Solving differential equations by means of Hermite transform

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Abstract

Let L be a linear differential operator with polynomial coefficients. We show that there is an isomorphism of differential operators $\mathcal{D}_\alpha$ and an integral transform $\mathcal{H}_\alpha$ (called the Hermite transform) on functions that enable us to find a solution of $Ly = 0$ provided that we know the solution of $(\mathcal{D}_\alpha L)y = 0$. A detailed algorithm is described to find the Hermite transform of a rational function.

1 Introduction

One of the well-known methods for solving linear differential equations is by means of definite integrals [5]. The aim is to obtain a definite integral of the form

$$y(x) = \int_\alpha^\beta f(t)K(x,t)dt, \tag{1}$$

where x enters as a parameter, to satisfy the given linear differential equation

$$L_x y = 0$$

(the index x indicating that L involves only x and $\partial/\partial x$). If we are able to determine

– the function $K(x,t)$, known as the *nucleus* of the definite integral, so that it will satisfy a partial differential equation of the form

$$L_x(K) = M_t(K),$$

– the function $f(t)$, a solution of the equation

$$\overline{M}_t(f) = 0, \tag{2}$$

where $\overline{M}_t$ is the operator adjoint to M_t,

– the limits of integration, α and β, such that

$$[P(K,t)]_{t=\alpha}^{t=\beta} = 0$$

identically, where P, known as the *bilinear concomitant*, is an expression depending on K, f and its derivatives with respect to t,

then (1) will satisfy the equation $L_x y = 0$. Examples of functions which can be chosen as nuclei are e^{xt} (the Laplace transform), $(x-t)^{-\nu-1}$ (the Euler transform), arbitrary function of the product xt (the Mellin transform). The Laplace transform is adaptable to any linear differential equation while in the other two cases the equation has to be of a special form. The success of the method as a means of obtaining an explicit solution depends upon the ability to obtain a solution of (2).

Another method is Frobenius' method of power series, which is one of the best-known methods for solving linear differential equations. In symbolic computation, other types of polynomial series have also been considered [2]. Rebillard [11] in his thesis considered Fourier series of families of hypergeometric polynomials. Here we use his approach on the family of generalized Hermite polynomials and show that there is an isomorphism $\mathcal{D}_\alpha$ of differential operators so that differential equation $Ly = 0$ has a formal Hermite series solution whenever $\mathcal{D}_\alpha y = 0$ has a formal power series solution with identical coefficients (see also [12]). We use this to construct an integral transform $\mathcal{H}_\alpha$ (called the Hermite transform) which enables us to find solutions of a given differential equation $Ly = 0$ if we know a solution of another differential equation $(\mathcal{D}_\alpha L)y = 0$. We exhibit a simple procedure to find the Hermite transform of polynomials and rational functions. In the last section we describe an algorithm that finds $n+1$ linearly independent solutions of $Ly = 0$ provided that $(\mathcal{D}_\alpha L)y = 0$ has a rational solution with n distinct poles.

Let K be a field of characteristic zero and $\mathbb{N}$ the set of nonnegative integers. Let E be the shift operator acting on sequences over K. We will denote by $K[n, E, E^{-1}]$ the algebra of recurrence operators and by $K[x, D]$ the Weyl algebra of linear differential operators with polynomial coefficients where $Dp(x) = \frac{d}{dx}p(x)$. If we write an element L of $K[x, D]$ in the monomial form

$$L = \sum_{i,k} a_{ik} x^i D^k,$$

then $\max_{a_{ik}\neq 0} k$ is the *order of* L and $\max_{a_{ik}\neq 0}(i+k)$ is the *total degree of* L. By L_a we will denote

$$L_a = \sum_{i,k} a_{ik}(x+a)^i D^k.$$

Note that $L_a f(x) = 0$ iff $Lf(x-a) = 0$.

A polynomial family $\mathcal{P} = \{P_n(x)\}_{n=0}^{\infty}$ is called *hypergeometric* if each P_n is of degree exactly n and satisfies the hypergeometric differential equation

$$\sigma(x)y'' + \tau(x)y' + \lambda_n y = 0 \tag{3}$$

where σ and τ are polynomials of degree at most two, respectively one, and $\lambda_n = -n\left(\frac{n-1}{2}\sigma''(x) + \tau'(x)\right)$. With σ and τ the family is determined up to a multiplicative constant. To get a complete family – this means that for every $n \in \mathbb{N}$ the equation (3) has a polynomial solution of degree n – the function $n \mapsto \lambda_n$ has to be injective. Thus τ has to be of degree one, and if $\sigma'' \neq 0$ then $2\tau'/\sigma'' \notin \{0, -1, -2, \ldots\}$. For every hypergeometric family there exist recurrence operators $\mathbf{X}$ and $\mathbf{B}$ in $K[n, E, E^{-1}]$ such that for each P_n we have

$$
\begin{aligned}
P_n(x) &= \mathbf{B}P_n'(x) = \beta_1(n)P_{n+1}'(x) + \beta_0(n)P_n'(x) + \beta_{-1}(n)P_{n-1}'(x), && (4)\\
xP_n(x) &= \mathbf{X}P_n(x) = x_1(n)P_{n+1}(x) + x_0(n)P_n(x) + x_{-1}(n)P_{n-1}(x), && (5)
\end{aligned}
$$

where $x_i, \beta_i \in K(n)$ and depend only on σ and τ [8, 9, 10, 11, 14]. If P is a family of Appell polynomials [3], i.e. $P_n = \beta_1(n)P_{n+1}'$, then it is more convenient to use $\tilde{\mathbf{B}} = \mathbf{B}^{-1}$ acting on P_n by

$$\tilde{\mathbf{B}}P_n(x) = P_n'(x) = \frac{1}{\beta_1(n-1)}P_{n-1}(x). \qquad (6)$$

