, x_{|\lambda|}$, namely let

$$g(\lambda; \mathbf{x}) := g(\lambda; x_1, x_2, \ldots, x_{|\lambda|}) := \prod_{j=1}^{l(\lambda)} \frac{1}{1 - \prod_{k=1}^{\lambda_j} x_{s(\lambda;j,k)}}, \tag{1.5}$$

with

$$s(\lambda; j, k) = k + \sum_{r=1}^{j-1} \lambda_r.$$

Of necessity, the notation used in defining (1.5) for a general partition λ makes a simple idea rather opaque. To understand immediately how to construct $g(\lambda; \mathbf{x})$ for any partition λ, simply consider, for example, for the five partitions of 4, we have the following associated "g-functions":

$$\begin{aligned}
g\big((4); x_1, x_2, x_3, x_4\big) &= \frac{1}{1 - x_1x_2x_3x_4},\\
g\big((3,1); x_1, x_2, x_3, x_4\big) &= \frac{1}{(1 - x_1x_2x_3)(1-x_4)},\\
g\big((2,2); x_1, x_2, x_3, x_4\big) &= \frac{1}{(1 - x_1x_2)(1-x_3x_4)},\\
g\big((2,1,1); x_1, x_2, x_3, x_4\big) &= \frac{1}{(1 - x_1x_2)(1-x_3)(1-x_4)},\\
g\big((1,1,1,1); x_1, x_2, x_3, x_4\big) &= \frac{1}{(1 - x_1)(1-x_2)(1-x_3)(1-x_4)}.
\end{aligned}$$

We denote the symmetric group of degree n by $\mathfrak{S}_n$. The application of a permutation $\sigma \in \mathfrak{S}_{|\lambda|}$ to $g(\lambda; \mathbf{x})$, will be written as $\sigma g(\lambda; \mathbf{x})$, with the intended meaning

$$\sigma g(\lambda; \mathbf{x}) = g(\lambda; \sigma\mathbf{x}) = g\big(\lambda; x_{\sigma(1)}, x_{\sigma(2)}, \ldots, x_{\sigma(|\lambda|)}\big).$$

Let $\mathscr{O}_\lambda$ (respectively, H_λ) denote the orbit (resp. stabilizer) of $g(\lambda; \mathbf{x})$ under the action of $\mathfrak{S}_{|\lambda|}$.

Thus,

$$|H_\lambda| = N\,(\lambda!)\,N\big(f(\lambda)!\big), \tag{1.6}$$

or, by the orbit–stabilizer theorem,

$$|\mathscr{O}_\lambda| = \frac{|\lambda|!}{N(\lambda!)\,N\big(f(\lambda)!\big)}. \tag{1.7}$$

Remark 1.14. Notice that

$$N((\lambda - \mathbf{1})!)\,|\mathscr{O}_\lambda| = \frac{|\lambda|!}{N(\lambda)\,N\big(f(\lambda)!\big)},$$

which is $|\lambda|!$ times the coefficient of the term indexed by λ in the MacMahon decomposition.

1.3. Statement of Main Result

The goal is to understand (1.1) combinatorially. This will be accomplished by proving the following natural multivariate generalization of (1.1):

Theorem 1.15.

$$\sum_{\sigma\in\mathfrak{S}_k} \sigma \frac{1}{(1-x_1)(1-x_1x_2)(1-x_1x_2x_3)\cdots(1-x_1x_2\cdots x_k)}$$
$$= \sum_{\lambda\vdash k} N((\lambda-\mathbf{1})!) \sum_{\phi(\mathbf{x})\in\mathscr{O}_\lambda} \phi(\mathbf{x}), \tag{1.8}$$

where $\mathscr{O}_\lambda$ is the orbit of $g(\lambda;\mathbf{x})$ under the action of $\mathfrak{S}_{|\lambda|}$, and $g(\lambda;\mathbf{x})$ is defined in (1.5).

2. Partial Fraction Decompositions

It is well known that for a fixed positive integer k, the generating function for $p_k(n)$ is

$$F_k(x) := \sum_{n\geq 0} p_k(n)x^n = \prod_{j=1}^{k} \frac{1}{1-x^j}. \tag{2.1}$$

Since the right-hand side of (2.1) is a rational function, it can be decomposed into ordinary partial fractions, as considered, e.g., by Cayley [4] and Rademacher [10, p. 302], or into q-partial fractions, as studied by Munagi [8,9].

In examining the ordinary partial fraction decompositions of, say, $F_4(x)$,

$$\begin{aligned} F_4(x) &= \frac{-17/72}{x-1} + \frac{59/288}{(x-1)^2} + \frac{1/8}{(x-1)^3} + \frac{1/24}{(x-1)^4} + \frac{1/8}{x+1} + \frac{1/32}{(x+1)^2} \\ &\quad + \frac{(x+1)/9}{x^2+x+1} + \frac{1/8}{x^2+1} \\ &= \frac{-17/72}{x-1} + \frac{59/288}{(x-1)^2} + \frac{1/8}{(x-1)^3} + \frac{1/24}{(x-1)^4} + \frac{1/8}{x+1} + \frac{1/32}{(x+1)^2} \\ &\quad + \frac{(2+\omega^2)/27}{x-\omega} + \frac{(2+\omega)/27}{x-\omega^2} + \frac{-i/16}{x-i} + \frac{i/16}{x+i}, \end{aligned} \tag{2.2}$$

where $\omega := \exp(2\pi i/3)$, we notice immediately the apparent arbitrariness of the coefficients that arise in the expansion.

For Munagi's q-partial fractions, the coefficients are nicer, but still not transparent:

$$F_4(x) = \frac{25/144}{(1-x)^2} + \frac{1/8}{(1-x)^3} + \frac{1/24}{(1-x)^4} + \frac{1/16}{1-x^2} + \frac{1/8}{(1-x^2)^2} + \frac{(x+2)/9}{1-x^3} + \frac{1/4}{1-x^4}.$$

MacMahon's partial fraction decomposition of $F_k(x)$,

$$\prod_{j=1}^{k} \frac{1}{1-x^j} = \sum_{\lambda \vdash m} \frac{g(\lambda; x, x, x, \ldots, x)}{N\big(f(\lambda)!\big) N(\lambda)}, \tag{2.3}$$

thus has the distinct advantage that the coefficients are known a priori and, furthermore, these coefficients are "combinatorial numbers" in the sense that they are products of integer exponential and factorial expressions.

To begin to understand (2.3) combinatorially, we shall multiply both sides of (2.3) by $k!$ and observe that $g(\lambda; x, x, x, \ldots, x)$ is the generating function for the sequence that counts a certain class of restricted weak k-compositions defined below.

Equation (2.3) together with (1.7), after some investigation, suggested the generalization of MacMahon's partial fraction decomposition presented above as Theorem 1.15.

3. Proof of Theorem 1.15

Starting with the left member of (1.8), we have

$$\begin{aligned}
&\sum_{\sigma \in \mathfrak{S}_k} \sigma\left(\frac{1}{(1-x_1)(1-x_1x_2)(1-x_1x_2x_3)\cdots(1-x_1x_2x_3\cdots x_k)}\right)\\
&\quad= \sum_{\sigma \in \mathfrak{S}_k} \sigma\left(\sum_{a_1,a_2,\ldots,a_k \geq 0} x_1^{a_1}(x_1x_2)^{a_2}(x_1x_2x_3)^{a_3}\cdots(x_1x_2\cdots x_k)^{a_k}\right)\\
&\quad= \sum_{\sigma \in \mathfrak{S}_k} \sigma\left(\sum_{a_1,a_2,\ldots,a_k \geq 0} x_1^{a_1+a_2+\cdots+a_k} x_2^{a_2+a_3+\cdots+a_k}\cdots x_{k-1}^{a_{k-1}+a_k} x_k^{a_k}\right)\\
&\quad= \sum_{\sigma \in \mathfrak{S}_k} \sigma\left(\sum_{w_1 \geq w_2 \geq \cdots \geq w_k \geq 0} x_1^{w_1} x_2^{w_2}\cdots x_k^{w_k}\right)\\
&\quad= \sum_{\sigma \in \mathfrak{S}_k} \sigma\left(\sum_{w \in \mathscr{W}_k} x_1^{w_1} x_2^{w_2}\cdots x_k^{w_k}\right)\\
&\quad= \sum_{\gamma \in \mathscr{C}_k} N\big(f(\gamma)!\big) x_1^{\gamma_1} x_2^{\gamma_2}\cdots x_k^{\gamma_k}.
\end{aligned}$$

Thus, we see that the left member of (1.8) generates every weak k-composition (where the jth part appears as the exponent of x_j) exactly $N\big(f(\gamma)!\big)$ times.

Now let us consider the right member of (1.8),

$$\sum_{\lambda\vdash m} N\left((\lambda-\mathbf{1})!\right) \sum_{\phi(\mathbf{x})\in\mathscr{O}_\lambda} \phi(\mathbf{x}), \tag{3.1}$$

where $\mathscr{O}_\lambda$ is the orbit of $g(\lambda;\mathbf{x})$ under the (transitive) action of $\mathfrak{S}_{|\lambda|}$.

Pick an arbitrary weak k-composition γ. We need to show that the term $x_1^{\gamma_1}x_2^{\gamma_2}\cdots x_k^{\gamma_k}$ appears in the expansion of (3.1) with coefficient $N\big(f(\gamma)!\big)$. Associated with γ is the frequency sequence $f(\gamma)=(f_0(\gamma),f_1(\gamma),f_2(\gamma),\dots)$. Permute the nonzero terms of $f(\gamma)$ into nondecreasing order to form a partition λ of weight k, and we may write $\lambda=\lambda(\gamma)$ since the partition λ is uniquely determined by γ. Thus, it must be the case that there exists $\sigma\in\mathfrak{S}_k$ such that the weak k-composition $\sigma(\gamma)$ is of type $[c_1^{\lambda_1}c_2^{\lambda_2}\cdots c_l^{\lambda_l}]$ for some distinct nonnegative integers $c_1,c_2,\dots,c_l$.

For a given $\lambda\vdash k$ of length l, we have, by expanding (1.5) as a series,

$$\begin{aligned} g(\lambda;\mathbf{x}) = \sum_{c_1,c_2,\dots,c_l\geq 0} &(x_1x_2\cdots x_{\lambda_1})^{c_1}(x_{\lambda_1+1}x_{\lambda_1+2}\cdots x_{\lambda_1+\lambda_2})^{c_2}\cdots \\ &\times(x_{\lambda_1+\lambda_2+\cdots\lambda_{l-1}+1}x_{\lambda_1+\lambda_2+\cdots+\lambda_{l-1}+2}\cdots x_{\lambda_1+\lambda_2+\cdots+\lambda_l})^{c_l}, \end{aligned}$$

so $g(\lambda;\mathbf{x})$ is the generating function for weak k-compositions of type

$$[c_1^{\lambda_1}c_2^{\lambda_2}\cdots c_l^{\lambda_l}].$$

Now the orbit $\mathscr{O}_\lambda$ of $g(\lambda;\mathbf{x})$ under the action of $\mathfrak{S}_k$ contains the terms that generate all permutations of weak k-compositions of type $[c_1^{\lambda_1}c_2^{\lambda_2}\cdots c_l^{\lambda_l}]$.

The terms $x_1^{\gamma_1}x_2^{\gamma_2}\cdots x_k^{\gamma_k}$ are generated by those terms of (3.1) in the orbit of $g(\mu;\mathbf{x})$ for multipartitions $\mu\in MD\,(\lambda)$, where $\lambda=\lambda(\gamma)$.

For each weak k-composition γ, and the corresponding partition

$$\lambda=\lambda(\gamma)=(\lambda_1,\lambda_2,\dots,\lambda_{l(\lambda)}),$$

we generate all the associated multipartitions in $MD\,(\lambda)$. Let μ_k^{ij} denote the kth part in the partition μ^{ij}, where μ^{ij} is the ith partition of λ_j, the jth part of λ.

Note that i runs from 1 through $p(\lambda_1)p(\lambda_2)\cdots p(\lambda_{l(\lambda)})$ where some ordering has been imposed on the multipartitions (any ordering will do). Of course, j runs from 1 to $l(\lambda)$, and k runs from 1 to $l(\mu^{ij})$.

For example, if we wish to calculate the number of times the weak 5-composition $\gamma=(7,7,4,7,4)$ is generated by the right-hand side of (1.8), i.e., the number of times the expression $x_1^7x_2^7x_3^4x_4^7x_5^4$ appears, we see, by symmetry, that this must be the same as the number of times $(7,7,7,4,4)$ appears.

The weak 5-partition $(7,7,7,4,4)$ is clearly obtained from $(7,7,4,7,4)$ by allowing the permutation $\sigma=(3,4)$ to act on it. Then $\lambda(\gamma)=\lambda(\sigma\gamma)=(3,2)$ because part 7 appears three times and part 4 appears two times in both γ and $\sigma\gamma$.

We notice that a certain number of copies of $(7,7,7,4,4)$ are generated by each of the terms

$$\frac{2}{(1-x_1x_2x_3)(1-x_4x_5)}, \frac{2}{(1-x_1x_2x_3)(1-x_4)(1-x_5)},$$
$$\frac{1}{(1-x_1x_2)(1-x_3)(1-x_4x_5)}, \frac{1}{(1-x_1x_2)(1-x_3)(1-x_4)(1-x_5)},$$
$$\frac{1}{(1-x_1)(1-x_2)(1-x_3)(1-x_4x_5)}, \frac{1}{(1-x_1)(1-x_2)(1-x_3)(1-x_4)(1-x_5)}, \tag{3.2}$$

and by no other terms. To aid our analysis we consider the multipartition dissection of the partition $\lambda = (3,2)$:

$$\begin{aligned} MD\,((3,2)) &= \{\text{all partitions of } 3\} \times \{\text{all partitions of } 2\} \\ &= \Big\{\Big((3),(2)\Big), \Big((3),(1,1)\Big), \Big((2,1),(2)\Big), \Big((2,1),(1,1)\Big), \\ &\quad \Big((1,1,1),(2)\Big), \Big((1,1,1),(1,1)\Big)\Big\}, \end{aligned} \tag{3.3}$$

because each of these six multipartitions indexes a term that generates some number of copies of $\sigma\gamma = (7,7,7,4,4)$. In this example, we have $\mu^1 = ((3),(2))$, $\mu^2 = ((3),(1,1))$, $\mu^3 = ((2,1),(2))$, $\mu^4 = ((2,1),(1,1))$, $\mu^5 = ((1,1,1),(2))$, and $\mu^6 = ((1,1,1),(1,1))$; $\mu^{11} = \mu^{21} = (3)$, $\mu^{12} = \mu^{32} = \mu^{52} = (2)$, $\mu^{22} = \mu^{42} = \mu^{62} = (1,1)$, and $\mu^{51} = \mu^{61} = (1,1,1)$.

We use elementary combinatorial reasoning to count how many copies of $(7,7,7,4,4)$ are generated by each of the six terms. That number is a consequence of the commutativity of ordinary multiplication. For example, consider the third term in (3.2)

To generate $(7,7,7,4,4)$, we may do so by any of the following permutations of this third term:

$$\frac{1}{(1-x_1x_2)(1-x_3)(1-x_4x_5)},$$
$$\frac{1}{(1-x_1x_3)(1-x_2)(1-x_4x_5)},$$
$$\frac{1}{(1-x_2x_3)(1-x_1)(1-x_4x_5)},$$

which are indexed by the multipartition $\mu^3 = ((2,1),(2))$.

This clearly lists all elements in the Cartesian product of the two orbits: one is the orbit of $\frac{1}{(1-x_1x_2)(1-x_3)}$ under the action of $\mathfrak{S}_3 = \mathfrak{S}_{\{1,2,3\}}$, (the permutations of $\{1,2,3\}$), and the other is the orbit of $\frac{1}{1-x_4x_5}$ under the action of $\mathfrak{S}_{\{4,5\}}$. Since each term generates one copy of $(7,7,7,4,4)$, the total contribution of these terms is given by

$$\begin{aligned} &(\mu^{31} - \mathbf{1})!(\mu^{32} - \mathbf{1})!\,\left|\mathscr{O}_{\mu^{31}}\right|\,\left|\mathscr{O}_{\mu^{32}}\right| \\ &\quad = \frac{\lambda_1!}{f_1(\mu^{31})!f_2(\mu^{31})!\cdots\mu_1^{31}\mu_2^{31}} \cdot \frac{\lambda_2!}{f_1(\mu^{32})!f_2(\mu^{32})!\cdots\mu_1^{32}}, \end{aligned} \tag{3.4}$$

where we have applied Remark 1.14.

Of course, to generate all copies of $(7,7,7,4,4)$, we must sum over all of the terms indexed by the six members of $MD\,((3,2))$, employing the analogous counting formula in each case.

In the general case, the preceding combinatorial argument yields

$$\prod_{i,j}(\mu^{ij}-\mathbf{1})!\,\left|\mathscr{O}_{\mu^{ij}}\right| = \sum_{i=1}^{p(\lambda_1)p(\lambda_2)\cdots p(\lambda_{l(\lambda)})}\prod_{j=1}^{l(\lambda)}\frac{\lambda_j!}{N\big(f(\mu^{ij})!\big)N\,(\mu^{ij})}.$$

Thus, all that remains to prove Theorem 1.15 is to establish:

$$N\big(f(\gamma)!\big) = \sum_i\prod_j\frac{\lambda_j!}{N\big(f(\mu^{ij})!\big)N\,(\mu^{ij})}. \tag{3.5}$$

Since $N\big(f(\gamma)!\big) = N\,(\lambda!) = \prod_j \lambda_j!$, we immediately see that (3.5) is equivalent to the assertion

$$1 = \sum_i\prod_j\frac{1}{N\big(f(\mu^{ij})!\big)N\,(\mu^{ij})},$$

which is exactly (1.4), and thus Theorem 1.15 is established. $\square$

4. Example: The Case $k = 4$

Before concluding, let us examine the $k = 4$ case in some detail. Our main result, Theorem 1.15, in the case $k = 4$ asserts

$$\begin{aligned}
&\sum_{\sigma\in\mathfrak{S}_4}\sigma\left(\frac{1}{(1-x_1)(1-x_1x_2)(1-x_1x_2x_3)(1-x_1x_2x_3x_4)}\right)\\
&\quad=\frac{6}{1-x_1x_2x_3x_4}+2\left(\frac{1}{(1-x_1x_2x_3)(1-x_4)}+\frac{1}{(1-x_1x_2x_4)(1-x_3)}\right.\\
&\qquad\left.+\frac{1}{(1-x_1x_3x_4)(1-x_2)}+\frac{1}{(1-x_1)(1-x_2x_3x_4)}\right)\\
&\qquad+\left(\frac{1}{(1-x_1x_2)(1-x_3x_4)}+\frac{1}{(1-x_1x_3)(1-x_2x_4)}\right.\\
&\qquad\left.+\frac{1}{(1-x_1x_4)(1-x_2x_3)}\right)\\
&\qquad+\left(\frac{1}{(1-x_1x_2)(1-x_3)(1-x_4)}+\frac{1}{(1-x_1x_3)(1-x_2)(1-x_4)}\right.\\
&\qquad+\frac{1}{(1-x_1x_4)(1-x_2)(1-x_3)}+\frac{1}{(1-x_1)(1-x_2x_3)(1-x_4)}\\
&\qquad\left.+\frac{1}{(1-x_1)(1-x_2x_4)(1-x_3)}+\frac{1}{(1-x_1)(1-x_2)(1-x_3x_4)}\right)\\
&\qquad+\frac{1}{(1-x_1)(1-x_2)(1-x_3)(1-x_4)}.
\end{aligned}\tag{4.1}$$

In the left member of (4.1), we have

$$\frac{1}{(1-x_1)(1-x_1x_2)(1-x_1x_2x_3)(1-x_1x_2x_3x_4)},$$

which generates every weak 4-partition $w = (w_1, w_2, w_3, w_4)$ exactly once. The cardinality of the orbit of the action of $\mathfrak{S}_4$ on w is

$$\frac{4!}{f_0(w)!f_1(w)!f_2(w)!\cdots},$$

i.e., there are $4!/(f_0(w)!f_1(w)!f_2(w)!\cdots)$ distinct weak 4-compositions of type w. Or equivalently, a given weak 4-composition γ which equals σw for some permutation $\sigma \in \mathfrak{S}_4$, is generated $N\big(f(\gamma)!\big) = f_0(\gamma)!f_1(\gamma)!f_2(\gamma)!\cdots$ times.

The generation of weak 4-compositions on the right side of (4.1) is more subtle. Notice that the terms of the right side are grouped according to the partitions of 4 (which index the sum on the right side) in the order (4), (3, 1), (2, 2), (2, 1, 1), (1, 1, 1, 1). For a given weak 4-composition γ, the multiplicities of the parts determine which of the terms of the right side contribute to its generation.

A detailed summary is provided in Table 1. To make sure the table is clear, let us look at one line of it in detail. Observe the case with $\lambda = (2, 2)$ and form of γ as *abab*. The *abab* means we are considering weak 4-compositions where the first and third parts are the same, and the second and fourth parts are the same, but the first and second parts are different. The corresponding terms from the right member of (4.1) are equivalent to

$$(23)g(22, \mathbf{x}) + (23)g(211, \mathbf{x}) + (14)g(211, \mathbf{x}) + g(1111, \mathbf{x}). \tag{4.2}$$

Since

$$g(22, \mathbf{x}) = \frac{1}{(1-x_1x_2)(1-x_3x_4)},$$

the first term $(23)g(22, \mathbf{x})$ of (4.2) is

$$\frac{1}{(1-x_1x_3)(1-x_2x_4)}. \tag{4.3}$$

Expand each factor of the right side of (4.3) as a geometric series to find that weak 4-compositions (w_1, w_2, w_3, w_4) are generated (in the exponents of the x_i's) in which $w_1 = w_3$ and $w_2 = w_4$, i.e., compositions of the type (a, b, a, b). Is this the only way that compositions of type (a, b, a, b) may be generated? No. Consider the second term of (4.2), $(23)g(211, \mathbf{x})$, which is

$$\frac{1}{(1-x_1x_3)(1-x_2)(1-x_4)}.$$

This term generates weak 4-compositions (w_1, w_2, w_3, w_4), in which $w_1 = w_3$. Some of the weak compositions generated by this term will happen to have $w_2 = w_4$, and thus these will be of the form (a, b, a, b) as well, i.e., this term generates compositions of the general form (a, b, a, c); on those occasions that it happens to be the case that $b = c$, we have a weak composition of the form considered by this particular line of the table. Similarly, compositions of type

TABLE 1. The letters a, b, c, d represent distinct nonnegative integers

λ	Form of γ	Generating terms of RHS of (4.1)
(4)	$aaaa$	$6g(4;\mathbf{x}) + 2\Big(() + (34) + (24) + (14)\Big)g(31;\mathbf{x})$ $+\Big(() + (23) + (24)\Big)g(22;\mathbf{x})$ $+\Big(() + (13) + (23) + (24) + (14) + (13)(24)\Big)g(211;\mathbf{x})$ $+g(1111;\mathbf{x})$
(31)	$aaab$	$2g(31;\mathbf{x}) + \Big(() + (13) + (23)\Big)g(211;\mathbf{x}) + g(1111;\mathbf{x})$
	$aaba$	$2(34)g(31;\mathbf{x}) + \Big(() + (24) + (14)\Big)g(211;\mathbf{x}) + g(1111;\mathbf{x})$
	$abaa$	$2(24)g(31;\mathbf{x}) + \Big((23) + (24) + (13)(24)\Big)g(211;\mathbf{x}) + g(1111;\mathbf{x})$
	$abbb$	$2(14)g(31;\mathbf{x}) + \Big((13) + (14) + (13)(24)\Big)g(211;\mathbf{x}) + g(1111;\mathbf{x})$
(22)	$aabb$	$g(22;\mathbf{x}) + \Big(() + (13)(24)\Big)g(211;\mathbf{x}) + g(1111;\mathbf{x})$
	$abab$	$(23)g(22;\mathbf{x}) + \Big((23) + (14)\Big)g(211;\mathbf{x}) + g(1111;\mathbf{x})$
	$abba$	$(24)g(22;\mathbf{x}) + \Big((13) + (24)\Big)g(211;\mathbf{x}) + g(1111;\mathbf{x})$
(211)	$aabc$	$g(211;\mathbf{x}) + g(1111;\mathbf{x})$
	$abac$	$(23)g(211;\mathbf{x}) + g(1111;\mathbf{x})$
	$abca$	$(24)g(211;\mathbf{x}) + g(1111;\mathbf{x})$
	$abbc$	$(13)g(211;\mathbf{x}) + g(1111;\mathbf{x})$
	$abcb$	$(14)g(211;\mathbf{x}) + g(1111;\mathbf{x})$
	$abcc$	$(13)(24)g(211;\mathbf{x}) + g(1111;\mathbf{x})$
(1111)	$abcd$	$g(1111;\mathbf{x})$

Permutations are presented in cycle notation

(a, b, a, b) also can be generated by the third and fourth terms of (4.2). The other lines of the table may be interpreted similarly.

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Andrew V. Sills
Department of Mathematical Sciences
Georgia Southern University
Statesboro
GA 30458
USA
e-mail: ASills@georgiasouthern.edu

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Twin Composites, Strange Continued Fractions, and a Transformation that Euler Missed (Twice)

Dedicated to George E. Andrews on the occasion of his 80th birthday

Kenneth B. Stolarsky

Abstract. We introduce a polynomial $E(d, t, x)$ in three variables that comes from the intersections of a family of ellipses described by Euler. For fixed odd integers $t \geq 3$, the sequence of $E(d, t, x)$ with d running through the integers produces, conjecturally, sequences of "twin composites" analogous to the twin primes of the integers. This polynomial and its lower degree relative $R(d, t, x)$ have strikingly simple discriminants and resolvents. Moreover, the roots of R for certain values of d have continued fractions with at least two large partial quotients, the second of which mysteriously involves the 12th cyclotomic polynomial. Various related polynomials whose roots also have conjecturally strange continued fractions are also examined.

Mathematics Subject Classification. 11A55, 11C08, 11R32, 12D05, 12D10.

Keywords. Continued fraction, Cyclotomic polynomial, Discriminant, Ellipse, Pythagorean triple, Resolvent, Resultant, Twin composites.

1. Introduction

Our object is to introduce a special polynomial $E = E(d, t, x)$ in three variables whose discriminant with respect to x and whose cubic resolvent (see [3, p. 316]) with respect to x are remarkably transparent. We also establish the geometric meaning of this polynomial, and make a series of conjectures regarding its properties and its connection to some strange continued fractions. These include a "twin composites" conjecture and a connection to the expression:

$$w(k, t) = \frac{4t(t^{12k} - 1)}{(t^2 - 1)(1 - t^2 + t^4)}$$

involving the 12th cyclotomic polynomial. The E polynomial arises from transformations of simpler polynomials that were almost ignored (but for good reason) by Euler.

