Project Summary

The overarching aim of this project is to develop newly-emerging structural aspects of the theory of permutation classes, to harmonize these with the existing structural framework, to employ the resulting expanded toolbox to solve major outstanding problems in the field, and to investigate these advances in the more general framework of relational structures.

The proposal builds upon the PI's recent characterization of the "small permutation classes", those of growth rate less than $\kappa \approx 2.20557$ (a root of a specific cubic polynomial). The machinery of geometric grid classes and the substitution decomposition were essential in this characterization, and the PI contends that these two tools play a central role in describing the structure of all permutation classes. Four of the open problems to be targeted are to: establish that all permutation classes have proper growth rates, determine the set of these growth rates, characterize the strongly rational permutation classes, and characterize the partially well-ordered permutation classes.

All aspects of this work have analogues for other combinatorial objects such as: graphs, ordered graphs, posets, integer or set partitions, and tournaments. The theory of relational structures provides the appropriate conceptual framework to study these objects all at once, and the PI proposes to extend his expertise in permutation classes into this unifying context. This change in perspective will lead to a truly systematic approach to these problems and should open up deep and novel areas for exploration.

Intellectual Merit. The study of permutation patterns has connections to algebra (e.g., the classification of Schubert varieties), computational molecular biology (e.g., sorting by reversals), model theory (in particular, the study of relational structures, and the joint embedding property for classes thereof), and to formal languages and automata. The PI anticipates that the tools developed in this project will have consequences for these neighboring disciplines. The proposed work will not only solve important and long-standing conjectures about permutation classes, but will also, via the extensions to relational structures, shed light on other types of object. With over 20 papers on the topics of this proposal, and a unique interdisciplinary network of collaborators, the PI is uniquely qualified to carry out the proposed research.

Broader Impacts. The proposal provides training at both graduate and undergraduate levels. At the undergraduate level, the proposal includes salary and travel support for two part-time undergraduate researchers. The PI will strive to recruit a diverse team and will apply for REU supplements if additional students are interested in participating. When possible, the PI will aim to support these undergraduate students during their third years, so that the research experience can inform their decisions about whether to apply to graduate school and form the basis for their honors theses. It is expected that the supported graduate student will be involved in this activity, and that this vertical integration will give the graduate student valuable mentoring experience.

The PI has been active in conference organization, as described in the Biographical Sketch. In particular, the PI was a local organizer for the annual *Permutation Patterns* conference in both 2007 and 2010. These conferences strongly encourage the participation of both graduate and undergraduate students, and have been instrumental in forming research networks. The PI co-edited the proceedings of the 2007 meeting (which form the book *Permutation Patterns* published by the Cambridge University Press) and is currently co-editing the proceedings of the 2010 meetings (which will be published in a special issue of the journal *Pure Mathematics and Applications*).

The Structure of Permutation Classes

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Project Description

The overarching aim of this project is to develop newly-emerging structural aspects of the theory of permutation classes, to harmonize these with the existing structural framework, to employ the resulting expanded toolbox to solve major outstanding problems in the field, and to investigate these advances in the more general framework of relational structures.

The fundamental object in the study of permutation classes is the partial order on the set of all finite permutations defined by $\sigma \leqslant \pi$ if and only if π contains a subsequence of entries which are order isomorphic to σ, in which case we also say that π *contains* σ (otherwise π *avoids* σ; here and in what follows we think of permutations in one-line notation, i.e., as sequences of images, and say that two sequences are *order isomorphic* if they have the same pairwise comparisons). For example, $\pi = 391867452$ contains $\sigma = 51342$, as can be seen by considering the subsequence $\pi(2)\pi(3)\pi(5)\pi(6)\pi(9) = 91672$.

Consideration of this order began in earnest with Knuth's *The Art of Computer Programming* [36], which showed that a permutation can be sorted by a first-in, last-out data structure called a *stack* if and only if it does not contain the permutation 231. The area has blossomed in the forty years since. Beyond the original motivating example of sorting machines, connections have been found to algebra (e.g., the classification of Schubert varieties), computational molecular biology (e.g., sorting by reversals), model theory (in particular, the study of relational structures, and the joint embedding property for classes thereof), and to formal languages and automata.

We refer to a downset in this permutation containment order as a *permutation class*. Our central goal is to develop tools for studying permutation classes which lead to general results. While it is difficult to disentangle the various threads involved, an attempt has been made with this proposal's organization. The proposal is arranged as follows: three major aspects of permutation classes are treated in separate sections,

- Section 2: Asymptotic Enumeration,
- Section 3: Exact Enumeration,
- Section 6: Partial Well-Order,

while two sections are devoted to our key approaches for deepening our understanding of the aspects above,

- Section 4: The Substitution Decomposition,
- Section 5: Geometric Grid Classes.

Permutations are but one type of combinatorial object. All notions discussed in this proposal have analogues for other kinds of combinatorial structure. The longer term developmental goal of this project is to extend the results obtained for permutation classes to these other types of structure. Several attractive targets for such extension are discussed in

- Section 7: Expanding the Vista: Combinatorial Objects as Relational Structures.

A major thesis of this project is that much can be gained by translating problems about combinatorial objects into problems about *languages*, i.e., sets of words over an alphabet. Once this translation has been made, the mature tools of formal language theory can then be brought to bear. From this viewpoint, the substitution decomposition of Section 4 translates certain permutation classes into context-free languages, the geometric grid classes of Section 5 translate certain permutation classes into regular languages, and the grid pin sequences of Section 6 also translate many (conjecturally, all) minimal infinite antichains of permutations into languages.

This project will entail extensive collaboration. In addition to the graduate student and undergraduate students for whom funding is requested, the principal collaborators will be[1]:

- Michael Albert, Associate Professor in the Department of Computer Science at the University of Otago, Dunedin, New Zealand,
- Robert Brignall, Lecturer (the U.K. equivalent of Assistant Professor) in the Department of Mathematics and Statistics at the Open University, Milton Keynes, England, and
- Nik Ruškuc, Professor and Head of the School (the U.K. equivalent of Chair) of Mathematics and Statistics at the University of St. Andrews, St. Andrews, Scotland.

2 Asymptotic Enumeration

Much of the early work on permutation classes was motivated by the Stanley-Wilf Conjecture, which stated that every nontrivial permutation class has a finite *upper growth rate*,

$$\overline{\mathrm{gr}}(\mathcal{C}) = \limsup_{n\to\infty} \sqrt[n]{|\mathcal{C}_n|},$$

where $\mathcal{C}_n$ denotes the set of permutations of length n in the permutation class $\mathcal{C}$.

Over five years ago, Marcus and Tardos [39] resolved this conjecture in the affirmative with an elegant proof of a conjecture of Füredi and Hajnal [26] concerning 0/1 matrices (the connection between the Füredi-Hajnal Conjecture and the Stanley-Wilf Conjecture had already been established by Klazar [33]). Although their proof shows that these upper growth rates exist and are finite for all nontrivial classes, we know very little else about these numbers. In particular, the following two questions are still open.

Question 2.1. *Does every nontrivial permutation class have a* proper growth rate*, i.e., does the limit of $\sqrt[n]{|\mathcal{C}_n|}$ as $n \to \infty$ exist for every proper permutation class $\mathcal{C}$?*

Question 2.2. *Which numbers can be realized as (upper) growth rates of permutation classes?*

On the small end of the spectrum, improving on the results of Kaiser and Klazar [32], the PI proved the following.

Theorem 2.3 (Vatter [49])**.** *Let κ denote the unique positive root of $x^3 - 2x^2 - 1$, approximately* 2.20557*. If the upper growth rate of $\mathcal{C}$ is less than κ then $\mathcal{C}$ has a proper growth rate which is either* 0, 2*, a root of one of the four polynomials*

[1] All three major collaborators have acknowledged their participation in this project with letters, which are attached to the proposal.