Consider a linear differential operator of order r

$$L = \sum_{k=0}^{r} p_k(x)D^k$$

with polynomial coefficients. In order to apply L to a formal series of hypergeometric polynomials $\sum_{n=0}^{\infty} c_n P_n(x)$ we need to know how differentiating and multiplying by x affect the series. Let us recall that for a recurrence operator $\mathbf{F} \in K[n, E, E^{-1}]$

$$\mathbf{F} = \sum_{k=r}^{s} f_k(n)E^k, \quad r, s \in \mathbb{Z},$$

its *adjoint operator* f is defined by

$$F = \sum_{k=-s}^{-r} f_{-k}(n+k)E^k.$$

If $(a_n)_{n=-\infty}^{\infty}$ and $(b_n)_{n=-\infty}^{\infty}$ are two sequences of functions, then $\sum\limits_{n=-\infty}^{\infty} a_n\mathbf{F}(b_n) = \sum\limits_{n=-\infty}^{\infty} F(a_n)b_n$. Therefore differentiating and multiplying a series by x can be expressed by the action of the adjoint operator on the coefficient sequence of the series [11]

$$
\begin{aligned}
x\sum c_nP_n(x) &= \sum X(c_n)P_n(x),\\
\sum c_nP_n'(x) &= \sum \tilde{B}(c_n)P_n(x)
\end{aligned}
$$

where $\tilde{B}$ and X are the adjoints of $\tilde{\mathbf{B}}$ and $\mathbf{X}$ from (4) and (5). Hence every operator L of order r induces a recurrence operator $\mathcal{L}$ such that

$$L\left(\sum_{n=0}^{\infty} c_nP_n(x)\right) = \sum_{n=0}^{\infty} \mathcal{L}(c_n)P_n(x),$$

and the hypergeometric family $\mathcal{P} = \{P_n(x)\}_{n=0}^{\infty}$ gives rise to the transformation $\mathcal{R}_{\mathcal{P}} : K[x, D] \to K[n, E, E^{-1}]$ defined by

$$\mathcal{R}_{\mathcal{P}} : L = \sum_{k=0}^{r} p_k(x) D^k \mapsto \mathcal{L} = \sum_{k=0}^{r} p_k(X) \tilde{B}^k.$$

Note that $L(\sum c_n P_n) = 0$ iff $\mathcal{R}_{\mathcal{P}}(L)(c_n) = 0$.

The *Hermite polynomials*, $H_n(x)$, form a hypergeometric family. They are defined by the Rodrigues' type formula [4, 13]

$$H_n(x) = (-1)^n e^{x^2} \frac{d^n}{dx^n} e^{-x^2}. \tag{7}$$

$y(x) = H_n(x)$ is a polynomial of degree n whose leading coefficient is $k_n = 2^n$ and satisfies the hypergeometric differential equation

$$y''(x) - 2xy'(x) + 2ny(x) = 0, \quad n = 0, 1, 2, \ldots \tag{8}$$

The most common alternative to the above terminology is to replace e^{-x^2} by $e^{-x^2/2}$ in (7) yielding a polynomial $He_n(x)$, which is called the Hermite polynomial by Jackson [6] and seems to be the preferred form for applications to statistics. $He_n(x)$ is a monic polynomial of degree n and satisfies the differential equation

$$y''(x) - xy'(x) + ny(x) = 0,$$

which is similar to (8). These observations thus suggest a further generalization which we use in the next section.

2 The isomorphism $\mathcal{D}_\alpha$

Let us focus on the hypergeometric differential equation

$$y''(x) - \alpha xy'(x) + \alpha ny(x) = 0 \tag{9}$$

depending on a parameter $\alpha > 0$. For $\alpha = 2$ this differential equation is satisfied by $H_n(x)$ and for $\alpha = 1$ by $He_n(x)$. As $\sigma(x) = 1$ and $\tau(x) = -\alpha x$, equation (9) has, by Rodrigues' type formula, a polynomial solution for every $n \in \mathbb{N}$

$$H_n^{(\alpha)}(x) = \frac{B_n}{\rho} \frac{d^n}{dx^n} \rho(x)$$

where B_n is a normalization coefficient and ρ is a solution of the Pearson differential equation

$$(\sigma\rho)' = \tau\rho,$$

therefore $\rho(x) = e^{-\alpha x^2/2}$. Put $B_n = (-1)^n$. Applying the formulas in [8, 9, 11] to the family $H^{(\alpha)} = \{H_n^{(\alpha)}; n \in \mathbb{N}\}$, we obtain the corresponding operators $\mathbf{X}_\alpha$ and $\mathbf{B}_\alpha$

$$\begin{aligned}\mathbf{X}_\alpha &= \frac{1}{\alpha}E + nE^{-1},\\ \mathbf{B}_\alpha &= \frac{1}{\alpha(n+1)}E,\end{aligned}$$

hence

$$\tilde{\mathbf{B}}_\alpha = \alpha n E^{-1}.$$

In this case the transformation $\mathcal{R}_{H^{(\alpha)}} : K[x, D] \to K[n, E, E^{-1}]$ is defined by

$$\mathcal{R}_{H^{(\alpha)}} : \sum_{k=0}^{r} p_k(x)D^k \longmapsto \sum_{k=0}^{r} p_k(X_\alpha)\tilde{B}_\alpha^k,$$

where

$$\begin{aligned}X_\alpha &= (n+1)E + 1/\alpha E^{-1},\\ \widetilde{B}_\alpha &= \alpha(n+1)E\end{aligned}$$

are the adjoints of $\mathbf{X}_\alpha$ and $\widetilde{\mathbf{B}}_\alpha$, respectively.