One may say that the theory of elliptic functions began when Euler studied the lemniscate bisection paper of Fagnano and then proceeded by analogy with trigonometry to produce the:

$$r = \frac{u\sqrt{1-v^4} + v\sqrt{1-u^4}}{1+u^2v^2}$$

addition formula for the lemniscate integral. As recounted in [4, pp. 15–16], Euler wrote to Goldbach that the integral

$$x^2 + y^2 = c^2 + 2xy\sqrt{1-c^4} - c^2x^2y^2$$

of

$$\frac{dy}{\sqrt{1-y^4}} = \frac{dx}{\sqrt{1-x^4}}$$

that led to his new addition formula was analogous to the integral:

$$x^2 + y^2 = c^2 + 2xy\sqrt{1-c^2} \tag{1.1}$$

for

$$\frac{dy}{\sqrt{1-y^2}} = \frac{dx}{\sqrt{1-x^2}}.$$

This integral can be written as:

$$y = x\sqrt{1-c^2} \pm c\sqrt{1-x^2}.$$

Euler thus began the theory of elliptic functions with no further emphasis on (1.1), which is the equation of an ellipse. He took the most important path. However, there is yet something special about the ellipse (1.1) with parameter c that calls for further investigation. The title of this paper echoes the title of the famous short paper of Apostol [1] in which it is shown that the "forty-five degree" orthogonal change of variable in a simple double integral easily yields a proof of Euler's formula for $\zeta(2)$. In fact, this very same change of variable leads to a convenient form of the ellipse (1.4), now viewed as a family of ellipses with parameter c. Also, this family has a simple elegant envelope related to this change of variable. Moreover, it has internal intersection properties for c running through sets of rational numbers that lead to deeper investigations.

The change of variable alluded to above is:

$$x = \frac{x^* - y^*}{\sqrt{2}}, \quad y = \frac{x^* + y^*}{\sqrt{2}}. \tag{1.2}$$

This produces (we now omit the "star" superscript) the family of ellipses:

$$x^2(1 - \sqrt{1-c^2}) + y^2(1 + \sqrt{1-c^2}) = c^2. \tag{1.3}$$

Each has area πc, so as c varies from 1 to 0, the area goes from π (the unit circle case) to 0 for $c = 0$. The foci are at $\pm\sqrt{2}(1-c^2)^{1/4}$. One also easily verifies that as $c \to 0$, the ellipse approaches the line segment $[-\sqrt{2}, \sqrt{2}]$. Now, define

$$T(x, y) = ((x - y)^2 - 2)((x + y)^2 - 2),$$

and

$$F(c, x, y) = x^2(1 - \sqrt{1 - c^2}) + y^2(1 + \sqrt{1 - c^2}) - c^2.$$

To determine the envelope of the ellipses for $0 \leq c \leq 1$, we must eliminate c from:

$$F = 0 \quad \text{and} \quad \frac{\partial F}{\partial c} = 0. \tag{1.4}$$

The result is $T(x, y) = 0$. The homogeneous form of this equation is essentially the assertion that at least one of the transformations of (1.2) is valid! Geometrically, it says that the envelope of the ellipses for $0 \leq c \leq 1$ is the boundary of a square with vertices at $(\pm\sqrt{2}, 0)$ and $(0, \pm\sqrt{2})$.

Now, let $E(c)$ denote the ellipse of (1.3). For $0 < c_1 < c_2 < 1$, the ellipses $E(c_1)$ and $E(c_2)$ have four real intersections. Let t and d be positive integers. We shall henceforth focus on $E(d, t, x)$, a polynomial with constant term t^4 that is quartic in t and octic in d and x. (In fact, it is quadratic in t^2 and quartic in both d^2 and x^2). Our geometric interest is in the positive x coordinate of the intersection of $E\left(\frac{1}{d}\right)$ and $E\left(\frac{t}{d}\right)$. We call it $x(d, t)$ or simply x. We show in Sect. 2 that $E(d, t, x)$ is divisible by the minimal polynomial of this $x(d, t)$. We then examine $E(d, t, x)$ from the point of view of discriminants and resolvents (Sect. 3), reducibility (Sect. 3), and continued fractions (Sect. 4).

The historical investigations of the lemniscate were bound up with compass and ruler constructions and the solution of equations by radicals. This led to considerations of reducibility. In fact, studies of this nature by Eisenstein led to his well-known irreducibility criterion [3, pp. 496–497]. Here, we shall pose a, perhaps, new type of question about polynomials, motivated by the common observation that polynomials in $\mathbb{Z}[x]$ are "usually" irreducible, while members of $\mathbb{Z}$, the integers, are "usually" composite. It is a form of the "infinitude of twin primes" conjecture, except that the roles of reducible (= composite) and irreducible (= prime) are reversed.

We ask if there are two-variable polynomials $p(u, x) \in \mathbb{Z}[u, x]$, such that the sequence $\{p(m, x)\}$, where $m \geq 1$ runs through the integers, has the following three properties.

1. The reducible elements in this sequence are rare, and, in fact, exponentially sparse.
2. There are infinitely many reducible elements.
3. To each reducible $p(m, k)$, there is another reducible element $p(k, x)$, such that

$$|m - k| < C,$$

where C is a fixed constant.

More precisely, our conjecture is that each $E(d, t, x)$ polynomial, with $t \geq 3$ a fixed odd number, provides an example. Curiously, the detailed conjecture (see Sect. 3) involves simple exact formulas, something certainly not available for the classical twin prime conjecture.

It is notable that $E(d,t,x)$ for a fixed odd integer $t \geq 3$ and d large has two roots whose ratio is very close to t. In Sect. 4, we conjecture a simple form for a polynomial $R = R(d,t,x)$ of degree 4 divisible by the minimal polynomial of this ratio. We then examine the ratio from the point of view of continued fractions. In Sect. 5, we study the root ratios of the polynomial R. Here, we have theorems rather than conjectures, and a curious reducibility connection with the set of Pythagorean triples. In Sect. 6, we prove some properties of generalizations of R and connect them with some further strange continued fractions.

2. The $E(d,t,x)$ Polynomial

Our central object of study is the $E(d,t,x)$ polynomial.

Definition 2.1. Let

$$E(d,t,x) = t^4 - 8t^2d^4x^2 + 8d^4(-t^2 + 2(t^2+1)d^2)x^4 - 32d^8x^6 + 16d^8x^8.$$

Theorem 2.2. *Let $d \geq 1$ and $t \geq 2$ be integers with $t \leq d$. Then, the x coordinates of the intersection of the ellipses $E\left(\frac{1}{d}\right)$ and $E\left(\frac{t}{d}\right)$ are roots of $E(d,t,x)$.*

Indication of Proof. It is straightforward to find formulas involving square roots for the coordinates of the intersection points of $E\left(\frac{1}{d}\right)$ and $E\left(\frac{t}{d}\right)$. Also, since $E(d,t,x)$ is quartic in x^2, it is solvable by radicals. Thus, one can verify that the roots of $E(d,t,x) = 0$ agree with the formulas for the x-coordinates coming from the intersection computation above. This is an elementary but somewhat tedious process, since these formulas are somewhat awkward. We suppress the details.

To simply see that the theorem is plausible, set $t = \lambda d$ where $0 \leq \lambda \leq 1$. Then, as a polynomial in d, the coefficient of the lead term is:

$$16x^4(\lambda^2 - 2x^2 + x^4).$$

Say d is very large, so this term dominates. If λ is close to 0, then there is a root very close to $\sqrt{2}$. If λ is close to 1, then there is a root very close to 1. For λ close to zero, we have two very thin ellipses reaching almost to $\sqrt{2}$. For λ close to 1, we have the very thin ellipse $E\left(\frac{1}{d}\right)$, reaching almost to $\sqrt{2}$, intersecting $E(\lambda)$, which is almost the unit circle.

If $E(d,t,x)$ is irreducible, this theorem implies of course that it is the minimal polynomial of the x coordinates. □

3. The Properties of E

The discriminant of $E = E(d,t,x)$ with respect to x is remarkably transparent. The same holds for the cubic resolvent of $E(d,t,\sqrt{x})$ with respect to x which also factors completely. Computer algebra yields the following.

Proposition 3.1. *We have*

$$\text{Discriminant } (E,x) = (2^{15}(1-d^2)(1-t^2)(d^2-t^2)t^3d^{16})^4.$$

All factors here are linear and each corresponds to an "obvious extreme value" of a parameter. The discriminant with respect to d is also somewhat simple, despite a non-obvious factor.

Proposition 3.2. *We have*

$$\text{Discriminant } (E,d) = 2^{44}(1-t^2)^4t^{20}x^{34}(x^2-2)P(x,t)^2,$$

where

$$P(x,t) = -8t^2 + 3(9-2t^2+9t^4)x^2 - 96t^2x^4 + 64t^2x^6.$$

Note the x^2-2 factor here, which clearly fits the geometry of our family of ellipses.

Proposition 3.3. *The cubic resolvent of $E(d,t,\sqrt{x})$ with respect to x is*

$$(-t^2+2d^4x)(-2d^2+t^2+2d^4x)(t^2-2d^2t^2+2d^4x)/(8d^{12}).$$

The question of when $E(d,t,x)$ is reducible leads, as indicated previously, to some curious conjectures. We begin with a special case.

Conjecture 3.4. *Let $t=3$. Let*

$$\sum_{n=0}^{\infty} a(n)x^n = \frac{1-2x+5x^2}{(1-x)(1-6x+x^2)} = 1+5x+33x^2+197x^3+\cdots$$

and

$$\sum_{n=0}^{\infty} b(n)x^n = \frac{1+8x-x^2}{(1-x)(1-6x+x^2)} = 1+15x+97x^2+575x^3+\cdots.$$

Then, $E(d,t,x)=E(d,3,x)$ is reducible whenever $d=a(n)$ or $a(n)+2$ and also when $d=b(n)$ or $b(n)+4$.

For example, when $d=197$, we have $E=p(x)p(-x)$ where

$$p(x) = (9+330960x+\cdots+6024553924x^4),$$

and when $d=199$, we have $E=p(x)p(-x)$ where

$$p(x) = (9+334320x+\cdots+6272956804x^4).$$

Note that the roots of the denominators of the above rational functions lie in $Q(\sqrt{2})$. It is also the case that

$$\frac{a_n}{b_n} \to 6-4\sqrt{2} \approx .3431\ldots \quad \text{and} \quad \frac{a_{n+1}}{b_n} \to 2,$$

so the two types of "twin composites" alternate when n is large.

In the above example, we note that 2 and 4 are $t-1$ and $t+1$, respectively. We now state a much more general conjecture.

Conjecture 3.5. *Set*

$$\frac{1+(t-5)x+(t+2)x^2}{(1-x)(1-6x+x^2)}=\sum_{n=0}^{\infty} r(n)x^n$$

and

$$\frac{(t-2)+(t+5)x-x^2}{(1-x)(1-6x+x^2)}=\sum_{n=0}^{\infty} s(n)x^n.$$

Fix an odd $t \geq 3$. Then, for d large, say that $d \geq d(t)$, we have the following:

1. *The numbers d for which $E(d,t,x)$ factors and such that $E(d+t-1,t,x)$ also factors, are exactly the $r(m)$ for $m \geq m_0$ for some positive m_0.*
2. *The numbers d for which $E(d,t,x)$ factors, and such that $E(d+t+1,t,x)$ also factors, are exactly the $s(m)$ for $m \geq m_0$ for some positive m_0.*
3. *All other $E(d,t,x)$ for $d \geq d_0$ are irreducible octic polynomials.*

We remark that the numerators of the rational functions in the above conjecture cannot have any factor in common with the denominators for $t \geq 3$. In fact, the resultants of each numerator with its denominator are, respectively, $-16(1-t)(1+t)^2$ and $16(1-t)^2(1+t)$.

There appear to be similar phenomena for t even, but here it seems harder to formulate a clear conjecture. The reader is welcome to tabulate the "twin composites" for $t=6$ and to attempt to describe in detail their pattern of occurrence.

4. Root Ratios and Continued Fractions Involving *E*

Given an odd $t \geq 3$, there seem to be always two roots of $E(d,t,x)$ whose ratio is close to t, and remarkably close if d is large. For $E(10^5,5,x)$, consider its two right-most roots lying in the interval $(0,1)$. Their ratio is

$$5.000\ 000\ 003\ 000\ 000\ 00\ 3825\ 000\ 00\ 6003\cdots.$$

As a continued fraction, this is

$$[5,\ 333\ 333\ 332, 1, 9, 1, 9, 1, 554630, 3, 1, 4, \ldots].$$

For $E(10^7,5,x)$, one gets

$$[5,\ 333\ 333\ 333\ 332, 1, 9, 1, 9, 1, 5546311701, 1, 4\ldots].$$

It is notable that the second and eighth partial quotients here are far larger than probability would suggest. It is also notable that their ratio seems to rapidly approach $601 = 1-5^2+5^4$, as d goes to infinity. Also, for $d=10^{14}$, there is a very large 30th partial quotient.

To understand the above, we would like to know about the minimal polynomial of the ratio.

Definition 4.1. Let

$$R = R(d,t,x) = -4d^2x(t-x)(1-tx)+t^2(1-x^2)^2. \tag{4.1}$$

Theorem 4.2. *For $t \geq 3$ odd, the minimal polynomial of the above root ratio for $E(d,t,x)$ is a divisor of $R(d,t,x)$.*

Indication of Proof. This follows by applying the formula for the cubic resolvent of E as is done for R and S in Sect. 5. We leave the details to the "Appendix". □

The polynomial R need not be irreducible. In fact, for $t = d$, we have:

$$R(d,d,x) = d^2(1 - 2dx + x^2)^2.$$

We remark further that having an at most degree 4 minimal polynomial for the ratio of roots of an asymmetric eighth degree polynomial is in itself notable. For a random example, the minimal polynomial of the ratio of the largest roots of $1 - 11x^2 + x^3$ has degree 6.

The $R(d,t,x)$ is again an example of a polynomial with both a remarkably transparent discriminant and a remarkably transparent cubic resolvent with respect to x.

Proposition 4.3. *We have*

$$\text{Discriminant } (R,x) = 2^{12}((1-d^2)d^2)^2((1-t^2)t^2)^2(d^2-t^2)^2. \tag{4.2}$$

Note that although R is very far from being symmetric in d and t, its discriminant does have this symmetry.

Proposition 4.4. *The cubic resolvent of $R(d,t,x)$ with respect to x is:*

$$\frac{(2-4d^2+x)(2t^2-4d^2+t^2x)(x-2)}{t^2}.$$

The large size of the second partial quotient in the above example is somewhat explained by the following. Make the change of variable $x = t + s$ with t fixed and s the new variable. The equation $R = 0$ becomes

$$14400 - 450(d^2-25)s + (3700 - 196d^2)5^2 - 20(d^2-25)5^2 + 25s^4 = 0.$$

For s small, one may neglect terms of order s^2 or higher. The resulting linear polynomial has a solution s given by:

$$\frac{1}{s} = \frac{d^2-25}{30}.$$

This suggests that the second partial quotient has quadratic growth in d.

It is in fact obvious from the form of R that for d large, there is a root very close to t. In fact, if we tie d to t by setting $d = t^{6k+1}$, this closeness takes on an elegant form.

Conjecture 4.5. *The equation*

$$R(t^{6k+1}, t, x) = 0, \quad k = 1, 2, \ldots \tag{4.3}$$

has

$$u(t) = t + \frac{t(1-t^2)}{1 + 2t^2 - 4t^{2(6k+1)}}$$

as an excellent approximation to the root near t in the sense that as t goes to infinity through the integers:

$$R(t^{6k+1}, t, u(t)) = 0(t^{6-24k}).$$

Note that as a continued fraction $u(t)$ has the form:

$$u(t) = \left\{t, \frac{4t(1-t^{12k})}{1-t^2}, \frac{t-1}{2}, 2, t-1, 2, \frac{t-1}{2}, g\right\},$$

(note the internal symmetry) where $g = \infty$.

The conjectural reason that $u(t)$ is such a good approximation is that for t large and odd the next partial quotient of the continued fraction expansion of the actual root, i.e., the actual value of the g of (4.5), is $w(k,t) - 1$ where $w(k,t)$ is the expression defined in the introduction.

Conjecture 4.6. *For $k \geq 1$, and $t \geq 3$, t odd, the eighth partial quotient of the root of (4.3) nearest to t is:*

$$\frac{4t(t^{12k}-1)}{(t^2-1)(1-t^2+t^4)} - 1 = w(k,t) - 1.$$

The above conjecture indicates that the eighth partial quotient is much smaller than the second owing to the presence of the 12th cyclotomic polynomial in its denominator, but it is very large nonetheless. It is likely that slightly more involved expansions exist for $d = t^{6k+a}$ with a not congruent to 1 mod 6. For example, when $d = t^{22}$, one can conjecture that the first large partial quotient of the continued fraction for the root is

$$b_2 = \frac{4t(t^{22}-1)}{t^2-1}$$

and that the eighth partial quotient is

$$b_8 = \frac{b_2 + 4t(1-t^2)}{1-t^2+t^4},$$

where the cubic polynomial added to b_2 in the numerator is the negative of the remainder of b_2 after polynomial division by $1 - t^2 + t^4$.

5. The Root Ratios for R

One might ask if the fourth-degree polynomial $R(t,d,x)$ has two roots whose ratio satisfies a second-degree polynomial.

Theorem 5.1. *Let t and d be fixed integers with $1 < t < d$. Then, the two largest roots of the polynomial $R(d,t,x)$ have a ratio whose minimal polynomial is a divisor of S, where*

$$\begin{aligned} S = S(t,d,x) &= t^2 + 2(t^2 - 2d^2)x + t^2x \\ &= -4d^2x + t^2(1+x)^2. \end{aligned}$$

Proof. Since

$$R = t^2 - 4d^2tx + 2(2d^2 - t^2 + 2d^2t^2)x^2 - 4d^2tx^3 + t^2x^4,$$

it is self-reciprocal. From (4.1), we see that it cannot have negative roots. Moreover, if we solve $R = 0$ for x by radicals (tedious to write out), we can easily confirm that all roots are real (one does need the observation that

$$(d^2 - 1) + (d^2 - t) - 2\sqrt{d^2 - 1}\sqrt{t^2 - 1}$$

is positive by the inequality of the arithmetic and geometric means). Thus, we can label the 4 roots by r, r_2, r_3, r_4, so that

$$0 < r_1 \le r_2 \le r_3 \le r_4.$$

The self-reciprocal property tells us that $r_1r_4 = 1$, and hence, we also have $r_2r_3 = 1$. This explains the $x - 2$ factor in the cubic resolvent of R, since one of the roots of this resolvent is $r_1r_4 + r_2r_3$. The other roots are $r_1r_3 + r_2r_4$ and $r_1r_2 + r_3r_4$. Since

$$r_1r_2 + r_3r_4 \ge r_1r_3 + r_2r_4,$$

it follows from the complete factorization of the cubic resolvent that besides

$$r_1r_4 + r_2r_3 = 2,$$

we also have

$$r_1r_3 + r_2r_4 = 4\left(\frac{d^2}{t^2}\right) - 2$$

and

$$r_1r_2 + r_3r_4 = 4d^2 - 2.$$

Next, division by $r_1r_4 = 1$ yields

$$\frac{r_3}{r_4} + \frac{r_4}{r_3} = 4\left(\frac{d^2}{t^2}\right) - 2.$$

Thus, for the ratio $p = r_4/r_3$, we have

$$(x - p)\left(x - \frac{1}{p}\right) = x^2 - \left(\frac{4d^2}{t^2} - 2\right)x + 1,$$

and the result follows. □

Proposition 5.2. *We have*

$$\text{Discriminant }(S, x) = 16d^2(d^2 - t^2).$$

Corollary 5.3. *The polynomial $S = S(d, t, x)$ is reducible if and only if d and t are two elements of a Pythagorean triple with d as the largest element.*

6. More Strange Continued Fractions

How robust is the conjecture about the continued fraction expansion of the root of (4.3) closest to t? Something does remain if the 4 is replaced by 5 or 7. We shall give a related conjecture below, and then proceed to prove properties about polynomials more general than R that display some new conjectural properties.

Conjecture 6.1. *Let $a_n(t)$ denote the nth partial quotient of the root closest to t of*

$$-ct^{12k+2}(t-x)x(1-xt)+t^2(1-x^2)^2=0.$$

If $c=5$ or 7:

$$t \equiv \pm 2 \bmod 6,$$

and t is large, then

$$a_2(t)=ct\frac{t^{12k}-1}{t^2-1} \quad \textit{and} \quad a_{10}(t)=\frac{a_2(t)}{1-t^2+t^4}-1.$$

Numerical evidence suggests that the above is false at least for $c=3,6,9$ and 11, and also false for $t \not\equiv \pm 2$ mod 6. Thus, the simplicity of the case $c=4$ is special.

We now widen our point of view.

Definition 6.2. For positive integers a, m, and t, define

$$H\ (a,m,t,x)=4ax(x-t)(1-xt)+m(1-x^2)^2.$$

We still have a polynomial whose discriminant is transparent:

Proposition 6.3.

$$\text{Discriminant }(H,x)=2^{12}a^2(m-a)^2(1-t^2)^2(m-at^2)^2.$$

Also, if H is normalized to become monic, its cubic resolvent factors completely and becomes:

$$\frac{(x-2)(mx-4a+2m)(mx-4at^2+2m)}{m^2}.$$

Observe that previously, we examined cases in which the first term was weighted. We now consider heavily weighting the second term that involves $(1-x^2)^2$.

Theorem 6.4. *Assume that $a \geq 1$ and $t \geq 2$ are positive integers. Then, for $m \geq at^2$, the polynomial H has all roots on the unit circle $\mho$.*

Proof. For $m=2at^2$, the equation $H=0$ becomes

$$2at^2-4atx+4ax^2-4atx^3+2at^2x^4=0.$$

If we replace x by $-x$, we have on the left a self-inversive polynomial whose coefficients starting with the constant term decrease with a minimum at the central term and then increase with the second appearance of the maximal coefficient at the lead term. By a known result (see [2]), all zeros lie on the unit circle. If we now vary m, we see by the symmetry of the roots that no root can leave the unit circle without causing the discriminant to be zero. Hence, by the formula for the discriminant in the above Proposition, all roots must lie on the unit circle, no matter how the real parameter m varies, provided $m \geq at^2$. ☐

Theorem 6.5. *Let a, m, and t be positive integers with $a \geq 1, t \geq 2$ and $m \geq at^2$. Then, the real parts of the roots of $H = 0$ satisfy the equation $M(x) = 0$, where*

$$M(x) = mx^2 - 2atx - (m - a(1 + t^2)).$$

The imaginary parts satisfy $N(x^2) = 0$, where

$$N(x) = a^2(t^2 - 1)^2 - 2a(m(1 + t^2) - 2at^2)x + m^2x^2.$$

Proof. Observe that if h is a root of $N(x^2)$, then ih satisfies $N^*(x) = 0$, where

$$N^*(x) = a^2(t^2 - 1)^2 + 2a(m(1 + t^2) - 2at^2)x^2 + m^2x^4.$$

Note the sign of the middle term above. Now

$$\text{Resultant } (M(x), N^*(y - x), x)$$

is a polynomial in y whose roots are all possible sums of r_j and ih_k where the r_j are the roots of M and the h_k are the roots of $N(x^2)$. By computer algebra, the above resultant has the factorization:

$$m^3H(a, m, t, y)K(a, m, t, y),$$

where K is a polynomial of degree 4 in y having 14 terms. (Curiously, the ratio of the y discriminant of the above K to that of the above H is exactly m^{18}.) Since, as calculation reveals, K is not self-reciprocal, it has the "wrong" combination of real and imaginary parts. Thus, we see that H is formed from the roots of M and N as desired. ☐

Proposition 6.6. *For $M(x)$, we have*

$$\text{Discriminant } (M, x) = 4(a - m)(at^2 - m)$$

and

$$\text{Discriminant } (M, t) = -4a(a - m)(1 - x^2).$$

For $N(x)$, we have

$$\text{Discriminant } (N, x) = 16a^2(m - at^2)(m - a)t^2,$$
$$\text{Discriminant } (N, t) = 2^{12}a^8(m - a)^2x^2(1 - x)^2(mx - a)^2$$

and

$$\text{Discriminant } (N, m) = 16a^2t^2(1 - x)x^2.$$

We now come to perhaps our most curious conjecture, which we state rather loosely. If m is "very large", the continued fraction of the largest root of $M(x)$ has many extremely large partial quotients. Their distribution among the partial quotients is "irregular", but if listed in order of size, say $b < b_2 < b_3 < \cdots$, their ratios b_{i+1}/b_i are remarkably close to squares of rational numbers with relatively small numerators and denominators. For $2 \le a < t < m$ with $m = t^A$ where A is "large", these approximating squares are conjecturally all composed of integers that divide $a^2t(1-t^2)^2$. The square integers have density 0, so this seems, a priori, most unlikely.

Example 6.7. Let $a = 7, t = 12$, and $m = 12^{42}$. Then, of the first 806 partial quotients, 6 exceed 10^{10}. The smallest of these is 4 947 868 451 216. Of those less than 10^{10}, the largest is 4 937 987. Of the ratios formed by the 6 largest, the first exceeds $(7 \cdot 11 \cdot 13)^2$ by less than $2 \cdot 10^{-7}$. The remaining ratios are much closer to $(7 \cdot 11 \cdot 13)^2$. We add that the large partial quotients are the a_n for $n = 3, 13, 33, 65, 105$, and 157.

The author does not know of any discussion of the continued fraction representations of the ratios of partial quotients occurring in a given continued fraction. However, also in the next examples, they seem of interest.

Example 6.8. Take $a = 2, t = 13$, and $m = 10^{50}$. Here, m is not a power of t, so one expects less regularity. Among the first 308 partial quotients, there are 13 that exceed 10^9. We now expand their 12 ratios into continued fractions. In each case, we find one or more "small" partial quotients followed by an "obviously large" partial quotient. After truncating, we find the following list of lists of initial small parts:

$$\begin{gathered}\{1,2,1,3,3\},\{26,2,4,2,2\},\{36\},\{1296\},\\ \{1,2,1,3,3\},\{26,2,4,2,2\},\{36\},\{1296\},\\ \{1,2,1,3,3\},\{26,2,4,2,2\},\{36\},\{1764\}.\end{gathered}$$

Upon converting these continued fractions to ordinary fractions, we obtain

$$\left(\frac{7}{6}\right)^2, \left(\frac{36}{7}\right)^2, 6^2, 36^2, \left(\frac{7}{6}\right)^2, \left(\frac{36}{7}\right)^2, 6^2, 36^2, \left(\frac{7}{6}\right)^2, \left(\frac{36}{7}\right)^2, 6^2, 42^2.$$

Note that in this case, $a^2(1-t^2)^2 = 2^8 3^2 7^2$. We add that the large partial quotients are the a_n with

$$n = 3, 9, 21, 37, 51, 69, 97, 117, 145, 173, 205, 245, 275.$$

Example 6.9. Let $a = 2, t = 13$, and $m = 13^{44}$. Then, each ratio is close to $(2^2 \cdot 3 \cdot 7)^2$, a divisor of $(2 \cdot 12 \cdot 14)^2$.

