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Asymptotic normality of coefficients of P-recursive polynomial sequences

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Abstract

In recent years, the asymptotic normality of some famous combinatorial sequences has been the subject of extensive study. However, the methods used to prove the asymptotic normality of various combinatorial sequences differ significantly. In this paper, we present a sufficient condition for establishing the asymptotic normality of the coefficients of a general P-recursive polynomial sequence. Additionally, we provide two examples that illustrate the application of this sufficient condition.

Keywords: P-recursive sequences, asymptotic normality, the central limit theorem, the local limit theorem.

1 Introduction

Suppose that $\{f_n(x)\}_{n \geq 0}$ is a polynomial sequence with nonnegative coefficients $a(n, k)$, that is,

$$f_n(x) = \sum_{k=0}^n a(n, k)x^k. \quad (1.1)$$

Let

$$p(n, k) = \frac{a(n, k)}{\sum_{j=0}^n a(n, j)}$$

be the normalized probabilities. We say that the coefficient $a(n, k)$ is *asymptotically normal with mean μ_n and variance σ_n^2 by a central limit theorem* if

$$\lim_{n \rightarrow \infty} \sup_{x \in \mathbb{R}} \left| \sum_{k \leq \mu_n + x\sigma_n} p(n, k) - \frac{1}{\sqrt{2\pi}} \int_{-\infty}^x \exp(-t^2/2) dt \right| = 0. \quad (1.2)$$

We say that the coefficient $a(n, k)$ is *asymptotically normal with mean μ_n and variance σ_n^2 by a local limit theorem* on the real set \mathbb{R} if

$$\lim_{n \rightarrow \infty} \sup_{x \in \mathbb{R}} \left| \sigma_n p(n, \lfloor \mu_n + x\sigma_n \rfloor) - \frac{1}{\sqrt{2\pi}} e^{-x^2/2} \right| = 0. \quad (1.3)$$

It is obvious that (1.3) can lead to (1.2), see Bender [1] and Canfield [2] for details.

In recent years, the asymptotic normality of various combinatorial sequences has been extensively studied. This includes the Eulerian numbers [3], the coefficients of q -Catalan numbers [7], the q -derangement numbers [8], the Stirling numbers of the first kind [11], the Stirling numbers of the second kind [13], the resonant sextet numbers [16], the Laplacian coefficients of graphs [21] and the Baxter permutations [24]. Most of their proofs are given based on the properties of specific sequences. Next, we will present a systematic method to prove the asymptotic normality of P-recursive polynomial sequences $\{f_n(x)\}_{n \geq 0}$.

Recall that a *P-recursive sequence* $\{a_n\}_{n \geq 0}$ of order d satisfies a recurrence relation of the form

$$p_0(n)a_n + p_1(n)a_{n+1} + \cdots + p_d(n)a_{n+d} = 0, \quad \forall n \geq 1,$$

where $p_i(n)$ are polynomials in n (see [20, Section 6.4]). Wimp and Zeilberger [23] (see also [12, Sec. VIII.7]) showed that a P-recursive sequence $\{a_n\}_{n \geq 0}$ is asymptotically equal a linear combination of terms in the form of

$$e^{Q(\rho, n)} s(\rho, n), \quad (1.4)$$

where

$$Q(\rho, n) = \mu_0 n \log n + \sum_{j=1}^{\rho} \mu_j n^{j/\rho}, \quad (1.5)$$

$$s(\rho, n) = n^r \sum_{j=0}^{t-1} (\log n)^j \sum_{s=0}^{\infty} b_{sj} n^{-s/\rho}, \quad (1.6)$$

where ρ, t being positive integers and μ_j, r, b_{sj} being complex numbers. In this paper, we focus on the case where $\mu_0 = 0$ in (1.5) and $t = 1$ in (1.6).

Suppose that $\{f_n(x)\}_{n \geq 0}$ is a P-recursive polynomial sequence with respect to the variable x . According to the properties of the least common left multiple of operators, we can obtain that both $\{f'_n(x)\}_{n \geq 0}$ and $\{f''_n(x)\}_{n \geq 0}$ are also P-recursive polynomial sequences. Thus, they all have asymptotic expressions similar to the above. By performing a ratio operation on their asymptotic forms, we finally obtain the mean and variance of the coefficient sequence of $\{f_n(x)\}_{n \geq 0}$. Given the condition that the known P-recursive polynomial sequences have only real roots, we can apply this method to demonstrate that their coefficients are asymptotically normal.