Proposition 1 *Let $\mathcal{P}$ be a hypergeometric polynomial family and X and $\tilde{B}$ the adjoints of $\mathbf{X}$ and $\tilde{\mathbf{B}}$, respectively, defined in (5) and (6). If X and $\tilde{B}$ satisfy the commutation rule*

$$\tilde{B}X = X\tilde{B} + 1, \tag{10}$$

then the transformation $\mathcal{R}_{\mathcal{P}} : K[x, D] \to K[n, E, E^{-1}]$

$$\mathcal{R}_{\mathcal{P}} : \sum_{k=0}^{r} p_k(x)D^k \longmapsto \sum_{k=0}^{r} p_k(X)\tilde{B}^k$$

is a homomorphism of K-algebras.

Proof Let $L_1 = \sum_{k=0}^{r_1} p_k(x)D^k$ and $L_2 = \sum_{j=0}^{r_2} q_j(x)D^j$. Clearly $\mathcal{R}_{\mathcal{P}}(\lambda_1 L_1 + \lambda_2 L_2) = \lambda_1 \mathcal{R}_{\mathcal{P}}(L_1) + \lambda_2 \mathcal{R}_{\mathcal{P}}(L_2)$. Using Leibniz' rule

$$D^j p(x) = \sum_{i=0}^{j} \binom{j}{i} p^{(i)}(x) D^{j-i}$$

it follows that

$$\begin{aligned}\mathcal{R}_{\mathcal{P}}(L_2L_1) &= \mathcal{R}_{\mathcal{P}}\left(\sum_{j=0}^{r_2}\sum_{k=0}^{r_1} q_j(x)D^j p_k(x)D^k\right)\\ &= \mathcal{R}_{\mathcal{P}}\left(\sum_{j=0}^{r_2}\sum_{k=0}^{r_1}\sum_{i=0}^{j}\binom{j}{i} q_j(x)p_k^{(i)}(x)D^{j-i+k}\right)\\ &= \sum_{j=0}^{r_2}\sum_{k=0}^{r_1}\sum_{i=0}^{j}\binom{j}{i} q_j(X)p_k^{(i)}(X)\tilde{B}^{j-i+k}.\end{aligned}$$

Since operators X and $\tilde{B}$ satisfy the same commutation rule as x and D, we can replace in Leibnitz' rule D and x by $\tilde{B}$ and X, respectively, and we find out that

$$\begin{aligned}\mathcal{R}_{\mathcal{P}}(L_2)\mathcal{R}_{\mathcal{P}}(L_1) &= \sum_{j=0}^{r_2}\sum_{k=0}^{r_1} q_j(X)\tilde{B}^j p_k(X)\tilde{B}^k \\ &= \sum_{j=0}^{r_2}\sum_{k=0}^{r_1}\sum_{i=0}^{j}\binom{j}{i} q_j(X)p_k^{(i)}(X)\tilde{B}^{j-i+k},\end{aligned}$$

hence $\mathcal{R}_{\mathcal{P}}(L_2L_1) = \mathcal{R}_{\mathcal{P}}(L_2)\mathcal{R}_{\mathcal{P}}(L_1)$. $\square$

Let $P = \{x^n\}$ be the power family. Since $xx^n = Ex^n$ and $Dx^n = nE^{-1}x^n$, the adjoint operators are $X_P = E^{-1}$ and $\widetilde{B}_P = (n+1)E$. Both families, $H^{(\alpha)}$ and P, satisfy the commutation rule (10), hence $\mathcal{R}_{H^{(\alpha)}}$ and $\mathcal{R}_P$ are homomorphisms.

We observe that

$$\begin{aligned}X_\alpha &= \widetilde{B}_P + \tfrac{1}{\alpha}X_P, & X_P &= \alpha X_\alpha - \widetilde{B}_\alpha, \\ \widetilde{B}_\alpha &= \alpha\widetilde{B}_P, & \widetilde{B}_P &= \tfrac{1}{\alpha}\widetilde{B}_\alpha,\end{aligned}$$

meaning there is an isomorphism $\mathcal{D}_\alpha : K[x,D] \to K[x,D]$ such that $\mathcal{D}_\alpha L$ applied to a formal power series $\sum c_n x^n$ induces the same recurrence operator as L acting on a formal Hermite series $\sum c_n H_n^{(\alpha)}(x)$, therefore

$$L\left(\sum c_n H_n^{(\alpha)}(x)\right) = 0 \Longleftrightarrow (\mathcal{D}_\alpha L)\left(\sum c_n x^n\right) = 0.$$

To describe $\mathcal{D}_\alpha$ on $K[x,D]$, it suffices to give it on the two generators x and D

$$\begin{aligned}\mathcal{D}_\alpha : \quad x &\longmapsto \frac{1}{\alpha}x + D, \\ D &\longmapsto \alpha D.\end{aligned}$$

The inverse is given by

$$\begin{aligned}\mathcal{D}_\alpha^{-1} : \quad x &\longmapsto \alpha x - D, \\ D &\longmapsto \frac{1}{\alpha}D.\end{aligned}$$

3 The Hermite transform $\mathcal{H}_\alpha$

For $\alpha > 0$

$$e^{-\frac{\alpha}{2}x^2} = \sqrt{\frac{\alpha}{2\pi}}\int_{-\infty}^{\infty} e^{-\frac{\alpha}{2}t^2 + i\alpha tx}dt.$$