Example 6.10. Return to Example 6.7 but now consider the largest root of $N(x)$; this is the square of the imaginary part of a root. Proceed as in Example 6.7 to find the partial quotients that exceed 10^4. Then form the continued fractions of their ratios. We find that each has a first quite large partial quotient followed by small ones that lead to another large one. We truncate these continued fractions by removing the second large partial quotient and all partial quotients that follow it. The resulting truncated continued fractions are:

$$\{27833, 2, 1, 3, 3\}, \{16032018\}, \{563625, 1, 1, 3, 2\},$$
$$\{445, 1, 3, 1, 1\}, \{250500, 3, 1\}, \{24843\}.$$

Here, the corresponding ordinary fractions are

$$\left(\frac{1001}{6}\right)^2, (4 \cdot 1001)^2, \left(\frac{3 \cdot 1001}{4}\right)^2, \left(\frac{2 \cdot 1001}{3}\right)^2, \left(\frac{1001}{2}\right)^2, 3 \cdot (7 \cdot 13)^2.$$

All prime factors of the above are (see Example 6.7) either $7, 11, 13$ or a divisor of $t = 12$. Only the last ratio fails to be a perfect square. We remark that the large partial quotients used to construct the ratios are the a_n for $n = 2, 13, 29, 55, 93, 139$, and 201. Only the first and largest of these has n even; if we exclude it we exclude the one fraction above that is not a perfect square.

7. Further Remarks

The square envelope result mentioned in the introduction is in fact a special case of a quartic envelope of a three-parameter family of ellipses. We give the details here, and then return to describe in greater detail the geometry of the ellipses inside the original square envelope. Finally, we indicate a direct connection between E and the set of Pythagorean triples.

Let b and h be real parameters and consider the three-parameter family of ellipses:

$$(b^2 - 1 - \sqrt{1 - c^2})x^2 + (h^2 - 1 + \sqrt{1 - c^2})y^2 = c^2.$$

The standard procedure for determining its envelope yields:

$$(x^2 - y^2)^2 - 4(x^2(b^2 - 1) + y^2(h^2 - 1) - 1) = 0.$$

For x and y large, the second term above is small, and we have curves that resemble two hyperbolas, one with foci on the x-axis and one with foci on the y-axis. For x and y small, the first term is small and we have an oval that resembles an ellipse centered at the origin. When $b = \sqrt{2}$, the left side factors and becomes

$$(x^2 - 2hy - y^2 - 2)(x^2 + 2hy - y^2 - 2) = 0.$$

In this case, for $h > \sqrt{2}$, the horizontal extremities of the inner "oval" are sharp corners, as are those of the "hyperbola-like" curve that intersects the x-axis. These curves touch at the two points $(\pm\sqrt{2}, 0)$. (Here, the "hyperbola-like" curve has the "wrong" concavity, and is a union of the infinite segments, lying in $|x| \geq \sqrt{2}$, of two distinct actual hyperbolas.)

The case studied in the previous sections of this paper is $b = h = \sqrt{2}$. Here, we have the further factorization:

$$\begin{aligned}&(\sqrt{2} - x - y)(\sqrt{2} + x - y)(\sqrt{2} - x + y)(\sqrt{2} + x + y)\\&\quad = ((x - y)^2 - 2)((x + y)^2 - 2).\end{aligned}$$

In this case, the "oval" has become a square, and the "hyperbolas" have become pairs of perpendicular rays emanating from the four vertices of the square, namely $(\pm\sqrt{2}, 0)$ and $(0, \pm\sqrt{2})$. The configuration as a whole, as the above factorization indicates, consists of two pairs of parallel lines, each orthogonal to the other.

There is more to say about both the $E(c)$ ellipses and how they fit into S, the convex hull of the enveloping square. The four points at which $E(c)$ touches the boundary of S are:

$$\left(\pm\frac{1+\sqrt{1-c^2}}{\sqrt{2}},\ \pm\frac{1-\sqrt{1-c^2}}{\sqrt{2}} \right).$$

Its foci are at $\pm\sqrt{2}(1-c^2)^{1/4}$, and its eccentricity is

$$e = \frac{\sqrt{2}(1-c^2)^{1/4}}{\sqrt{(1+\sqrt{1-c^2})}}.$$

Here, the eccentricity may be verified to be monotonically decreasing as c goes from 0 to 1. Hence, this family of ellipses contains, up to similarity, exactly one copy of every ellipse (or two copies if the transverse family generated by interchanging x and y is included).

We now indicate how the ellipses fill the convex hull S.

Lemma 7.1. *Let $0 \le x \le y$ and $x + y \le \sqrt{2}$. Then*

$$x^2(1-u) + y^2(1+u) = 1 - u^2$$

implies that u is real and $|u| \le 1$.

Proof. The discriminant D of

$$(x^2 + y^2 - 1) + (y^2 - x^2)u + u^2$$

with respect to u is

$$\begin{aligned} D &= 4 + (x^2 - y^2)^2 - 4(x^2 + y^2) \\ &= (2 + x^2 - y^2)^2 - 8x^2 \\ &= (2 - (x+y)^2)(2 - (y-x)^2) \ge 0, \end{aligned}$$

so u must be real. Now, observe that

$$2 + x^2 - y^2 \ge 2 - y^2 \ge 0.$$

Since

$$D \le (2 + x^2 - y^2)^2,$$

we have

$$|u| = |x^2 - y^2 \pm \sqrt{D}|/2 \le (y^2 - x^2 + (2 + x^2 - y^2))/2 = 1$$

and the result follows. □

Now, define c by $u^2+c^2=1$ to see that when $0 \le x \le y$ and $x+y \le \sqrt{2}$, there exists a value of c with $0 \le c \le 1$, such that

$$x^2(1-\sqrt{1-c^2})+y^2(1+\sqrt{1-c^2})=c^2.$$

By applying essentially the same argument to other sectors of S, we have the following result.

Theorem 7.2. *For any point in S, there is either (1) a value of c with $0 \le c \le 1$, such that an ellipse of the family passes through it, or (2) a value of c with $0 \le c \le 1$, such that an ellipse of the transverse family*

$$x^2(1+\sqrt{1-c^2})+y^2(1-\sqrt{1-c^2})=c^2$$

passes through it.

We can provide a bit more detail here. Rewrite the equation of a typical $E(c)$ ellipse as $P=P(x,y,c)=0$, where

$$P=h^2+h((y^2-x^2)^2-2(x^2+y^2))+4x^2y^2$$

and $h=c^2$. Although this has degree 4 in c, the numbers c and $-c$ play the same role for our purposes, so we may as well work with the above quadratic equation. Of course, a quadratic equation has either one root or two depending upon its discriminant. The discriminant of P with respect to h is

$$(x-y)^2(x+y)^2((x-y)^2-2)((x+y)^2-2).$$

From the points at which this is zero, we draw the following conclusions.

(1) Any point at which an ellipse of our family touches the boundary of S is a point that uniquely determines the ellipse.
(2) Any point at which an ellipse of our family intersects a line of the form $y=\pm x$ is a point that uniquely determines the ellipse.
(3) At any other point V inside S, there are exactly two ellipses of the family (or of the transverse family) that pass through V.

We conclude here by observing that if (d,t,u) is a Pythagorean triple with $d>t$ and $d>u$, then $E(d,t,x)$ factors. Set

$$\begin{aligned} P_y &= P_y(h,k,x) \\ &= (h+k)^4+4(h+k)^2(h^2+k^2)^3x+4(h^2+k^2)^4x^2. \end{aligned}$$

Then, for $d=k^2+h^2$ and $t=k^2-h^2$, we have

$$E(d,t,x)=P_y(h,k,x^2)\cdot P_y(h,-k,x^2).$$

Also

$$\text{Discriminant }(P_y,x)=16(h+k)^4(h^2+k^2-1)(h^2+k^2+1)(h^2+k^2)^4,$$

where every non-constant factor is obvious. There are similar formulas for $t=2hk$.

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Appendix: More About *E*

Here, we provide more details about the nature of $E = E(d,t,x)$ and its relation to $R = R(d,t,x)$.

Theorem 7.3. *Let* $1 < t < d$. *Then, all roots of the quartic*

$$E(d,t,\sqrt{x}) = 0$$

are real and positive.

Proof. Inspection of the unwieldy solution by radicals formula shows that all roots are real. For $d = t$, the discriminant is 0, and we have

$$E(t,t,x) = t^4(1 - 4t^2x + 4t^2x^2)^2.$$

The roots of the above are all positive, since $t > 1$. It has two distinct roots, each of multiplicity two. As d increases from t, these multiple roots immediately separate into distinct positive roots lying on the real axis, since the roots must remain real. Since the discriminant does not change sign for d in the open interval (t,∞), the roots remain henceforth distinct and real. If, for some $d > t$, one of them becomes negative, then, by continuity, one of them became 0 for a smaller value of d. However

$$E(d,t,0) = t^4,$$

so this cannot happen. The result follows. $\square$

Remark 7.4. One can show by implicit differentiation that as d increases from t, each of the two double roots separates horizontally into two simple roots at a rate that is initially infinite. However, the ratio of the rate of separation of the larger root to that of the smaller root is the larger root of

$$1 - (4t^2 - 1)x + x^2 = 0.$$

We now establish the relationship between E and R.

Theorem 7.5. *If we arrange the roots* s_i *of* $E(d,t,\sqrt{x})$, *so that* $0 \le s_1 \le s_2 \le s_3 \le s_4$, *then the positive square root of* s_3/s_2 *is a root of* $R(d,t,x)$.

Proof. Proceed as in Sect. 5. In $E(d,t,\sqrt{x})$, replace x by $\frac{tx}{(2d^2)}$ and call the new roots the s_i'. This change of variable results in a self-reciprocal polynomial, so $s_1's_4' = s_2's_3' = 1$. Thus, for the original roots, we have

$$s_1s_4 = s_2s_3 = t^2/(4d^4).$$

We now apply the cubic resolvent of Sect. 3 much as we did with a cubic resolvent in Sect. 5. We obtain

$$s_1s_2 + s_3s_4 = \frac{t^2(2d^2-1)}{2d^4},$$
$$s_1s_3 + s_2s_4 = \frac{2d^2-t^2}{2d^4},$$

and

$$s_1s_4 + s_2s_3 = \frac{t^2}{2d^4}.$$

Division of the second of the above equations by $s_2s_3 = s_1s_4$ yields

$$(s_1/s_2) + (s_2/s_1) = 2(2d^2-t^2)/t^2,$$

and similarly, the third yields

$$(s_1/s_3) + (s_3/s_1) = 2(2d^2-1).$$

Now, $x+(1/x) = r$ is equivalent to $x^2-rx+1 = 0$ and a simple calculation with resultants shows that if also $y^2-sy+1 = 0$, then xy is a root of $T = T(r,s,z)$, where

$$T = 1 - rsz + (r^2+s^2-2)z2 - rsz^3 + z^4.$$

Thus, to find a multiple of the minimal polynomial of

$$(s_3/s_2) = (s_3/s_1)(s_1/s_2),$$

we simply use the values of r and s provided by the above equations. Computer algebra then yields the factorization

$$T(r,s,z^2) = R(d,t,z)R(d,t,-z).$$

Thus, the positive square root of the ratio is a root of $R(d,t,z)$ and the result follows. □

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Kenneth B. Stolarsky
Mathematics Department
University of Illinois
Urbana
IL 61801
USA
e-mail: stolarsky.ken@gmail.com

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On the Andrews–Yee Identities Associated with Mock Theta Functions

Dedicated to Professor George Andrews on the occasion of his 80th birthday

Jin Wang and Xinrong Ma

Abstract. In this paper, we generalize the Andrews–Yee identities associated with the third-order mock theta functions $\omega(q)$ and $\nu(q)$. We obtain some q-series transformation formulas, one of which gives a new Bailey pair. Using the classical Bailey lemma, we derive a product formula for two ${}_2\phi_1$ series. We also establish recurrence relations and transformation formulas for two finite sums arising from the Andrews–Yee identities.

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Keywords. Mock theta functions, Bailey pair, The WZ method, Transformation formulas.

1. Introduction

Throughout this paper, we adopt the standard notation and terminology for q-series in [15]: For $|q| < 1$ and $n \geq 1$, the q-shifted factorials are defined by

$$\begin{aligned}(x;q)_0 &= 1,\\ (x;q)_n &= (1-x)(1-xq)\cdots(1-xq^{n-1}),\\ (x;q)_\infty &= \lim_{n\to\infty}(x;q)_n.\end{aligned}$$

The third-order mock theta functions $\omega(q)$ and $\nu(q)$ due to Ramanujan [19] and Watson [23] are defined by

$$\omega(q) = \sum_{n=0}^{\infty}\frac{q^{2n^2+2n}}{(q;q^2)_{n+1}^2}, \quad \nu(q) = \sum_{n=0}^{\infty}\frac{q^{n^2+n}}{(-q;q^2)_{n+1}}.$$

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Andrews [1] introduced the following two variable generalizations:

$$\omega(z;q)=\sum_{n=0}^{\infty}\frac{z^nq^{2n^2+2n}}{(q,zq;q^2)_{n+1}},\quad \nu(z;q)=\sum_{n=0}^{\infty}\frac{q^{n^2+n}}{(-zq;q^2)_{n+1}}. \tag{1.1}$$

Notice that for $z=1$, $\omega(z;q)$ and $\nu(z;q)$ reduce to $\omega(q)$ and $\nu(q)$. Andrews, Dixit and Yee [8] discovered that $\omega(q)$ and $\nu(q)$ serve as the generating functions for special integer partitions.

Andrews and Yee [11] obtained the following q-series identities associated with $\omega(q)$ and $\nu(q)$:

Lemma 1.1. ([11, Theorem 1, Eq. (6)/Eq. (7); Eq. (8)]). *We have*

$$\sum_{n=0}^{\infty}\frac{z^nq^{2n^2+2n+1}}{(q,zq;q^2)_{n+1}}=\sum_{n=1}^{\infty}\frac{z^{n-1}q^n}{(q;q^2)_n}, \tag{1.2}$$

$$\sum_{n=0}^{\infty}q^n(-zq^{n+1};q)_n(zq^{2n+2};q^2)_\infty=\sum_{n=0}^{\infty}\frac{z^nq^{n^2+n}}{(q;q^2)_{n+1}}, \tag{1.3}$$

$$\sum_{n=1}^{\infty}\frac{q^n}{(zq^n;q)_{n+1}(zq^{2n+2};q^2)_\infty}=\sum_{n=1}^{\infty}\frac{z^{n-1}q^n}{(q;q^2)_n}. \tag{1.4}$$

In this paper, we generalize the identities (1.3) and (1.4) and give a new proof of (1.2).

Theorem 1.2. *For $|y|<1$, we have*

$$\sum_{n=0}^{\infty}y^n(-zq^{n+1};q)_n(zq^{2n+2};q^2)_\infty$$
$$=\sum_{n=0}^{\infty}z^nq^{n^2+n}\frac{(-y;q)_n}{(yq^n;q)_{n+1}}\sum_{k=0}^{n}q^{2k}\frac{(y/q;q)_{2k}}{(q^2,y^2;q^2)_k}, \tag{1.5}$$

$$\sum_{n=1}^{\infty}\frac{y^{n-1}}{(zq^n;q)_{n+1}(zq^{2n+2};q^2)_\infty}=\sum_{n=0}^{\infty}\frac{(-y;q)_n(qz)^n}{(yq^n;q)_{n+1}}\sum_{k=0}^{n}\frac{(y/q;q)_{2k}q^k}{(q^2,y^2;q^2)_k}. \tag{1.6}$$

In the process of proving Theorem 1.2, we obtain the following identities. Recall that the q-binomial coefficients are defined by

$$\begin{bmatrix}n\\k\end{bmatrix}_q=\frac{(q;q)_n}{(q;q)_{n-k}(q;q)_k}.$$

Theorem 1.3. *For any $n\geq 0$ and any complex number y not of the form $-q^{-m}$ with $m\geq 0$, the following identities hold:*

$$\sum_{k=0}^{n}q^k\frac{(yq^n;q)_k}{(q^2;q^2)_k}=(-y;q)_n\sum_{k=0}^{n}q^k\frac{(y/q;q)_{2k}}{(q^2,y^2;q^2)_k}, \tag{1.7}$$

$$\sum_{k=0}^{n}\begin{bmatrix}n\\k\end{bmatrix}_q y^kq^{k(k-1)/2}\frac{(q;q)_k}{(yq^n;q)_{k+1}}=\frac{(q^2,y^2;q^2)_n}{(y;q)_{2n+1}}\sum_{k=0}^{n}q^{2k}\frac{(y/q;q)_{2k}}{(q^2,y^2;q^2)_k}. \tag{1.8}$$

In particular, we have

$$\sum_{k=0}^{\infty} y^k q^{k(k-1)/2} = (q^2;q^2)_\infty(-y;q)_\infty\, {}_2\phi_1\left[\begin{matrix} y/q, y \\ y^2 \end{matrix}; q^2, q^2\right], \tag{1.9}$$

and for any $r \geq 0$,

$$(-q;q)_{2r}\sum_{k=0}^{\infty} q^{\frac{k(k+1)}{2}+2rk} = \frac{(q^2;q^2)_\infty}{(q;q^2)_\infty}\sum_{k=0}^{\infty} q^{2k}\frac{(q^{2r};q)_{2k}}{(q^2,q^{4r+2};q^2)_k}. \tag{1.10}$$

It should be noted that Andrews and Warnaar [10] used the Bailey transformation to deduce the following identity that is different from (1.10):

$$(-q;q)_{2r}\sum_{k=0}^{\infty} q^{\frac{k(k+1)}{2}+2rk} = \sum_{k=0}^{\infty}\frac{(q^{2r+1};q)_{2k}q^k}{(q^2,q^{4r+2};q^2)_k}. \tag{1.11}$$

We find that (1.8) gives a new Bailey pair:

$$\begin{cases} \alpha_n(y) = y^n q^{n(n-1)/2}, \\ \beta_n(y) = \dfrac{(-q,-y;q)_n}{(yq;q)_{2n}}\displaystyle\sum_{k=0}^{n}\frac{q^{2k}(y/q;q)_{2k}}{(q^2,y^2;q^2)_k}. \end{cases} \tag{1.12}$$

Applying the Bailey lemma to the Bailey pair $(\alpha_n(y), \beta_n(y))$, we get

Theorem 1.4. *Assume that* y *is not of the form* $-q^{-m}$ *with* $m \geq 0$. *Assume that* $|yq/ab| < 1$. *Then*

$$\sum_{n=0}^{\infty}\frac{(a,b;q)_n}{(yq/a,yq/b;q)_n}\left(\frac{y^2}{ab}\right)^n q^{n(n+1)/2}$$
$$= \frac{(yq,yq/ab;q)_\infty}{(yq/a,yq/b;q)_\infty}\sum_{n=0}^{\infty}\frac{(-q,-y,a,b;q)_n}{(yq;q)_{2n}}\left(\frac{yq}{ab}\right)^n\sum_{k=0}^{n}\frac{(y/q;q)_{2k}q^{2k}}{(q^2,y^2;q^2)_k}. \tag{1.13}$$

We also derive from (1.8) a formula for the product of two ${}_2\phi_1$ series in terms of a ${}_4\phi_3$ series. Other formulas for the products of two ${}_2\phi_1$ series can be found in [20]. The ${}_{r+1}\phi_r$ series is given by

$${}_{r+1}\phi_r\left[\begin{matrix} a_1, a_2, \ldots, a_{r+1} \\ b_1, b_2, \ldots, b_r \end{matrix}; q, x\right] = \sum_{n=0}^{\infty}\frac{(a_1,a_2,\ldots,a_{r+1};q)_n}{(q,b_1,b_2,\ldots,b_r;q)_n}x^n.$$

Theorem 1.5. *For any complex numbers* a *and* b *not of the form* $-q^{-m}$ *with* $m \geq 0$, *we have*

$${}_2\phi_1\left[\begin{matrix} a, a/q \\ a^2 \end{matrix}; q^2, q^2\right]{}_2\phi_1\left[\begin{matrix} b, b/q \\ b^2 \end{matrix}; q^2, q^2\right]$$
$$= \frac{(q;q^2)_\infty}{(q^2;q^2)_\infty}\,{}_4\phi_3\left[\begin{matrix} (ab)^{1/2}, -(ab)^{1/2}, (ab/q)^{1/2}, -(ab/q)^{1/2} \\ -a, -b, ab/q \end{matrix}; q, q\right]. \tag{1.14}$$

This paper is organized as follows: In Sect. 2, we prove Theorems 1.2–1.5, and give another proof of the identity (1.2). In Sect. 3, we define two finite sums $U_m(x)$ and $S_m(x,y)$ related to (1.2) and (1.3). We derive recurrence relations satisfied by $U_m(x)$ and deduce a transformation formula for $S_m(x,y)$.

2. Proofs of the Main Theorems

In this section, we give the proofs of Theorems 1.2–1.5 and present a proof of the identity (1.2) using the Lagrange inversion formula.

2.1. Proofs of Theorems 1.2 and 1.3

For notational convenience, we write $\tau(n)$ for $(-1)^n q^{n(n-1)/2}$. In addition, given any formal power series

$$f(x)=\sum_{n=0}^{\infty} a_n x^n,$$

we adopt the common notation

$$[x^n]f(x)=a_n,$$

for $n\geq 0$. To prove Theorem 1.2, we need the following transformation formulas:

Lemma 2.1. ([15, p. 360, (III.11)/(III.13)]).

$$ {}_3\phi_2\left[\begin{matrix} q^{-n},b,c \\ d,e \end{matrix};q,q\right]=\frac{(de/bc;q)_n}{(e;q)_n}\left(\frac{bc}{d}\right)^n {}_3\phi_2\left[\begin{matrix} q^{-n},d/b,d/c \\ d,de/bc \end{matrix};q,q\right], \tag{2.1}$$

$$ {}_3\phi_2\left[\begin{matrix} q^{-n},b,c \\ d,e \end{matrix};q,\frac{deq^n}{bc}\right]=\frac{(e/c;q)_n}{(e;q)_n} {}_3\phi_2\left[\begin{matrix} q^{-n},c,d/b \\ d,cq^{1-n}/e \end{matrix};q,q\right]. \tag{2.2}$$

Now we give the proof of Theorem 1.2.

Proof of Theorem 1.2. We first prove (1.5). Let

$$\sum_{n=0}^{\infty} f_n(y)z^n=\sum_{n=0}^{\infty} y^n(-zq^{n+1};q)_n(zq^{2n+2};q^2)_\infty. \tag{2.3}$$

Then (1.5) is equivalent to

$$f_m(y)=q^{m^2+m}\frac{(y^2;q^2)_m}{(y;q)_{2m+1}}\sum_{k=0}^{m}\frac{q^{2k}(y/q;q)_{2k}}{(q^2,y^2;q^2)_k}. \tag{2.4}$$

Thus we only need to prove (2.4). It is easy to see that

$$(-zq^{n+1};q)_n(-zq^{2n+2};q^2)_\infty=\frac{(-zq^{n+1};q)_\infty}{(-zq^{2n+1};q^2)_\infty}. \tag{2.5}$$

It follows from (2.3) and (2.5) that

$$\sum_{n=0}^{\infty} f_n(y)z^n=\sum_{n=0}^{\infty} y^n\frac{(-zq^{n+1};q)_\infty}{(-zq^{2n+1};q^2)_\infty}. \tag{2.6}$$

By the q-binomial theorem [15, p. 354, (II.1) and (II.2)], we obtain

$$\sum_{n=0}^{\infty} f_n(y)z^n=\sum_{n=0}^{\infty} y^n\sum_{i=0}^{\infty}\frac{\tau(i)}{(q;q)_i}(-zq^{n+1})^i\sum_{j=0}^{\infty}\frac{(-zq^{2n+1})^j}{(q^2;q^2)_j}. \tag{2.7}$$

Comparing the coefficients of z^m on both sides of (2.7), we see that

$$\begin{aligned} f_m(y) &= \sum_{n=0}^{\infty} y^n \sum_{i+j=m} (-1)^i \frac{\tau(i)q^{ni+i}}{(q;q)_i} \frac{(-1)^j(q^{2n+1})^j}{(q^2;q^2)_j} \\ &= (-1)^m q^m \sum_{i+j=m} \frac{\tau(i)}{(q;q)_i(q^2;q^2)_j} \sum_{n=0}^{\infty} q^{n(i+2j)} y^n \\ &= (-1)^m q^m \sum_{i+j=m} \frac{\tau(i)}{(q;q)_i(q^2;q^2)_j} \frac{1}{1-yq^{m+j}}. \end{aligned} \tag{2.8}$$

Multiplying both sides of (2.8) by $(-1)^m q^{-m}(1-yq^m)(q;q)_m$ yields

$$(-1)^m q^{-m}(1-yq^m)(q;q)_m f_m(y) = \sum_{j=0}^{m} \begin{bmatrix} m \\ m-j \end{bmatrix}_q \tau(m-j) \frac{(yq^m;q)_j}{(-q, yq^{m+1};q)_j}. \tag{2.9}$$

Noting that

$$\begin{bmatrix} n \\ k \end{bmatrix}_q \tau(k) = \frac{(q^{-n};q)_k}{(q;q)_k} q^{nk} \tag{2.10}$$

(see [15, p. 353, (I.42)]) and

$$\frac{(a;q)_{n-k}}{(b;q)_{n-k}} = \frac{(a;q)_n}{(b;q)_n} \frac{(q^{1-n}/b;q)_k}{(q^{1-n}/a;q)_k} \left(\frac{b}{a}\right)^k \tag{2.11}$$

(see [15, p. 351, (I.11)]), we obtain

$$\begin{aligned} \begin{bmatrix} m \\ m-j \end{bmatrix}_q \tau(m-j) &= \frac{(q^{-m};q)_{m-j}}{(q;q)_{m-j}} q^{m(m-j)} \\ &= \frac{(q^{-m};q)_m}{(q;q)_m} \frac{(q^{-m};q)_j}{(q;q)_j} q^{m(m-j)+(m+1)j} \\ &= \tau(m) q^j \frac{(q^{-m};q)_j}{(q;q)_j}. \end{aligned}$$

Thus (2.9) reduces to

$$(-1)^m q^{-m}(1-yq^m)(q;q)_m f_m(y) = \tau(m) {}_3\phi_2 \left[\begin{matrix} q^{-m}, yq^m, 0 \\ -q, yq^{m+1} \end{matrix}; q, q \right]. \tag{2.12}$$

Taking

$$(n, b, c, d, e) = (m, yq^m, 0, yq^{m+1}, -q)$$

in the transformation formula (2.1), we get

$${}_3\phi_2 \left[\begin{matrix} q^{-m}, yq^m, 0 \\ -q, yq^{m+1} \end{matrix}; q, q \right] = \frac{q^{m(m+1)/2}}{(-q;q)_m} {}_2\phi_1 \left[\begin{matrix} q^{-m}, q \\ yq^{m+1} \end{matrix}; q, -yq^m \right]. \tag{2.13}$$

Substituting (2.13) into (2.12) gives

$$(-1)^m q^{-m}(1-yq^m)(q;q)_m f_m(y) = \frac{(-1)^m q^{m^2}}{(-q;q)_m}\sum_{k=0}^{m}\begin{bmatrix} m \\ k \end{bmatrix}_q y^k q^{k(k-1)/2}\frac{(q;q)_k}{(yq^{m+1};q)_k},$$

or, equivalently,

$$q^{-m^2-m}(q^2;q^2)_m f_m(y) = \sum_{k=0}^{m}\begin{bmatrix} m \\ k \end{bmatrix}_q y^k q^{k(k-1)/2}\frac{(q;q)_k}{(yq^m;q)_{k+1}}. \tag{2.14}$$

To compute $f_m(y)$, we define

$$S_m(k) = \begin{bmatrix} m \\ k \end{bmatrix}_q y^k q^{k(k-1)/2}\frac{(q;q)_k}{(yq^m;q)_{k+1}} \quad \text{and} \quad T_m = \sum_{k=0}^{m} S_m(k).$$

By the WZ method (see [18]), we find that $S_m(k)$ satisfies the following recurrence relation

$$S_{m+1}(k) - \frac{(1-q^{2m+2})(1-y^2q^{2m})}{(1-yq^{2m+1})(1-yq^{2m+2})}S_m(k) = G(m,k) - G(m,k-1), \tag{2.15}$$

where

$$G(m,k) = \frac{yq^{k+m+1}+q^{k+1}-q^{m+1}-1}{(1-yq^{2m+1})\,(1-yq^{2m+2})} \times \frac{(q;q)_{k+1}}{(yq^{m+1};q)_{k+1}}\begin{bmatrix} m+1 \\ k+1 \end{bmatrix}_q y^{k+1}q^{\frac{k(k-1)}{2}+m}.$$

Summing both sides of (2.15) over k from 0 to m, we get

$$T_m - \frac{(1-q^{2m})(1-y^2q^{2m-2})}{(1-yq^{2m-1})(1-yq^{2m})}T_{m-1} = \frac{q^{2m-1}(q-y)}{(1-yq^{2m-1})\,(1-yq^{2m})}, \tag{2.16}$$

where $T_0 = 1/(1-y)$. Therefore,

$$T_m = \frac{(q^2,y^2;q^2)_m}{(y;q)_{2m+1}}\sum_{k=0}^{m}\frac{q^{2k}(y/q;q)_{2k}}{(q^2,y^2;q^2)_k}. \tag{2.17}$$

Substituting (2.17) into (2.14) yields (2.4).