(P1) $x^3 - x^2 - x - 3$,
(P2) $x^4 - x^3 - x^2 - 2x - 3$,
(P3) $x^4 - x^3 - x^2 - 3x - 1$,
(P4) $x^5 - x^4 - x^3 - 2x^2 - 3x - 1$,

or a root of one of the three families of polynomials

(F1) $x^{k+1} - 2x^k + 1$,
(F2) $(x^3 - 2x^2 - 1)x^{k+\ell} + x^\ell + 1$, *or*
(F3) $(x^3 - 2x^2 - 1)x^k + 1$

for integers $k \geqslant 1$ *and* $\ell \geqslant 0$.

We refer to classes with $\overline{\mathrm{gr}}(\mathcal{C}) < \kappa$ as *small permutation classes*. Note that the number κ in Theorem 2.3 is far from arbitrary; it is at this growth rate that permutation classes can contain infinite antichains (infinite sets of pairwise incomparable elements). The possibility of such infinite antichains allows for much more complicated permutation classes, which have yet to be adequately analyzed. In particular, this is the first point on the spectrum where we encounter uncountably many permutation classes.

Just beyond κ, the set of growth rates undergoes a striking qualitative shift in behavior, as established by the PI building on previous work by Albert and Linton [5].

Theorem 2.4 (Vatter [50])**.** *Let* λ *denote the unique positive root of* $x^5 - 2x^4 - 2x^2 - 2x - 1$, *approximately* 2.48187. *Every real number at least* λ *is the proper growth rate of a permutation class.*

Theorem 2.4 thoroughly refuted a conjecture of Balogh, Bollobás, and Morris [8]; their conjecture would have implied that the set of growth rates of permutation classes contains no accumulation points from above. However, Theorems 2.3 and 2.4 leave open a gap:

Question 2.5. *What numbers between* κ *and* λ *can be realized as (upper) growth rates of permutation classes?*

The structure of these growth rates is also of interest. Note that the growth rates from family (F2) accumulate at the growth rates from family (F3), which themselves accumulate at κ, making κ a *second order accumulation point*. It is therefore natural to conjecture that the set of growth rates between κ and λ contains mth order accumulation points for all m, and that these growth rates themselves accumulate at λ. In addition to Question 2.5, we could also ask roughly how prevalent such growth rates are.

Question 2.6. *What is the Lebesgue measure of the set of (upper) growth rates between* κ *and* λ*?*

Another fruitful direction in which to investigate growth rates would be to study subclasses of a particular class. Outside of certain nearly trivial cases, such investigations have yet to be conducted. An obvious target would be subclasses of the separable permutations (a popular "toy model" for well-behaved permutation classes). The *separable permutations* are those which can be built from the trivial permutation 1 by repeatedly applying two operations, known as *direct sum*

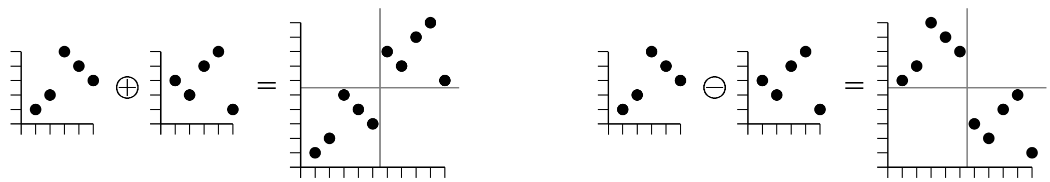

Figure 1: An example of a direct sum and a skew sum, shown as plots of the corresponding permutations.

(or simply, *sum*) and *skew sum* (or simply, *skew*) which are defined, respectively, on permutations π of length m and σ of length n by

$$(\pi \oplus \sigma)(i) = \begin{cases} \pi(i) & \text{if } 1 \leqslant i \leqslant m, \\ \sigma(i-m)+m & \text{if } m+1 \leqslant i \leqslant m+n, \end{cases}$$

$$(\pi \ominus \sigma)(i) = \begin{cases} \pi(i)+n & \text{if } 1 \leqslant i \leqslant m, \\ \sigma(i-m) & \text{if } m+1 \leqslant i \leqslant m+n, \end{cases}$$

(The operations $\oplus$ and $\ominus$ are best understood by considering the plots of the permutations, as in Figure 1.)

Question 2.7. *What are the possible upper growth rates of subclasses of the separable permutations? Are all such growth rates proper?*

3 Exact Enumeration

There has also been considerable interest in solving the exact enumeration problem for various permutation classes, which amounts to finding the *generating function*

$$\sum_{\pi \in \mathcal{C}} x^{|\pi|} = \sum_{n \geqslant 0} |\mathcal{C}_n| x^n$$

for the class $\mathcal{C}$. It is known that there are classes with rational generating functions, with algebraic (and nonrational) generating functions, and with holonomic (and nonalgebraic) generating functions[2]. A natural goal would be to characterize those permutation classes with a particular type of enumeration[3].

There has been one notable success in this area: Huczynska and the PI, building on the work of Kaiser and Klazar [32] and using the (geometric) grid classes of Section 5, characterized the permutation classes whose enumeration is given by a polynomial (for large n) in [31]. However, this result about extremely small permutation classes appears to be the exception rather than the rule. Relating asymptotic growth to exact enumeration, the PI has made the following conjecture.

Conjecture 3.1 (Vatter [49])**.** *Every small permutation class has a rational generating function.*

[2]It is not known if there are classes defined by a finite number of forbidden permutations with nonholonomic generating functions; indeed, Noonan and Zeilberger [42] once conjectured that this could not occur, although Zeilberger has since reversed his opinion (see Elder and Vatter [22]), conjecturing that the generating function for the class of permutations not containing 1324 is non-holonomic.

[3]Zeilberger was the first to explicitly state this goal, at the *Permutation Patterns 2005* conference (again see Elder and Vatter [22]).

Recall that the small permutation classes, defined in the previous section, are those with growth rate less than κ. One of the first and most concrete goals of the proposed project is to prove Conjecture 3.1, and a possible argument is outlined in Section 5.

A natural aim would be to characterize the permutation classes with rational generating functions. However, there are known to be permutation classes with rational generating functions which exhibit a great deal of disorder[4]. Given this, we propose to study permutation classes which satisfy a "hereditary rationality" condition (c.f. the definition of perfect graphs [29]).

Definition 3.2. *The permutation class $\mathcal{C}$ is* strongly rational *if $\mathcal{C}$ and every subclass of $\mathcal{C}$ have rational generating functions.*

Standard techniques imply that the union and intersection of two strongly rational classes is again strongly rational (this fact does not hold for classes which are merely rational). One of the goals of this project is to answer the following problem.

Problem 3.3. *Characterize the strongly rational permutation classes.*

The notion of strong rationality is very recent, so we have few examples of such classes. The most notable example concerns subclasses of the separable permutations. This class also contains the stack-sortable permutations (i.e., by Knuth's seminal result from [36], the class of permutations not containing 231), which are counted by the Catalan numbers. Therefore, if $\mathcal{C}$ is a subclass of the separable permutations which contains the stack-sortable permutations, then $\mathcal{C}$ is not strongly rational. Recently, the PI, Albert, and Atkinson have established that this obvious necessary condition is also sufficient.

Theorem 3.4 (Albert, Atkinson, and Vatter [4])**.** *The subclass $\mathcal{C}$ of the separable permutations is strongly rational if and only if $\mathcal{C}$ does not contain the* 231*-avoiding permutations or any symmetry of this class (i.e., the* 132-*,* 213-*, or* 312*-avoiding permutations).*

The proof of Theorem 3.4 is sketched in Section 5. It is known that every subclass of the separable permutations has (at worst) an algebraic generating function (this fact follows from the techniques of Section 4). Thus another interesting challenge regarding this toy model would be to develop rules which determine the degree of the generating function of a given subclass of the separable permutations over $\mathbb{Q}(x)$ (without actually computing the generating function). Albert [private communication] has proved that this degree must be a power of 2, and, furthermore, that all powers of 2 occur.