Since the integral can be repeatedly differentiated with respect to x, we obtain

$$\frac{d^n}{dx^n}e^{-\frac{\alpha}{2}x^2} = \sqrt{\frac{\alpha}{2\pi}}\int_{-\infty}^{\infty}(i\alpha t)^n e^{-\frac{\alpha}{2}t^2 + i\alpha tx}dt,$$

hence

$$H_n^{(\alpha)}(x) = (-1)^n e^{\frac{\alpha}{2}x^2}\sqrt{\frac{\alpha}{2\pi}}\int_{-\infty}^{\infty}(i\alpha t)^n e^{-\frac{\alpha}{2}t^2+i\alpha tx}dt \tag{11}$$

$$= \sqrt{\frac{\alpha}{2\pi}}\int_{-\infty}^{\infty}(-i\alpha t)^n e^{-\frac{\alpha}{2}(t-ix)^2}dt. \tag{12}$$

We observe that the change of variable $u = \sqrt{\alpha/2}\,t$ converts (11) into

$$H_n^{(\alpha)}(x) = \left(\frac{\alpha}{2}\right)^{n/2}\frac{1}{\sqrt{\pi}}\int_{-\infty}^{\infty}(-2iu)^n e^{-(u-i\sqrt{\frac{\alpha}{2}}x)^2}du$$

and we find a connection between $H_n^{(\alpha)}$ and H_n

$$H_n^{(\alpha)}(x) = \left(\frac{\alpha}{2}\right)^{n/2} H_n(\sqrt{\alpha/2}\,x).$$

Definition 1 *Let f be a complex function and $\alpha > 0$. If there is a (finite or infinite) interval (a,b) such that the integral*

$$\mathcal{H}_\alpha(f)(x) = \sqrt{\frac{\alpha}{2\pi}}\int_{-\infty}^{\infty} f(-i\alpha t)e^{-\frac{\alpha}{2}(t-ix)^2}dt \tag{13}$$

is convergent for all $x \in (a,b)$, then we will call the function $\mathcal{H}_\alpha(f)(x)$ the Hermite transform *of f with parameter α.*

As

$$\mathcal{H}_\alpha(f)(x) = \frac{1}{\sqrt{2\pi\alpha}}e^{\alpha x^2/2}\mathcal{F}(f(-it)e^{-t^2/(2\alpha)})(x),$$

where $\mathcal{F}(f)(x) = \int_{-\infty}^{\infty} f(t)e^{itx}\,dt$ is the Fourier transform of f, we have the inversion formula

$$f(t) = \sqrt{\frac{\alpha}{2\pi}}\int_{-\infty}^{\infty}\mathcal{H}_\alpha(f)(x)e^{-\alpha/2(x-t/\alpha)^2}dx,$$

whenever the integral on the right exists.

Evidently, not every function has the Hermite transform, so we will be concerned with questions of existence. For instance, if $f(t) = e^{t^4}$, it is obvious that the integral

$$\int_{-\infty}^{\infty} e^{\alpha^4 t^4}e^{-\frac{\alpha}{2}(t-ix)^2}dt$$

does not exists for any $\alpha > 0$ and $x \in \mathbb{R}$. On the other hand, the Hermite transform of e^{t^2} exists for all positive α and it is defined on $\mathbb{R}$.

Definition 2 *Let $\alpha > 0$. A complex function f satisfies the $\mathcal{H}_\alpha$-*condition *if there exist a real number $k < \alpha/2$, a nonnegative number N and a polynomial p such that*

1. $|f(-i\alpha t)| \le p(t)e^{kt^2}$ *for* $|t| \ge N$,

2. $f(-i\alpha t)$ is absolutely integrable on $(-N, N)$.

Let f satisfy the $\mathcal{H}_\alpha$-condition. We write (13) in the form

$$e^{\frac{\alpha}{2}x^2}\int_{-\infty}^{\infty} f(-i\alpha t)e^{-\frac{\alpha}{2}t^2}e^{i\alpha xt}dt \tag{14}$$

and split the integral in three parts

$$\int_{-\infty}^{\infty} f(-i\alpha t)e^{-\frac{\alpha}{2}t^2}e^{i\alpha xt}dt = I_1 + I_2 + I_3,$$

where

$$\begin{aligned}
I_1 &= \int_{-N}^{N} f(-i\alpha t)e^{-\frac{\alpha}{2}t^2}e^{i\alpha xt}dt,\\
I_2 &= \int_{-\infty}^{-N} f(-i\alpha t)e^{-\frac{\alpha}{2}t^2}e^{i\alpha xt}dt,\\
I_3 &= \int_{N}^{\infty} f(-i\alpha t)e^{-\frac{\alpha}{2}t^2}e^{i\alpha xt}dt.
\end{aligned}$$

Then

$$\begin{aligned}
|I_1| &\leq \int_{-N}^{N} |f(-i\alpha t)|dt < \infty,\\
|I_2| &\leq \int_{-\infty}^{-N} |p(t)|e^{(k-\frac{\alpha}{2})t^2}dt < \infty,\\
|I_3| &\leq \int_{N}^{\infty} |p(t)|e^{(k-\frac{\alpha}{2})t^2}dt < \infty,
\end{aligned}$$

hence the integral (13) is absolutely convergent on the real axis and uniformly convergent on any finite interval.