This paper is organized as follows. In Section 2, we provide a sufficient condition for the asymptotic normality of the coefficients of a P-recursive polynomial sequence. In Section 3, we give two examples of the asymptotic normality of specific sequences. Moreover, we list the means and variances of some other sequences.

2 A sufficient condition for asymptotic normality

In this section, we consider the asymptotic normality of the coefficient of a P-recursive polynomial sequence $\{f_n(x)\}_{n \geq 0}$, where $f_n(x)$ is expressed in the form shown in (1.1). From (1.4), we observe that it is asymptotically equal to

$$e^{Q(\rho, n)(x)} s(\rho, n)(x). \quad (2.1)$$

Firstly, we explore the relationships among $f_n(x)$, $f'_n(x)$ and $f''_n(x)$.

Lemma 2.1. Assuming that $\{f_n(x)\}_{n \geq 0}$ is a P-recursive polynomial sequence in the variable x , then $\{f'_n(x)\}_{n \geq 0}$ and $\{f''_n(x)\}_{n \geq 0}$ are also.

Proof. Since $\{f_n(x)\}_{n \geq 0}$ is a P-recursive polynomial sequence in terms of x , there exists an operator L_1 such that

$$L_1 f_n(x) = 0. \quad (2.2)$$

By taking the derivative of both sides of (2.2) with respect to x , we find that there exists an operator L_2 such that

$$L_1 f'_n(x) + L_2 f_n(x) = 0. \quad (2.3)$$

According to Theorem 8 of [19], there exist operators U and V such that

$$UL_1 = VL_2 = lclm(L_1, L_2),$$

where $lclm(L_1, L_2)$ denotes the least common left multiple of L_1 and L_2 . Applying the operator V to both sides of (2.3) yields

$$\begin{aligned} & VL_1 f'_n(x) + VL_2 f_n(x) \\ &= VL_1 f'_n(x) + UL_1 f_n(x) \\ &= VL_1 f'_n(x) \\ &= 0. \end{aligned}$$

Consequently, we obtain that $\{f'_n(x)\}_{n \geq 0}$ is also a P-recursive polynomial sequence with respect to x . Using the same method, we can conclude that this holds for $\{f''_n(x)\}_{n \geq 0}$. \square

Remark 1. Suppose that $\{f_n(x)\}_{n \geq 0}$ is a P-recursive polynomial sequence. According to (2.1), we can assume that the asymptotic expression of $f_n(x)$ is as follows,

$$f_n(x) = C_1 \cdot e^{Q_1(\rho, n)(x)} s_1(\rho, n)(x), \quad (2.4)$$

where C_1 is a constant.

Then we differentiate both sides of (2.4) with respect to x and obtain the asymptotic expression of $f'_n(x)$,

$$f'_n(x) = C_1 \cdot e^{Q_1(\rho, n)(x)} (Q'_1(\rho, n)(x) s_1(\rho, n)(x) + s'_1(\rho, n)(x)). \quad (2.5)$$

And it can be known from Lemma 2.1 that $\{f'_n(x)\}_{n \geq 0}$ is also a P-recursive polynomial sequence. Consequently, we can further assume that its asymptotic expression is as follows,

$$f'_n(x) = C_2 \cdot e^{Q_2(\rho, n)(x)} s_2(\rho, n)(x), \quad (2.6)$$

where C_2 is a constant.

By comparing (2.5) and (2.6), we can obtain that

$$Q_1(\rho, n)(x) = Q_2(\rho, n)(x)$$

and

$$C_1 \cdot (Q'_1(\rho, n)(x) s_1(\rho, n)(x) + s'_1(\rho, n)(x)) = C_2 \cdot s_2(\rho, n)(x).$$

Therefore, we can derive that the asymptotic expansion of $f'_n(x)$,

$$f'_n(x) = C_2 \cdot e^{Q_1(\rho, n)(x)} s_2(\rho, n)(x).$$

Similarly, we can also obtain the asymptotic expression of $f''_n(x)$ as follows,

$$f''_n(x) = C_3 \cdot e^{Q_1(\rho, n)(x)} s_3(\rho, n)(x),$$

where C_3 is a constant.