4 The Substitution Decomposition

To describe the substitution decomposition we need a few definitions. An *interval* in the permutation π is a set of contiguous indices $I = \{a, a+1, \dots, b\}$ such that the set of values $\{\pi(a), \pi(a+1), \dots, \pi(b)\}$ is also contiguous. The permutation π of length n is then said to be *simple* if it

[4]One such example is the class of 3-*bounded permutations*, studied by Albert, Atkinson, and Ruškuc [3]. This is the class of permutations π such that for all i, there are at most 2 elements below and to the right of $\pi(i)$, i.e., there are at most 2 entries $\pi(j)$ with $j \geqslant i$ and $\pi(j) \leqslant \pi(i)$. While this class (and all its subclasses defined by excluding only finitely many additional permutations) has a rational generating function, it also contains an infinite antichain, and thus, by an elementary counting argument, contains a subclass with a non-holonomic generating function.

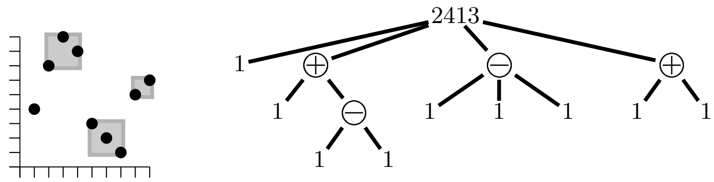

Figure 2: A plot of the permutation 479832156 together with its substitution decomposition tree.

contains only the trivial intervals of sizes 0, 1, and n. Finally, given a permutation σ of length m and nonempty permutations $\alpha_1, \ldots, \alpha_m$, the *inflation* of σ by $\alpha_1, \ldots, \alpha_m$ — denoted $\sigma[\alpha_1, \ldots, \alpha_m]$ — is the permutation obtained by replacing each entry $\sigma(i)$ by an interval that is order isomorphic to α_i in such a way that the intervals are order isomorphic to σ. For example,

$$2413[1, 132, 321, 12] = 4\ 798\ 321\ 56.$$

(see the plot on the left of Figure 2). It follows readily that every permutation is the inflation of a unique simple permutation, called the *simple quotient* and, moreover, that the intervals in such an inflation are unique unless the simple quotient is 12 or 21. Note that we have already introduced special terminology for these two cases: $12[\alpha_1, \alpha_2] = \alpha_1 \oplus \alpha_2$ and $21[\alpha_1, \alpha_2] = \alpha_1 \ominus \alpha_2$. We therefore see that every permutation π may be decomposed uniquely in one of three ways:

- as $\pi = \sigma[\alpha_1, \ldots, \alpha_m]$ where σ is a nonmonotone simple permutation,
- as $\pi = \alpha_1 \oplus \cdots \oplus \alpha_m$ where each α_i is *sum indecomposable* (cannot be written as the sum of two shorter permutations), or
- as $\pi = \alpha_1 \ominus \cdots \ominus \alpha_m$ where each α_i is *skew indecomposable* (cannot be written as the skew sum of two shorter permutations).

Utilizing this observation, Albert and Atkinson [1] proved the following result.

Theorem 4.1 (Albert and Atkinson [1])**.** *If a permutation class contains only finitely many simple permutations then it has an algebraic generating function.*

There are several different proofs of Theorem 4.1, but the one most in keeping with the spirit of this proposal shows that such classes correspond to context-free languages. This argument is still unpublished, although we are optimistic that it can be generalized to prove the much stronger Conjecture 5.5. Theorem 4.1 was generalized to the context of "query-complete sets of properties" by Brignall, Huczynska, and the PI [15], an idea which later became crucial in the proof of Theorem 3.4.

By recursively decomposing each interval α_i in the permutation π, we obtain a rooted tree which we call the *substitution decomposition tree* of π; an example is shown in Figure 2. The *substitution depth* of π is the height of its substitution decomposition tree, so for example, the substitution depth of the permutation from Figure 2 is 3, while the substitution depth of any simple (or monotone) permutation is 1. Another example of strongly rational classes are the following.

Proposition 4.2 (Folklore)**.** *If the permutation class $\mathcal{C}$ contains only finitely many simple permutations and has bounded substitution depth then it is strongly rational.*

Brignall, Huczynska, and the PI proved a Ramsey-type result for simple permutations themselves:

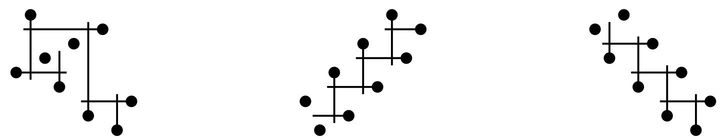

Figure 3: Three proper pin sequences. Permutations of the form shown in the middle and on the right are called *increasing oscillations* and *decreasing oscillations*, respectively. The increasing oscillations are also known as *Gollan permutations* in computational molecular biology, because they are the most difficult permutations to sort by reversals (this was conjectured by Gollan and proved by Bafna and Pevzner [7]).

Theorem 4.3 (Brignall, Huczynska, and Vatter [14])**.** *Every sufficiently long simple permutations contains a parallel alternation, a simple wedge alternation of type* 1 *or* 2*, or a proper pin sequence of length at least* $2k$ *(and all such permutations are themselves simple).*

The first three of the families in Theorem 4.3 are quite small and easily described. The third family, of *proper pin sequences*, seems to be much more interesting and requires further investigation. The first two points (called *pins*) of a proper pin sequence are arbitrary, and therefore are order isomorphic to either 12 or 21. The later pins are chosen so as to separate the two pins before them. Thus later pins are in one of four directions, left, right, up, or down, and pin sequences can be encoded by words over the alphabet $\{1, 2, L, R, U, D\}$. For example, the three pin sequences shown in Figure 3 are encoded as $12DLURDRD$, $21RURURUR$, and $12DRDRDRD$, respectively.

If a class happens to contain only finitely many simple permutations, then this fact can (at least in theory, if not in practice) be verified automatically, using the following result.

Theorem 4.4 (Schmerl and Trotter [48])**.** *Every simple permutation of length* n *contains a simple permutation of length* $n-1$ *or* $n-2$*.*

The PI, Brignall, and Ruškuc in [16] used Theorem 4.3 to prove that it is algorithmically decidable whether a permutation class contains infinitely many simple permutations, an algorithm which was improved to polynomial-time recently by Bassino, Bouvel, Pierrot, and Rossin [11, 10]. Brignall's recent extension (in [13]) of pin sequences to *grid pin sequences* in order to describe infinite antichains of permutations is discussed in Section 6. Finally, we note that Fairbanks (an undergraduate student working with the PI) has very recently proved an analogue of Theorem 4.3 for complete ordered matchings in [23].

It is known that the simple permutations have density $1/e^2$ among all permutations (which can be proved, among other ways, with the Poisson paradigm, see Corteel, Louchard, and Pemantle [19]). Atkinson conjectured (at the problem session of *Permutation Patterns 2010*) that in any particular permutation class, the density of simple permutations is 0. A proof of this conjecture would seem to have deep ramifications for the exact enumeration problem in general.

5 Geometric Grid Classes

We begin this section with a slight change in perspective: the notion of order isomorphism can be extended to point sets in the plane. Two sets S and T of points in the plane are said to be *order isomorphic* if the axes can be stretched and shrunk in some manner to map one of the sets onto the other. In other words, if $S \subseteq A \times B \subseteq \mathbb{R}^2$, then S is order isomorphic to T if there are increasing

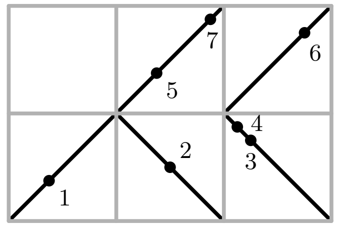

Figure 4: The permutation 1527436 lies in the geometric grid class of $\begin{pmatrix} 0 & 1 & 1 \\ 1 & -1 & -1 \end{pmatrix}$ — note that in order to align these matrices with the plots of permutations we have adopted a nonstandard indexing system.

injections $\phi_x : A \to \mathbb{R}$ and $\phi_y : B \to \mathbb{R}$ such that

$$\phi(S) = \{(\phi_x(a), \phi_y(b)) \,:\, (a,b) \in S\} = T.$$

Our interest is in point sets in which no two points share a coordinate, which we refer to as *independent*. Each finite independent point set in the plane is order isomorphic to the plot, $\{(i, \pi(i))\}$, of some permutation π.