Theorem 1 *Let f satisfy the $\mathcal{H}_\alpha$-condition for some $\alpha > 0$. Then the Hermite transform of f with parameter α is defined for every $x \in \mathbb{R}$. The integral (13) converges uniformly on any finite interval and*

$$\frac{d^k}{dx^k}\int_{-\infty}^{\infty} f(-i\alpha t)e^{-\frac{\alpha}{2}(t-ix)^2}dt = \int_{-\infty}^{\infty} f(-i\alpha t)\frac{d^k}{dx^k}e^{-\frac{\alpha}{2}(t-ix)^2}dt.$$

Proof We have to justify that

$$\int_{-\infty}^{\infty} f(-i\alpha t)e^{-\frac{\alpha}{2}t^2}e^{i\alpha xt}dt$$

can be repeatedly differentiated under the integral sign. Since

$$\frac{d^k}{dx^k}e^{i\alpha xt} = (i\alpha t)^k e^{i\alpha xt}$$

and observing that $t^k f$ satisfies the $\mathcal{H}_\alpha$-condition if f does, the integral

$$\int_{-\infty}^{\infty} f(-i\alpha t) e^{-\frac{\alpha}{2}t^2} \frac{d^k}{dx^k} e^{i\alpha xt} dt$$

converges uniformly on $\mathbb{R}$, which concludes the proof. $\square$

Proposition 2 *If f satisfies the $\mathcal{H}_\alpha$-condition then*

1. $\mathcal{H}_\alpha(f')(x) = \frac{1}{\alpha} D\mathcal{H}_\alpha(f)(x)$
2. $\mathcal{H}_\alpha(tf)(x) = (\alpha x - D)\mathcal{H}_\alpha(f)(x)$

Proof Using integration by parts and Theorem 1, we obtain

$$\begin{aligned}
\mathcal{H}_\alpha(f')(x) &= i\sqrt{\frac{\alpha}{2\pi}} \int_{-\infty}^{\infty} (t - ix) f(-i\alpha t) e^{-(\alpha/2)(t-ix)^2} dt \\
&= \sqrt{\frac{\alpha}{2\pi}} \frac{1}{\alpha} \int_{-\infty}^{\infty} f(-i\alpha t) \frac{d}{dx} e^{-(\alpha/2)(t-ix)^2} dt = \frac{1}{\alpha} D\mathcal{H}_\alpha(f)(x),
\end{aligned}$$

which proves the first statement. For the second statement, we compute

$$\begin{aligned}
\mathcal{H}_\alpha(tf)(x) &= \sqrt{\frac{\alpha}{2\pi}} \int_{-\infty}^{\infty} (\alpha x - \alpha x - i\alpha t) f(-i\alpha t) e^{-(\alpha/2)(t-ix)^2} dt \\
&= \alpha x \mathcal{H}_\alpha(f)(x) - D\mathcal{H}_\alpha(f)(x).
\end{aligned}$$

$\square$

Theorem 2 *Let L be a linear differential operator with polynomial coefficients, $\alpha > 0$ and let f satisfy the $\mathcal{H}_\alpha$-condition. If $(\mathcal{D}_\alpha L)f(x) = 0$ then $L\mathcal{H}_\alpha(f)(x) = 0$.*

Proof Since $\mathcal{D}_\alpha$ is an isomorphism on $K[x, D]$, we can write $\mathcal{D}_\alpha L = \sum_{i,k} a_{ik} x^i D^k$, hence $L = \sum_{i,k} a_{ik} (\alpha x - D)^i \alpha^{-k} D^k$. Then using Proposition 2 and linearity of $\mathcal{H}_\alpha$, we have

$$\begin{aligned}
L\mathcal{H}_\alpha(f)(x) &= \sum_{i,k} a_{ik} (\alpha x - D)^i \alpha^{-k} D^k \mathcal{H}_\alpha(f)(x) \\
&= \sum_{i,k} a_{ik} (\alpha x - D)^i \mathcal{H}_\alpha(f^{(k)})(x) \\
&= \sum_{i,k} a_{ik} \mathcal{H}_\alpha(t^i f^{(k)})(x) \\
&= \mathcal{H}_\alpha \left(\sum_{i,k} a_{ik} t^i f^{(k)} \right)(x) = 0.
\end{aligned}$$

$\square$

Example 1 Let $L = D^2 - 4xD + 3x^2 + 2n - 1$. Then

$$\mathcal{D}_\alpha L = (\alpha^2 - 4\alpha + 3)D^2 + (-4x + (6/\alpha)x)D + (3/\alpha^2)x^2 + 3/\alpha + 2n - 1$$

and choosing $\alpha = 1$ a solution of $\mathcal{D}_1 Ly = 0$ is $f(x) = e^{-3x^2/4}x^{-n-1}$ for which $\mathcal{H}_1(f)$ does not exist. For $\alpha = 3$ we get $\mathcal{D}_3 L = -2xD + x^2/3 + 2n$. A solution of $\mathcal{D}_3 Ly = 0$ is $g(x) = x^n e^{x^2/12}$ and $\mathcal{H}_3(g)(x) = \sqrt{\frac{\alpha}{2\pi}}e^{x^2/2}H_n(x)$ is a solution of $Ly = 0$, where H_n are Hermite polynomials.

Therefore each solution f of a differential equation $\mathcal{D}_\alpha Ly = 0$ gives us a solution of $Ly = 0$ provided that the Hermite transform $\mathcal{H}_\alpha(f)$ exists. Sometimes $\mathcal{H}_\alpha(f)$ does not exist but $\mathcal{H}_\alpha(f(t+\alpha d))(x)$ exists for some d. Then we can still obtain a solution of $Ly = 0$ from a solution f of $(\mathcal{D}_\alpha L)y = 0$.