Next, we present a lemma proposed by Bender [1] to determine whether the coefficients of a polynomial sequence are asymptotic normal.

Lemma 2.2. *Suppose that $\{f_n(x)\}_{n \geq 0}$ is a sequence of real-rooted polynomial with non-negative coefficients defined by (1.1). Let*

$$\mu_n = \frac{f'_n(1)}{f_n(1)}, \quad \sigma_n^2 = \frac{f''_n(1)}{f_n(1)} + \mu_n - \mu_n^2. \quad (2.7)$$

If $\sigma_n^2 \rightarrow +\infty$ as $n \rightarrow +\infty$, then the coefficients of $f_n(x)$, that is, the numbers $a(n, k)$ are asymptotically normal by local and central limit theorems with mean μ_n and variance σ_n^2 .

Combined with the above Remark 1 and Lemma 2.2, we present the main result of this paper.

To make the theorem concise, we call that a P-recursive polynomial sequence $\{f_n(x)\}_{n \geq 0}$ is *well-defined* if the asymptotic form of $f_n^{(i)}(1)$ is expressed as follows,

$$f_n^{(i)}(1) = C'_i \cdot e^{Q(\rho, n)} n^{r_i} \left(\sum_{s=0}^M b_s^{(i)} n^{-s/\rho} + o(n^{-M/\rho}) \right),$$

where $f_n^{(i)}(x)$ denotes the i -th derivative of $f_n(x)$ ($i = 0, 1, 2$), C_i' is a constant, $Q(\rho, n) = \sum_{j=1}^{\rho} \mu_j n^{j/\rho}$, ρ, M are positive integers, $\mu_j, r_i, b_s^{(i)'} are real numbers, $r_1 < r_2 < r_3$ and $b_0^{(i)'} > 0$.$

Theorem 2.3. Suppose that $\{f_n(x)\}_{n \geq 0}$ defined by (1.1) is well-defined and is a sequence of real-rooted polynomial with non-negative coefficients. Let

$$\mu_n = \frac{f_n'(1)}{f_n(1)}, \quad \sigma_n^2 = \frac{f_n''(1)}{f_n(1)} + \mu_n - \mu_n^2. \quad (2.8)$$

If the following two conditions are satisfied, then $a(n, k)$ is asymptotically normal.

- (1) The limits $\lim_{n \rightarrow +\infty} \frac{f_n'(1)}{f_n(1) \cdot n^{r_2 - r_1}}$ and $\lim_{n \rightarrow +\infty} \frac{f_n''(1)}{f_n(1) \cdot n^{r_3 - r_1}}$ both exist, denoted by a and b respectively.
- (2) There exists an integer m such that $1/\rho \leq m \leq r_3 - r_1$. For indices $0 \leq s \leq m - 1$, the coefficients of n^s in σ_n^2 are zero, while the coefficient of n^m in σ_n^2 is positive.

Proof. Let $C_i = C_i' b_0^{(i)'}$, $b_s^{(i)} = \frac{b_s^{(i)'}}{b_0^{(i)'}}$, then the asymptotic expression of $f_n^{(i)}(1)$ can be reformulated as follows,

$$f_n^{(i)}(1) = C_i \cdot e^{Q(\rho, n)} n^{r_i} \left(1 + \sum_{s=1}^M b_s^{(i)} n^{-s/\rho} + o(n^{-M/\rho}) \right).$$

Consequently, we can obtain that

$$\mu_n = \frac{f_n'(1)}{f_n(1)} = \frac{C_2}{C_1} \cdot n^{r_2 - r_1} \left(1 + \frac{A_1}{n^{1/\rho}} + \cdots + \frac{A_M}{n^{M/\rho}} + o\left(\frac{1}{n^{M/\rho}}\right) \right), \quad (2.9)$$

where A_s is a polynomial in $b_1^{(0)}, \dots, b_s^{(0)}, b_1^{(1)}, \dots, b_s^{(1)}$.