This change of perspective allows us to define *geometric grid classes*. Given a finite $0/\pm 1$ matrix M, the *standard figure* of M is the point set in $\mathbb{R}^2$ consisting of:

- the line segment from $(k-1, \ell-1)$ to (k, ℓ) if $\mathcal{M}_{k,\ell} = 1$ and
- the line segment from $(k-1, \ell)$ to $(k, \ell-1)$ if $\mathcal{M}_{k,\ell} = -1$.

The geometric grid class of $\mathcal{M}$ is the set of all permutations which are order isomorphic to finite independent subsets of the standard figure of $\mathcal{M}$. Figure 4 shows an example.

Given a finite alphabet Σ, we denote by Σ^* the set of all finite words over Σ. A slight generalization of a result of the PI and Waton [51] shows that for every $0/\pm 1$ matrix M, there is a length- and order-preserving map φ from some language Σ^* ordered by the (not necessarily contiguous) subword order onto the geometric grid class of M.

We say that the class $\mathcal{G}$ is *geometrically griddable* if it is contained in some geometric grid class. By restricting the domain of the map φ above, we see that every geometrically griddable class is the image of a length- and order-preserving map from a regular language. However, permutations in the class will typically have multiple preimages. The PI and collaborators have recently shown how to avoid this ambiguity, establishing the following.

Theorem 5.1 (Albert, Atkinson, Bouvel, Ruškuc, and Vatter [2])**.** *Every geometrically griddable class is in bijection with a regular language, and thus has a rational generating function.*

The characterization of strongly rational subclasses of the separable permutations (Theorem 3.4) used one particular geometric grid class extensively, the geometric grid class of

$$\begin{pmatrix} -1 & 1 \\ 1 & -1 \end{pmatrix},$$

which we denote by $\mathcal{X}$. Given another class $\mathcal{U}$, the $\mathcal{X}$*-inflation* of $\mathcal{U}$ is defined as

$$\{\chi[\mu_1, \ldots, \mu_m] \,:\, \chi \in \mathcal{X}_m \text{ and } \mu_1, \ldots, \mu_m \in \mathcal{U}\}.$$

This operation preserves strong rationality, in the sense described in the following result.

Theorem 5.2 (Albert, Atkinson, and Vatter [4])**.** *If $\mathcal{U}$ is strongly rational then $\mathcal{X}[\mathcal{U}]$ is also strongly rational.*

We now have all the machinery necessary to sketch the proof of Theorem 3.4. Suppose to the contrary that the theorem were false, and choose a minimal counterexample $\mathcal{C}$ (that a minimal counterexample exists follows from the fact that the class of separable permutations has no infinite antichain, and thus its subclasses satisfy a descending chain condition). By this minimality and the fact that the union of two strongly rational classes is strongly rational, we see that $\mathcal{C}$ cannot be expressed as a finite union of proper subclasses. The major argument of the proof then that this property implies that $\mathcal{C} \subseteq \mathcal{X}[\mathcal{U}]$ for a *proper* subclass $\mathcal{U} \subsetneq \mathcal{C}$, at which point Theorem 5.2 provides a contradiction, completing the proof.

Another initial goal of this project is to extend Theorem 5.2 to arbitrary geometrically griddable classes:

Conjecture 5.3. *If $\mathcal{G}$ is a geometrically griddable permutation class and $\mathcal{U}$ is a strongly rational permutation class then $\mathcal{G}[\mathcal{U}]$ is strongly rational.*

The situation described in Conjecture 5.3 also arises when in the characterization of the small permutation classes. Given a class $\mathcal{C}$ of permutations, let $\langle\mathcal{C}\rangle$ denote the *substitution completion* of $\mathcal{C}$, which is defined to be the smallest class containing $\mathcal{C}$ which is closed under inflation. We further let $\langle\mathcal{C}\rangle^{\leqslant d}$ denote the subclass of $\langle\mathcal{C}\rangle$ consisting of permutations with substitution depth at most d. Letting $\mathcal{O}_k$ denote the class of all permutations contained in increasing and decreasing oscillations of length at most k (these are defined in Figure 3) the central structural result of the PI's characterization of small permutation classes can be stated as follows.

Theorem 5.4 (Vatter [49, Theorems 4.4 and 5.4])**.** *For every small permutation class $\mathcal{C}$, there is a geometrically griddable class $\mathcal{G}$ and integers d and k such that $\mathcal{C} \subseteq \mathcal{G}[\langle\mathcal{O}_k\rangle^{\leqslant d}]$.*

Therefore, a proof of Conjecture 5.3 would imply that all small permutation classes have rational generating functions (Conjecture 3.1) and thus are strongly rational.

Moving beyond merely substituting once (as in constructions of the form $\mathcal{G}[\mathcal{U}]$), we have the following conjecture about the substitution completions of geometrically griddable classes, which would represent a powerful generalization of Theorem 4.1.

Conjecture 5.5. *If $\mathcal{G}$ is a geometrically griddable permutation class then every subclass of its substitution completion, $\langle\mathcal{G}\rangle$, has an algebraic generating function.*

Given the potent role that geometrically griddable classes seem to play in the structural theory of permutation classes, another important goal is to characterize these classes.

Problem 5.6. *Characterize the geometrically griddable permutation classes.*

6 Partial Well-Order

Three celebrated results in combinatorics show that in certain partial (and quasi-) orders of combinatorial structures, infinite antichains do not exist: Higman's Theorem on ordering by divisibility [30], Kruskal's Tree Theorem [38], and Robertson and Seymour's Graph Minor Theorem [47]. For many other structures and orders, however, infinite antichains do exist, and in some cases they exist in abundance. Driven by the impact of results such as the Graph Minor Theorem, the study of how and when infinite antichains occur has drawn continual attention ever since Higman essentially founded the area in the 1950s.

It has been known since Pratt [46] that there are infinite antichains of permutations (i.e., infinite sets of pairwise incomparable permutations) under the permutation containment order. If a permutation class does not contain an infinite antichain, then it is said to be *partially well-ordered (pwo)*. A straight-forward application of Higman's Theorem, along with the order preserving maps discussed in Section 5 establishes the following.

Theorem 6.1 (Vatter and Waton [51], essentially)**.** *Every geometric grid class is pwo.*

By invoking Kruskal's Tree Theorem (or by more elementary arguments), we also see that classes without too many simple permutations are well-behaved:

Theorem 6.2 (Albert and Atkinson [1])**.** *If a permutation class contains only finitely many simple permutations then it is pwo.*

Finally, a simple counting argument verifies that the strongly rational classes are well-behaved.

Proposition 6.3 (Albert, Atkinson, and Vatter [4])**.** *Every strongly rational class is pwo.*

The PI proved in [49] that the small permutation classes are pwo and thus the smallest (as measured by growth rate) non-pwo permutation class has growth rate precisely κ. (Indeed this turns out to be the major difficulty in extending the techniques used in the proof of this theorem to handle classes with growth rates in the gap between κ and λ.) Therefore partial well-order appears to be a strong indication that a class is well-structured, and the following is a long-term goal.

Question 6.4. *Does every pwo permutation class have an algebraic (or rational) generating function?*

Regardless of the answer to Question 6.4, it is frequently important to determine if a class is pwo because one may then appeal to the descending chain condition for its subclasses. Therefore another long-term goal is the following.

Problem 6.5. *Characterize the pwo permutation classes.*

In order to characterize the pwo permutation classes it is necessary to construct antichains on demand. One of the first systematic methods for such constructions was introduced by Murphy and the PI in [41]. This has since been greatly extended by the *grid pin sequences* of Brignall [13]. Roughly, a grid pin sequence is like the pin sequences of Section 4, except that some liberties are allowed when a pin crosses into another cell. Given an infinite chain of grid pin sequences, we obtain an infinite antichain either by inflating the first and last pins of every sequence (creating what are referred to as *anchors*) or by adding one additional point which *ties together* the beginning and end of the pin sequences. Three examples are shown in Figure 5.