Corollary 1 *Let $d \in \mathbb{C}$ be such that $f(t+\alpha d)$ satisfies the $\mathcal{H}_\alpha$-condition. If $(\mathcal{D}_\alpha L)f(x) = 0$ then $L\mathcal{H}_\alpha(f(t+\alpha d))(x-d) = 0$.*

Proof Note that $(\mathcal{D}_\alpha L)f(x) = 0$ implies $(\mathcal{D}_\alpha L)_c f(x+c) = 0$ for every $c \in \mathbb{C}$. Since $(\mathcal{D}_\alpha L)_c = \mathcal{D}_\alpha L_{c/\alpha}$ if follows that $(\mathcal{D}_\alpha L_{c/\alpha})f(x+c) = 0$ for every c. Taking $c = \alpha d$ and using Theorem 2 we obtain $L_d\mathcal{H}_\alpha(f(t+\alpha d))(x) = 0$ and finally $L\mathcal{H}_\alpha(f(t+\alpha d))(x-d) = 0$. $\square$

Proposition 3

1. $\mathcal{H}_\alpha\left(\sum_{k=0}^{d} a_k t^k\right)(x) = \sum_{k=0}^{d} a_k H_k^{(\alpha)}(x)$,
2. $\mathcal{H}_\alpha((t+\alpha d)^k)(x-d) = \mathcal{H}_\alpha(t^k)(x)$ *for every* $d \in \mathbb{C}$, $k \in \mathbb{N}$.

Proof By (12) and (13),

$$\mathcal{H}_\alpha(t^n)(x) = H_n^{(\alpha)}(x). \tag{15}$$

Therefore the first assertion follows by linearity of $\mathcal{H}_\alpha$. To prove the second statement, we integrate an analytic function $f(t) = (-i\alpha t)^k e^{-\frac{\alpha}{2}(t-ix)^2}$ over the rectangle with the vertices R, $R+id$, $-R+id$, $-R$ where $R > 0$. By the residue theorem the integral is zero for every $R > 0$. On the other hand, letting $R \to \infty$, the integrals over the vertical sides tend to zero, therefore

$$\int_{-\infty}^{\infty} f(t)dt = \int_{-\infty+id}^{\infty+id} f(t)dt.$$

The integral on the left is equal to $\sqrt{2\pi/\alpha}\mathcal{H}_\alpha(t^k)(x)$. To conclude the proof we introduce a new variable $u = t - id$ in the integral on the right and obtain

$$\mathcal{H}_\alpha(t^k)(x) = \sqrt{\frac{\alpha}{2\pi}}\int_{-\infty}^{\infty}(-i\alpha u + \alpha d)^k e^{-\frac{\alpha}{2}(u-i(x-d))^2}du = \mathcal{H}_\alpha((t+\alpha d)^k)(x-d).$$

$\square$

Using the partial fraction decomposition of a rational function f it is easy to see that the Hermite transform of f exists as soon as f has no poles with zero real part. Namely, if $\Re b \neq 0$ then $g(t) = 1/(t-b)^k$ satisfies the $\mathcal{H}_\alpha$-condition for every $\alpha > 0$, for $g(-i\alpha t)$ is bounded on $\mathbb{R}$. For simple fractions we have the following result.

Proposition 4 *For a complex number b with nonzero real part the following is true*

1. $\mathcal{H}_\alpha(\frac{1}{t-b})(x) = e^{\alpha x^2/2 - bx}(C - \int_0^x e^{-\alpha s^2/2+bs} ds)$ *where* $C = \mathcal{H}_\alpha(\frac{1}{t-b})(0)$,
2. $\mathcal{H}_\alpha((\frac{1}{t-b})^{k+1})(x) = (-\frac{1}{\alpha})^k \frac{1}{k!} D^k \mathcal{H}_\alpha(\frac{1}{t-b})(x)$.

Proof Let us write $f(t) = (t-b)^{-1}$. As $\mathcal{H}_\alpha(1)(x) = 1$ and

$$\mathcal{H}_\alpha((t-b)f)(x) = \mathcal{H}_\alpha(tf)(x) - b\mathcal{H}_\alpha(f)(x),$$

by Proposition 2 we get that $y = \mathcal{H}_\alpha(f)(x)$ satisfies $(\alpha x - b)y - y' = 1$, therefore

$$\mathcal{H}_\alpha(f)(x) = e^{\alpha x^2/2 - bx}\left(C - \int_0^x e^{-\alpha s^2/2+bs} ds\right).$$

By putting $x = 0$ we conclude the proof of the first statement. To prove the second identity we use induction on k and Proposition 2, and we compute

$$\begin{aligned}
\mathcal{H}_\alpha\left(\left(\frac{1}{t-b}\right)^{k+1}\right)(x) &= -\frac{1}{\alpha k} D\mathcal{H}_\alpha\left(\left(\frac{1}{t-b}\right)^{k}\right)(x) = \\
&= (-1/\alpha)^k \frac{1}{k!} D^k \mathcal{H}_\alpha\left(\frac{1}{t-b}\right)(x).
\end{aligned}$$

$\square$

Corollary 2 *Let f be a rational function with a partial fraction decomposition*

$$f(t) = q(t) + \sum_{i=1}^{n}\sum_{j=1}^{m_i} \frac{c_{ij}}{(t-b_i)^j}, \tag{16}$$

where q is a polynomial and b_i, $i = 1, \ldots, n$, are distinct poles of f. Then for every $d \neq (1/\alpha)\Re b_i$, $i = 1, \ldots, n$,

$$\mathcal{H}_\alpha(f(t+\alpha d))(x-d) = \mathcal{H}_\alpha(q(t))(x) + \sum_{i=1}^{n} \tilde{C}_i(d) u_i(x) F_i(x) - u_i(x) G_i(x) - v_i(x),$$

where for $i = 1, \ldots, n$

$$\begin{aligned}
F_i(x) &= e^{\alpha x^2/2 - b_i x}, \\
G_i(x) &= F_i(x) \int_0^x \frac{1}{F_i(s)} ds,
\end{aligned}$$

$\tilde{C}_i(d)$ are constants and u_i, v_i are some polynomials.