From this, we find that

$$\mu_n^2 = \frac{C_2^2}{C_1^2} \cdot n^{2r_2 - 2r_1} \left(1 + \frac{D_1}{n^{1/\rho}} + \cdots + \frac{D_M}{n^{M/\rho}} + o\left(\frac{1}{n^{M/\rho}}\right) \right) \quad (2.10)$$

and

$$\frac{C_2}{C_1} = \lim_{n \rightarrow +\infty} \frac{f'_n(1)}{f_n(1) \cdot n^{r_2-r_1}} = a,$$

where D_s is a polynomial in A_1, \dots, A_s .

By similar reasoning, we establish that

$$\frac{f''_n(1)}{f_n(1)} = \frac{C_3}{C_1} \cdot n^{r_3-r_1} \left(1 + \frac{B_1}{n^{1/\rho}} + \dots + \frac{B_M}{n^{M/\rho}} + o\left(\frac{1}{n^{M/\rho}}\right) \right) \quad (2.11)$$

and

$$\frac{C_3}{C_1} = \lim_{n \rightarrow +\infty} \frac{f''_n(1)}{f_n(1) \cdot n^{r_3-r_1}} = b,$$

where B_s is a polynomial in $b_1^{(0)}, \dots, b_s^{(0)}, b_1^{(2)}, \dots, b_s^{(2)}$.

Given that $r_3 > r_2 > r_1$, it follows that $r_3 - r_1 > r_2 - r_1$. Without loss of generality, we assume that $r_3 - r_1 \geq 2r_2 - 2r_1$. Consequently, there exists an integer l_1 such that $r_3 - r_1 - l_1/\rho = 2r_2 - 2r_1$, and another an integer l_2 such that $r_3 - r_1 - l_2/\rho = r_2 - r_1$. Thus, by (2.9), (2.10) and (2.11), we derive that

$$\begin{aligned} \sigma_n^2 &= \frac{f''_n(1)}{f_n(1)} + \mu_n - \mu_n^2 \\ &= \frac{C_3}{C_1} \cdot n^{r_3-r_1} \left(1 + \frac{B_1}{n^{1/\rho}} + \dots + \frac{B_M}{n^{M/\rho}} + o\left(\frac{1}{n^{M/\rho}}\right) \right) \\ &\quad + \frac{C_2}{C_1} \cdot n^{r_2-r_1} \left(1 + \frac{A_1}{n^{1/\rho}} + \dots + \frac{A_M}{n^{M/\rho}} + o\left(\frac{1}{n^{M/\rho}}\right) \right) \\ &\quad - \frac{C_2^2}{C_1^2} \cdot n^{2r_2-2r_1} \left(1 + \frac{D_1}{n^{1/\rho}} + \dots + \frac{D_M}{n^{M/\rho}} + o\left(\frac{1}{n^{M/\rho}}\right) \right) \\ &= b \cdot n^{r_3-r_1} + b \cdot B_1 n^{r_3-r_1-1/\rho} + \dots + b \cdot B_{l_1-1} n^{r_3-r_1-(l_1-1)/\rho} \\ &\quad + (bB_{l_1} - a^2) \cdot n^{r_3-r_1-l_1/\rho} + (bB_{l_1+1} - a^2 D_1) \cdot n^{r_3-r_1-(l_1+1)/\rho} \\ &\quad + \dots + (bB_{l_2-1} - a^2 D_{l_2-l_1-1}) \cdot n^{r_3-r_1-(l_2-1)/\rho} \\ &\quad + (bB_{l_2} - a^2 D_{l_2-l_1} + a) \cdot n^{r_3-r_1-l_2/\rho} \\ &\quad + (bB_{l_2+1} - a^2 D_{l_2-l_1+1} + aA_1) \cdot n^{r_3-r_1-(l_2+1)/\rho} + \dots \\ &\quad + (bB_{(r_3-r_1)\rho-1} - a^2 D_{(r_3-r_1)\rho-l_1-1} + aA_{(r_3-r_1)\rho-l_2-1}) \cdot n^{1/\rho} + C + o(1). \end{aligned}$$

According to condition (2), as $n \rightarrow +\infty$, it follows that $\sigma_n^2 \rightarrow +\infty$. Therefore, $a(n, k)$ is asymptotically normal. \square

3 The asymptotic normality of some sequences

In this section, we utilize Theorem 2.3 to demonstrate the asymptotic normality of two specific sequences. Moreover, we list the means and variances of some other combinatorial sequences.