Next we proceed to prove (1.6). Let

$$\sum_{n=0}^{\infty} g_n(y)z^n = \sum_{n=1}^{\infty}\frac{y^n}{(zq^n;q)_{n+1}(zq^{2n+2};q^2)_\infty}. \tag{2.18}$$

In order to prove (1.6), it suffices to show that

$$g_m(y) = yq^m\frac{(-y;q)_m}{(yq^m;q)_{m+1}}\sum_{k=0}^{m} q^k\frac{(y/q;q)_{2k}}{(q^2,y^2;q^2)_k}. \tag{2.19}$$

Observing that

$$\frac{1}{(zq^n;q)_{n+1}(zq^{2n+2};q^2)_\infty} = \frac{(zq^{2n+1};q^2)_\infty}{(zq^n;q)_\infty},$$

we get

$$\sum_{n=0}^{\infty} g_n(y)z^n = \sum_{n=1}^{\infty} y^n \frac{(zq^{2n+1};q^2)_\infty}{(zq^n;q)_\infty}. \tag{2.20}$$

Comparing the coefficients of z^m on both sides of (2.20) yields

$$g_m(y) = \sum_{n=1}^{\infty} y^n [z^m] \frac{(zq^{2n+1};q^2)_\infty}{(zq^n;q)_\infty}. \tag{2.21}$$

Invoking the q-binomial theorem [15, p. 354, (II.3)], we get

$$\frac{(zq^{2n+1};q^2)_\infty}{(zq^n;q)_\infty} = \sum_{i=0}^{\infty} \frac{(zq^n)^i}{(q;q)_i} \sum_{j=0}^{\infty} \frac{(-1)^j q^{j^2-j}(zq^{2n+1})^j}{(q^2;q^2)_j}. \tag{2.22}$$

Combining (2.21) and (2.22), we have

$$\begin{aligned} g_m(y) &= \sum_{n=1}^{\infty} y^n \sum_{i+j=m} \frac{q^{ni}}{(q;q)_i} \frac{(-1)^j q^{j^2-j}(q^{2n+1})^j}{(q^2;q^2)_j} \\ &= \sum_{i+j=m} \frac{(-1)^j q^{j^2}}{(q;q)_i (q^2;q^2)_j} \sum_{n=1}^{\infty} (yq^{i+2j})^n \\ &= \frac{yq^m}{1-yq^m} \sum_{i+j=m} \frac{(-1)^j q^{j^2+j}}{(q;q)_i (q;q)_j (-q;q)_j} \frac{(yq^m;q)_j}{(yq^{m+1};q)_j}. \end{aligned} \tag{2.23}$$

Now, the identity (2.23) can be written as

$$(1-yq^m)(q;q)_m g_m(y) = yq^m \lim_{b\to\infty} {}_3\phi_2\left[\begin{matrix} q^{-m}, yq^m, b \\ -q, yq^{m+1} \end{matrix}; q, -\frac{q^{m+2}}{b}\right]. \tag{2.24}$$

Taking

$$(n, b, c, d, e) = (m, b, yq^m, -q, yq^{m+1})$$

in the transformation formula (2.2), we get

$${}_3\phi_2\left[\begin{matrix} q^{-m}, yq^m, b \\ -q, yq^{m+1} \end{matrix}; q, -\frac{q^{m+2}}{b}\right] = \frac{(q;q)_m}{(yq^{m+1};q)_m} {}_3\phi_2\left[\begin{matrix} q^{-m}, yq^m, -q/b \\ -q, q^{-m} \end{matrix}; q, q\right]. \tag{2.25}$$

Substituting (2.25) into (2.24) gives

$$(1-yq^m)(q;q)_m g_m(y) = yq^m \frac{(q;q)_m}{(yq^{m+1};q)_m} \sum_{k=0}^{m} \frac{(yq^m;q)_k}{(q^2;q^2)_k} q^k,$$

which simplifies to

$$\frac{(yq^m;q)_{m+1}}{yq^m} g_m(y) = \sum_{k=0}^{m} \frac{(yq^m;q)_k}{(q^2;q^2)_k} q^k. \tag{2.26}$$

In order to compute $g_m(y)$, we define

$$S_m(k) = \frac{(yq^m;q)_k}{(q^2;q^2)_k} q^k \quad \text{and} \quad T_m = \sum_{k=0}^{m} S_m(k).$$

By the WZ method, we find that $S_m(k)$ admits the following recurrence relation:

$$S_{m+1}(k)-(1+yq^m)S_m(k)=G(m,k)-G(m,k-1), \tag{2.27}$$

where

$$G(m,k)=-yq^m\frac{(yq^{m+1};q)_k}{(q^2;q^2)_k}.$$

Summing both sides of (2.27) over k from 0 to m, we obtain that

$$T_{m+1}-(1+yq^m)T_m=\frac{q^m(q-y)(yq^{m+1};q)_m}{(q^2;q^2)_{m+1}},$$

where $T_0=1$. It follows that

$$T_m=(-y;q)_m\sum_{k=0}^{m}q^k\frac{(y/q;q)_{2k}}{(q^2,y^2;q^2)_k}. \tag{2.28}$$

Combining (2.26) and (2.28), we reach (2.19). □

Using the relations in the proof of Theorem 1.2, we give a proof of Theorem 1.3.

Proof of Theorem 1.3. Combining (2.19) and (2.26), we obtain (1.7). Substituting (2.4) into (2.14) yields (1.8). Taking the limitation of (1.8) as n tends to infinity, we arrive at (1.9). Setting $y=q^{2r+1}$ in (1.9), we get (1.10). □

2.2. Proofs of Theorems 1.4 and 1.5

Theorem 1.4 can be proved using (1.8). We need the definition of a Bailey pair, which can be found in Andrews [3, p. 26, Remark].

Definition 2.2. A pair of sequences $\{\alpha_n(t)\}_{n\geq 0}$ and $\{\beta_n(t)\}_{n\geq 0}$ that satisfies

$$\beta_n(t)=\sum_{k=0}^{n}\frac{\alpha_k(t)}{(q;q)_{n-k}(tq;q)_{n+k}} \tag{2.29}$$

is called a Bailey pair relative to t.

As demonstrated by Andrews and Berndt [6, pp. 251–259, Sect. 11.5] and [7, pp. 97–112, Chap. 5], the Bailey lemma is a basic tool to study Ramanujan's mock theta function identities. The Bailey lemma [12, Eq. (3.1)] can be stated as follows. See also [3, p. 27, Eq. (3.33)], [21, p. 99, Eq. (3.4.9)] and [22]:

Lemma 2.3. *(Bailey's Lemma). For any Bailey pair $(\alpha_n(t),\beta_n(t))$, we have*

$$\sum_{n=0}^{\infty}\frac{(a,b;q)_n}{(tq/a,tq/b;q)_n}\left(\frac{tq}{ab}\right)^n\alpha_n(t)$$
$$=\frac{(tq,tq/ab;q)_\infty}{(tq/a,tq/b;q)_\infty}\sum_{n=0}^{\infty}(a,b;q)_n\left(\frac{tq}{ab}\right)^n\beta_n(t). \tag{2.30}$$

Proof of Theorem 1.4. Consider the sequences $\alpha_n(y)$ and $\beta_n(y)$ given in (1.12):

$$\alpha_n(y) = y^n q^{n(n-1)/2}, \tag{2.31}$$

$$\beta_n(y) = \frac{(-q,-y;q)_n}{(yq;q)_{2n}} \sum_{k=0}^{n} \frac{q^{2k}(y/q;q)_{2k}}{(q^2,y^2;q^2)_k}. \tag{2.32}$$

By virtue of (1.8), we see that

$$\beta_n(y) = \sum_{k=0}^{n} \frac{\alpha_k(y)}{(q;q)_{n-k}(yq;q)_{n+k}}.$$

Thus $(\alpha_n(y), \beta_n(y))$ forms a Bailey pair. Utilizing Lemma 2.3, we obtain that

$$\sum_{n=0}^{\infty} \frac{(a,b;q)_n}{(yq/a,yq/b;q)_n} \left(\frac{y^2q}{ab}\right)^n q^{n(n-1)/2}$$
$$= \frac{(yq,yq/ab;q)_\infty}{(yq/a,yq/b;q)_\infty} \sum_{n=0}^{\infty} (a,b;q)_n \left(\frac{yq}{ab}\right)^n \frac{(-q,-y;q)_n}{(yq;q)_{2n}} \sum_{k=0}^{n} \frac{q^{2k}(y/q;q)_{2k}}{(q^2,y^2;q^2)_k}.$$

This completes the proof. ☐

As a consequence of Theorem 1.4, we obtain

Corollary 2.4. *For* $|q^2/ab| < 1$,

$$\sum_{n=0}^{\infty} \frac{(a,b;q)_n}{(q^2/a,q^2/b;q)_n} \left(\frac{1}{ab}\right)^n q^{n(n+5)/2}$$
$$= \frac{(q^2,q^2/ab;q)_\infty}{(q^2/a,q^2/b;q)_\infty} {}_3\phi_2\left[\begin{matrix} a,b,-q \\ q^{3/2},-q^{3/2} \end{matrix}; q, \frac{q^2}{ab}\right]. \tag{2.33}$$

In particular,

$$\sum_{n=0}^{\infty} q^{3n(n+1)/2} = (q;q)_\infty \sum_{n=0}^{\infty} \frac{(-q;q)_n}{(q;q)_n(q;q^2)_{n+1}} q^{n(n+1)}, \tag{2.34}$$

$$\sum_{n=0}^{\infty} \frac{(a,-q^{3/2};q)_n}{(q^2/a,-q^{1/2};q)_n} \left(-\frac{1}{a}\right)^n q^{n(n+2)/2} = \frac{(q^2,q^{3/2}/a;q)_\infty}{(q^2/a,q^{3/2};q)_\infty}. \tag{2.35}$$

Proof. Taking $y = q$ in (1.13) yields (2.33). As a, b in (2.33) tend to infinity, we are led to (2.34). Considering $b = -q^{3/2}$ in (2.33) and then applying the q-Gauss ${}_2\phi_1$ sum [15, p. 354, (II.8)], we obtain (2.35). ☐

Using (2.33), we establish the following transformation formula:

Corollary 2.5. *For* $|ab| > 1$,

$$\sum_{n=-\infty}^{\infty} \frac{(aq,bq;q)_n}{(q/a,q/b;q)_n} \left(\frac{1}{ab}\right)^n q^{n(n+1)/2}$$
$$= 2\frac{(q^2,1/ab;q)_\infty}{(q/a,q/b;q)_\infty} {}_3\phi_2\left[\begin{matrix} aq,bq,-q \\ q^{3/2},-q^{3/2} \end{matrix}; q, \frac{1}{ab}\right]. \tag{2.36}$$

Proof. Notice that

$$\sum_{n=-\infty}^{\infty} \frac{(aq,bq;q)_n}{(q/a,q/b;q)_n}\left(\frac{1}{ab}\right)^n q^{n(n+1)/2}$$
$$=\left\{\sum_{n=0}^{\infty}+\sum_{n=-\infty}^{-1}\right\}\frac{(aq,bq;q)_n}{(q/a,q/b;q)_n}\left(\frac{1}{ab}\right)^n q^{n(n+1)/2}. \tag{2.37}$$

Using (2.11), we have

$$\sum_{n=-\infty}^{-1} \frac{(aq,bq;q)_n}{(q/a,q/b;q)_n}\left(\frac{1}{ab}\right)^n q^{n(n+1)/2}$$
$$=\sum_{n=1}^{\infty}\frac{(a,b;q)_n}{(1/a,1/b;q)_n}\left(\frac{1}{ab}\right)^n q^{n(n-1)/2}$$
$$\overset{n-1\to n}{=}\sum_{n=0}^{\infty}\frac{(aq,bq;q)_n}{(q/a,q/b;q)_n}\left(\frac{1}{ab}\right)^n q^{n(n+1)/2}. \tag{2.38}$$

Substituting (2.38) into (2.37), we find that

$$\sum_{n=-\infty}^{\infty} \frac{(aq,bq;q)_n}{(q/a,q/b;q)_n}\left(\frac{1}{ab}\right)^n q^{n(n+1)/2}=2\sum_{n=0}^{\infty}\frac{(aq,bq;q)_n}{(q/a,q/b;q)_n}\left(\frac{1}{ab}\right)^n q^{n(n+1)/2}.$$

Applying (2.33) to the above identity with a,b replace by aq,bq, respectively, we obtain (2.36). $\square$

We are now ready to prove Theorem 1.5.

Proof of Theorem 1.5. Let

$$\theta(q,x)=\sum_{n=0}^{\infty}\tau(n)x^n.$$

For $y=-a,-b$, the identity (1.9) in Theorem 1.3 yields

$$\theta(q,a)=(q^2;q^2)_\infty(a;q)_\infty\,{}_2\phi_1\left[\begin{matrix}-a/q,-a\\ a^2\end{matrix};q^2,q^2\right],$$
$$\theta(q,b)=(q^2;q^2)_\infty(b;q)_\infty\,{}_2\phi_1\left[\begin{matrix}-b/q,-b\\ b^2\end{matrix};q^2,q^2\right].$$

It follows that

$$\theta(q,a)\theta(q,b)=(q^2;q^2)_\infty^2(a,b;q)_\infty$$
$$\times\,{}_2\phi_1\left[\begin{matrix}-a/q,-a\\ a^2\end{matrix};q^2,q^2\right]{}_2\phi_1\left[\begin{matrix}-b/q,-b\\ b^2\end{matrix};q^2,q^2\right]. \tag{2.39}$$

On the other hand, Andrews and Warnaar [9, Theorem 1.1] have shown that

$$\theta(q,a)\theta(q,b)=(q,a,b;q)_\infty\sum_{n=0}^{\infty}\frac{(ab/q;q)_{2n}}{(q,a,b,ab/q;q)_n}q^n. \tag{2.40}$$

Combining (2.39) and (2.40), we have

$$(q^2;q^2)_\infty^2(a,b;q)_\infty {}_2\phi_1\left[\begin{matrix}-a/q,-a\\a^2\end{matrix};q^2,q^2\right]{}_2\phi_1\left[\begin{matrix}-b/q,-b\\b^2\end{matrix};q^2,q^2\right]$$
$$=(q,a,b;q)_\infty\sum_{n=0}^{\infty}\frac{(ab/q;q)_{2n}}{(q,a,b,ab/q;q)_n}q^n. \tag{2.41}$$

Replacing a,b by $-a,-b$ in (2.41), we arrive at (1.14), as claimed. □

2.3. Proof of Identity (1.2)

Andrews [2] utilized the transformation formula for ${}_3\phi_2$ series to give a proof of (1.2), and Chern [13] provided a combinatorial proof. We now present a proof of (1.2) by using the Lagrange inversion formula.

Lemma 2.6. *(The Lagrange inversion formula* [16, Example 2.2]*). Let* $\{x_n\}_{n\geq 0}$ *be a complex sequences such that* $|1-x_nz|\neq 0$ *and* $F(z)$ *be an analytic function. If* $F(z)$ *has the following expansion:*

$$F(z)=\sum_{n=0}^{\infty}a_n\frac{z^n}{\prod_{i=1}^{n+1}(1-x_iz)}, \tag{2.42a}$$

then

$$a_n=[z^n]F(z)\prod_{i=1}^{n}(1-x_iz). \tag{2.42b}$$

Proof. By Lemma 2.6, we see that (1.2) is equivalent to

$$\frac{q^{2n^2+2n+1}}{(q;q^2)_{n+1}}=[z^n]\left(\sum_{k=1}^{\infty}\frac{q^kz^{k-1}}{(q;q^2)_k}(zq;q^2)_n\right). \tag{2.43}$$

We claim that (2.43) is equivalent to the following identity:

$$\sum_{k=0}^{n}\left[\begin{matrix}1+2n\\2k\end{matrix}\right]_q(q;q^2)_k(-1)^kq^{k(k-1)}=q^{2n^2+n}. \tag{2.44}$$

Note that the right-hand side of (2.43) equals

$$\sum_{k=0}^{n}\frac{q^{k+1}}{(q;q^2)_{k+1}}[z^{n-k}]\frac{(zq;q^2)_\infty}{(zq^{2n+1};q^2)_\infty},$$

and so

$$\frac{q^{2n^2+2n+1}}{(q;q^2)_{n+1}}=\sum_{k=0}^{n}\frac{q^{k+1}}{(q;q^2)_{k+1}}[z^{n-k}]\frac{(zq;q^2)_\infty}{(zq^{2n+1};q^2)_\infty}. \tag{2.45}$$

Employing the relations (2.10) and (2.11), we have

$$[z^{n-k}]\frac{(zq^{1/2};q)_\infty}{(zq^{n+1/2};q)_\infty}=[z^{n-k}]\sum_{i=0}^{\infty}\frac{(q^{-n};q)_i}{(q;q)_i}(zq^{n+1/2})^i$$
$$=\frac{(q^{-n};q)_{n-k}}{(q;q)_{n-k}}q^{(n+1/2)(n-k)}$$

$$
\begin{aligned}
&= \frac{(q^{-n};q)_n}{(q;q)_n}\frac{(q^{-n};q)_k}{(q;q)_k}q^{(k+2n^2+n)/2} \\
&= \begin{bmatrix} n \\ k \end{bmatrix}_q (-1)^{n-k}q^{(n-k)^2/2}. \qquad (2.46)
\end{aligned}
$$

Replacing q by q^2 in (2.46), we obtain that

$$
[z^{n-k}]\frac{(zq;q^2)_\infty}{(zq^{2n+1};q^2)_\infty} = \frac{(q^2;q^2)_n}{(q^2;q^2)_k(q^2;q^2)_{n-k}}(-1)^{n-k}q^{(n-k)^2}. \qquad (2.47)
$$

Plugging (2.47) into (2.45), we find that

$$
q^{2n^2+2n+1} = \sum_{k=0}^{n} q^{k+1}\frac{(q;q^2)_{n+1}(q^2;q^2)_n}{(q;q^2)_{k+1}(q^2;q^2)_k}\frac{(-1)^{n-k}q^{(n-k)^2}}{(q^2;q^2)_{n-k}}. \qquad (2.48)
$$

Dividing by q^{n+1} on both sides of (2.48), we have

$$
\begin{aligned}
q^{2n^2+n} &= \sum_{k=0}^{n} q^{k-n}\frac{(q;q)_{2n+1}}{(q;q)_{2k+1}(q;q)_{2(n-k)}}\frac{(-1)^{n-k}q^{(n-k)^2}(q;q)_{2n-2k}}{(q^2;q^2)_{n-k}} \\
&\overset{k\to n-k}{=} \sum_{k=0}^{n}\begin{bmatrix} 1+2n \\ 2k \end{bmatrix}_q \frac{(-1)^k q^{k^2-k}(q;q)_{2k}}{(q^2;q^2)_k} \\
&= \sum_{k=0}^{n}\begin{bmatrix} 1+2n \\ 2k \end{bmatrix}_q (q;q^2)_k(-1)^k q^{k^2-k}.
\end{aligned}
$$

So the claim is confirmed.

Therefore, we only need to prove (2.44). Recall the q-Kummer (Bailey-Daum) sum [15, p. 351, (II.9)]:

$$
\sum_{k=0}^{\infty}\frac{(a,b;q)_k}{(q,aq/b;q)_k}(-q/b)^k = \frac{(-q;q)_\infty(aq,aq^2/b^2;q^2)_\infty}{(-q/b,aq/b;q)_\infty}. \qquad (2.49)
$$

Setting $a=q^{-n}$ and letting b tend to infinity in (2.49), we find that

$$
\sum_{k=0}^{n}\begin{bmatrix} n \\ k \end{bmatrix}_q \tau(n-k)Y(k) = X(n), \qquad (2.50)
$$

where

$$
X(n) = \begin{cases} (-1)^k q^{k^2-k}(q;q^2)_k, & \text{if} \quad n=2k, \\ 0, & \text{if} \quad n=2k+1, \end{cases}
$$

and $Y(n) = q^{n(n-1)/2}$. Note that (2.50) is equivalent to

$$
\sum_{k=0}^{n}\begin{bmatrix} n \\ k \end{bmatrix}_q X(k) = Y(n). \qquad (2.51)
$$

Hence (2.44) holds, and the proof is complete. □

3. Two Finite q-Series Sums

In this section, we introduce two finite sums related to the Andrews–Yee identities in Lemma 1.1. As noted by Andrews and Yee [11], the proofs of (1.2) and (1.3) rely on the identities

$$\sum_{k=0}^{m} \frac{(q;q)_{m+k}}{(q^2;q^2)_k} q^k = (q^2;q^2)_m, \tag{3.1}$$

$$\sum_{k=0}^{m} \frac{(q;q)_{m+k}}{(q^2;q^2)_k} q^{2k} = (q;q^2)_{m+1} + q^{m+1}(q^2;q^2)_m. \tag{3.2}$$

For $m \geq 0$, we define

$$U_m(x) = \sum_{k=0}^{m} \begin{bmatrix} m+k \\ k \end{bmatrix}_q \frac{x^k}{(-q;q)_k}, \tag{3.3}$$

$$S_m(x,y) = \sum_{k=0}^{m} \begin{bmatrix} m+k \\ k \end{bmatrix}_q \frac{(y;q)_k}{(x;q)_k} q^k. \tag{3.4}$$

When $x = q$ and $x = q^2$, $U_m(x)$ becomes the sums in (3.1) and (3.2) subject to a factor. More precisely, we have

$$U_m(q) = (-q;q)_m, \tag{3.5}$$

$$U_m(q^2) = \frac{(q;q^2)_{m+1}}{(q;q)_m} + q^{m+1}(-q;q)_m. \tag{3.6}$$

Notice that $S_m(-q,0) = U_m(q)$. We shall establish recurrence relations for $U_m(x)$ and a transformation formula for $S_m(x,y)$.

3.1. q-Difference Equations and a Three-Term Recurrence Relation for $U_m(x)$

We find that $U_m(x)$ satisfies the following recurrence relations:

Lemma 3.1. *For $m \geq 0$, we have*

$$U_m(xq^2) = U_m(x) - (1-q^{m+1})xU_{m+1}(x) + (x+1)x^{m+1}\frac{(q;q^2)_{m+1}}{(q;q)_m}, \tag{3.7}$$

$$q^{m+1}U_m(xq) = U_m(x) - (1-q^{m+1})U_{m+1}(x) + x^{m+1}\frac{(q;q^2)_{m+1}}{(q;q)_m}. \tag{3.8}$$

Proof. We first prove (3.7). Note that

$$\begin{aligned}(1-q^m)xU_m(x) &= \sum_{k=0}^{m} \frac{(q^m;q)_{k+1}}{(q^2;q^2)_{k+1}} x^{k+1}(1-q^{2k+2}) \\ &= \sum_{k=1}^{m+1} \frac{(q^m;q)_k}{(q^2;q^2)_k} x^k - \sum_{k=1}^{m+1} \frac{(q^m;q)_k}{(q^2;q^2)_k} (xq^2)^k. \end{aligned} \tag{3.9}$$

It is easy to check that the right-hand side of (3.9) equals

$$U_{m-1}(x) - U_{m-1}(xq^2) + (x+1)x^m \frac{(q;q^2)_m}{(q;q)_{m-1}}. \tag{3.10}$$

Hence

$$U_{m-1}(xq^2) = U_{m-1}(x) - (1-q^m)xU_m(x) + (x+1)x^m \frac{(q;q^2)_m}{(q;q)_{m-1}}.$$

Shifting m to $m+1$, we get (3.7).

To prove (3.8), let

$$V_m(x) = (q;q)_m U_m(x),$$

so that

$$\begin{aligned}
V_m(x) - V_{m-1}(x) &= \sum_{k=0}^{m} \frac{(q;q)_{m+k}}{(q^2;q^2)_k} x^k - \sum_{k=0}^{m-1} \frac{(q;q)_{m+k-1}}{(q^2;q^2)_k} x^k \\
&= \frac{(q;q)_{2m}}{(q^2;q^2)_m} x^m + \sum_{k=0}^{m-1} \frac{(q;q)_{m+k} - (q;q)_{m+k-1}}{(q^2;q^2)_k} x^k \\
&= (q;q^2)_m x^m - q^m \sum_{k=0}^{m-1} \frac{(q;q)_{m+k-1}}{(q^2;q^2)_k} (xq)^k \\
&= (q;q^2)_m x^m - q^m V_{m-1}(xq). \qquad (3.11)
\end{aligned}$$

Replacing m with $m+1$ in (3.11) yields (3.8). □

Example 3.2. The initial value of $U_m(x)$ is given in (3.5), that is, $U_m(q) = (-q;q)_m$. Combining (3.5) and (3.8), we can compute $U_m(q^k)$ for $k \geq 2$. For instance, setting $x = q$ in (3.8), we have

$$\begin{aligned}
q^{m+1} U_m(q^2) &= U_m(q) - (1-q^{m+1}) U_{m+1}(q) + q^{m+1} \frac{(q;q^2)_{m+1}}{(q;q)_m} \\
&= (-q;q)_m - (1-q^{m+1})(-q;q)_{m+1} + q^{m+1} \frac{(q;q^2)_{m+1}}{(q;q)_m} \\
&= (-q;q)_m q^{2m+2} + q^{m+1} \frac{(q;q^2)_{m+1}}{(q;q)_m},
\end{aligned}$$

which gives (3.6).