Given an infinite antichain, one can make other, arbitrarily complicated antichains simply by adding more elements to each member. In order to narrow our focus to the fundamental reason "why" an antichain is an antichain, we define an order on infinite antichains: $A \preceq B$ if for every $\beta \in B$, there exists an $\alpha \in A$ with $\alpha \leqslant \beta$. An antichain is *minimal* if it is minimal under this order. Remarkably, every known minimal infinite antichain of permutations can be produced with Brignall's techniques, suggesting the following conjecture.

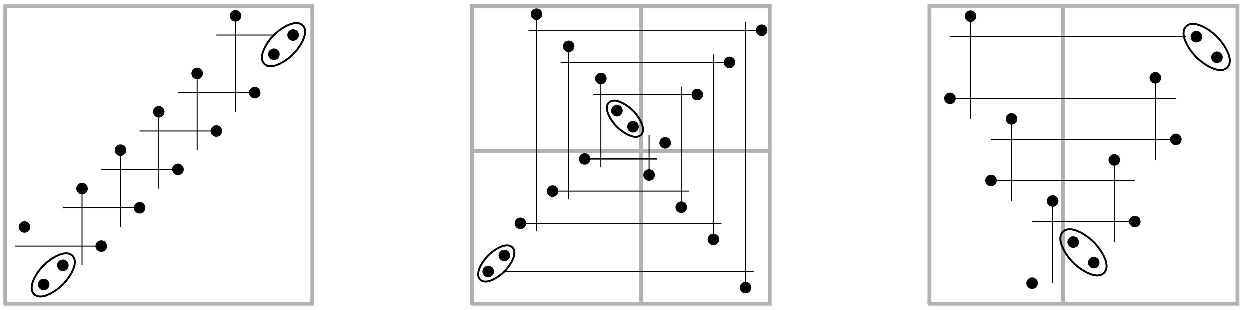

Figure 5: Typical elements of the three fundamental permutation antichains in Λ_1 (up to symmetry). From left to right: the increasing oscillating antichain, Widdershins and the antichain V.

Conjecture 6.6. *Every minimal infinite antichain of permutations can be obtained from grid pin sequences.*

Fairbanks and the PI are currently investigating the analogue of Conjecture 6.6 for complete ordered matchings. This context should be more tractable than the permutation case because the notions of pin sequences and grid pin sequences are identical for matchings.

7 Expanding the Vista: Combinatorial Objects as Relational Structures

All aspects of this work can be translated to other combinatorial objects, such as graphs, ordered graphs, posets, integer or set partitions, and tournaments. While each type of object has its own unique flavor, I contend that their shared features make it of paramount importance that we, as combinatorialists, adopt a more consolidated approach toward studying them, which would reflect this underlying unity. Such a change in approach will open up deep and novel areas for exploration. A major long-term goal of this project is to set a foundation for the "combinatorics of relational structures."

The term *combinatorial structure* can take many different meanings. In order to narrow the focus to a more manageable size, let us focus on *relational structures*, by which we mean ground sets of vertices (*domains*) together with one or more relations (specified by a *signature*). From this perspective: a graph is a ground set with a symmetric, irreflexive, binary relation; an ordered graph is a graph equipped with a linear order; a poset is a ground set with a reflexive, antisymmetric, transitive binary relation; an integer partition is a ground set with an equivalence relation; a set partition is an integer partition equipped with a linear order; a tournament is a ground set with a complete, irreflexive, anti-symmetric relation; and finally, a permutation is a ground set with two linear orders.

It seems in practice that the study of relational structures often provides the "correct" level of generality to consider many different phenomena, and in which to tie together results proved about particular types of structure, allowing for beneficial cross-fertilization of ideas. In the rest of this section we survey the possibility of raising several of the threads introduced in the preceding sections to this more general level.

Geometric grid classes. As we defined them in Section 5, geometric grid classes seem to depend on the geometric perspective of permutations. This presentation, however, is merely an aesthetic convenience, and we conjecture that the essential features of geometric grid classes can be captured with a definition that does not depend on geometry at all.

In order to frame the forthcoming discussion properly, let us say that a *class* of relational structures is a downward-closed (under the induced substructure order) set of relational structures, all of the same signature. If we wish to forbid automorphisms (or for other reasons), we may wish to restrict our purview to *linear structures*, which are relational structures with at least one linear order (thereby including ordered graphs, set partitions, and permutations).

The fundamental property of geometrically griddable classes is that for each such class, $\mathcal{C}$, there is a subword-closed language $\mathcal{L}$ and an order-preserving map $\varphi : \mathcal{L} \to \mathcal{C}$. In fact, these maps satisfy an even stronger condition.

Definition 7.1. *Let $\mathcal{L}$ be a subword-closed language and $\mathcal{C}$ a class of relational structures. We say that the map $\varphi : \mathcal{L} \to \mathcal{C}$ is a* natural encoding *if it is length- and order-preserving and if whenever $R \leqslant \varphi(w)$ for $w \in \mathcal{L}$, there is a subword $u \leqslant w$ such that $\varphi(u) = R$.*

Although many permutation classes are images of maps from subword-closed languages, we do not know of any others which satisfy this strong property. An obvious question, for which an answer to Problem 5.6 most likely serves as a prerequisite, is therefore:

Question 7.2. *If the permutation class $\mathcal{C}$ admits a natural encoding, is $\mathcal{C}$ necessarily geometrically griddable?*

Preliminary research makes us strongly suspect that the answer to this question is "yes", which further justifies the contention that they play such a central role in the structural theory of permutation classes. However, even a negative answer would be interesting, as then there would be other permutation classes which have similar structure to the geometrically griddable classes, and it would be important to understand these classes.

No matter what the answer to Question 7.2 turns out to be, the investigation of naturally encoded classes of relational structures should prove to be interesting. I know of only one piece of work which studies this concept (albeit, inevitably, under a different name): Petkovšek's investigation of "letter graphs" in [43]. As the pwo result generalizes trivially to all images of natural encodings, the next most natural question is enumeration (thereby, hopefully, generalizing Theorem 5.1).

Question 7.3. *If the class $\mathcal{C}$ of linear structures admits a natural encoding, is $\mathcal{C}$ in bijection with a regular language? (This would imply that $\mathcal{C}$ has a rational generating function.)*

It may even be the case that the "linear" hypothesis in Question 7.3 can be removed. An excellent context in which to investigate this possibility would be the case of graphs, which could reveal an interesting link with Pólya theory. Indeed, Petkovšek [43] showed (by checking all possible cases) that Question 7.3 is true for classes of graphs which admit a natural encoding from a subword-closed language with at most 2 letters.

The substitution decomposition. The substitution decomposition (under a great variety of names) dates back at least to a 1953 talk of Fraïssé [24], and was notably employed by Gallai [27, 28] in his investigations of transitive orientations of graphs. The definition of a *simple relational structure* is analogous to that of a simple permutations. Schmerl and Trotter proved Theorem 4.4 in the context of binary relational structures; one natural question is therefore how their theorem generalizes to relational structure of any arity.

Conjecture 7.4. *Every n-element simple relational structure with relations have arity at most r contains a simple proper substructure with at least $n - r$ elements.*

Another natural target for study are analogues of the Ramsey-type result for permutations (Theorem 4.3); in particular the graph case remains open:

Question 7.5. *Is there an analogue of Theorem 4.3 for graphs?*

A straightforward translation of the proof in the permutation case fails, essentially because this proof relies on the geometry of the plane (and in the complete ordered matching case, it relies on the geometry of the real line). Some partial results in this direction were established by Pouzet and Zaguia [45].