Recall that the hypergeometric polynomial of degree n is expressed in the following form,

$${}_pF_q \left(\begin{matrix} a_1, a_2, \dots, a_p \\ b_1, \dots, b_q \end{matrix}; x \right) = \sum_{k=0}^n \frac{(a_1)_k (a_2)_k \dots (a_p)_k}{(b_1)_k \dots (b_q)_k} \cdot \frac{x^k}{k!},$$

where $(a)_n = a(a+1) \dots (a+n-1)$ is the Pochhammer symbol. We begin by presenting a lemma established by Driver *et al.* [10, Theorem 7], which proves that the roots of the hypergeometric polynomials ${}_3F_2$ are all real.

Lemma 3.1. *Let $a, b > 0$ and let $l, m, n \in \mathbb{N}$. Then the polynomials*

$${}_3F_2 \left(\begin{matrix} -n, -m, a+l \\ a, b \end{matrix}; x \right)$$

has only negative real zeros.

Let

$$f_n(x) = \sum_{k=0}^n f(n, k) x^k = \sum_{k=0}^n \binom{n}{k}^2 \binom{n+k}{k} x^k$$

be the Apéry polynomial. Chen and Xia [9] studied the 2-log-convexity of $f_n(1)$, while Mao and Pei [18] proved the asymptotic log-convexity of Apéry-like numbers. Next, we will prove the asymptotic normality of $f(n, k)$. Let the mean be $\mu_n = \frac{f'_n(1)}{f_n(1)}$ and the variance be $\sigma_n^2 = \frac{f''_n(1)}{f_n(1)} + \mu_n - \mu_n^2$.

Theorem 3.2. *The coefficients $f(n, k)$ of the Apéry polynomial $f_n(x)$ are asymptotically normal by local and central limit theorems with $\mu_n \sim \frac{-1+\sqrt{5}}{2}n$, $\sigma_n^2 \sim \frac{5-2\sqrt{5}}{5}n$.*

Proof. By transforming $f_n(x)$, we can express it as follows,

$$f_n(x) = {}_3F_2 \left(\begin{matrix} -n, -n, n+1 \\ 1, 1 \end{matrix}; x \right).$$

Consequently, based on Lemma 3.1, we can know that $f_n(x)$ has only negative real roots.

Using the Maple package APCI developed by Hou [14], we can derive the following recurrence relations,

$$\begin{aligned} f_{n+2}(1) &= \frac{(11n^2 + 33n + 25)}{(n+2)^2} f_{n+1}(1) + \frac{(n+1)^2}{(n+2)^2} f_n(1), \\ f'_{n+2}(1) &= \frac{55n^3 + 187n^2 + 190n + 48}{(5n+2)(n+2)(n+1)} f'_{n+1}(1) + \frac{n(5n+7)}{(n+1)(5n+2)} f'_n(1), \\ f''_{n+2}(1) &= \frac{55n^4 + 154n^3 + 110n^2 - 7n - 12}{n^2(n+2)(5n-1)} f''_{n+1}(1) \\ &\quad + \frac{(n+1)(n+3)(5n+4)(n-1)}{n^2(n+2)(5n-1)} f''_n(1). \end{aligned}$$