By Lemma 3.1, we find that $U_m(x)$ satisfies a second-order q-difference equation.

Theorem 3.3. *We have*

$$U_m(xq^2) = (1-x)U_m(x) + xq^{m+1} U_m(xq) + x^{m+1} \frac{(q;q^2)_{m+1}}{(q;q)_m}. \qquad (3.12)$$

Proof. Rewriting (3.8) as

$$(1-q^{m+1}) U_{m+1}(x) = U_m(x) - q^{m+1} U_m(xq) + x^{m+1} \frac{(q;q^2)_{m+1}}{(q;q)_m}, \qquad (3.13)$$

and substituting (3.13) into (3.7), we find that

$$U_m(xq^2) = U_m(x) - x \left(U_m(x) - q^{m+1} U_m(xq) + x^{m+1} \frac{(q;q^2)_{m+1}}{(q;q)_m} \right)$$

$$+ (x+1)x^{m+1}\frac{(q;q^2)_{m+1}}{(q;q)_m}$$
$$= (1-x)U_m(x) + xq^{m+1}U_m(xq) + x^{m+1}\frac{(q;q^2)_{m+1}}{(q;q)_m}.$$

This completes the proof. □

Employing Lemma 3.1, we deduce a three-term recurrence relation for $\{U_m(x)\}_{m\geq 0}$.

Theorem 3.4. *For $m \geq 0$, we have*

$$(1-q^{m+2})U_{m+2}(x) + (xq^{2m+3}-q-1)U_{m+1}(x) + q(1+q^{m+1})U_m(x)$$
$$= (x-q)\frac{(q^{m+2};q)_m}{(q^2;q^2)_m}x^{m+1}. \tag{3.14}$$

Proof. It follows from (3.8) that

$$q^{2m+3}U_m(xq^2) = q^{m+2}U_m(xq) - (1-q^{m+1})q^{m+2}U_{m+1}(xq)$$
$$+ x^{m+1}q^{2m+3}\frac{(q;q^2)_{m+1}}{(q;q)_m}.$$

Applying (3.8) to $U_m(xq)$ and $U_{m+1}(xq)$, we have

$$q^{2m+3}U_m(xq^2) = qU_m(x) - (1+q)(1-q^{m+1})U_{m+1}(x)$$
$$+ (1-q^{m+1})(1-q^{m+2})U_{m+2}(x)$$
$$+ qx^{m+1}\frac{(q;q^2)_{m+1}}{(q;q)_m} - x^{m+2}\frac{(q;q^2)_{m+2}}{(q;q)_m}$$
$$+ x^{m+1}q^{2m+3}\frac{(q;q^2)_{m+1}}{(q;q)_m}. \tag{3.15}$$

Multiplying both sides of (3.7) by q^{2m+3}, we get

$$q^{2m+3}U_m(xq^2) = q^{2m+3}U_m(x) - q^{2m+3}(1-q^{m+1})xU_{m+1}(x)$$
$$+ q^{2m+3}(x+1)x^{m+1}\frac{(q;q^2)_{m+1}}{(q;q)_m}. \tag{3.16}$$

Subtracting (3.16) from (3.15), we have

$$0 = q(1-q^{2m+2})U_m(x) + (xq^{2m+3}-1-q)(1-q^{m+1})U_{m+1}(x)$$
$$+ (1-q^{m+1})(1-q^{m+2})U_{m+2}(x) + qx^{m+1}\frac{(q;q^2)_{m+1}}{(q;q)_m}$$
$$- x^{m+2}\frac{(q;q^2)_{m+1}}{(q;q)_m},$$

which gives (3.14). □

Remark 3.5. Let $V_m(k;x)$ be the summand of $U_m(x)$, that is,

$$V_m(k;x) = \begin{bmatrix} m+k \\ k \end{bmatrix}_q \frac{x^k}{(-q;q)_k}.$$

In light of the WZ method, we find that $V_m(k;x)$ satisfies the recurrence relation:

$$V_{m+2}(k;x)+\frac{-xq^{2m+3}+q+1}{q^{m+2}-1}V_{m+1}(k;x)+\frac{q(q^{m+1}+1)}{1-q^{m+2}}V_m(k;x)$$
$$=V_m(k;x)R(m,k;x)-V_m(k-1;x)R(m,k-1;x), \tag{3.17}$$

where

$$R(m,k;x)=\frac{xq^{2m+3}(1-q^{m+k+1})}{(1-q^{m+1})(1-q^{m+2})}.$$

The recurrence relation (3.14) can be derived by summing both sides of (3.17) over k from 0 to m.

3.2. A Transformation Formula for $S_m(x,y)$

We derive the following transformation formula for $S_m(x,y)$ by the WZ method.

Theorem 3.6. *For any $m\geq 0$,*

$$\frac{(yq^2/x;q)_m}{(q^2/x;q)_m}\sum_{k=0}^{m}\begin{bmatrix}m+k\\k\end{bmatrix}_q\frac{(y;q)_k}{(x;q)_k}q^k$$
$$=1+\frac{q+x}{x-yq}\sum_{k=1}^{m}\begin{bmatrix}2k-1\\k\end{bmatrix}_q\frac{(y,yq/x;q)_k}{(x,q^2/x;q)_k}q^k$$
$$-\frac{yq}{x-yq}\sum_{k=1}^{m}\begin{bmatrix}2k\\k\end{bmatrix}_q\frac{(y,yq/x;q)_k}{(x,q^2/x;q)_k}q^{2k}. \tag{3.18}$$

Proof. Using the WZ method, we find that $\{S_m(x,y)\}_{m\geq 0}$ satisfies the first-order recurrence relation:

$$S_m(x,y)-\frac{x-q^{m+1}}{x-yq^{m+1}}S_{m-1}(x,y)$$
$$=q^m\frac{q+x-q^{m+1}(1+q^m)y}{x-q^{m+1}y}\begin{bmatrix}2m-1\\m\end{bmatrix}_q\frac{(y;q)_m}{(x;q)_m},$$

and thus

$$S_m(x,y)=\frac{(q^2/x;q)_m}{(yq^2/x;q)_m}\Bigg(1+\frac{1}{x-yq}\sum_{k=1}^{m}q^k(q+x-y(q^k+1)q^{k+1})$$
$$\times\begin{bmatrix}2k-1\\k\end{bmatrix}_q\frac{(y,yq/x;q)_k}{(x,q^2/x;q)_k}\Bigg),$$

which leads to (3.18). ☐

Setting $x=-q$ and $y=0$ in Theorem 3.6, respectively, we arrive at the following identities:

Corollary 3.7. *For any $m\geq 0$,*

$$\frac{(-yq;q)_m}{(-q;q)_m}\sum_{k=0}^{m}\frac{(q^{m+1},y;q)_k}{(q^2;q^2)_k}q^k=1+\frac{y}{1+y}\sum_{k=1}^{m}\begin{bmatrix}2k\\k\end{bmatrix}_q\frac{(y^2;q^2)_k}{(-q;q)_k^2}q^{2k},$$

$$\frac{1}{(q^2/x;q)_m}\sum_{k=0}^{m}\begin{bmatrix}m+k\\k\end{bmatrix}_q\frac{q^k}{(x;q)_k}=1+(1+q/x)\sum_{k=1}^{m}\begin{bmatrix}2k-1\\k\end{bmatrix}_q\frac{q^k}{(x,q^2/x;q)_k}.$$

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Jin Wang
Department of Mathematics
Zhejiang Normal University
JinHua 321004
People's Republic of China
e-mail: jinwang@zjnu.edu.cn

Xinrong Ma
Department of Mathematics
Soochow University
Suzhou 215006
People's Republic of China
e-mail: xrma@suda.edu.cn

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Congruences for q-Binomial Coefficients

To George Andrews, with warm q-wishes and well-looking q-congruences

Wadim Zudilin

Abstract. We discuss q-analogues of the classical congruence $\binom{ap}{bp} \equiv \binom{a}{b}$ (mod p^3), for primes $p > 3$, as well as its generalisations. In particular, we prove related congruences for (q-analogues of) integral factorial ratios.

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Keywords. Congruence, q-Binomial coefficient, Cyclotomic polynomial, Radial asymptotics.

1. Introduction

For a non-negative integer a, a standard q-environment includes the q-numbers $[a] = [a]_q = (1-q^a)/(1-q) \in \mathbb{Z}[q]$, the q-factorials $[a]! = [1][2]\cdots[a] \in \mathbb{Z}[q]$ and the q-binomial coefficients

$$\begin{bmatrix} a \\ b \end{bmatrix} = \begin{bmatrix} a \\ b \end{bmatrix}_q = \frac{[a]!}{[b]!\,[a-b]!} \in \mathbb{Z}[q], \quad \text{where } b = 0, 1, \dots, a.$$

One also adopts the cyclotomic polynomials

$$\Phi_n(q) = \prod_{\substack{j=1 \\ (j,n)=1}}^{n} (q - e^{2\pi i j/n}) \in \mathbb{Z}[q]$$

as q-analogues of prime numbers, because these are the only factors of the q-numbers which are irreducible over $\mathbb{Q}$.

Arithmetically significant relations often possess several q-analogues. While looking for q-extensions of the classical (Wolstenholme–Ljunggren) congruence

$$\binom{ap}{bp} \equiv \binom{a}{b} \pmod{p^3} \quad \text{for any prime } p > 3, \tag{1.1}$$

more precisely, at a 'q-microscope setup' (when q-congruences for truncated hypergeometric sums are read off from the asymptotics of their non-terminating versions, usually equipped with extra parameters, at roots of unity, see [5]) for Straub's q-congruence [8], [9, Theorem 2.2],

$$\begin{bmatrix} an \\ bn \end{bmatrix}_q \equiv \begin{bmatrix} a \\ b \end{bmatrix}_{q^{n^2}} - b(a-b)\binom{a}{b}\frac{n^2-1}{24}(q^n-1)^2 \pmod{\Phi_n(q)^3}, \tag{1.2}$$

this author accidentally arrived at

$$\begin{bmatrix} an \\ bn \end{bmatrix}\sigma_n^b q^{\binom{bn}{2}} \equiv \binom{a-1}{b} + \binom{a-1}{a-b}\sigma_n^a q^{\binom{an}{2}} \pmod{\Phi_n(q)^2}, \tag{1.3}$$

where the notation

$$\sigma_n = (-1)^{n-1}$$

is implemented. Notice that the expression on the right-hand side is a sum of two q-monomials. The q-congruence (1.3) may be compared with another q-extension of (1.1),

$$\begin{bmatrix} an \\ bn \end{bmatrix}_q \equiv \sigma_n^{b(a-b)} q^{b(a-b)\binom{n}{2}} \begin{bmatrix} a \\ b \end{bmatrix}_{q^n} \pmod{\Phi_n(q)^2} \tag{1.4}$$

for any $n > 1$. This is given by Andrews in [2] for primes $n = p > 3$ only; though proved modulo $\Phi_p(q)^2$, a complimentary result from [2] demonstrates that (1.1) in its full modulo p^3 strength can be derived from (1.4). More directly, Pan [7] shows that (1.4) can be generalised further to

$$\begin{bmatrix} an \\ bn \end{bmatrix}_q \equiv \sigma_n^{b(a-b)} q^{b(a-b)\binom{n}{2}} \begin{bmatrix} a \\ b \end{bmatrix}_{q^n} + ab(a-b)\binom{a}{b}\frac{n^2-1}{24}(q^n-1)^2 \pmod{\Phi_n(q)^3}. \tag{1.5}$$

It is worth mentioning that the transition from $\Phi_n(q)^2$ to $\Phi_n(q)^3$ (or, from p^2 to p^3) is significant because the former has a simple combinatorial proof (resulting from the q-Chu–Vandermonde identity) whereas no combinatorial proof is known for the latter.

Since $q^{\binom{bn}{2}} \sim \sigma_n^b$ as $q \to \zeta$, a primitive n-th root of unity, the congruence (1.3) is seen to be an extension of the trivial (q-Lucas) congruence

$$\begin{bmatrix} an \\ bn \end{bmatrix} \equiv \binom{a}{b} = \binom{a-1}{b} + \binom{a-1}{a-b} \pmod{\Phi_n(q)}.$$

The principal goal of this note is to provide a modulo $\Phi_n(q)^4$ extension of (1.3) (see Lemma 2.1 below) as well as to use the result for extending the congruences (1.2) and (1.5). In this way, our theorems provide two q-extensions of the congruence

$$\binom{ap}{bp} \equiv \binom{a}{b} + ab(a-b)\binom{a}{b}p\sum_{k=1}^{p-1}\frac{1}{k} \pmod{p^4} \quad \text{for prime } p > 3.$$

The latter can be continued further to higher powers of primes [6], and our 'mechanical' approach here suggests that one may try—with a lot of effort!—to deduce corresponding q-analogues.

Theorem 1.1. *The congruence*

$$\begin{bmatrix} an \\ bn \end{bmatrix}_q \equiv \begin{bmatrix} a \\ b \end{bmatrix}_{q^{n^2}} - b(a-b)\binom{a}{b}(q^n-1)\biggl(a\sum_{k=1}^{n-1}\frac{q^k}{1-q^k}+\frac{a(n-1)}{2} + \frac{(a+1)(n^2-1)}{24}(q^n-1)+\frac{(b(a-b)n-a-2)(n^2-1)}{48}(q^n-1)^2\biggr) \tag{1.6}$$

holds modulo $\Phi_n(q)^4$ *for any* $n>1$.

Theorem 1.2. *For any* $n>1$, *we have the congruence*

$$\begin{bmatrix} an \\ bn \end{bmatrix}_q \equiv \sigma_n^{b(a-b)} q^{b(a-b)\binom{n}{2}} \begin{bmatrix} a \\ b \end{bmatrix}_{q^n} - ab(a-b)\binom{a}{b}(q^n-1)\biggl(\sum_{k=1}^{n-1}\frac{q^k}{1-q^k}+\frac{n-1}{2} - \frac{(b(a-b)n-1)(n^2-1)}{48}(q^n-1)^2\biggr) \pmod{\Phi_n(q)^4}. \tag{1.7}$$

We point out that a congruence $A_1(q) \equiv A_2(q) \pmod{P(q)}$ for rational functions $A_1(q), A_2(q) \in \mathbb{Q}(q)$ and a polynomial $P(q) \in \mathbb{Q}[q]$ is understood as follows: the polynomial $P(q)$ is relatively prime with the denominators of $A_1(q)$ and $A_2(q)$, and $P(q)$ divides the numerator $A(q)$ of the difference $A_1(q)-A_2(q)$. The latter is equivalent to the condition that for each zero $\alpha \in \mathbb{C}$ of $P(q)$ of multiplicity k, the polynomial $(q-\alpha)^k$ divides $A(q)$ in $\mathbb{C}[q]$; in other words, $A_1(q) - A_2(q) = O\bigl((q-\alpha)^k\bigr)$ as $q \to \alpha$. This latter interpretation underlies our argument in proving the results. For example, the congruence (1.3) can be established by verifying that

$$\begin{bmatrix} an \\ bn \end{bmatrix}_q (1-\varepsilon)^{\binom{bn}{2}} = \binom{a-1}{b} + \binom{a-1}{a-b}\sigma_n^a(1-\varepsilon)^{\binom{an}{2}} + O(\varepsilon^2) \quad \text{as } \varepsilon \to 0^+, \tag{1.8}$$

when $q = \zeta(1-\varepsilon)$ and ζ is any primitive n-th root of unity.

Our approach goes in line with [5] and shares similarities with the one developed by Gorodetsky in [4], who reads off the asymptotic information of binomial sums at roots of unity through the q-Gauss congruences. It does not seem straightforward to us but Gorodetsky's method may be capable of proving Theorems 1.1 and 1.2. Furthermore, the part [4, Sect. 2.3] contains a survey on q-analogues of (1.1).

After proving an asymptotical expansion for q-binomial coefficients at roots of unity in Sect. 2 [essentially, the $O(\varepsilon^4)$-extension of (1.8)], we perform a similar asymptotic analysis for q-harmonic sums in Sect. 3. The information gathered is then applied in Sect. 4 to proving Theorems 1.1 and 1.2. Finally, in Sect. 5, we generalise the congruences (1.2) and (1.5) in a different direction, to integral factorial ratios.

2. Expansions of q-Binomials at Roots of Unity

This section is exclusively devoted to an asymptotical result, which forms the grounds of our later arithmetic analysis. We moderate its proof by highlighting principal ingredients (and difficulties) of derivation and leaving some technical details to the reader.

Lemma 2.1. *Let ζ be a primitive n-th root of unity. Then, as $q = \zeta(1-\varepsilon) \to \zeta$ radially,*

$$\begin{aligned}\begin{bmatrix} an \\ bn \end{bmatrix}_q \sigma_n^b q^{\binom{bn}{2}} - \binom{a-1}{b} - \binom{a-1}{a-b}\sigma_n^a q^{\binom{an}{2}} \\ = b(a-b)\binom{a}{b}\bigl(-\varepsilon^2 n^2 \rho_0(a,n) + \varepsilon^3 n^2 \rho_1(a,b,n) + \varepsilon^3\, anS_{n-1}(\zeta)\bigr) + O(\varepsilon^4),\end{aligned} \tag{2.1}$$

where

$$\begin{aligned}\rho_0(a,n) &= \frac{3(an-1)^2 - an^2 - 1}{24}, \\ \rho_1(a,b,n) &= \frac{abn^2(an-1)(an-n-2) + (an+2)(an-1)^2(an-3) + an^2 + a + 2}{48}\end{aligned}$$

and

$$S_{n-1}(q) = \frac{1}{2}\sum_{k=1}^{n-1} \frac{kq^k((k+1)q^k + k - 1)}{(1-q^k)^3}.$$

Proof. It follows from the q-binomial theorem [3, Chap. 10] that

$$(x;q)_N = \sum_{k=0}^{N} \begin{bmatrix} N \\ k \end{bmatrix}_q (-x)^k q^{\binom{k}{2}}. \tag{2.2}$$

Taking $N = an$, for a primitive n-th root of unity $\zeta = \zeta_n$, we have

$$\frac{1}{n}\sum_{j=1}^{n}(\zeta^j x;q)_{an} = \sum_{\substack{k=0 \\ n|k}}^{an} \begin{bmatrix} an \\ k \end{bmatrix}(-x)^k q^{k(k-1)/2} = \sum_{b=0}^{a} \begin{bmatrix} an \\ bn \end{bmatrix}(-x)^{bn} q^{bn(bn-1)/2}. \tag{2.3}$$

When $q = \zeta(1-\varepsilon)$, we get $\mathrm{d}/\mathrm{d}\varepsilon = -\zeta\,(\mathrm{d}/\mathrm{d}q)$. If

$$f(q) = (x;q)_{an} \quad \text{and} \quad g(q) = \frac{\mathrm{d}}{\mathrm{d}q}\log f(q) = -\sum_{\ell=1}^{an-1} \frac{\ell q^{\ell-1}x}{1-q^\ell x},$$

then $f(q)|_{\varepsilon=0} = (1-x^n)^a$ and

$$\frac{\mathrm{d}f}{\mathrm{d}q} = fg, \quad \frac{\mathrm{d}^2 f}{\mathrm{d}q^2} = f\left(g^2 + \frac{\mathrm{d}g}{\mathrm{d}q}\right), \quad \frac{\mathrm{d}^3 f}{\mathrm{d}q^3} = f\left(g^3 + 3g\frac{\mathrm{d}g}{\mathrm{d}q} + \frac{\mathrm{d}^2 g}{\mathrm{d}q^2}\right).$$

In particular,

$$\begin{aligned}\left.\frac{\mathrm{d}f}{\mathrm{d}\varepsilon}\right|_{\varepsilon=0} &= (1-x^n)^a \sum_{\ell=1}^{an-1} \frac{\ell\zeta^\ell x}{1-\zeta^\ell x}, \\ \left.\frac{\mathrm{d}^2 f}{\mathrm{d}\varepsilon^2}\right|_{\varepsilon=0} &= (1-x^n)^a\left(\left(\sum_{\ell=1}^{an-1} \frac{\ell\zeta^\ell x}{1-\zeta^\ell x}\right)^2 - \sum_{\ell=1}^{an-1}\left(\frac{\ell^2\zeta^{2\ell}x^2}{(1-\zeta^\ell x)^2} + \frac{\ell(\ell-1)\zeta^\ell x}{1-\zeta^\ell x}\right)\right)\end{aligned}$$

and

$$\frac{\mathrm{d}^3 f}{\mathrm{d}\varepsilon^3}\bigg|_{\varepsilon=0} = (1-x^n)^a\Bigg(\bigg(\sum_{\ell=1}^{an-1}\frac{\ell\zeta^\ell x}{1-\zeta^\ell x}\bigg)^3 - 3\sum_{\ell=1}^{an-1}\frac{\ell\zeta^\ell x}{1-\zeta^\ell x}\sum_{\ell=1}^{an-1}\bigg(\frac{\ell^2\zeta^{2\ell}x^2}{(1-\zeta^\ell x)^2}+\frac{\ell(\ell-1)\zeta^\ell x}{1-\zeta^\ell x}\bigg) + \sum_{\ell=1}^{an-1}\bigg(\frac{2\ell^3\zeta^{3\ell}x^3}{(1-\zeta^\ell x)^3}+\frac{3\ell^2(\ell-1)\zeta^{2\ell}x^2}{(1-\zeta^\ell x)^2}+\frac{\ell(\ell-1)(\ell-2)\zeta^\ell x}{1-\zeta^\ell x}\bigg)\Bigg).$$

Now observe the following summation formulae:

$$\frac{1}{n}\sum_{j=1}^{n}\frac{x}{1-x}\bigg|_{x\mapsto\zeta^j x} = \frac{x^n}{1-x^n},$$

$$\frac{1}{n}\sum_{j=1}^{n}\bigg(\frac{x}{1-x}\bigg)^2\bigg|_{x\mapsto\zeta^j x} = \frac{nx^n}{(1-x^n)^2}-\frac{x^n}{1-x^n},$$

$$\frac{1}{n}\sum_{j=1}^{n}\frac{x}{1-x}\,\frac{\zeta^k x}{1-\zeta^k x}\bigg|_{x\mapsto\zeta^j x} = -\frac{x^n}{1-x^n}\quad\text{for } k\not\equiv 0 \pmod n,$$

$$\frac{1}{n}\sum_{j=1}^{n}\bigg(\frac{x}{1-x}\bigg)^3\bigg|_{x\mapsto\zeta^j x} = \frac{n^2x^n(1+x^n)}{2(1-x^n)^3}-\frac{3nx^n}{2(1-x^n)^2}+\frac{x^n}{1-x^n},$$

$$\frac{1}{n}\sum_{j=1}^{n}\frac{x}{1-x}\,\frac{\zeta^k x}{1-\zeta^k x}\,\frac{\zeta^\ell x}{1-\zeta^\ell x}\bigg|_{x\mapsto\zeta^j x} = \frac{x^n}{1-x^n}\quad\text{for } k\not\equiv 0,\ \ell\not\equiv 0,\ k\not\equiv\ell \pmod n,$$

and

$$\frac{1}{n}\sum_{j=1}^{n}\bigg(\frac{x}{1-x}\bigg)^2\frac{\zeta^k x}{1-\zeta^k x}\bigg|_{x\mapsto\zeta^j x} = \frac{nx^n}{(1-x^n)^2}\,\frac{\zeta^k}{1-\zeta^k}+\frac{x^n}{1-x^n}\quad\text{for } k\not\equiv 0 \pmod n.$$

Implementing this information in (2.3), we obtain

$$\sum_{b=0}^{a}\begin{bmatrix}an\\bn\end{bmatrix}(-x)^{bn}q^{bn(bn-1)/2}\bigg|_{q=\zeta(1-\varepsilon)} = (1-x^n)^a\Bigg(1+\varepsilon\frac{x^n}{1-x^n}\sum_{\ell=1}^{an-1}\ell - \frac{\varepsilon^2}{2}\,\frac{x^n}{1-x^n}\bigg(\sum_{\ell=1}^{an-1}\ell\bigg)^2 + \frac{\varepsilon^2}{2}\,\frac{nx^n}{(1-x^n)^2}\sum_{\substack{\ell_1,\ell_2=1\\ \ell_1\equiv\ell_2 \pmod n}}^{an-1}\ell_1\ell_2 - \frac{\varepsilon^2}{2}\bigg(\frac{nx^n}{(1-x^n)^2}-\frac{x^n}{1-x^n}\bigg)\sum_{\ell=1}^{an-1}\ell^2 - \frac{\varepsilon^2}{2}\,\frac{x^n}{1-x^n}\sum_{\ell=1}^{an-1}\ell(\ell-1)$$

$$+\cdots+\frac{\varepsilon^3}{2}\,\frac{nx^n}{(1-x^n)^2}\sum_{\substack{\ell_1,\ell_2,\ell_3=1\\ \ell_1\equiv\ell_2\not\equiv\ell_3 \pmod n}}^{an-1}\ell_1\ell_2\ell_3\,\frac{\zeta^{\ell_3-\ell_1}}{1-\zeta^{\ell_3-\ell_1}}+\cdots$$

$$+\cdots-\frac{\varepsilon^3}{2}\,\frac{nx^n}{(1-x^n)^2}\sum_{\substack{\ell_1,\ell_2=1\\ \ell_1\not\equiv\ell_2 \pmod n}}^{an-1}\ell_1^2\ell_2\,\frac{\zeta^{\ell_2-\ell_1}}{1-\zeta^{\ell_2-\ell_1}}+\cdots\Bigg)+O(\varepsilon^4),$$

where we intentionally omit all ordinary ε^3-terms—those that sum up to polynomials in a and n multiplied by powers of $x^n/(1-x^n)$, like the ones appearing as ε- and ε^2-terms. The exceptional ε^3-summands are computed separately:

$$\sum_{\substack{\ell_1,\ell_2,\ell_3=1\\ \ell_1\equiv\ell_2\not\equiv\ell_3 \pmod n}}^{an-1}\ell_1\ell_2\ell_3\,\frac{\zeta^{\ell_3-\ell_1}}{1-\zeta^{\ell_3-\ell_1}}=-a^3\sum_{k=1}^{n-1}\frac{k\zeta^k((k+1)\zeta^k+k-1)}{(1-\zeta^k)^3}$$
$$-\frac{a^3n(n-1)(3an(an-1)(an-2)+n^2(an-2a-1)-2)}{48}$$

and

$$\sum_{\substack{\ell_1,\ell_2=1\\ \ell_1\not\equiv\ell_2 \pmod n}}^{an-1}\ell_1^2\ell_2\,\frac{\zeta^{\ell_2-\ell_1}}{1-\zeta^{\ell_2-\ell_1}}=-a^2\sum_{k=1}^{n-1}\frac{k\zeta^k((k+1)\zeta^k+k-1)}{(1-\zeta^k)^3}$$
$$-\frac{a^2n(n-1)(an(an-1)(2an-3)-an^2-1)}{24}.$$

The finale of our argument is comparison of the coefficients of powers of x^n on both sides of the relation obtained; this way we arrive at the asymptotics in (2.1). □

3. A q-Harmonic Sum

Again, the notation ζ is reserved for a primitive n-th root of unity. For the sum

$$H_{n-1}(q)=\sum_{k=1}^{n-1}\frac{q^k}{1-q^k},$$

we have

$$\frac{\mathrm{d}H_{n-1}}{\mathrm{d}q}=\sum_{k=1}^{n-1}\frac{kq^{k-1}}{(1-q^k)^2},$$
$$\frac{\mathrm{d}^2H_{n-1}}{\mathrm{d}q^2}=\sum_{k=1}^{n-1}\frac{kq^{k-2}((k+1)q^k+k-1)}{(1-q^k)^3}=2q^{-2}S_{n-1}(q),$$

where $S_{n-1}(q)$ is defined in Lemma 2.1. It follows that, for $q=\zeta(1-\varepsilon)$,

$$H_{n-1}(q)=\sum_{k=1}^{n-1}\frac{\zeta^k}{1-\zeta^k}-\varepsilon\sum_{k=1}^{n-1}\frac{k\zeta^k}{(1-\zeta^k)^2} \tag{3.1}$$

$$
\begin{aligned}
&+\frac{\varepsilon^2}{2}\sum_{k=1}^{n-1}\frac{k\zeta^k((k+1)\zeta^k+k-1)}{(1-\zeta^k)^3}+O(\varepsilon^3)\\
&=-\frac{n-1}{2}+\frac{(n^2-1)n}{24}\,\varepsilon+S_{n-1}(\zeta)\varepsilon^2+O(\varepsilon^3)\\
&=-\frac{n-1}{2}-\frac{n^2-1}{24}\,(q^n-1)+\frac{(n-1)(n^2-1)}{48n}(q^n-1)^2\\
&+\frac{1}{n^2}S_{n-1}(\zeta)(q^n-1)^2+O(\varepsilon^3) \qquad (3.2)
\end{aligned}
$$

as $\varepsilon \to 0$, where we use

$$\varepsilon=-\frac{1}{n}\,(q^n-1)+\frac{n-1}{2n^2}\,(q^n-1)^2+O(\varepsilon^3) \quad \text{as } \varepsilon\to 0.$$

The latter asymptotics implies that

$$
\begin{aligned}
\sum_{k=1}^{n-1}\frac{q^k}{1-q^k} &\equiv -\frac{n-1}{2}-\frac{n^2-1}{24}\,(q^n-1)+\frac{(n-1)(n^2-1)}{48n}(q^n-1)^2\\
&+\frac{(q^n-1)^2}{2n^2}\sum_{k=1}^{n-1}\frac{kq^k((k+1)q^k+k-1)}{(1-q^k)^3} \pmod{\Phi_n(q)^3},
\end{aligned}
$$

which may be viewed as an extension of

$$\sum_{k=1}^{n-1}\frac{q^k}{1-q^k}\equiv-\frac{n-1}{2}-\frac{n^2-1}{24}\,(q^n-1) \pmod{\Phi_n(q)^2}$$

recorded, for example, in [6].

