and line 6, *read*:

$$\text{so that } B_n^m = (-1)^{n-m}\binom{n}{m} \quad \text{and} \quad B_n^m = -B_{n-1}^m + B_{n-1}^{m-1}.$$

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# ON ESTIMATES IN NUMBER THEORY

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In many problems in number theory, one is interested in estimating the number of solutions in integers of certain equations. It often happens that one does not need the most refined estimate for the number of solutions, but an estimate somewhat finer than a trivial estimate is needed. It is the purpose of this paper to present a very general method for dealing with such problems. The main result will be a consequence of a recently discovered theorem in the geometry of numbers [2, p. 270].

THEOREM. *If $f(x)$ is continuous and twice differentiable with continuous second derivative in $[a, b]$, $M=\max_{a\le x\le b} f(x)$, $m=\min_{a\le x\le b} f(x)$, and $f''(x)=0$ has $d$ solutions in $(a, b)$, then*

$$N < c(d+1)\{(M-m)(b-a)\}^{1/3},$$

*where $N$ is the number of solutions in integers of $y=f(x)$ and $c>0$ is an absolute constant.*

*Proof.* It is sufficient to prove the theorem when $d=0$. For assume it has been proved in this case. Consider the intervals $(a, x_1)$, $(x_1, x_2)$, $\cdots$, $(x_d, b)$ where $x_j$ is the $j$th zero of $f''(x)$ in $(a, b)$ (there are at most $d+1$ such intervals). Let $N_i$

denote the number of solutions in integers of $y=f(x)$ in $(x_{i-1}, x_i)$ with $a=x_0$ and $b=x_{d+1}$. Then

$$\begin{aligned} N &\leqq \sum_{j=1}^{d+1} N_j \\ &< c \sum_{j=1}^{d+1} \left\{(M - m)(x_j - x_{j-1})\right\}^{1/3} \\ &\leqq c \sum_{j=1}^{d+1} \left\{(M - m)(b - a)\right\}^{1/3} \\ &= c(d + 1)\cdot\left\{(M - m)(b - a)\right\}^{1/3}. \end{aligned}$$

If $d=0$, then the curve $y=f(x)$ forms an arc which is strictly convex since now $f''(x)$ must be of constant sign [**5**, pp. 172–3]. Also the region, $C_1$, bounded by $y=f(x)$ and the straight line $L$ through $P_1$: $(a, f(a))$ and $P_2$: $(b, f(b))$ lies entirely within the rectangle $R$ given by $m \leqq y \leqq M$, $a \leqq x \leqq b$.

Hence if $A_1$ is the area of $C_1$ and $A_0$ is the area of $R$, then $A_1 < A_0$.

Let us now pass a circle of radius $\lambda$ through the two points $P_1$ and $P_2$ with center on the perpendicular bisector of $\overline{P_1P_2}$ and on the same side of $L$ as the graph of $y=f(x)$. Let $\alpha$ denote the arc of this circle which lies on the opposite side of $L$ from the graph of $y=f(x)$ and has as end points $P_1$ and $P_2$. Provided we take $\lambda$ sufficiently large, we have that $\alpha$ and the graph of $y=f(x)$ bound a strictly convex body, $C_2$, with area as near that of $C_1$ as we please. Also we may take $\lambda$ large enough so that if $A_2$ is the area of $C_2$, then $A_2 < A_0$.

By the main theorem in [**2**, p. 270],

$$A_2 > \mathcal{K}(2)N^{(2+1)/(2-1)} = \mathcal{K}(2)N^3,$$

where $\mathcal{K}(2) > 0$ is an absolute constant. Hence there exists an absolute constant $c > 0$, such that

$$\begin{aligned} N &< cA_2^{1/3} \\ &< cA_0^{1/3} \\ &= c\left\{(M - m)(b - a)\right\}^{1/3}. \end{aligned}$$

This concludes the proof of the theorem.

We now show how this theorem may be applied.

If $d(n)$ denotes the number of divisors of $n$, then trivially $d(n) = O(n)$. In most texts in elementary number theory, improved results are obtained only after carefully studying properties of multiplicative functions [**3**, p. 260]. By our main theorem we may quickly get a nontrivial result.

THEOREM. $d(n) = O(n^{2/3})$.

*Proof.* Let $f(x) = n/x$ with $a=1$, $b=n$, then $M=n$, $m=1$, $N=d(n)$, and $f''(x) = 2n/x^3 > 0$ in $(1, n)$. Hence

$$d(n) = N < c\{(n-1)(n-1)\}^{1/3}$$
$$< cn^{2/3}.$$

By a somewhat more subtle use of the main theorem, we show how a more refined result may be obtained.

THEOREM. $d(n) = O(n^{1/3} \log n)$.

*Proof.* Consider the $r$ intervals $(1, n^{1/r})$, $(n^{1/r}, n^{2/r})$, $\cdots$, $(n^{(r-1)/r}, n)$. Let $f(x) = n/x$ as before. Let $N_j$ be the number of lattice points on the curve in the $j$th interval $(n^{(j-1)/r}, n^{j/r})$. Then in the $j$th interval $M - m \leqq M = n^{1-(j-1)/r}$, and $n^{j/r} - n^{(j-1)/r} < n^{j/r}$. Hence $(M-m)(n^{j/r} - n^{(j-1)/r}) < n^{1+(1/r)}$ in the $j$th interval. Thus

$$d(n) = N \leqq \sum_{j=1}^{r} N_j$$
$$< c \cdot r \cdot (n^{1+(1/r)})^{1/3}.$$

Taking $r = [\log n] + 1$, we obtain the desired result.

Of course we come nowhere near the truth that $d(n) = O(n^{\epsilon})$ for any $\epsilon > 0$ [3, p. 260], but we obtained our results without any use of the arithmetic properties of $d(n)$.

As one sees, very seldom will best possible results be obtained. It is clear, however, that this method is useful when something stronger than a nontrivial estimate is required, and a minimal amount of information about the arithmetic properties of the equation under consideration is available.

Finally, we note that it is not possible to improve our main theorem substantially. Taking $f(x) = x^2$, $a = 0$, $b = n$, we find that the theorem implies

$$N < c(n^2 \cdot n)^{1/3} = c \cdot n.$$

Actually we see by inspection that $N = n+1$, so that the exponent $\frac{1}{3}$ cannot be replaced by any smaller number.

Professor I. J. Schoenberg has pointed out to me the close connection between the main result in this paper and the result in [4]. However, Jarnik's result is weaker; for example, his result will imply $d(n) = O(n^{2/3})$, but will not imply $d(n) = O(n^{1/3} \log n)$. Actually the result of Jarnik implies the result in [1] in the two-dimensional case.

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