Then, utilizing the Mathematica package P-rec.m provided by Hou and Zhang [15], we can obtain the following asymptotic expressions,

$$\begin{aligned} f_n(1) &= C_1 \cdot \left(\frac{11+5\sqrt{5}}{2} \right)^n n^{-1} \left(1 + \frac{-5+\sqrt{5}}{10n} + \frac{13-5\sqrt{5}}{50n^2} + o\left(\frac{1}{n^2}\right) \right), \\ f'_n(1) &= C_2 \cdot \left(\frac{11+5\sqrt{5}}{2} \right)^n \left(1 - \frac{-1+\sqrt{5}}{5n} + \frac{-1+\sqrt{5}}{25n^2} + o\left(\frac{1}{n^2}\right) \right), \\ f''_n(1) &= C_3 \cdot \left(\frac{11+5\sqrt{5}}{2} \right)^n n \left(1 + \frac{9-11\sqrt{5}}{10n} - \frac{18(-3+\sqrt{5})}{25n^2} + o\left(\frac{1}{n^2}\right) \right), \end{aligned}$$

where C_1, C_2, C_3 are all constants. Furthermore, we find that

$$\lim_{n \rightarrow +\infty} \frac{f_{n+1}(1)}{f_n(1)} = \frac{11+5\sqrt{5}}{2}. \quad (3.1)$$

Next, based on the Maple package APCI, we derive the following recurrence relations,

$$f'_n(1) = \frac{(n+1)^2}{5n} f_{n+1}(1) - \frac{(n+1)(8n+3)}{5n} f_n(1), \quad (3.2)$$

$$f''_n(1) = -\frac{(n^2+3n-1)(n+1)^2}{5n(n-1)} f_{n+1}(1) + \frac{(n+1)(13n^3+27n^2+n-3)}{5n(n-1)} f_n(1). \quad (3.3)$$

Thus, according to Theorem 2.3, Equations (3.1) and (3.2), we can conclude that

$$\begin{aligned}\frac{C_2}{C_1} &= \lim_{n \rightarrow +\infty} \frac{f'_n(1)}{f_n(1) \cdot n} \\ &= \lim_{n \rightarrow +\infty} \frac{(n+1)^2}{5n^2} \frac{f_{n+1}(1)}{f_n(1)} - \frac{(n+1)(8n+3)}{5n^2} \\ &= \frac{-1 + \sqrt{5}}{2}.\end{aligned}$$

Similarly, we can also derive that

$$\frac{C_3}{C_1} = \frac{3 - \sqrt{5}}{2}.$$

Hence, we obtain the mean as follows,

$$\begin{aligned}\mu_n &= \frac{f'_n(1)}{f_n(1)} \\ &= \frac{-1 + \sqrt{5}}{2} \cdot n \left(1 + \frac{7 - 3\sqrt{5}}{10n} + \frac{5 - 2\sqrt{5}}{25n^2} + o\left(\frac{1}{n^2}\right) \right), \\ &\sim \frac{-1 + \sqrt{5}}{2} n\end{aligned} \quad (3.4)$$

and

$$\frac{f''_n(1)}{f_n(1)} = \frac{3 - \sqrt{5}}{2} \cdot n^2 \left(1 + \frac{7 - 6\sqrt{5}}{5n} - \frac{2(-40 + 17\sqrt{5})}{25n^2} + o\left(\frac{1}{n^2}\right) \right). \quad (3.5)$$

From (3.4) and (3.5), we can finally get the variance,

$$\begin{aligned}\sigma_n^2 &= \frac{f''_n(1)}{f_n(1)} + \mu_n - \mu_n^2 \\ &= \frac{3 - \sqrt{5}}{2} \cdot n^2 \left(1 + \frac{7 - 6\sqrt{5}}{5n} - \frac{2(-40 + 17\sqrt{5})}{25n^2} + o\left(\frac{1}{n^2}\right) \right) \\ &\quad + \frac{-1 + \sqrt{5}}{2} \cdot n \left(1 + \frac{7 - 3\sqrt{5}}{10n} + \frac{5 - 2\sqrt{5}}{25n^2} + o\left(\frac{1}{n^2}\right) \right) \\ &\quad - \frac{3 - \sqrt{5}}{2} \cdot n^2 \left(1 + \frac{7 - 3\sqrt{5}}{5n} + \frac{67 - 29\sqrt{5}}{50n^2} + o\left(\frac{1}{n^2}\right) \right)\end{aligned}$$

$$\begin{aligned}
&= \frac{5 - 2\sqrt{5}}{5}n + C + o(1) \\
&\sim \frac{5 - 2\sqrt{5}}{5}n.
\end{aligned}$$

Therefore, according to Theorem 2.3, we can conclude that $f(n, k)$ is asymptotically normal. \square

Next, we provide another example to illustrate the application of Theorem 2.3. To begin, we present a lemma proposed by Driver *et al.* [10, Theorem 9] that establishes the reality of the roots of the hypergeometric function ${}_3F_2$.