A different consequence of (3.2) is the following fact.

Lemma 3.1. *The term $\varepsilon^2S_{n-1}(\zeta)$ appearing in the expansion (2.1) can be replaced with*

$$H_{n-1}(q)+\frac{n-1}{2}+\frac{n^2-1}{24}\,(q^n-1)-\frac{(n-1)(n^2-1)}{48n}(q^n-1)^2+O(\varepsilon^3)$$

when $q=\zeta(1-\varepsilon)$ and $\varepsilon\to 0$.

4. Proof of the Theorems

To prove Theorems 1.1 and 1.2, we need to produce 'matching' asymptotics for

$$\begin{bmatrix} a\\ b\end{bmatrix}_{q^{n^2}} \quad\text{and}\quad \sigma_n^{b(a-b)}q^{b(a-b)\binom{n}{2}}\begin{bmatrix} a\\ b\end{bmatrix}_{q^n},$$

respectively. These happen to be easier than that from Lemma 2.1 because $q^{n^2}=(1-\varepsilon)^{n^2}$ and $q^n=(1-\varepsilon)^n$ do not depend on the choice of primitive n-th root of unity ζ when $q=\zeta(1-\varepsilon)$.

Lemma 4.1. *As $q=\zeta(1-\varepsilon)\to\zeta$ radially,*

$$\begin{aligned}&\begin{bmatrix}a\\b\end{bmatrix}_{q^{n^2}}\sigma_n^b q^{\binom{bn}{2}}-\binom{a-1}{b}-\binom{a-1}{a-b}\sigma_n^a q^{\binom{an}{2}}\\&\quad=b(a-b)\binom{a}{b}\big(-\varepsilon^2n^2\hat\rho_0(a,n)+\varepsilon^3n^2\hat\rho_1(a,b,n)\big)+O(\varepsilon^4),\end{aligned}$$

where

$$\begin{aligned}\hat\rho_0(a,n)&=\frac{3(an-1)^2-(a+1)n^2}{24},\\ \hat\rho_1(a,b,n)&=\frac{bn(an-1)((an-1)^2-(a+1)n^2)}{48}\\&\quad+\frac{an(an-1)^3-6(an-1)^2+2(a+1)n^2}{48}.\end{aligned}$$

Proof. For $N=a$ in (2.2), take $x^nq^{\binom{n}{2}}$ and q^{n^2} for x and q:

$$(x^nq^{\binom{n}{2}};q^{n^2})_a=\sum_{b=0}^{a}\begin{bmatrix}a\\b\end{bmatrix}_{q^{n^2}}\sigma_n^b(-x)^{bn}q^{\binom{bn}{2}}.$$

Then, for $q=\zeta(1-\varepsilon)$, we write $y=\sigma_nx^n$ to obtain

$$\begin{aligned}(x^nq^{\binom{n}{2}};q^{n^2})_a&=(y(1-\varepsilon)^{\binom{n}{2}};(\varepsilon)^{n^2})_a\\&=\prod_{\ell=0}^{a-1}\Big(1-y(1-\varepsilon)^{\ell n^2+\binom{n}{2}}\Big)\\&=(1-y)^a\prod_{\ell=0}^{a-1}\left(1-\frac{y}{1-y}\sum_{i=1}^{\ell n^2+\binom{n}{2}}\binom{\ell n^2+\binom{n}{2}}{i}(-\varepsilon)^i\right).\end{aligned}$$

To conclude, we apply the same argument as in the proof of Lemma 2.1. □

Proof of Theorem 1.1. Combining the expansions in Lemmas 2.1–4.1, we find that

$$\begin{aligned}&\begin{bmatrix}an\\bn\end{bmatrix}_q\sigma_n^bq^{\binom{bn}{2}}-\begin{bmatrix}a\\b\end{bmatrix}_{q^{n^2}}\sigma_n^bq^{\binom{bn}{2}}\\&\quad=-b(a-b)\binom{a}{b}(q^n-1)\bigg(a\sum_{k=1}^{n-1}\frac{q^k}{1-q^k}+\frac{a(n-1)}{2}+\frac{(a+1)(n^2-1)}{24}(q^n-1)\\&\qquad+\frac{(b(a-b)n-a-2)(n^2-1)}{48}(q^n-1)^2\bigg)+O(\varepsilon^4)\end{aligned}$$

as $q=\zeta(1-\varepsilon)\to\zeta$ radially. This means that the difference of both sides is divisible by $(q-\zeta)^4$ for any n-th primitive root of unity ζ, hence by $\Phi_n(q)^4$. The latter property is equivalent to the congruence (1.6). □

Proof. We first use Lemma 2.1 with $n = 1$:

$$\begin{aligned}\begin{bmatrix} a \\ b \end{bmatrix}_q q^{\binom{b}{2}} - \binom{a-1}{b} - \binom{a-1}{a-b} q^{\binom{a}{2}} \\ = b(a-b)\binom{a}{b}\big(-(1-q)^2\rho_0(a,1) + (1-q)^3\rho_1(a,b,1)\big) + O\big((1-q)^4\big)\end{aligned}$$

as $q \to 1$. Now, take $n > 1$ arbitrarily and apply this relation with q replaced with q^n, where $q = \zeta(1-\varepsilon)$, $0 < \varepsilon < 1$ and ζ is a primitive n-th root of unity:

$$\begin{aligned}&\sigma_n^{b(a-b)} q^{b(a-b)\binom{n}{2}} \begin{bmatrix} a \\ b \end{bmatrix}_{q^n} \\ &= \binom{a-1}{b}(1-\varepsilon)^{b(a-b)\binom{n}{2}-\binom{b}{2}n} + \binom{a-1}{a-b}(1-\varepsilon)^{b(a-b)\binom{n}{2}+\binom{a}{2}n-\binom{b}{2}n} \\ &\quad + b(a-b)\binom{a}{b}\big(-(1-(1-\varepsilon)^n)^2\rho_0(a,1) + \varepsilon^3 n^3 \rho_1(a,b,1)\big) \\ &\quad \times (1-\varepsilon)^{b(a-b)\binom{n}{2}-\binom{b}{2}n} + O(\varepsilon^4) \\ &= \binom{a-1}{b}(1-\varepsilon)^{-\binom{bn}{2}+ab\binom{n}{2}} + \binom{a-1}{a-b}(1-\varepsilon)^{\binom{an}{2}-\binom{bn}{2}-a(a-b)\binom{n}{2}} \\ &\quad + b(a-b)\binom{a}{b}\big(-\varepsilon^2 n^2\rho_0(a,1) + \varepsilon^3 n^2\big((n-1)\rho_0(a,1) + n\rho_1(a,b,1)\big)\big) \\ &\quad \times \left(1 - \frac{((a-b)n-a+1)bn}{2}\varepsilon + O(\varepsilon^2)\right) + O(\varepsilon^4)\end{aligned}$$

as $\varepsilon \to 0$. At the same time, from Lemma 2.1, we have

$$\begin{aligned}\begin{bmatrix} an \\ bn \end{bmatrix}_q &= \binom{a-1}{b}(1-\varepsilon)^{-\binom{bn}{2}} + \binom{a-1}{a-b}(1-\varepsilon)^{\binom{an}{2}-\binom{bn}{2}} \\ &\quad + b(a-b)\binom{a}{b}\big(-\varepsilon^2 n^2\rho_0(a,n) + \varepsilon^3 n^2\rho_1(a,b,n) + \varepsilon^3\, anS_{n-1}(\zeta)\big) \\ &\quad \times \left(1 + \binom{bn}{2}\varepsilon + O(\varepsilon^2)\right) + O(\varepsilon^4)\end{aligned}$$

as $\varepsilon \to 0$. Using

$$(1-\varepsilon)^N = 1 - N\varepsilon + \binom{N}{2}\varepsilon^2 - \binom{N}{3}\varepsilon^3 + O(\varepsilon^4) \quad \text{as } \varepsilon \to 0$$

for $N = -\binom{bn}{2} + ab\binom{n}{2}$, $\binom{an}{2} - \binom{bn}{2} - a(a-b)\binom{n}{2}$, $-\binom{bn}{2}$ and $\binom{an}{2} - \binom{bn}{2}$ we deduce from the two expansions and Lemma 3.1 that, for $q = \zeta(1-\varepsilon)$,

$$\begin{aligned}
&\begin{bmatrix} an \\ bn \end{bmatrix}_q - \sigma_n^{b(a-b)} q^{b(a-b)\binom{n}{2}} \begin{bmatrix} a \\ b \end{bmatrix}_{q^n} \\
&\quad = -ab(a-b)\binom{a}{b}(q^n-1)\Bigg(\sum_{k=1}^{n-1}\frac{q^k}{1-q^k}+\frac{n-1}{2} \\
&\qquad - \frac{(b(a-b)n-1)(n^2-1)}{48}(q^n-1)^2\Bigg) + O(\varepsilon^4)
\end{aligned}$$

as $\varepsilon \to 0$. This implies the congruence in (1.7). □

5. q-Rious Congruences

In this final part, we look at the binomial coefficients as particular instances of integral ratios of factorials, also known as Chebyshev–Landau factorial ratios. In the q-setting, these are defined by

$$D_n(q) = D_n(\boldsymbol{a}, \boldsymbol{b}; q) = \frac{[a_1 n]! \cdots [a_r n]!}{[b_1 n]! \cdots [b_s n]!},$$

where $\boldsymbol{a} = (a_1, \dots, a_r)$ and $\boldsymbol{b} = (b_1, \dots, b_s)$ are tuples of positive integers satisfying

$$a_1 + \cdots + a_r = b_1 + \cdots + b_s \tag{5.1}$$

and

$$\lfloor a_1 x \rfloor + \cdots + \lfloor a_r x \rfloor \ge \lfloor b_1 x \rfloor + \cdots + \lfloor b_s x \rfloor \quad \text{for all } x > 0 \tag{5.2}$$

(see, for example, [10]), $\lfloor \cdot \rfloor$ denotes the integer part of a number. Then $D_n(q) \in \mathbb{Z}[q]$ are polynomials with values

$$D_n(1) = \frac{(a_1 n)! \cdots (a_r n)!}{(b_1 n)! \cdots (b_s n)!}$$

at $q = 1$, and the congruences (1.2) and (1.5) generalise as follows.

Theorem 5.1. *In the notation*

$$c_i = c_i(\boldsymbol{a}, \boldsymbol{b}) = \binom{a_1}{i} + \cdots + \binom{a_r}{i} - \binom{b_1}{i} - \cdots - \binom{b_s}{i} \quad \textit{for } i = 2, 3,$$

the congruences

$$D_n(q) \equiv D_1(q^{n^2}) - D_1(1)\frac{c_2\,(n^2-1)}{24}(q^n-1)^2 \pmod{\Phi_n(q)^3} \tag{5.3}$$

and

$$D_n(q) \equiv \sigma_n^{c_2} q^{c_2\binom{n}{2}} D_1(q^n) + D_1(1)\frac{(c_2+c_3)\,(n^2-1)}{12}(q^n-1)^2 \pmod{\Phi_n(q)^3} \tag{5.4}$$

are valid for any $n \ge 1$.

Observe that when $n = p > 3$ and $q \to 1$, one recovers from any of these two congruences

$$D_p(1) \equiv D_1(1) \pmod{p^3},$$

of which (1.1) is a special case. Furthermore, it is tempting to expect that these two families of q-congruences may be generalised even further in the spirit of Theorems 1.1 and 1.2, and that the polynomials $D_n(q)$ satisfy the q-Gauss relations from [4]. We do not pursue this line here.

Proof of Theorem 5.1. Though the congruences (5.3) and (5.4) are between polynomials rather than rational functions, we prove the theorem without the assumption (5.2): in other words, the congruences remain true for the rational functions $D_n(q)$ provided that the balancing condition (5.1) (equivalently, $c_1(\boldsymbol{a}, \boldsymbol{b}) = 0$ in the above notation for c_i) is satisfied. In turn, this more general statement follows from its validity for particular cases

$$D_n(q) = \frac{[an]!}{[bn]!\,[(a-b)n]!} \quad \text{and} \quad \tilde{D}_n(q) = \frac{[bn]!\,[(a-b)n]!}{[an]!}$$

by induction (on $r + s$, say). Indeed, the inductive step exploits the property of both (5.3) and (5.4) to imply the congruence for the product

$$D_n(\boldsymbol{a}, \boldsymbol{b}; q) D_n(\tilde{\boldsymbol{a}}, \tilde{\boldsymbol{b}}; q)$$

whenever it is already known for the individual factors; we leave this simple fact to the reader and only discuss its other appearance when dealing with $\tilde{D}_n(q)$ below. Notice that $\left[\begin{smallmatrix} an \\ bn \end{smallmatrix}\right] \equiv \binom{an}{bn} \not\equiv 0 \pmod{\Phi_n(q)}$, so that $\tilde{D}_n(q) = \left[\begin{smallmatrix} an \\ bn \end{smallmatrix}\right]^{-1}$ is well-defined modulo of any power of $\Phi_n(q)$.

For $D_n(q) = \left[\begin{smallmatrix} an \\ bn \end{smallmatrix}\right]$, we have $c_2 = b(a-b)$ and $c_2 + c_3 = ab(a-b)/2$; hence, (5.3) and (5.4) follow from (1.2) and (1.5), respectively.

Turning to $q = \zeta(1-\varepsilon)$, where $0 < \varepsilon < 1$ and ζ is a primitive n-th root of unity, write the congruences (1.2) and (1.5) as the asymptotic relation

$$\begin{bmatrix} an \\ bn \end{bmatrix} = B(q) + cB(1)\varepsilon^2 + O(\varepsilon^3) \quad \text{as } \varepsilon \to 0,$$

in which

$$B(q) = \begin{bmatrix} a \\ b \end{bmatrix}_{q^{n^2}}, \quad c = -\frac{b(a-b)n^2(n^2-1)}{24} \quad \text{for the case (1.2)}$$

and

$$B(q) = \sigma_n^{b(a-b)} q^{b(a-b)\binom{n}{2}} \begin{bmatrix} a \\ b \end{bmatrix}_{q^n}, \quad c = \frac{ab(a-b)n^2(n^2-1)}{24} \quad \text{for the case (1.5)}.$$

Then

$$\begin{aligned} \tilde{D}_n(q) = \begin{bmatrix} an \\ bn \end{bmatrix}^{-1} &= B(q)^{-1}\left(1 + cB(1)B(q)^{-1}\varepsilon^2 + O(\varepsilon^3)\right)^{-1} \\ &= B(q)^{-1} - cB(1)B(q)^{-2}\varepsilon^2 + O(\varepsilon^3) \\ &= B(q)^{-1} - cB(1)^{-1}\varepsilon^2 + O(\varepsilon^3), \end{aligned}$$

because we have $B(q) = B(1) + O(\varepsilon)$ as $\varepsilon \to 0$ for our choices of $B(q)$. The resulting expansion implies the truth of (5.3) and (5.4) for $\tilde{D}_n(q) = D_n((b, a-b), (a); q)$ in view of

$$c_i((b, a-b), (a)) = -c_i((a), (b, a-b)) \quad \text{for } i = 2, 3.$$

As explained above, this also establishes the general case of (5.3) and (5.4). $\square$

For related Lucas-type congruences satisfied by the q-factorial ratios $D_n(q)$, see [1].

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Wadim Zudilin
Department of Mathematics
IMAPP, Radboud University
PO Box 9010
6500 GL Nijmegen
The Netherlands
e-mail: w.zudilin@math.ru.nl

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The Final Problem: A Series Identity from the Lost Notebook

Bruce C. Berndt, Junxian Li, and Alexandru Zaharescu

In celebration of the 80th birthday of George Andrews.

Abstract When Ramanujan's lost notebook (Ramanujan, The lost notebook and other unpublished papers, Narosa, New Delhi, 1988) was published in 1988, accompanying it were other unpublished notes and partial manuscripts by Ramanujan. In one of these previously unpublished partial manuscripts, Ramanujan offered two elegant identities, associated, respectively, with the classical *circle* and *divisor problems*. In fact, they are two-variable analogues, but not generalizations, of classical identities associated with these two famous problems. The origin and history of this partial manuscript is unclear. We do know that after Ramanujan died in 1920, the University of Madras on 30 August, 1923 sent to G.H. Hardy a parcel of Ramanujan's unpublished work, probably containing the lost notebook and the previously mentioned fragment (Berndt and Rankin, Ramanujan: essays and surveys, American Mathematical Society, Providence, 2001; London Mathematical Society, London, 2001, p. 266). Unfortunately, we do not have any record of what was included in this package. If this fragment was included in the mailing, then it is possible that Ramanujan wrote it at the end of his life in either 1919 or 1920. On the other hand, from Hardy's paper (Hardy, Q J Math, 46:263–283, 1915) on the *circle problem* published in 1915, it is evident that by early in his stay in England, Ramanujan had a strong interest in the *circle* and *divisor problems*, and so the

B. C. Berndt (✉)
Department of Mathematics, University of Illinois, Urbana, IL, USA
e-mail: berndt@illinois.edu

J. Li
Max Planck Institute for Mathematics, Bonn, Germany
e-mail: jli135@mpim-bonn.mpg.de

A. Zaharescu
Department of Mathematics, University of Illinois, Urbana, IL, USA

Simion Stoilow Institute of Mathematics of the Romanian Academy, Bucharest, Romania
e-mail: zaharesc@illinois.edu

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fragment may emanate from this period. In 2013, the first and third present authors and S. Kim published a proof (Berndt et al., Adv Math, 236:24–59, 2013) of the identity from the fragment connected with the *circle problem*. In this paper, a proof of the second identity is briefly sketched.

1 Introduction

After returning to India in 1919, Ramanujan wrote to Hardy only once before he died on 26 April 1920. Ramanujan began his letter by announcing a new discovery, which he called *mock ϑ-functions*.

> I am extremely sorry for not writing you a single letter up to now .... I discovered very interesting functions recently which I call "Mock" ϑ-functions. Unlike the "False" ϑ-functions (studied by Prof. Rogers in his interesting paper) they enter into mathematics as beautifully as the ordinary ϑ-functions. I am sending you with this letter some examples ....

Part of the original letter has apparently been lost. A copy of an evidently small portion of the letter can be found in Ramanujan's *Collected Papers* [17, pp. xxxi–xxxii, 354–355]. A much larger portion of the letter was photocopied and published with Ramanujan's "lost notebook" [18, pp. 127–131]. As the ellipses ... above indicate, it is highly likely that a portion of the first paragraph or some of the text immediately following the first paragraph is missing. For further commentary on the completeness (or lack thereof) of this letter, see [10, pp. 223–224].

G.N. Watson, titled his final Presidential Address to the London Mathematical Society on 14 November, 1935, **The Final Problem: An Account of the Mock Theta Functions**. His published paper [20], [11, pp. 325–347] was an expanded version of his address. The subtitle, *The Final Problem*, is borrowed from Sir Arthur Conan Doyle's last memoir on Sherlock Holmes. We also note that Sherlock Holmes' famous sidekick was Dr. Watson (no relation).

The present authors are clearly borrowing from G.N. Watson who had borrowed from Sir Arthur Conan Doyle and Sherlock Holmes. However, for us, the "final problem" is the final entry from the "lost notebook" that remained to be proved. As readers have noted above, this paper is devoted to George Andrews who discovered the "lost notebook" in the spring of 1976 in the library of Trinity College, Cambridge [1], [11, pp. 165–184]. Since that time, he has been proving and elucidating the contents of the "lost notebook," in particular, during the past 20 years, with the first author of this paper [2–6]. However, one entry remained impenetrable, and a proof was not given in [2–6]. We emphasize that we are using the term "lost notebook" broadly here, for with the publication of the "lost notebook" [18], as mentioned at the beginning of our abstract, are partial unpublished manuscripts and other fragments, mostly from the last year of Ramanujan's life, which the authors also address in their volumes [2–6]. The entry that remained unproved was the second of a pair of identities in an isolated fragment in [18, p. 335] connected, respectively, with the *circle* and *divisor problems*. The present authors recently constructed a proof of this identity [9]. The purpose of this paper is to share this identity with readers and to convince them that this is indeed a beautiful, and possibly important, identity. We provide a brief sketch of the proof at the end of the present paper, and refer readers to a complete proof in [9].

Perhaps it is therefore wise to begin with very brief descriptions of the *circle* and *divisor problems*. More complete and recent surveys of these problems can be found in a paper with S. Kim by the first and third authors [8].

2 The Circle and Divisor Problems

Let $r_2(n)$ denote the number of representations of the positive integer n as a sum of two squares, where representations with different orders and different signs on the summands being squared are regarded as distinct. Thus, $r_2(7) = 0$ and $r_2(13) = 8$, since $13 = (\pm 2)^2 + (\pm 3)^2 = (\pm 3)^2 + (\pm 2)^2$. Set $r_2(0) = 1$. Each representation of n as a sum of two squares can be identified with a lattice point in the plane. It is therefore not difficult to see that

$$\sum_{0\le n\le x} r_2(n)$$

is equal to the number of lattice points in a circle of radius $\sqrt{x}$, and that this number is approximately equal to πx. Define the "error" term $P(x)$, for $x > 0$, by

$$R(x) := \sideset{}{'}\sum_{0<n\le x} r_2(n) = \pi x + P(x), \tag{2.1}$$

where the prime $'$ on the summation sign indicates that if x is an integer, then only $\frac{1}{2}r_2(x)$ is counted. Finding the correct order of magnitude for $P(x)$, as $x \to \infty$, is known as the *circle problem*.

We remark that an elementary representation of $R(x)$, depending on an elementary formula for $r_2(n)$ [15, p. 150], can be given, namely,

$$\begin{aligned} R(x) &= 4 \sideset{}{'}\sum_{0<n\le x} \sum_{d|n} \sin\left(\frac{\pi d}{2}\right) \\ &= 4 \sideset{}{'}\sum_{0<dj\le x} \sin\left(\frac{\pi d}{2}\right) \\ &= 4 \sideset{}{'}\sum_{0<d\le x} \left[\frac{x}{d}\right] \sin\left(\frac{\pi d}{2}\right), \end{aligned} \tag{2.2}$$

where $[x]$ is the greatest integer less than or equal to x. We shall return to this formula shortly.

Let $d(n)$ denote the number of positive divisors of the positive integer n. For example, $d(6) = 4$, since $1, 2, 3$ and 6 comprise all of the divisors of 6. Each divisor d of $n \le x$ can be associated with a lattice point $(d, n/d)$ in the plane. For any $x > 0$, by an elementary argument counting these lattice points and involving a familiar estimate for a partial sum of a harmonic series [16, p. 102, Theorem 42], we can estimate, as $x \to \infty$,

$$D(x) := \sum_{n\le x}{}' d(n) = x(\log x + (2\gamma - 1)) + O(\sqrt{x}), \tag{2.3}$$

where the prime $'$ on the summation sign indicates that if x is an integer, then only $\frac{1}{2}d(x)$ is counted, and where γ denotes Euler's constant. The error term $\Delta(x)$ has historically been defined by

$$D(x) = x(\log x + (2\gamma - 1)) + \frac{1}{4} + \Delta(x), \tag{2.4}$$

where the discrepancy $\frac{1}{4}$ between (2.3) and (2.4) arises because of a representation for $\Delta(x)$ to be given in the sequel. Determining the correct order of magnitude of $\Delta(x)$ as x tends to ∞ is known as the *divisor problem*.

In finding bounds for the error terms $P(x)$ and $\Delta(x)$, representations in terms of infinite series of Bessel functions are of central importance; we describe these now.

In reference to (2.1), the error term $P(x)$, $x > 0$, has the representation

$$\sum_{0<n\le x}{}' r_2(n) = \pi x + \sum_{n=1}^{\infty} r_2(n) \left(\frac{x}{n}\right)^{1/2} J_1(2\pi\sqrt{nx}), \tag{2.5}$$

where the ordinary Bessel function $J_\nu(z)$ is defined by

$$J_\nu(z) := \sum_{n=0}^{\infty} \frac{(-1)^n}{n!\Gamma(\nu+n+1)} \left(\frac{z}{2}\right)^{\nu+2n}, \qquad 0 < |z| < \infty, \qquad \nu \in \mathbb{C}. \tag{2.6}$$

The identity (2.5) apparently first appeared in Hardy's paper [13], [14, pp. 243–263], where he wrote "The form of this equation was suggested to me by Mr. S. Ramanujan, ..." We can therefore conclude that (2.5) was first proved by Ramanujan, although we do not have access to his proof.