Growth rates. Growth rates have been studied for a variety of combinatorial structures; see Bollobás [12] and Klazar [35] for surveys. One closely related context is that of ordered graphs, because every permutation class can be viewed as a downset of ordered graphs (but not vice versa). Balogh, Bollobás, and Morris [8, 9] have shown that the sub-2 growth rates of downsets of ordered graphs are precisely the same as the sub-2 growth rates of permutation classes. However, there is a downset of ordered graphs whose growth rate is approximately 2.03166, which Theorem 2.3 shows is not the growth rate of any permutation class.

A natural problem would be to adapt the techniques used to prove Theorem 2.3 to the context of ordered graphs:

Problem 7.6. *Extend the characterization of growth rates of ordered graphs up to the constant κ.*

We anticipate that the study of naturally encoded ordered graphs will play a crucial role in the resolution of Problem 7.6, just as grid classes played a crucial role in the proof of Theorem 2.3. More generally, one could ask the following.

Question 7.7. *What can be said about growth rates of classes of linear structures in general?*

For example, fix a signature, and consider all classes of linear structure of this signature. Is it true that the first (if any) accumulation point of growth rates of these classes is reached from below? Some work along these lines has been done by Klazar [34].

Partial well-order. In the more general context of relational structures, partial well-order has been extensively studied, and we refer to Cherlin [17] for a recent survey. Much of this research has focused on characterizing the minimal infinite antichains of a certain type. The most general result about these antichains is the following (which we have specialized to our context).

Theorem 7.8 (Cherlin and Latka [18])**.** *For any type of relational structure and every integer k, there is a finite set Λ_k of minimal infinite antichains such that any class of structures defined by at most k minimal forbidden elements is pwo if and only if has finite intersection with every antichain in Λ_k.*

The strength of this result lies in the evidence it provides that the antichains of combinatorial objects are "nice": Λ_1 is known for graphs [20], tournaments [18], and permutations [6] (in fact, Λ_1 for permutations consists of the three antichains in Figure 5). Moreover, the sets Λ_1 in each of these three cases are incredibly similar, which lends hope to the idea that minimal antichains could be characterized in general.

At the relational structure level, an obvious goal is the following.

Problem 7.9. *Generalize Brignall's grid pin sequences to arbitrary relational structures.*

Following on from Problem 7.9, we would ask the following generalization of Conjecture 6.6.

Question 7.10. *Can all minimal infinite antichains of relational structures be constructed from grid pin sequences?*

We conclude by recalling two attractive long-standing conjectures about minimal infinite antichains of combinatorial objects. Although the first conjecture was made in the context of permutations, we see no reason not to generalize it.

Conjecture 7.11 (Murphy [40])**.** *Let A be a minimal infinite antichain of relational structures. Then there are only finitely many integers n such that A contains more than one n-element structure.*

The second conjecture, open since 1972, was originally stated for graphs, but again we extend it to the context of relational structures. Given k-colored structures R and S, we write $R \leqslant_k S$ if R has a color-preserving embedding into S. A (monochromatic) class $\mathcal{C}$ of relational structures is k-pwo if the set consisting of all k-colorings of structures from $\mathcal{C}$ is pwo when viewed as a downset in the $\leqslant_k$ ordering. The following conjecture can be interpreted as saying that we never need to use more than one type of anchor to construct an antichain.

Conjecture 7.12 (Pouzet [44]; see also [25, 37])**.** *A class of relational structures is 2-pwo if and only if it is k-pwo for all k.*

Specializing to permutations briefly, we note that if a permutation class contains an infinite grid pin sequence, then it is not 2-pwo. Therefore, the permutation case of Conjecture 7.12 may follow from Conjecture 6.6 and some additional work. In the long-term, we hope to prove Conjecture 7.12 for all types of relational structure by answering Question 7.10 affirmatively (once a sensible definition of grid pin sequences is found in general).

8 Results From Prior NSF Support

The PI was received funding (as a Co-PI) on NSF grant DMS-1003908, which provided \$14,460 to support participants of the *Permutation Patterns 2010* conference. The week-long conference, held at Dartmouth College, was attended by 11 undergraduate students, 19 graduate students, 18 junior researchers (postdocs and pre-tenure faculty), and 13 senior researchers (tenured faculty). There were 34 talks, and the PI is currently co-editing the proceedings, which will be published as a special issue in the journal *Pure Mathematics and Applications*.

Outside of this grant, the PI has not been funded by the NSF.

9 Summary

The major goals of this project, and more generally this line of research, are to answer the following major questions, problems, and conjectures.

- **Question 2.1.** *Does every nontrivial permutation class have a proper growth rate?*

- **Question 2.2.** *What numbers between κ and λ can be realized as (upper) growth rates of permutation classes?*
- **Problem 3.3.** *Characterize the strongly rational permutation classes.*
- **Problem 6.5.** *Characterize the pwo permutation classes.*
- **Question 7.3.** *If the class $\mathcal{C}$ of linear structures admits a natural encoding, is $\mathcal{C}$ in bijection with a regular language?*
- **Conjecture 7.4.** *Every n-element simple relational structure with relations have arity at most r contains a simple proper substructure with at least $n-r$ elements.*
- **Question 7.10.** *Can all minimal infinite antichains of relational structures be constructed from grid pin sequences?*
- **Conjecture 7.11.** *Let A be a minimal infinite antichain of relational structures. Then there are only finitely many integers n such that A contains more than one n-element structure.*
- **Conjecture 7.12.** *A class of relational structures is 2-pwo if and only if it is k-pwo for all k.*

The major tools we will use to investigate these questions are automata and formal languages, geometric grid classes (and their generalization as natural encodings), the substitution decomposition, pin sequences, and Brignall's method of constructing infinite antichains from grid pin sequences.

It may seem odd, given the viewpoint of Section 7, that so much consideration has been given to one particular type of object, namely permutations. One of the reasons for this is simply the observation that it is easier to develop intuition by focusing your field of vision, and the research team has considerable experience in the context of permutations. Another explanation is that the permutation context offers a unique perspective, which brings different features to the fore (for example, the geometric perspective taken in Section 5). Finally, we believe that it is precisely this interplay between the specific and the general which will lead to new ideas.

The budget includes support for two undergraduate students working part-time each year. The undergraduate students will conduct research in this area under the direction of the PI and the graduate student. This represents a continuation of my undergraduate mentoring efforts. Last year, when they were both in their penultimate years of undergraduate study, I began working with Tim Dwyer and James Fairbanks. Both have done very well; they have each written a paper [21, 23] and are applying to graduate school. Tim is currently investigating grid classes while James is studying infinite antichains of complete ordered matchings. I expect future collaborations with undergraduate students to take roughly the same form, although in the future I will encourage the students to work together on a single project. I will recruit students from the undergraduate Combinatorics sequence and from the Pi Mu Epsilon undergraduate mathematics club.

The budget also requests funds to support a graduate student each year of the grant. I expect to recruit this student from the graduate Combinatorics sequence. This funding will allow the graduate student to conduct full-time research during the year and provide travel expenses to attend conferences and foster collaborations, both of which can be crucial for the career development of Ph.D. students. This graduate student will be my second: Dr. Robert Brignall was co-supervised by myself and Nik Ruškuc at the University of St. Andrews. In addition, I was heavily involved in the training of Dr. Steve Waton, who also received his Ph.D. at the University of St. Andrews under the direction of Nik Ruškuc.

References Cited

[1] Albert, M. H., and Atkinson, M. D. Simple permutations and pattern restricted permutations. *Discrete Math. 300*, 1-3 (2005), 1–15.

[2] Albert, M. H., Atkinson, M. D., Bouvel, M., Ruškuc, N., and Vatter, V. Geometric grid classes of permutations. *Trans. Amer. Math. Soc.*, submitted, August 31, 2011, arXiv:1108.6319v1 [math.CO].

[3] Albert, M. H., Atkinson, M. D., and Ruškuc, N. Regular closed sets of permutations. *Theoret. Comput. Sci. 306*, 1-3 (2003), 85–100.