Lemma 3.3. *Let $a, b > 0$ and let $l, m, n \in \mathbb{N}$. Then the polynomial*

$${}_3F_2 \left(\begin{matrix} -n, -m, -l \\ a, b \end{matrix}; x \right)$$

has only positive real zeros.

Let

$$g_n(x) = \sum_{k=0}^n g(n, k)x^k = \sum_{k=0}^n \binom{n}{k}^3 x^k$$

be the Franel polynomial. And let the mean be $\mu_n = \frac{g'_n(1)}{g_n(1)}$ and the variance be $\sigma_n^2 = \frac{g''_n(1)}{g_n(1)} + \mu_n - \mu_n^2$.

Theorem 3.4. *The coefficients $g(n, k)$ of the Franel polynomial $g_n(x)$ are asymptotically normal by local and central limit theorems with $\mu_n \sim \frac{n}{2}$, $\sigma_n^2 \sim \frac{n}{12}$.*

Proof. Since the Franel polynomial is defined as

$$g_n(x) = \sum_{k=0}^n \binom{n}{k}^3 x^k = {}_3F_2 \left(\begin{matrix} -n, -n, -n \\ 1, 1 \end{matrix}; -x \right),$$

it follows from Lemma 3.3 that $g_n(x)$ has only negative real roots.

Utilizing the Maple package APCI, we obtain the following recurrence

relations,

$$\begin{aligned}
g_{n+2}(1) &= \frac{7n^2 + 21n + 16}{(n+2)^2} g_{n+1}(1) + \frac{8(n+1)^2}{(n+2)^2} g_n(1), \\
g'_{n+3}(1) &= \frac{2(3n+8)(3n^2+12n+11)}{(3n+4)(n+3)(n+2)} g'_{n+2}(1) \\
&\quad + \frac{45n^3 + 243n^2 + 432n + 256}{(3n+4)(n+3)(n+2)} g'_{n+1}(1) \\
&\quad + \frac{8(n+1)^2(3n+7)}{(3n+4)(n+3)(n+2)} g'_n(1), \\
g''_{n+3}(1) &= \frac{3(n+2)(18n^4 + 100n^3 + 197n^2 + 173n + 60)}{(n+3)(9n^2 + 11n + 4)(n+1)^2} g''_{n+2}(1) \\
&\quad + \frac{3(n+2)^2(5n+7)(9n+8)}{(n+3)(n+1)(9n^2 + 11n + 4)} g''_{n+1}(1) \\
&\quad + \frac{8(n+2)(9n^2 + 29n + 24)}{(n+3)(9n^2 + 11n + 4)} g''_n(1).
\end{aligned}$$

Next, using the Mathematica package `P-rec.m`, we can derive the corresponding asymptotic expansions,

$$\begin{aligned}
g_n(1) &= C_1 \cdot \frac{8^n}{n} \left(1 - \frac{1}{3n} + \frac{1}{27n^2} + o\left(\frac{1}{n^2}\right) \right), \\
g'_n(1) &= C_2 \cdot 8^n \left(1 - \frac{1}{3n} + \frac{1}{27n^2} + o\left(\frac{1}{n^2}\right) \right), \\
g''_n(1) &= C_3 \cdot 8^n n \left(1 - \frac{2}{n} + \frac{22}{27n^2} + o\left(\frac{1}{n^2}\right) \right),
\end{aligned}$$

where C_1, C_2, C_3 are all constants. Consequently, we have that

$$\lim_{n \rightarrow +\infty} \frac{g_{n+1}(1)}{g_n(1)} = \frac{g'_{n+1}(1)}{g'_n(1)} = \frac{g''_{n+1}(1)}{g''_n(1)} = 8. \quad (3.6)$$