An analogue for $\Delta(x)$ was established earlier in 1904 by G.F. Voronoï [19], and, from (2.4), is given by

$$D(x) = x\,(\log x + 2\gamma - 1) + \frac{1}{4} + \sum_{n=1}^{\infty} d(n) \left(\frac{x}{n}\right)^{1/2} I_1(4\pi\sqrt{nx}), \tag{2.7}$$

where $I_\nu(z)$ is defined by

$$I_\nu(z) := -Y_\nu(z) - \frac{2}{\pi} K_\nu(z), \tag{2.8}$$

and where $Y_\nu(z)$ is the Bessel function of the second kind defined by

$$Y_\nu(z) := \frac{J_\nu(z)\cos(\nu\pi) - J_{-\nu}(z)}{\sin(\nu\pi)}, \qquad |z| < \infty, \tag{2.9}$$

and $K_\nu(z)$ is the modified Bessel function defined by

$$K_\nu(z) := \frac{\pi}{2} \frac{e^{\pi i\nu/2} J_{-\nu}(iz) - e^{-\pi i\nu/2} J_\nu(iz)}{\sin(\nu\pi)}, \qquad -\pi < \arg z < \tfrac{1}{2}\pi, \qquad 0 < |z| < \infty. \tag{2.10}$$

If ν is an integer n, it is understood that we define the functions above by taking their limits as $\nu \to n$ in (2.9) and (2.10).

3 A Fragment Published with the Lost Notebook

The fragment [18, p. 335] published with the "lost notebook" contains two-variable analogues of (2.5) and (2.7). We shall adhere to Ramanujan's notation and first define

$$F(x) = \begin{cases} [x], & \text{if } x \text{ is not an integer,} \\ x - \frac{1}{2}, & \text{if } x \text{ is an integer.} \end{cases} \tag{3.1}$$

The first entry from [18, p. 335] is associated with the *circle problem*.

Entry 3.1 ([18, p. 335]) *Let* $F(x)$ *be defined by* (3.1)*, and recall that* $J_1(z)$ *is defined in* (2.6)*. If* $0 < \theta < 1$ *and* $x > 0$*, then*

$$\sum_{n=1}^{\infty} F\left(\frac{x}{n}\right) \sin(2\pi n\theta) = \pi x \left(\frac{1}{2} - \theta\right) - \frac{1}{4}\cot(\pi\theta) + \frac{1}{2}\sqrt{x} \sum_{m=1}^{\infty} \sum_{n=0}^{\infty} \left\{ \frac{J_1\left(4\pi\sqrt{m(n+\theta)x}\right)}{\sqrt{m(n+\theta)}} - \frac{J_1\left(4\pi\sqrt{m(n+1-\theta)x}\right)}{\sqrt{m(n+1-\theta)}} \right\}. \tag{3.2}$$

This formula is very interesting, for the left-hand side of (3.2) can be considered as a two-variable analogue of (2.2), while the right-hand side of (3.2) is a two-variable analogue of the right-hand side of (2.5). Entry 3.1 was first proved by the first and third authors [12] with the order of summation on the double sum *reversed* from that recorded by Ramanujan. A proof of (3.2) with the order of summation as given by Ramanujan was established 7 years later by the first author, S. Kim, and the third author [7]. They also proved a version of (3.2) with the product of the indices mn tending to ∞.

The second entry from the fragment [18, p. 335] is associated with the *divisor problem*.

Entry 3.2 ([18, p. 335]) *Let $F(x)$ be defined by (3.1). Then, for $x > 0$ and $0 < \theta < 1$,*

$$\sum_{n=1}^{\infty} F\left(\frac{x}{n}\right)\cos(2\pi n\theta) = \frac{1}{4} - x\log(2\sin(\pi\theta))$$
$$+\frac{1}{2}\sqrt{x}\sum_{m=1}^{\infty}\sum_{n=0}^{\infty}\left\{\frac{I_1\left(4\pi\sqrt{m(n+\theta)x}\right)}{\sqrt{m(n+\theta)}} + \frac{I_1\left(4\pi\sqrt{m(n+1-\theta)x}\right)}{\sqrt{m(n+1-\theta)}}\right\}, \tag{3.3}$$

where $I_1(z)$ is defined by (2.8).

Recalling the definition of $D(x)$ from (2.3), by an elementary argument we see that

$$D(x) = \sum_{n\le x}\sum_{d|n}{}' 1 = \sum_{dj\le x}{}' 1 = \sum_{d\le x}\sum_{1\le j\le x/d}{}' 1 = \sum_{d\le x}{}' \left[\frac{x}{d}\right]. \tag{3.4}$$

Thus, the left side of (3.3) is a generalization of (3.4), while the right-hand side is a two-variable analogue of the right-hand side of (2.7).

The first and third authors in collaboration with S. Kim [7] were able to prove (3.3), but only with the order of summation either *reversed* or with the product of indices mn tending to ∞. A proof of (3.3) with the order of summation as recorded by Ramanujan evidently required a more sophisticated argument than the one in [7]. Indeed, certain difficulties, that did not arise in the proof of (3.2) arose in their attempts to prove (3.3). Thus, Entry 3.2 has remained unproved and has had the distinction in recent times for being the only unproved claim of Ramanujan from [18]. Fortunately, to help celebrate the birthday of George Andrews, the present authors were able to devise a proof of (3.3) [9].

It is possible that Ramanujan devised Entries 3.1 and 3.2 in order to attack the *circle problem* and *divisor problem*, respectively. It is clear from Hardy's paper [13] that Ramanujan had developed an interest in the *circle problem*, and possibly also the *divisor problem*, either before he came to England in 1914 or shortly thereafter. In analytic number theory, often the introduction of an extra parameter or variable allows the researcher to employ additional ideas and techniques that would otherwise be unavailable in investigating the problem. Perhaps Ramanujan had ideas to attack the *circle problem* and *divisor problem* when he introduced another variable

θ to establish two-variable analogues of (2.5) and (2.7) in Entries 3.1 and 3.2, respectively.

4 Brief Sketch of the Proof of Entry 3.2

A primary difficulty in proving (3.3) is that when we apply the asymptotic formulas, as $z \to \infty$, for the Bessel functions (2.9) and (2.10), we readily find that the series on the right-hand side of (3.3) does not converge absolutely. We therefore consider a two-variable extension of this series defined by

$$G(x, \theta, s, w) := \sum_{m=1}^{\infty} \sum_{n=1}^{\infty} \left(\frac{a_1(x, \theta, m, n)}{m^s (n+\theta)^w} + \frac{a_2(x, \theta, m, n)}{m^s (n+1-\theta)^w} \right), \tag{4.1}$$

where

$$a_1(x, \theta, m, n) = I_1(4\pi \sqrt{m(n+\theta)x}), \tag{4.2}$$

$$a_2(x, \theta, m, n) = I_1(4\pi \sqrt{m(n+1-\theta)x}), \tag{4.3}$$

where $I_1(z)$ is defined by (2.8). We not only want to determine the values of (s, w) for which $G(x, \theta, s, w)$ converges, but we also want to obtain regions in the complex s and w planes where the series converges uniformly with respect to θ on any compact subset of (0, 1).

Theorem 4.1 *Let $G(x, \theta, s, w)$ be defined above. Assume that* $\mathrm{Re}(s) > \frac{1}{4}$ *and* $\mathrm{Re}(w) > \frac{1}{4}$. *Furthermore, if x is an integer, assume that* $\mathrm{Re}(s) + \mathrm{Re}(w) > \frac{25}{26}$, *while if x is not an integer, assume that* $\mathrm{Re}(s) + \mathrm{Re}(w) > \frac{5}{6}$. *Then the series* (4.1) *converges uniformly with respect to θ in any compact subinterval of* (0, 1).

It is helpful to work with continuous functions on [0, 1], instead of functions continuous only on (0, 1). We therefore first multiply both sides of the proposed identity (3.3) by $\sin^2(\pi\theta)$ in order to extend the domain of continuity $0 < \theta < 1$ to $0 \leq \theta \leq 1$. We then rewrite the amended proposed identity by isolating the series of Bessel functions on one side of the equation. Next, we calculate the Fourier series of each side of the new amended proposed identity and show that they are equal. Because each side of the proposed identity is continuous, we can thus conclude from the theory of Fourier series that the two sides are identical for $0 < \theta < 1$.

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A Family of Identities That Yields a Wide Variety of Partition Theorems

Louis W. Kolitsch and Stephanie Kolitsch

Dedicated to George Andrews in celebration of his 80th birthday.

Abstract In a recent paper, Andrews and Yee presented two identities that were described as surprising. In this paper, a family of similar identities will be presented and used to establish a wide variety of partition theorems.

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1 Introduction

In [4] Andrews and Yee presented the identities

$$\sum_{j=0}^{n} \frac{q^j (q;q)_{n+j}}{(q^2;q^2)_j} = (q^2;q^2)_n$$

and

$$\sum_{j=0}^{n} \frac{q^{2j} (q;q)_{n+j}}{(q^2;q^2)_j} = (q;q^2)_{n+1} + q^{n+1}(q^2;q^2)_n.$$

In this paper, a family of one-parameter series will be presented where the first two members of the family are connected in a similar manner. The first two members of the family are stated in Theorems 2.1 and 2.2 and the general case of the family is

L. W. Kolitsch (✉) · S. Kolitsch
Department of Mathematics and Statistics, The University of Tennessee at Martin, Martin, TN, USA
e-mail: lkolitsc@utm.edu; styler@utm.edu

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stated in Theorem 2.3. After proving these theorems, several examples for specific choices of the parameter z will be explored.

2 The Initial Theorems

Theorems 2.1 and 2.2 will be proved using the technique used by Andrews and Yee in [4]. Namely, let

$$S_n(k, z) = \sum_{j=0}^{n} \frac{q^{kj}(z;q)_j}{(q;q)_j}$$

and observe that the sum in Theorem 2.1 is $S_n(1, z)$ and the sum in Theorem 2.2 is $S_n(2, z)$.

Theorem 2.1 *For a non-negative integer n,*

$$\sum_{j=0}^{n} \frac{q^j(z;q)_j}{(q;q)_j} = \frac{(zq;q)_n}{(q;q)_n}.$$

Proof It is easy to see that the result is true for $n = 0$ and $n = 1$. Proceeding by mathematical induction, we find that

$$S_{n+1}(1, z) = \sum_{j=0}^{n+1} \frac{q^j(z;q)_j}{(q;q)_j}$$

$$= S_n(1, z) + \frac{q^{n+1}(z;q)_{n+1}}{(q;q)_{n+1}}$$

$$= \frac{(zq;q)_n}{(q;q)_n} + \frac{q^{n+1}(z;q)_{n+1}}{(q;q)_{n+1}}$$

$$= \frac{(zq;q)_n}{(q;q)_{n+1}}\left((1 - q^{n+1}) + q^{n+1}(1 - z)\right)$$

$$= \frac{(zq;q)_{n+1}}{(q;q)_{n+1}}.$$

Since the result for n implies the result for $n + 1$, this completes the proof of the theorem. □

Theorem 2.2 *For a positive integer n,*

$$\sum_{j=0}^{n} \frac{q^{2j}(z;q)_j}{(q;q)_j} = \frac{(zq^2;q)_n}{(q^2;q)_{n-1}} + \frac{q^{n+1}(zq;q)_n}{(q;q)_n}.$$

Proof To prove Theorem 2.2, we first note that for a positive integer n,

$$S_n(k,z) = \sum_{j=0}^{n} \frac{q^{kj}(z;q)_j}{(q;q)_j} = \sum_{j=0}^{n} \frac{q^{(k-1)j}(1-(1-q^j))(z;q)_j}{(q;q)_j}$$

$$= S_n(k-1,z) - \sum_{j=1}^{n} \frac{q^{(k-1)j}(z;q)_j}{(q;q)_{j-1}}$$

$$= S_n(k-1,z) - \sum_{j=0}^{n-1} \frac{q^{(k-1)(j+1)}(z;q)_{j+1}}{(q;q)_j}$$

$$= S_n(k-1,z) - q^{k-1}(1-z)S_{n-1}(k-1,zq).$$

Thus

$$S_n(2,z) = S_n(1,z) - q(1-z)S_{n-1}(1,zq)$$

$$= \frac{(zq;q)_n}{(q;q)_n} - q(1-z)\left(\frac{(zq^2;q)_{n-1}}{(q;q)_{n-1}}\right)$$

$$= \frac{(zq^2;q)_{n-1}}{(q;q)_n}\Big((1-zq) - q(1-z)(1-q^n)\Big)$$

$$= \frac{(zq^2;q)_{n-1}}{(q;q)_n}\Big((1-q) + q^{n+1}(1-z)\Big)$$

$$= \frac{(zq^2;q)_{n-1}}{(q;q)_n}\Big((1-zq^{n+1})(1-q) + q^{n+1}(1-zq)\Big)$$

$$= \frac{(zq^2;q)_n}{(q^2;q)_{n-1}} + \frac{q^{n+1}(zq;q)_n}{(q;q)_n}.$$

□

It should be noted that the sum in Theorem 2.1 is $F(zq^{-1},1,q,n)$ and the sum in Theorem 2.2 is $F(zq^{-1},1,q^2,n)$ from [2]. If we take the limit of the result in Theorem 2.1 as n goes to infinity, we obtain the q-binomial theorem (Theorem 2.1 in [3]) with t replaced by q and a replaced by z. Also, in a personal communication from Andrews and in a note from the editors, it was pointed out that Theorem 2.1 can be proved using the q-analog of the Chu-Vandemonde summation

$$\sum_{j=0}^{n} \frac{(q^{-n};q)_j (b;q)_j}{(q;q)_j (c;q)_j} = \frac{b^n (\frac{c}{b};q)_n}{(c;q)_n}$$

by letting $c = q^{-n}$.

The result of Theorem 2.1 is the $k = 1$ case of the following theorem while the $k = 2$ case of this theorem is the fourth line in the proof of the result for $S_n(2, z)$.

Theorem 2.3 *For a positive integer n and a positive integer $k \leq n + 1$,*

$$\sum_{j=0}^{n} \frac{q^{kj}(z;q)_j}{(q;q)_j} = \frac{(zq^k;q)_{n-k+1}}{(q;q)_n} \sum_{j=0}^{k-1} \left(q^{(n+1)j} (z;q)_j (q^{j+1};q)_{k-1-j} \right).$$

Proof We will use induction to show that the result is true for k if we assume the result is true for $k - 1$. Starting with the fact that

$$S_n(k, z) = S_n(k-1, z) - q^{k-1}(1-z)S_{n-1}(k-1, zq),$$

we have

$$S_n(k,z) = \frac{(zq^{k-1};q)_{n-k+2}}{(q;q)_n} \left(\sum_{j=0}^{k-2} q^{(n+1)j}(z;q)_j (q^{j+1};q)_{k-2-j} \right)$$

$$-q^{k-1}(1-z) \frac{(zq^k;q)_{n-k+1}}{(q;q)_{n-1}} \left(\sum_{j=0}^{k-2} q^{nj}(zq;q)_j (q^{j+1};q)_{k-2-j} \right)$$

$$= \frac{(zq^k;q)_{n-k+1}}{(q;q)_n} \left((1-zq^{k-1}) \sum_{j=0}^{k-2} q^{(n+1)j}(z;q)_j (q^{j+1};q)_{k-1-j} \right.$$

$$\left. -q^{k-1} \sum_{j=0}^{k-2} (q^{nj} - q^{n(j+1)})(z;q)_{j+1} (q^{j+1};q)_{k-2-j} \right)$$

$$= \frac{(zq^k;q)_{n-k+1}}{(q;q)_n} \sum_{j=0}^{k-2} \left(q^{nj}(z;q)_j (q^{j+1};q)_{k-2-j} (q^j - q^{k-1} + q^{k-1+n} \right.$$

$$\left. -zq^{k-1+n+j}) \right)$$

$$
\begin{aligned}
&= \frac{(zq^k;q)_{n-k+1}}{(q;q)_n} \sum_{j=0}^{k-2} \Big(q^{nj}(z;q)_j (q^{j+1};q)_{k-2-j} (q^j - q^{k-1+j} - q^{k-1} \\
&\qquad + q^{k-1+j} + q^{k-1+n} - zq^{k-1+n+j}) \Big) \\
&= \frac{(zq^k;q)_{n-k+1}}{(q;q)_n} \Bigg(\sum_{j=0}^{k-2} \Big(q^{(n+1)j}(z;q)_j (q^{j+1};q)_{k-1-j} \Big) \\
&\qquad - q^{k-1} \sum_{j=0}^{k-2} \Big(q^{nj}(z;q)_j (q^j;q)_{k-1-j} \Big) \\
&\qquad + q^{k-1} \sum_{j=0}^{k-2} \Big(q^{n(j+1)}(z;q)_{j+1} (q^j;q)_{k-2-j} \Big) \Bigg).
\end{aligned}
$$

When combined these last two sums are equal to $q^{k-1}q^{n(k-1)}(z;q)_{k-1}$ giving us

$$
S_n(k,z) = \frac{(zq^k;q)_{n-k+1}}{(q;q)_n} \sum_{j=0}^{k-1} \Big(q^{(n+1)j}(z;q)_j (q^{j+1};q)_{k-1-j} \Big)
$$

which completes the proof. □

3 Some Partition Theorems Associated with Theorem 2.1

In this section, we will interpret the result in Theorem 2.1 in terms of partitions for several choices of the parameter z. Each result will first be proved analytically using a generating function. A combinatorial proof of each result is also provided, along with an example illustrating the bijection in the combinatorial proof. For the first result we let $z = 0$ which gives us the following identity:

$$
\sum_{j=0}^{n} \frac{q^j}{(q;q)_j} = \frac{1}{(q;q)_n}.
$$

The product on the right-hand side is the generating function for partitions into parts less than or equal to n. The sum on the left-hand side simply sorts these partitions based on the largest part less than or equal to n that appears in the partition.

Our first theorem is:

Theorem 3.1 *The number of partitions of m into parts less than or equal to n is the same as the number of partitions of m into parts less than or equal to* $2n - 1$ *where the gap between the largest part less than n and the smallest part greater than or equal to n is at least n.*

Proof Replacing n with $n - 1$ and letting $z = q^n$ in Theorem 2.1, we obtain

$$\sum_{j=0}^{n-1} \frac{q^j (q^n; q)_j}{(q; q)_j} = \frac{(q^{n+1}; q)_{n-1}}{(q; q)_{n-1}}.$$

To interpret this in terms of partitions we divide each side of the equation by $(q^n; q)_n$ to obtain

$$\sum_{j=0}^{n-1} \frac{q^j}{(q^{n+j}; q)_{n-j} (q; q)_j} = \frac{1}{(q; q)_n}.$$

The product on the right-hand side is the generating function for partitions into parts less than or equal to n. The sum on the left-hand side is the generating function for partitions into parts less than or equal to $2n-1$ where the gap between the largest part less than n and the smallest part greater than or equal to n is at least n. (If there are no parts less than n, we take the largest part less than n to be 0.) □

To illustrate this theorem we have listed the partitions of $m = 7$ of the two types described in Theorem 3.1 for $n = 3$ in the table below. We have paired the partitions using the bijection that can be used to prove this theorem combinatorially. The combinatorial proof is presented immediately following the table.

To prove Theorem 3.1 combinatorially, we will describe a bijection between the two types of partitions.

Partitions of 7 into parts less than or equal to 3	Partitions of 7 into parts less than or equal to 5 where the gap between the largest part less than 3 and the smallest part greater than or equal to 3 is at least 3
3 + 3 + 1	4 + 3
3 + 2 + 2	5 + 2
3 + 2 + 1 + 1	5 + 1 + 1
3 + 1 + 1 + 1 + 1	4 + 1 + 1 + 1
2 + 2 + 2 + 1	2 + 2 + 2 + 1
2 + 2 + 1 + 1 + 1	2 + 2 + 1 + 1 + 1
2 + 1 + 1 + 1 + 1 + 1	2 + 1 + 1 + 1 + 1 + 1
1 + 1 + 1 + 1 + 1 + 1 + 1	1 + 1 + 1 + 1 + 1 + 1 + 1

Proof We will start with a partition, $a_1 \le a_2 \le \cdots \le a_r < b_1 \le b_2 \le \cdots \le b_s$, where the parts are all less than or equal to $2n-1$, the a-parts are less than n, the b-parts are greater than or equal to n, and $b_1 - a_r \ge n$. If the set of b-parts is empty, the partition is already a partition into parts less than n and we will not change the partition. If the set of a-parts is empty, we will simply subtract n from each of the b-parts and create a new partition consisting of s parts of size n and the parts b_1-n, $b_2-n, \ldots, b_s-n$. Note that some of the parts $b_i - n$ could be zero and will simply be deleted. If neither the set of a-parts nor the set of b-parts is empty, we will subtract n from each b-part and create a new partition, $a_1 \le a_2 \le \cdots a_r \le b_1 - n \le b_2 - n \le \cdots \le b_s - n < n \le n \le \cdots \le n$ where there are s parts of size n. It should be noted that $b_1 - n \ge a_r$ because of the gap condition and $b_s - n < n$ since the parts in our original partition are all less than or equal to $2n-1$. Clearly the transformations described above can be reversed and thus we have a bijection between the two sets of partitions in Theorem 3.1. □

Our second theorem is:

Theorem 3.2 *The number of partitions of m into parts less than or equal to* $2n+1$ *where the only odd parts that can appear are* $2n+1$ *is the same as the number of partitions of m into parts less than or equal to* $4n+1$ *where the even parts are less than or equal to* $2n$ *and the odd parts are at least* $2n+1$ *greater than the largest even part.*

Proof Replacing q by q^2 and letting $z = q^{2n-1}$ in Theorem 2.1, we have

$$\sum_{j=0}^{n} \frac{q^{2j}(q^{2n-1};q^2)_j}{(q^2;q^2)_j} = \frac{(q^{2n+1};q^2)_n}{(q^2;q^2)_n}.$$

To interpret this in terms of partitions we multiply each side of the equation by $\frac{(q;q^2)_n}{(q;q^2)_{2n+1}}$ to obtain

$$\sum_{j=0}^{n} \frac{q^{2j}}{(q^{2n+2j+1};q^2)_{n-j+1}(q^2;q^2)_j} = \frac{1}{(1-q^{2n+1})(q^2;q^2)_n}.$$

The product on the right-hand side is the generating function for partitions into parts less than or equal to $2n+1$ where the only odd parts that can appear are $2n+1$. The sum on the left-hand side is the generating function for partitions into parts less than or equal to $4n+1$ where the even parts are less than or equal to $2n$, the odd parts are greater than the largest even part, and the gap between the largest even part and the smallest odd part is at least $2n+1$. (If there are no even parts, we take the largest even part to be 0.) □

To illustrate this theorem we have listed the partitions of $m = 21$ of the two types described in Theorem 3.2 for $n = 3$ in the table below. We have paired the partitions using the bijection that can be used to prove this theorem combinatorially. The combinatorial proof is presented immediately following the table.

Partitions of 21 into parts less than or equal to 7 where the only odd parts that can appear are 7s	Partitions of 21 into parts less than or equal to 13 where the even parts are less than or equal to 6 and the odd parts are at least 7 more than the largest even part
7 + 7 + 7	7 + 7 + 7
7 + 6 + 6 + 2	13 + 6 + 2
7 + 6 + 4 + 4	13 + 4 + 4
7 + 6 + 4 + 2 + 2	13 + 4 + 2 + 2
7 + 6 + 2 + 2 + 2 + 2	13 + 2 + 2 + 2 + 2
7 + 4 + 4 + 4 + 2	11 + 4 + 4 + 2
7 + 4 + 4 + 2 + 2 + 2	11 + 4 + 2 + 2 + 2
7 + 4 + 2 + 2 + 2 + 2 + 2	11 + 2 + 2 + 2 + 2 + 2
7 + 2 + 2 + 2 + 2 + 2 + 2 + 2	9 + 2 + 2 + 2 + 2 + 2 + 2

To prove this theorem combinatorially, we will describe a bijection between the two types of partitions.

Proof We will start with a partition, $a_1 \le a_2 \le \cdots \le a_r < b_1 \le b_2 \le \cdots \le b_s$, where the parts are all less than or equal to $4n+1$, the a-parts are even and less than or equal to $2n$, the b-parts are odd and greater than or equal to $2n+1$, and $b_1 - a_r \ge 2n+1$. If the set of b-parts is empty, the partition is already a partition into even parts less than or equal to $2n$ and we will not change the partition. If the set of a-parts is empty, we will simply subtract $2n+1$ from each of the b-parts and create a new partition consisting of s parts of size $2n+1$ and the parts $b_1 - 2n - 1$, $b_2 - 2n - 1, \ldots, b_s - 2n - 1$. Note that some of the parts $b_i - 2n - 1$ could be zero and will simply be deleted. Also note that the parts $b_i - 2n - 1$ are even and less than or equal to $2n$. If neither the set of a-parts nor the set of b-parts is empty, we will subtract $2n+1$ from each b-part and create a new partition, $a_1 \le a_2 \le \cdots a_r \le b_1 - 2n - 1 \le b_2 - 2n - 1 \le \cdots \le b_s - 2n - 1 \le 2n+1 \le 2n+1 \le \cdots \le 2n+1$ where there are s parts of size $2n+1$. It should be noted that the parts $b_i - 2n - 1$ are all even and less than or equal to $2n$ and $b_1 - 2n - 1 \ge a_r$ because of the gap condition. Clearly the transformations described above can be reversed and thus we have a bijection between the two sets of partitions in Theorem 3.2. □

Our next theorem is:

Theorem 3.3 *The number of overpartitions of m where the overlined parts are greater than or equal to 2 and less than or equal to $n+1$ and the non-overlined parts are less than or equal to n is the same as the number of overpartitions of m into parts less than or equal to n where the overlined parts in the partition are less than or equal to the largest non-overlined part.*

Proof Letting $z = -q$ in Theorem 2.1, we have

$$\sum_{j=0}^{n} \frac{q^j(-q;q)_j}{(q;q)_j} = \frac{(-q^2;q)_n}{(q;q)_n}.$$

The product on the right-hand side is the generating function for overpartitions [5] where the overlined parts are greater than or equal to 2 and less than or equal to $n + 1$ and the non-overlined parts are less than or equal to n. The sum on the left-hand side is the generating function for overpartitions into parts less than or equal to n where the overlined parts in the partition are less than or equal to the largest non-overlined part. □

To illustrate this theorem we have listed the overpartitions of $m = 6$ of the two types described in Theorem 3.3 for $n = 2$ in the table below. We have paired the partitions using the bijection that can be used to prove this theorem combinatorially. The combinatorial proof is presented immediately following the table.