[4] Albert, M. H., Atkinson, M. D., and Vatter, V. Subclasses of the separable permutations. *Bull. Lond. Math. Soc.*, published online April 29, 2011, doi: 10.1112/blms/bdr022 (yet to be assigned to an issue).

[5] Albert, M. H., and Linton, S. Growing at a perfect speed. *Combin. Probab. Comput. 18* (2009), 301–308.

[6] Atkinson, M. D., Murphy, M. M., and Ruškuc, N. Partially well-ordered closed sets of permutations. *Order 19*, 2 (2002), 101–113.

[7] Bafna, V., and Pevzner, P. A. Genome rearrangements and sorting by reversals. *SIAM J. Comput. 25*, 2 (1996), 272–289.

[8] Balogh, J., Bollobás, B., and Morris, R. Hereditary properties of ordered graphs. In *Topics in discrete mathematics*, M. Klazar, J. Kratochvíl, M. Loebl, J. Matoušek, R. Thomas, and P. Valtr, Eds., vol. 26 of *Algorithms Combin.* Springer, Berlin, 2006, pp. 179–213.

[9] Balogh, J., Bollobás, B., and Morris, R. Hereditary properties of partitions, ordered graphs and ordered hypergraphs. *European J. Combin. 27*, 8 (2006), 1263–1281.

[10] Bassino, F., Bouvel, M., Pierrot, A., and Rossin, D. A polynomial algorithm for deciding the finiteness of simple permutations contained in permutation classes. In preparation.

[11] Bassino, F., Bouvel, M., Pierrot, A., and Rossin, D. Deciding the finiteness of simple permutations contained in a wreath-closed class is polynomial. *Pure Math. Appl. (PU.M.A.) 21* (2010), 119–135.

[12] Bollobás, B. Hereditary and monotone properties of combinatorial structures. In *Surveys in Combinatorics 2007*, A. Hilton and J. Talbot, Eds., no. 346 in London Mathematical Society Lecture Note Series. Cambridge University Press, 2007, pp. 1–39.

[13] Brignall, R. Grid classes and partial well order. *J. Combin. Theory Ser. A 119* (2012), 99–116.

[14] Brignall, R., Huczynska, S., and Vatter, V. Decomposing simple permutations, with enumerative consequences. *Combinatorica 28* (2008), 385–400.

[15] Brignall, R., Huczynska, S., and Vatter, V. Simple permutations and algebraic generating functions. *J. Combin. Theory Ser. A 115*, 3 (2008), 423–441.

[16] Brignall, R., Ruškuc, N., and Vatter, V. Simple permutations: decidability and unavoidable substructures. *Theoret. Comput. Sci. 391*, 1–2 (2008), 150–163.

[17] Cherlin, G. L. Forbidden substructures and combinatorial dichotomies: WQO and universality. *Discrete Math. 311*, 15 (2011), 1543–1584.

[18] Cherlin, G. L., and Latka, B. J. Minimal antichains in well-founded quasi-orders with an application to tournaments. *J. Combin. Theory Ser. B 80*, 2 (2000), 258–276.

[19] Corteel, S., Louchard, G., and Pemantle, R. Common intervals of permutations. *Discrete Math. Theor. Comput. Sci. 8*, 1 (2006), 189–214.

[20] Ding, G. Subgraphs and well-quasi-ordering. *J. Graph Theory 16*, 5 (1992), 489–502.

[21] Dokos, T., Dwyer, T., Johnson, B. P., Sagan, B. E., and Selsor, K. Permutation patterns and statistics. arXiv:1109.4976 [math.CO].

[22] Elder, M., and Vatter, V. Problems and conjectures presented at the Third International Conference on Permutation Patterns, University of Florida, March 7–11, 2005. arXiv:math.CO/0505504.

[23] Fairbanks, J. A Ramsey theorem for indecomposable matchings. In preparation.

[24] Fraïssé, R. On a decomposition of relations which generalizes the sum of ordering relations. *Bull. Amer. Math. Soc. 59* (1953), 389.

[25] Fraïssé, R. *Theory of relations*, revised ed., vol. 145 of *Studies in Logic and the Foundations of Mathematics*. North-Holland Publishing Co., Amsterdam, 2000.

[26] Füredi, Z., and Hajnal, P. Davenport-Schinzel theory of matrices. *Discrete Math. 103*, 3 (1992), 233–251.

[27] Gallai, T. Transitiv orientierbare Graphen. *Acta Math. Acad. Sci. Hungar 18* (1967), 25–66.

[28] Gallai, T. *A translation of T. Gallai's paper: "Transitiv orientierbare Graphen"*. Wiley-Intersci. Ser. Discrete Math. Optim. Wiley, Chichester, 2001, pp. 25–66.

[29] Golumbic, M. C. *Algorithmic graph theory and perfect graphs*, second ed., vol. 57 of *Annals of Discrete Mathematics*. Elsevier Science B.V., Amsterdam, 2004.

[30] Higman, G. Ordering by divisibility in abstract algebras. *Proc. London Math. Soc. (3) 2* (1952), 326–336.

[31] Huczynska, S., and Vatter, V. Grid classes and the Fibonacci dichotomy for restricted permutations. *Electron. J. Combin. 13* (2006), Research paper 54, 14 pp.

[32] Kaiser, T., and Klazar, M. On growth rates of closed permutation classes. *Electron. J. Combin. 9*, 2 (2003), Research paper 10, 20 pp.

[33] Klazar, M. The Füredi-Hajnal conjecture implies the Stanley-Wilf conjecture. In *Formal power series and algebraic combinatorics (Moscow, 2000)*. Springer, 2000, pp. 250–255.

[34] Klazar, M. On growth rates of permutations, set partitions, ordered graphs and other objects. *Electron. J. Combin. 15*, 1 (2008), Research paper 75, 22 pp.

[35] Klazar, M. Overview of some general results in combinatorial enumeration. In *Permutation Patterns* (2010), S. Linton, N. Ruškuc, and V. Vatter, Eds., vol. 376 of *London Mathematical Society Lecture Note Series*, Cambridge University Press, pp. 3–40.

[36] Knuth, D. E. *The art of computer programming. Volume 1.* Addison-Wesley Publishing Co., Reading, Mass., 1968. Fundamental Algorithms.

[37] Kříž, I., and Thomas, R. On well-quasi-ordering finite structures with labels. *Graphs Combin. 6*, 1 (1990), 41–49.

[38] Kruskal, J. B. Well-quasi-ordering, the Tree Theorem, and Vazsonyi's conjecture. *Trans. Amer. Math. Soc. 95* (1960), 210–225.

[39] Marcus, A., and Tardos, G. Excluded permutation matrices and the Stanley-Wilf conjecture. *J. Combin. Theory Ser. A 107*, 1 (2004), 153–160.

[40] Murphy, M. M. *Restricted Permutations, Antichains, Atomic Classes, and Stack Sorting.* PhD thesis, Univ. of St Andrews, 2002.

[41] Murphy, M. M., and Vatter, V. Profile classes and partial well-order for permutations. *Electron. J. Combin. 9*, 2 (2003), Research paper 17, 30 pp.

[42] Noonan, J., and Zeilberger, D. The enumeration of permutations with a prescribed number of "forbidden" patterns. *Adv. in Appl. Math. 17*, 4 (1996), 381–407.

[43] Petkovšek, M. Letter graphs and well-quasi-order by induced subgraphs. *Discrete Math. 244*, 1-3 (2002), 375–388. Algebraic and topological methods in graph theory (Lake Bled, 1999).

[44] Pouzet, M. Un bel ordre d'abritement et ses rapports avec les bornes d'une multirelation. *C. R. Acad. Sci. Paris Sér. A-B 274* (1972), A1677–A1680.

[45] Pouzet, M., and Zaguia, I. On minimal prime graphs and posets. *Order 26*, 4 (2009), 357–375.

[46] Pratt, V. R. Computing permutations with double-ended queues, parallel stacks and parallel queues. In *STOC '73: Proceedings of the Fifth Annual ACM Symposium on the Theory of Computing* (New York, NY, USA, 1973), ACM Press, pp. 268–277.