By utilizing the Maple package `APCI` once more, we obtain the following recurrence relations,

$$g_{n+1}(1) = \frac{11n^2 + 7n + 2}{(n+1)^2} g_n(1) - \frac{12(n-1)}{(n+1)^2} g'_n(1) + \frac{12}{(n+1)^2} g''_n(1), \quad (3.7)$$

$$g_n(1) = \frac{n+1}{6n^2 + 4n + 1} g'_{n+1}(1) + \frac{7n-5}{6n^2 + 4n + 1} g'_n(1) - \frac{6}{6n^2 + 4n + 1} g''_n(1). \quad (3.8)$$

According to Theorem 2.3, along with Equations (3.6) and (3.7), we get that

$$\begin{aligned}\frac{C_2}{C_1} &= \lim_{n \rightarrow +\infty} \frac{g'_n(1)}{g_n(1) \cdot n} \\ &= \lim_{n \rightarrow +\infty} \frac{\frac{g_{n+1}(1)}{g_n(1)} + (1 + \frac{g_{n+1}(1)}{g_n(1)})n}{2(1 + \frac{g'_{n+1}(1)}{g_n(1)})n} \\ &= \frac{1}{2}.\end{aligned}$$

Similarly, based on Theorem 2.3, as well as Equations (3.6) and (3.8), we have

$$\frac{C_3}{C_1} = \frac{1}{4}.$$

Thus, we know that the mean is given by

$$\mu_n = \frac{g'_n(1)}{g_n(1)} = \frac{n}{2} \quad (3.9)$$

and

$$\frac{g''_n(1)}{g_n(1)} = \frac{n^2}{4} \left(1 - \frac{5}{3n} + \frac{2}{9n^2} + o\left(\frac{1}{n^2}\right) \right). \quad (3.10)$$

Combining Equations (3.9) and (3.10), we find that the variance is

$$\begin{aligned}\sigma_n^2 &= \frac{g''_n(1)}{g_n(1)} + \mu_n - \mu_n^2 \\ &= \frac{n^2}{4} \left(1 - \frac{5}{3n} + \frac{2}{9n^2} + o\left(\frac{1}{n^2}\right) \right) + \frac{n}{2} - \frac{n^2}{4} \\ &\sim \frac{n}{12}.\end{aligned}$$

Therefore, $g(n, k)$ is asymptotically normal. \square

Finally, we list the asymptotic values of means and variances for some combinatorial sequences.

Table 1: The means and variances

	Means	Variances	References
The Narayana numbers	$\frac{n}{2}$	$\frac{n}{8}$	[4]
The unaerated Motzkin triangle*	$\frac{n}{3}$	$\frac{n}{18}$	[5]
The reversed unaerated Motzkin triangle	$\frac{n}{3}$	$\frac{n}{18}$	[5]
The even Motzkin triangle*	$\frac{n}{3}$	$\frac{n}{9}$	[5]
The reversed even Motzkin triangle*	$\frac{2n}{3}$	$\frac{n}{9}$	[5]
The odd Motzkin triangle*	$\frac{n}{3}$	$\frac{n}{9}$	[5]
The reversed odd Motzkin triangle*	$\frac{2n}{3}$	$\frac{n}{9}$	[5]
The Schröder triangle*	$\frac{(2-\sqrt{2})n}{2}$	$\frac{\sqrt{2}n}{8}$	[5]
The reversed Schröder triangle*	$\frac{\sqrt{2}n}{2}$	$\frac{\sqrt{2}n}{8}$	[5]
The generalized Narayana numbers	$\frac{n}{2}$	$\frac{n}{8}$	[6]
The central trinomial coefficient	$\frac{n}{3}$	$\frac{n}{18}$	[17]
The Delannoy numbers	$\frac{n}{2}$	$\frac{\sqrt{2}n}{8}$	[22]

Remark 2. The asterisk indicates that the specific numerical values were not given in the previous article. In this article, we have calculated the means and variances of the sequences.

Remark 3. Based on [5], we call an infinite lower triangular matrix $[a_{n,k}]_{n,k \geq 0}$ asymptotically normal if the numbers $a_{n,k}$ are asymptotically normal.

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