Proof If $d \neq (1/\alpha)\Re b_i$, then $\mathcal{H}_\alpha((t+\alpha d - b_i)^{-j})(x)$ exists for all $x \in \mathbb{R}$. By Proposition 4,

$$\mathcal{H}_\alpha\left(\frac{1}{t+\alpha d - b_i}\right)(x-d) = e^{\alpha(x-d)^2/2-(b_i-\alpha d)(x-d)}\left(C_i(d) - \int_0^{x-d} e^{-\alpha s^2/2+(b_i-\alpha d)s} ds\right),$$

where $C_i(d) = \mathcal{H}_\alpha(1/(t+\alpha d - b_i))(0)$. Introducing the variable $u = s + d$, we get

$$\mathcal{H}_\alpha\left(\frac{1}{t+\alpha d - b_i}\right)(x-d) = \tilde{C}_i(d)F_i(x) - G_i(x), \tag{17}$$

where we denoted

$$\tilde{C}_i(d) = C_i(d)e^{-\alpha d^2/2+b_i d} + \int_0^d e^{-\alpha u^2/2+b_i u} du.$$

One easily verifies that for a polynomial w

$$D^k(wF_i) = u_k F_i, \tag{18}$$
$$D^k(wG_i) = u_k G_i + v_k, \tag{19}$$

where u_k and v_k are some polynomials. Therefore by linearity of $\mathcal{H}_\alpha$, Proposition 3, Proposition 4, (17), (18) and (19) we obtain

$$\mathcal{H}_\alpha(f(t+\alpha d))(x-d) = \mathcal{H}_\alpha(q(t))(x) + \sum_{i=1}^n \tilde{C}_i(d)u_i(x)F_i(x) - u_i(x)G_i(x) - v_i(x),$$

where u_i, v_i are polynomials defined by

$$u_i(x)F_i(x) = \sum_{j=1}^{m_i}\left(-\frac{1}{\alpha}\right)^{j-1}\frac{c_{ij}}{(j-1)!}D^{j-1}F_i(x), \tag{20}$$
$$u_i(x)G_i(x) + v_i(x) = \sum_{j=1}^{m_i}\left(-\frac{1}{\alpha}\right)^{j-1}\frac{c_{ij}}{(j-1)!}D^{j-1}G_i(x). \tag{21}$$

□

Theorem 3 *Let f be as in (16) and L a linear differential operator with polynomial coefficients. If $(\mathcal{D}_\alpha L)f(x) = 0$ then $Ly = 0$ has $n+1$ linearly independent solutions of the form*

$$\sum_{j=1}^{m_i}\left(-\frac{1}{\alpha}\right)^{j-1}\frac{c_{ij}}{(j-1)!}D^{j-1}F_i(x), \quad i = 1, \ldots, n,$$
$$\mathcal{H}_\alpha(q)(x) - \sum_{i=1}^n\sum_{j=1}^{m_i}\left(-\frac{1}{\alpha}\right)^{j-1}\frac{c_{ij}}{(j-1)!}D^{j-1}G_i(x),$$

where F_i and G_i are as in Corollary 2.

Proof For every $d \neq (1/\alpha)\Re b_i,\ i=1,\dots,n$, by Corollary 1 $L\mathcal{H}_\alpha(f(t+\alpha d))(x-d)=0$. On the other hand, by Corollary 2,

$$L\mathcal{H}_\alpha(f(t+\alpha d))(x-d) = L\mathcal{H}_\alpha(q(t))(x)+\sum_{i=1}^{n}\tilde{C}_i(d)L(u_i(x)F_i(x))-L(u_i(x)G_i(x))-L(v_i(x)).$$

Using (18), (19) and denoting

$$\begin{aligned} L(u_i(x)F_i(x)) &= \tilde{u}_i(x)F_i(x), \\ L(u_i(x)G_i(x)) &= \tilde{u}_i(x)G_i(x)+\tilde{v}_i(x), \end{aligned}$$

we obtain

$$\sum_{i=1}^{n}\tilde{C}_i(d)\tilde{u}_i(x)F_i(x) = \sum_{i=1}^{n}\tilde{u}_i(x)G_i(x) + \text{ polynomial.}$$

As the right-hand side is independent of d, which is arbitrary, it follows that $\tilde{u}_i(x)F_i(x) = 0,\ i=1,\dots,n$. Therefore $u_i(x)F_i(x),\ i=1,\dots,n$, and $\mathcal{H}_\alpha(q)(x)-\sum_i u_i(x)G_i(x)+v_i(x)$ satisfy $Ly=0$ and by (20), (21) we conclude the proof. □

Example 2 It is known that the Weber differential equation

$$y''-xy'-ay=0$$

has a formal power series solution [5, 7]

$$\begin{aligned} y &= C_1\left(1+\sum_{k=1}^{\infty}\frac{a(a+2)\cdots(a+2k-2)}{(2k)!}x^{2k}\right) \\ &+ C_2\left(x+\sum_{k=1}^{\infty}\frac{(a+1)(a+3)\cdots(a+2k-1)}{(2k+1)!}x^{2k+1}\right). \end{aligned}$$