[47] Robertson, N., and Seymour, P. Graph minors I–XX. *J. Combinatorial Theory Ser. B* (1983–2004).

[48] Schmerl, J. H., and Trotter, W. T. Critically indecomposable partially ordered sets, graphs, tournaments and other binary relational structures. *Discrete Math. 113*, 1-3 (1993), 191–205.

[49] Vatter, V. Small permutation classes. *Proc. Lond. Math. Soc. (3)*, published online July 8, 2011, doi: 10.1112/plms/pdr017 (yet to be assigned to an issue).

[50] Vatter, V. Permutation classes of every growth rate above 2.48188. *Mathematika 56* (2010), 182–192.

[51] Vatter, V., and Waton, S. On partial well-order for monotone grid classes of permutations. *Order 28* (2011), 193–199.

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John Wesley Young Research Instructor, Dartmouth College, June 2008–August 2010

Appointments

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Research Fellow, University of St. Andrews, September 2005–June 2007

Five publications most closely related to proposed project

- M. H. Albert, M. D. Atkinson, M. Bouvel, N. Ruškuc, and V. Vatter, Geometric grid classes of permutations, *Trans. Amer. Math. Soc.*, submitted, August 31, 2011, arXiv:1108.6319v1 [math.CO].
- V. Vatter, Small permutation classes, *Proc. Lond. Math. Soc. (3)*, published online July 8, 2011, doi: 10.1112/plms/pdr017 (yet to be assigned to an issue).
- M. H. Albert, M. D. Atkinson, and V. Vatter, Subclasses of the separable permutations, *Bull. Lond. Math. Soc.*, published online April 29, 2011, doi: 10.1112/blms/bdr022 (yet to be assigned to an issue).
- V. Vatter, Permutation classes of every growth rate above 2.48188, *Mathematika* **56** (2010), 182–192.
- R. Brignall, S. Huczynska, and V. Vatter, Decomposing simple permutations, with enumerative consequences, *Combinatorica* **28** (2008), 385–400.

Five other recent significant publications

- R. Brignall, N. Ruškuc, and V. Vatter, Simple extensions of combinatorial structures, *Mathematika*, **57** (2011), 193–214.
- V. Vatter, Maximal independent sets and separating covers, *Amer. Math. Monthly*, **118** (2011), 418–423.
- R. Brignall, S. Huczynska, and V. Vatter, Simple permutations and algebraic generating functions, *J. Combin. Theory Ser. A* **115** (2008), 423–441.
- V. Vatter, Enumeration schemes for restricted permutations, *Combin. Probab. Comput.* **17** (2008), 137–159.
- B. E. Sagan and V. Vatter, The Möbius function of a composition poset, *J. Alg. Combin.* **24** (2006), 117–136.

Synergistic Activities

- Conference organization:
  - Member of the program committee for *FPSAC 2011*, June 2011, Reykjavik, Iceland,
  - Local organizer for *Permutation Patterns 2010*, August 2010, Dartmouth College,
  - Member of the organizing committee for *From* $A = B$ *to* $Z = 60$, May 2010, Rutgers University,
  - Local organizer for *Permutation Patterns 2007*, June 2007, University of St. Andrews,
- Editing:
  - Co-editor of a special volume of *Pure Mathematics and Applications* containing the conference proceedings of *Permutation Patterns 2010*.
  - Co-editor of the book *Permutation Patterns* (Cambridge University Press) containing the conference proceedings of *Permutation Patterns 2007*.
- Seminar organization:
  - Organizer of the *Combinatorics Seminar* at the University of Florida, Fall 2010–present,
  - Organizer of the *Combinatorics Seminar* at Dartmouth College, Winter and Spring, 2010.
- Co-author of the expository article "Of pancakes, mice and men" in *Plus Magazine* (online at `http://plus.maths.org/content/pancakes-mice-and-men`) and author of the expository article "Maximal independent sets and separating covers", *Amer. Math. Monthly*, **118** (2011), 418–423.
- Referee for *Advances in Applied Mathematics*, *Annals of Combinatorics*, *Ars Combinatoria*, *The Computer Journal*, *Discrete Mathematics*, *Discrete Applied Mathematics*, the *Electronic Journal of Combinatorics*, the *European Journal of Combinatorics*, *FPSAC* (Formal Power Series and Algebraic Combinatorics), the *Journal of Combinatorics*, *Mathematics Magazine*, the *Journal of Combinatorial Theory Series A*, the *Journal of Integer Sequences*, *Pure Mathematics and Applications*, and *Séminaire Lotharingien de Combinatoire*.
- Reviewer for *Mathematical Reviews*.

Collaborators

Collaborators within the last 48 months Michael Albert (University of Otago), Mike Atkinson (University of Otago), Miklós Bóna (University of Florida), Mathilde Bouvel (Université Bordeaux 1), Robert Brignall (Open University), Tim Dwyer (University of Florida), Sergi Elizalde (Dartmouth College), James Fairbanks (University of Florida), Cheyne Homberger (University of Florida), Sophie Huczynska (University of St. Andrews), Steve Linton (University of St. Andrews), Colva Roney-Dougal (University of St. Andrews), Nik Ruškuc (University of St. Andrews), Bruce Sagan (Michigan State University), Rebecca Smith (The College at Brockport), Andrew Vince (University of Florida).

Ph.D. Advisor. Doron Zeilberger (2001–2006, Rutgers University)

Postdoctoral Sponsors. Nik Ruškuc (2005–2007, University of St. Andrews) and Sergi Elizalde (2008–2010, Dartmouth College)

Ph.D. Students. Robert Brignall (2005–2007, University of St. Andrews, joint with Nik Ruškuc)

Undergraduate Project Students. Tim Dwyer and James Fairbanks (both 2010–2012, University of Florida)

Budget Justification

Salaries. Two months of summer salary per year for the PI at $7,556.50 per month is requested to perform the proposed research. Funds are also budgeted to support a graduate student each year of the grant, at the rate of $20,119 per year. Finally, funds are requested to support two undergraduate students for 10 hours per week at the rate of $14 per hour during the each academic year of the grant. All salaries include a 3% inflation increase each year.

Fringe Benefits. Fringe benefits are computed using the rates of 26.9% for faculty salaries, 8.3% for graduate student salaries, and 3.1% for undergraduate salaries. If this proposal is funded, the rates quoted above shall, at the time of funding, be subject to adjustment for any period subsequent to June 30, 2012. The University of Florida fringe rates are approved annually by the U.S. Department of Health and Human Services.

Domestic Travel. The $4,000 per year of domestic travel will allow the project members to disseminate the research at conferences in the U.S. and Canada, and to bring outside speakers to the University of Florida to speak at the Combinatorics Seminar and/or the Colloquium. The conferences attended will generally include the annual *Permutation Patterns* and *FPSAC* conferences (in years when they are held domestically) and meetings of the AMS and SIAM.

International Travel. The $6,000 per year of international travel will allow the project members to attend international conferences (in particular, *Permutation Patterns* and *FPSAC* when they are held internationally, and the *British Combinatorial Conference*), and to collaborate with international collaborators.

Materials and Supplies. The materials and supplies section of the proposed budget is intended primarily for the purchase of computer equipment, and includes a new computer in Year 1.

Tuition. Tuition for a graduate student project member has been computed using the approved tuition charge of $11,796 per year (24 credit hours). These charges are increased by 15% per year. (This is listed in the proposed budget as an "other direct cost".)

Facilities and Administrative (Indirect) Costs. The University of Florida provisional on-campus rate for research is 46.5%. Provisional rates are subject to adjustment when superseding government-approved rates are established.

Data Management Plan

The proposed project will result in research articles and software. When possible we will strive to ensure free, archived access to all outputs produced. In particular, with regard to the research articles we will take the following steps:

- All articles will be added to the `arXiv`.
- All articles will be linked to from the PI's homepage.
- All articles will be submitted for publication to an appropriate and well-established journal.

The software which results from this project will be published on the PI's homepage.