Overpartitions of 6 where the overlined parts are greater than or equal to 2 and less than or equal to 3 and the non-overlined parts are less than or equal to 2	Overpartitions of 6 into parts less than or equal to 2 where the overlined parts in the partition are less than or equal to the largest non-overlined part
$\overline{3}+2+1$	$2+2+\overline{1}+1$
$\overline{3}+\overline{2}+1$	$\overline{2}+2+\overline{1}+1$
$\overline{3}+1+1+1$	$2+\overline{1}+1+1+1$
$\overline{2}+2+2$	$\overline{2}+2+2$
$2+2+2$	$2+2+2$
$\overline{2}+2+1+1$	$\overline{2}+2+1+1$
$2+2+1+1$	$2+2+1+1$
$\overline{2}+1+1+1+1$	$\overline{1}+1+1+1+1+1$
$2+1+1+1+1$	$2+1+1+1+1$
$1+1+1+1+1+1$	$1+1+1+1+1+1$

To prove this theorem combinatorially, we will describe a bijection between the two types of partitions.

Proof We will start with a partition, $2 \le a_1 < a_2 < \cdots < a_r \le n+1$ and $1 \le b_1 \le b_2 \le \cdots \le b_s \le n$, where the a-parts are the overlined parts and the b-parts are the non-overlined parts. If the set of a-parts is empty or $a_r \le b_s$, the partition is already a partition in the other set and we will not change the partition. If $a_r > b_s$, we will delete a_r and replace it with the non-overlined part $a_r - 1$ and the overlined part $\overline{1}$ to get a partition in the other set. Clearly this transformation can be reversed and thus we have a bijection between the two sets of partitions in Theorem 3.3. □

In Corollary 1.2 in [5] the identity

$$\sum_{k=0}^{n} \frac{(\frac{-1}{a}; q)_k c^k a^k q^{k(k+1)/2}}{(cq; q)_k} \begin{bmatrix} n \\ k \end{bmatrix} = \frac{(-acq; q)_n}{(cq; q)_n}$$

was interpreted combinatorially. If we let $c = 1$ and $a = q$, the right-hand side of this identity is the same as the right-hand side of the identity we interpreted in Theorem 3.3. This gives us the following theorem.

Theorem 3.4 *The number of overpartitions of m into parts less than or equal to n where the overlined parts in the partition are less than or equal to the largest non-overlined part is the same as the number of Frobenius partitions of m where the number of columns is less than or equal to n, the partition in the top row is a partition into nonnegative, distinct parts where each part is less than or equal to $n-1$ and the partition in the bottom row is an overpartition where the overlined parts are nonnegative integers and the non-overlined parts are positive integers.*

Noticing that when $a = q$ in the above sum all of the terms for k greater than 1 will include a factor of 2 gives us the following corollaries.

Corollary 3.5 *The number of overpartitions of m into parts less than or equal to n where the overlined parts in the partition are less than or equal to the largest non-overlined part is even if and only if $m > n$.*

Corollary 3.6 *The number of Frobenius partitions of m where the number of columns is less than or equal to n, the partition in the top row is a partition into distinct parts where each part is less than or equal to $n-1$ and the partition in the bottom row is an overpartition where the overlined parts are nonnegative integers and the non-overlined parts are positive integers is even if and only if $m > n$.*

Corollary 3.7 *The number of overpartitions of m where the overlined parts are greater than or equal to 2 and less than or equal to $n+1$ and the non-overlined parts are less than or equal to n is even if and only if $m > n$.*

Our next theorem is:

Theorem 3.8 *The number of overpartitions of m into parts less than or equal to $2n$ where non-overlined parts are less than or equal to n and overlined parts must be at most n greater than the largest non-overlined part is the same as the number of overpartitions of m into parts less than or equal to $2n+1$ where $n+1$ does not appear as a part, parts less than or equal to n can be overlined or not, and parts greater than $n+1$ must be overlined.*

Proof Letting $z = -q^{n+1}$ in Theorem 2.1 gives us

$$\sum_{j=0}^{n} \frac{q^j(-q^{n+1};q)_j}{(q;q)_j} = \frac{(-q^{n+2};q)_n}{(q;q)_n}.$$

To interpret this in terms of partitions we multiply each side of the equation by $(-q;q)_n$ to obtain

$$\sum_{j=0}^{n} \frac{q^j(-q;q)_{n+j}}{(q;q)_j} = \frac{(-q;q)_{2n+1}}{(1+q^{n+1})(q;q)_n}.$$

The product on the right-hand side is the generating function for overpartitions into parts less than or equal to $2n+1$ where $n+1$ does not appear as a part, parts less than or equal to n can be overlined or not, and parts greater than $n+1$ must be overlined. The sum on the left-hand side is the generating function for overpartitions into parts less than or equal to $2n$ where non-overlined parts are less than or equal to n and overlined parts must be at most n greater than the largest non-overlined part. (If there are no non-overlined parts, we take the largest non-overlined part to be 0.) □

To illustrate this theorem we have listed the overpartitions of $m = 5$ of the two types described in Theorem 3.8 for $n = 2$ in the table below. We have paired the overpartitions using the bijection that can be used to prove this theorem combinatorially. The table appears immediately following the combinatorial proof.

Overpartitions of 5 into parts less than or equal to 4 where non-overlined parts are less than or equal to 2 and overlined parts must be at most 2 greater than the largest non-overlined part	Overpartitions of 5 into parts less than or equal to 5 where 3 does not appear as a part, parts less than or equal to 2 can be overlined or not, and parts greater than 3 must be overlined
$\overline{3}+2$	$\overline{5}$
$\overline{3}+1+1$	$\overline{4}+1$
$\overline{3}+\overline{1}+1$	$\overline{4}+\overline{1}$
$\overline{2}+2+1$	$\overline{2}+2+1$
$\overline{2}+2+\overline{1}$	$\overline{2}+2+\overline{1}$
$2+2+\overline{1}$	$2+2+\overline{1}$
$2+2+1$	$2+2+1$
$\overline{2}+\overline{1}+1+1$	$\overline{2}+\overline{1}+1+1$
$\overline{2}+1+1+1$	$\overline{2}+1+1+1$
$2+\overline{1}+1+1$	$2+\overline{1}+1+1$
$2+1+1+1$	$2+1+1+1$
$\overline{1}+1+1+1+1$	$\overline{1}+1+1+1+1$
$1+1+1+1+1$	$1+1+1+1+1$

To prove this theorem combinatorially, we will describe a bijection between the two types of partitions.

Proof We will start with an overpartition, λ, with parts $a_1 \leq a_2 \leq \cdots \leq a_r$ and $b_1 < b_2 < \cdots < b_s$, where the a-parts are the non-overlined parts less than or equal to n and the b-parts are the overlined parts that are greater than or equal to 1 and are all less than or equal to $a_r + n$. If there are no non-overlined parts a_r is taken to be 0. If λ is an overpartition with no part of size $n+1$ then λ is a partition in the other set. If λ contains a part $n+1$ then we will transform λ into an overpartition in the other set. We first note that there must be a non-overlined part in λ since if there is no non-overlined part then the largest non-overlined part would be 0 and the size condition on the overlined parts would force the overlined parts less than or equal to n. We will delete the overlined part $n+1$ and create the new overlined part $a_r + n + 1$ to create an overpartition in the other set. It should be noted that the part

$a_r + n + 1$ is the largest overlined part in the new overpartition and was not already in λ because of the size constraint on the overlined parts relative to the largest non-overlined part. Clearly the transformation described above can be reversed and thus we have a bijection between the two sets of overpartitions in Theorem 3.8. □

Our next theorem is:

Theorem 3.9 *The number of pod-partitions of m where the odd parts are greater than or equal to 3 and less than or equal to* $2n + 1$ *and the even parts are less than or equal to* $2n$ *is the same as the number of pod-partitions of m into parts less than or equal to* $2n$ *where the odd parts in the partition are less than the largest even part.*

Proof Letting $z = zq$ and then replacing q with q^2 and z with $-q^{-1}$ in Theorem 2.1, we have

$$\sum_{j=0}^{n} \frac{q^{2j}(-q;q^2)_j}{(q^2;q^2)_j} = \frac{(-q^3;q^2)_n}{(q^2;q^2)_n}.$$

The product on the right-hand side is the generating function for partitions with odd parts distinct (*pod*-partitions) [1] where the odd parts are greater than or equal to 3 and less than or equal to $2n + 1$ and the even parts are less than or equal to $2n$. The sum on the left-hand side is the generating function for *pod*-partitions into parts less than or equal to $2n$ where the odd parts in the partition are less than the largest even part. □

To illustrate this theorem we have listed the *pod*-partitions of $m = 10$ of the two types described in Theorem 3.9 for $n = 4$ in the table below. We have paired the partitions using the bijection that can be used to prove this theorem combinatorially. The combinatorial proof is presented immediately following the table.

pod-partitions of 10 where the odd parts are greater than or equal to 3 and less than or equal to 9 and the even parts are less than or equal to 8	*pod*-partitions of 10 into parts less than or equal to 8 where the odd parts in the partition are less than the largest even part
8 + 2	8 + 2
7 + 3	6 + 3 + 1
6 + 4	6 + 4
6 + 2 + 2	6 + 2 + 2
5 + 3 + 2	4 + 3 + 2 + 1
4 + 4 + 2	4 + 4 + 2
4 + 2 + 2 + 2	4 + 2 + 2 + 2
2 + 2 + 2 + 2 + 2	2 + 2 + 2 + 2 + 2

To prove this theorem combinatorially, we will describe a bijection between the two types of *pod*-partitions.

Proof We will start with a partition, $3 \le a_1 < a_2 < \cdots < a_r \le 2n+1$ and $2 \le b_1 \le b_2 \le \cdots \le b_s \le 2n$, where the a-parts are the odd parts and the b-parts are the even parts. If the set of a-parts is empty or $a_r < b_s$, the partition is already a partition in the other set and we will not change the partition. If $a_r > b_s$, we will delete a_r and replace it with the even part $a_r - 1$ and the odd part 1 to get a partition in the other set. Clearly this transformation can be reversed and thus we have a bijection between the two sets of partitions in Theorem 3.9. □

Again if we use a combinatorial interpretation of the identity from Corollary 2.1 in [5] with q replaced by q^2 and $c = 1$ and $a = q$, we obtain the right-hand side of the identity we interpreted in Theorem 3.9. This gives us the following theorem.

Theorem 3.10 *The number of pod-partitions of m where the odd parts are greater than or equal to 3 and less than or equal to* $2n+1$ *and the even parts are less than or equal to* $2n$ *is the same as the number of Frobenius partitions of m where the number of columns is less than or equal to n, the partition in the top row is a partition into distinct odd parts where each part is less than or equal to* $2n-1$ *and the partition in the bottom row is an overpartition where the overlined parts are nonnegative even integers and the non-overlined parts are odd positive integers.*

Our last theorem in this section is:

Theorem 3.11 *The number of pod-partitions of m where the even parts are less than or equal to* $2n$*, the odd parts are less than or equal to* $4n+1$*, and* $2n+1$ *does not appear as a part is the same as the number of pod-partitions of m where the even parts are less than or equal to* $2n$ *and the odd parts are at most* $2n-1$ *greater than the largest even part.*

Proof Replacing q by q^2 and letting $z = -q^{2n+1}$ in Theorem 2.1, gives us

$$\sum_{j=0}^{n} \frac{q^{2j}(-q^{2n+1}; q^2)_j}{(q^2; q^2)_j} = \frac{(-q^{2n+3}; q^2)_n}{(q^2; q^2)_n}.$$

To interpret this in terms of partitions we multiply each side of the equation by $(-q; q^2)_n$ to obtain

$$\sum_{j=0}^{n} \frac{q^{2j}(-q; q^2)_{n+j}}{(q^2; q^2)_j} = \frac{(-q; q^2)_{2n+1}}{(1+q^{2n+1})(q^2; q^2)_n}.$$

The product on the right-hand side is the generating function for *pod*-partitions where the even parts are less than or equal to $2n$, the odd parts are less than or equal to $4n+1$, and $2n+1$ does not appear as a part. The sum on the left-hand side is the generating function for *pod*-partitions into parts less than or equal to $4n-1$ where the odd parts are at most $2n-1$ greater than the largest even part. (If there are no even parts, we take the largest even part to be 0.) □

To illustrate this theorem we have listed the partitions of $m = 11$ of the two types described in Theorem 3.11 for $n = 3$ in the table below. We have paired the partitions using the bijection that can be used to prove this theorem combinatorially. The table appears immediately following the combinatorial proof.

pod-partitions of 11 where the even parts are less than or equal to 6, the odd parts are less than or equal to 13, and 7 does not appear as a part	*pod*-partitions of 11 where the even parts are less than or equal to 6 and the odd parts are at most 5 greater than the largest even part
11	7 + 4
9 + 2	7 + 2 + 2
6 + 5	6 + 5
6 + 4 + 1	6 + 4 + 1
6 + 3 + 2	6 + 3 + 2
6 + 2 + 2 + 1	6 + 2 + 2 + 1
5 + 4 + 2	5 + 4 + 2
5 + 3 + 2 + 1	5 + 3 + 2 + 1
5 + 2 + 2 + 2	5 + 2 + 2 + 2
4 + 4 + 3	4 + 4 + 3
4 + 4 + 2 + 1	4 + 4 + 2 + 1
4 + 3 + 2 + 2	4 + 3 + 2 + 2
4 + 2 + 2 + 2 + 1	4 + 2 + 2 + 2 + 1
3 + 2 + 2 + 2 + 2	3 + 2 + 2 + 2 + 2
2 + 2 + 2 + 2 + 2 + 1	2 + 2 + 2 + 2 + 2 + 1

To prove this theorem combinatorially, we will describe a bijection between the two types of partitions.

Proof We will start with a *pod*-partition, λ, with parts $a_1 \leq a_2 \leq \cdots \leq a_r$ and $b_1 < b_2 < \cdots < b_s$, where the a-parts are all even and less than or equal to $2n$ and the b-parts are all odd, greater than or equal to 1, and no more than $2n - 1$ greater than a_r. Note that if there are no even parts then a_r is taken to be 0. If λ is a *pod* partition with no parts of size $2n + 1$ then λ is a partition in the other set. If λ contains a part $2n + 1$ then we will transform λ into a *pod* partition in the other set. We first note that there must be an even part in λ since if there is no even part then the largest even part would be 0 and the size condition on the odd parts would force the odd parts less than or equal to $2n - 1$. We will delete the part $2n + 1$ and create the new part $a_r + 2n + 1$ to create a *pod* partition in the other set. It should be noted that the part $a_r + 2n + 1$ is the largest odd part in the new partition and was not already in λ because of the size constraint on the odd parts relative to the largest even part. Clearly the transformation described above can be reversed and thus we have a bijection between the two sets of partitions in Theorem 3.11. □

4 Some Partition Theorems Associated with Theorems 2.2 and 2.3

In this section, we will interpret the results in Theorems 2.2 and 2.3 in terms of partitions. For the first result we let $z = 0$ in Theorem 2.3 which leads us to the following theorem.

Theorem 4.1 *The number of partitions of m into parts less than or equal to n where the largest part is repeated at least k times is the same as the number of partitions of m into parts less than or equal to n where either there are no parts less than k or if $j > 0$ is the largest part less than k appearing in the partition then the partition contains at least j n's.*

Proof Letting $z = 0$ in Theorem 2.3, we have

$$\sum_{j=0}^{n} \frac{q^{kj}}{(q;q)_j} = \frac{1}{(q;q)_n} \sum_{j=0}^{k-1} \left(q^{(n+1)j} (q^{j+1};q)_{k-1-j} \right)$$

$$= \frac{1}{(q^k;q)_{n-k+1}} + \sum_{j=1}^{k-1} \left(\frac{q^{(n+1)j}}{(q;q)_j (q^k;q)_{n-k+1}} \right).$$

The sum on the left-hand side is the generating function for partitions into parts less than or equal to n where the largest part is repeated at least k times. The function on the right-hand side counts partitions of m where the parts are less than or equal to n and there are either no parts less than k or if a part less than k appears in the partition then the number of n's that appear in the partition is at least as large as the largest part less than k that appears. □

To illustrate this theorem we have listed the partitions of $m = 10$ of the two types described in Theorem 4.1 for $n = 3$ and $k = 2$ in the table below. We have paired the partitions using the bijection that can be used to prove this theorem combinatorially. The combinatorial proof is presented immediately following the table.

Partitions of 10 into parts less than or equal to 3 where the largest part is repeated at least 2 times	Partitions of 10 into parts less than or equal to 3 which contain no 1s or contain at least one 1 and at least one 3
3 + 3 + 3 + 1	3 + 3 + 3 + 1
3 + 3 + 2 + 2	3 + 3 + 2 + 2
3 + 3 + 2 + 1 + 1	3 + 3 + 2 + 1 + 1
3 + 3 + 1 + 1 + 1 + 1	3 + 3 + 1 + 1 + 1 + 1
2 + 2 + 2 + 2 + 2	2 + 2 + 2 + 2 + 2
2 + 2 + 2 + 2 + 1 + 1	3 + 2 + 2 + 2 + 1
2 + 2 + 2 + 1 + 1 + 1 + 1	3 + 2 + 2 + 1 + 1 + 1
2 + 2 + 1 + 1 + 1 + 1 + 1 + 1	3 + 2 + 1 + 1 + 1 + 1 + 1
1 + 1 + 1 + 1 + 1 + 1 + 1 + 1 + 1 + 1	3 + 1 + 1 + 1 + 1 + 1 + 1 + 1

To prove this theorem combinatorially, we will describe a bijection between the two types of partitions.

Proof We will start with a partition λ with parts, $a_1 \leq a_2 \leq \cdots \leq a_r < b_1 \leq b_2 \leq \cdots \leq b_s$, where the parts are all less than or equal to n, the a-parts are less than k, the b-parts are greater than or equal to k, and $b_s = b_{s-1} = \cdots = b_{s-k+1}$. If $r = 0$, we will do nothing which gives us the partitions counted by the $j = 0$ term on the right-hand side of the equation that have their largest part repeated k or more times. If $0 < r \leq n$ and $s = 0$, we will simply conjugate λ to get the partitions counted by the $j = 0$ term on the right hand side of the equation that contain fewer than k parts greater than or equal to k. If $r > n$ and $s = 0$, we will conjugate the block of parts, $a_r, a_{r-1}, \ldots, a_{r-n+1}$, and form the partition with the conjugate parts and the parts $a_{r-n}, \ldots, a_1$ to get the partitions counted by the $j = a_{r-n}$ term on the right hand side of the equation that contain at least a_{r-n} but fewer than k n's and contain fewer than k parts greater than or equal to k. If $r > 0$ and $b_s = n$, we will do nothing which accounts for the partitions counted by the $j = a_r$ term on the right-hand side of the equation that contain k or more n's. If $r > 0$, $b_s \neq n$, and $r + b_s \leq n$, we will conjugate the r parts less than k and add the conjugate parts successively to $b_s, b_{s-1}, \ldots, b_{s-a_r-1}$ to get the partitions counted by the $j = 0$ term on the right-hand side of the equation that have k or more parts greater than or equal to k and their largest part is repeated fewer than k times. If $r > 0$, $b_s \neq n$, and $r + b_s > n$, we will conjugate the block of parts, $a_r, a_{r-1}, \ldots, a_{r-n+b_s+1}$ and add the conjugate parts successively to the largest parts in λ giving us the partitions counted by the $j = a_{r-n+b_s}$ term on the right-hand side of the equation that contain at least a_{r-n+b_s} but fewer than k n's and contain k or more parts greater than or equal to k. Clearly these transformations give us each of the partitions counted by the sum on the right hand side of the equation in a unique way which gives us a bijection between the two sets of partitions in Theorem 4.1. □

For our second example in this section, we obtain the following result from Theorem 2.2.

Theorem 4.2 *For $n \geq 2$, the number of partitions of m into parts less than or equal to $2n-1$ where the largest part less than n is repeated and the gap between the largest part less than n and the smallest part greater than or equal to n is at least n is equal to the number of partitions of m into parts less than or equal to n where the partitions contain at least as many parts of size n as they do 1s or the partitions contain the part $n-1$ and the number of 1s in the partition is greater than the number of n's.*

Proof Replacing n by $n-1$ and letting $z = q^n$ in Theorem 2.2, we obtain

$$\sum_{j=0}^{n-1} \frac{q^{2j}(q^n;q)_j}{(q;q)_j} = \frac{(q^{n+2};q)_{n-1}}{(q^2;q)_{n-2}} + \frac{q^n(q^{n+1};q)_{n-1}}{(q;q)_{n-1}}.$$

To interpret this in terms of partitions, we will multiply each side of the equation by $\frac{(q;q)_{n-1}}{(q;q)_{2n-1}}$ to obtain

$$\sum_{j=0}^{n-1} \frac{q^{2j}}{(q^{n+j};q)_{n-j}(q;q)_j} = \frac{1-q^{2n}}{(q^2;q)_n} + \frac{q^n}{(q;q)_n}$$

$$= \frac{1}{(q^2;q)_{n-1}(1-q^{n+1})} + \frac{q^n}{(q;q)_{n-1}(1-q^{n+1})}.$$

The sum on the left-hand side is the generating function for partitions into parts less than or equal to $2n-1$ where the largest part less than n is repeated and the gap between the largest part less than n and the smallest part greater than or equal to n is at least n. Note that this function allows the largest part less than n to be 0 in which case the partitions generated are simply those into parts greater than or equal to n and less than or equal to $2n-1$. To interpret the right-hand side of the equation in terms of partitions, we will view the factor $1-q^{n+1}$ in the denominators as generating pairs of parts 1 and n and will view the factor q^n in the numerator as generating parts 1 and $n-1$. With this view, the first function on the right-hand side is the generating function for partitions into parts less than or equal to n where the number of parts of size n is greater than or equal to the number of 1s and the second function on the right-hand side is the generating functions for partitions into parts less than or equal to n which contain the part $n-1$ and the number of 1s in the partition is greater than the number of n's. □

To illustrate this theorem we have listed the partitions of $m = 7$ of the two types described in Theorem 4.2 for $n = 4$ in the table below. Unlike the previous examples, we are simply illustrating that there are the same number of partitions in each of the two sets.

Partitions of 7 into parts less than or equal to 7 where the largest part less than 4 is repeated and the gap between the largest part less than 4 and the smallest part greater than or equal to 4 is at least 4	Partitions of 7 into parts less than or equal to 4 where the partitions contain at least as many parts of size 4 as they do 1s or the partitions contain the part 3 and the number of 1s in the partition is greater than the number of 4s
7	4 + 3
5 + 1 + 1	4 + 2 + 1
3 + 3 + 1	3 + 3 + 1
2 + 2 + 2 + 1	3 + 2 + 2
2 + 2 + 1 + 1 + 1	3 + 2 + 1 + 1
1 + 1 + 1 + 1 + 1 + 1 + 1	3 + 1 + 1 + 1 + 1

We do not have an explicit combinatorial proof of Theorem 4.2. However, we can give a partial combinatorial proof.

Proof We will start by rewriting the generating function in the following form:

$$\sum_{j=0}^{n-1} \frac{q^{2j}}{(q^{n+j};q)_{n-j}(q;q)_j} = \frac{1}{(q;q)_n} - \frac{q}{(q;q)_{n-2}(1-q^{n+1})}.$$

The function on the right-hand side is now the generating function for partitions into parts less than or equal to n excluding those partitions that contain no parts of size $n-1$ and have at least as many 1s as parts of size n. We will now argue combinatorially why the number of partitions of an integer m into parts less than or equal to $2n-1$ where the largest part less than n is repeated and the gap between the largest part less than n and the smallest part greater than or equal to n is at least n is the same as the total number of partitions of m into parts less than or equal to n where the partitions contain at least as many parts of size n as they do 1s or the partitions contain the part $n-1$ and the number of 1s in the partition is greater than or equal to the number of n's.

We first note that the partitions of an integer m into parts less than or equal to $2n-1$ where the largest part less than n is repeated and the gap between the largest part less than n and the smallest part greater than or equal to n is at least n is a subset of the partitions in Theorem 3.1. To complete the combinatorial proof, we need to show that the partitions into parts less than or equal to $2n-1$ where the largest part less than n is greater than 0, is not repeated, and the gap between this largest part less than n and the smallest part greater than or equal to n is at least n are equinumerous with the partitions generated by $\frac{q}{(q;q)_{n-2}(1-q^{n+1})}$. To show this we will describe a bijection between these latter two sets of partitions. We will start with a partition λ given by $a_1 \le a_2 \le \cdots < a_r < b_1 \le b_2 \le \cdots \le b_s$, where the parts are all less than or equal to $2n-1$, the a-parts are less than n, the b-parts are greater than or equal to n, $b_1 - a_r \ge n$, and if $r > 1$, $a_{r-1} < a_r$. Note that r must be at least 1, but s could be zero. If $s = 0$ and $r = 1$, we will map λ to the partition

1 and $a_r - 1$. If $s = 0$ and $r > 1$ then $1 < a_r \leq n - 1$ and we can map λ to the partition consisting of $1, a_1, a_2, \ldots, a_r - 1$. If $s > 0$ and $r = 1$ then we can map λ to the partition consisting of $s + 1$ 1s, s n's, $a_r - 1$ and the parts $b_1 - n - 1, b_2 - n - 1, \ldots, b_s - n - 1$ where any parts of size 0 are ignored. If $s > 0$ and $r > 1$ ($a_r > 1$) then we can map λ to the partition consisting of $s + 1$ 1s, s n's, and the parts $a_1, a_2, \ldots, a_r - 1, b_1 - n - 1, b_2 - n - 1, \ldots, b_s - n - 1$.

To clearly show that this map is a bijection, we will describe how this map can be reversed by describing how a partition μ of m into parts less than or equal to n that contains no parts of size $n - 1$ and contains more 1s than n's can be mapped to a partition in the other set. Let $s \geq 0$ be the number of n's, t where $t \geq s + 1$ be the number of 1s, and $c_{t+1} \leq c_{t+2} \leq \cdots \leq c_{t+v}$ be the parts greater than 1 and less than $n - 2$ in μ. Let $c_1 = c_2 = \cdots = c_t = 1$. We will use s of the 1s together with the s parts of size n to create s $n + 1$'s. Note that there are $t - s$ parts of size 1 remaining. If $s \geq t - s - 1 + v$ we will map μ to the partition consisting of the parts $a_1 = c_1$, $s - t - v + 1$ $(n + 1)$'s (if $s - t - v + 1 > 0$, these are the parts $b_1, \ldots, b_{s-t-v+1}$), and $b_{s-t-v+2} = n + 1 + c_2, \ldots, b_s = n + 1 + c_{t+v}$. If $s < t - s - 1 + v$ we will map μ to the partition consisting of the parts $a_1 = c_2, a_2 = c_3, \ldots, a_{t+v-2-s} = c_{t+v-1-s}$, $a_{t+v-1-s} = c_{t+v-s} + c_1$, $b_1 = n + 1 + c_{t+v-s+1}$, $b_2 = n + 1 + c_{t+v-s+2}, \ldots$, $b_s = n + 1 + c_{t+v}$. $\square$

5 Conclusion

The results in Theorems 2.1–2.3 seem to be very versatile. We have shown how they are connected to a variety of partitions including ordinary partitions, overpartitions, and *pod*-partitions. A couple of questions for further research immediately come to mind.

Question 1: Is there a family of similar functions that include the series given by Andrews and Yee as a special case?

Question 2: Are there other such families or examples of functions of this type?

Question 3: Can an explicit bijection be found between the two sets of partitions in Theorem 4.2?

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