If $a=-n,\ n\in\mathbb{N}$, is even (odd) then the first (second) sum is a polynomial of degree n. With our method we can find explicit solutions when $a=n$. Writing $L=D^2-xD-n$ we get $\mathcal{D}_1 L = -xD-n$ and a rational solution $f(x)=x^{-n}$ of $(\mathcal{D}_1 L)y=0$ gives a solution of $Ly=0$

$$C_1\frac{d^{n-1}}{dx^{n-1}}e^{x^2/2}+C_2\frac{d^{n-1}}{dx^{n-1}}\left(e^{x^2/2}\int_0^x e^{-s^2/2}ds\right).$$

4 An application

Let L be a differential operator of total degree m written in the monomial form

$$L=\sum_{i=0}^{m}a_{i,m-i}x^iD^{m-i}+\sum_{i+k<m}a_{i,k}x^iD^k. \tag{22}$$

Recalling Theorem 3, we observe that if $(\mathcal{D}_\alpha L)y=0$ has a rational solution, we get at least two linearly independent solutions of $Ly=0$. It is known [1] that $(\mathcal{D}_\alpha L)y=0$ may have rational solutions only if $\mathcal{D}_\alpha L$ has singular points.
First let us prove two lemmas.

Lemma 1

$$(x+D)^i = \sum_{j=0}^{i} \binom{i}{j} x^j D^{i-j} + \textit{ terms of total degree } < i$$

Proof We use induction on i. From $Dx = xD + 1$ it follows that $D^n x = xD^n + nD^{n-1}$, hence

$$\begin{aligned} (x+D)^{i+1} &= \sum_{j=0}^{i} \binom{i}{j} x^{j+1} D^{i-j} + \sum_{j=0}^{i} \binom{i}{j} x^j D^{i+1-j} + \dots \\ &= \sum_{j=0}^{i+1} \binom{i+1}{j} x^j D^{i+1-j} + \dots . \end{aligned}$$

which concludes the proof. □

Lemma 2 *Let L be as in (22). Then $\mathcal{D}_1 L$ is of order m iff $\sum_{i=0}^{m} a_{i,m-i} \neq 0$. Additionally, if $\mathcal{D}_1 L$ is of order m then it has no singular points.*

Proof By Lemma 1 we obtain

$$\begin{aligned} \mathcal{D}_1 L &= \sum_{i=0}^{m} a_{i,m-i}(x+D)^i D^{m-i} + \sum_{i+k<m} a_{i,k}(x+D)^i D^k \\ &= \sum_{i=0}^{m} a_{i,m-i} \sum_{j=0}^{i} \binom{i}{j} x^j D^{m-j} + \sum_{i+k<m} b_{i,k} x^i D^k \\ &= \sum_{j=0}^{m} x^j D^{m-j} \sum_{i=j}^{m} a_{i,m-i} \binom{i}{j} + \sum_{i+k<m} b_{i,k} x^i D^k \end{aligned}$$

which concludes the proof. □

Let L be as in (22). First we construct a Laurent polynomial

$$p(t) = \sum_{i-0}^{m} a_{i,m-i} t^{m-2i}$$

and let β be its zero. Next we search for solutions of $Ly(x) = 0$ in the form $y(x) = u(\xi)$, where $\xi = \beta x$. We get a differential equation for the function u

$$\tilde{L}u(\xi) = \left(\sum_{i=0}^{m} a_{i,m-i} \beta^{m-2i} \xi^i D^{m-i} + \sum_{i+k<m} a_{i,k} \beta^{k-i} \xi^i D^k \right) u(\xi) = 0.$$

By Lemma 2, $\mathcal{D}_1 \tilde{L}$ has order less than m and $(\mathcal{D}_1 \tilde{L})y = 0$ may have rational solutions which can be found by an algorithm of Abramov [1]. Using the partial fraction decomposition of f by Proposition 2 and Theorem 3 we get solutions of $Ly = 0$.

Example 3 Let $L = D^3 - 4xD^2 + (4x^2 - 5)D + 8x$. Then the Laurent polynomial is $p(t) = t^{-1}(t^2 - 2)^2$. We write $y(x) = u(\xi)$, where $\xi = \sqrt{2}x$. The corresponding operator $\tilde{L}$ is

$$\tilde{L} = 2D^3 - 4\xi D^2 + (2\xi^2 - 5)D + 4\xi$$

and

$$\mathcal{D}_1 \tilde{L} = (2\xi^2 + 1)D + 4\xi.$$

Since $(\mathcal{D}_1\tilde{L})y = 0$ has a rational solution

$$f(\xi) = \frac{1}{2i}\left(\frac{1}{\sqrt{2}\xi - i} - \frac{1}{\sqrt{2}\xi + i}\right),$$

by Theorem 3, we obtain solutions of $\tilde{L}u = 0$

$$\begin{aligned} u_1(\xi) &= e^{\xi^2/2 - i\xi/\sqrt{2}}, \\ u_2(x) &= e^{\xi^2/2 + i\xi/\sqrt{2}}, \\ u_3(x) &= u_1(\xi)\int_0^\xi \frac{1}{u_1(s)}ds - u_2(\xi)\int_0^\xi \frac{1}{u_2(s)}ds, \end{aligned}$$

and putting $\xi = \sqrt{2}x$ we find that

$$\begin{aligned} y_1(x) &= e^{x^2 - ix}, \\ y_2(x) &= e^{x^2 + ix}, \\ y_3(x) &= e^{x^2 - ix}\int_0^x e^{-s^2 + is}ds - e^{x^2 + ix}\int_0^x e^{-s^2 - is}ds, \end{aligned}$$

are solutions of $Ly = 0$.

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