Indeed, by using (12),

$$t_{nm} = \sum_{k=0}^{m} \frac{2k+1}{m+k+1} \binom{n-1-k}{m-k} \binom{n+k}{m+k} t_{kk}$$
$$= \sum \frac{(2k+1)t_{kk}}{m+k+1} \sum_j \binom{n+j+k}{2m} \binom{m+k+1}{m-j-k} \binom{m-k-1}{j}$$
$$= \sum_j \binom{n+j}{2m} \sum_k \frac{2k+1}{m+k+1} \binom{m+k+1}{m-j} \binom{m-1-k}{j-k} t_{kk}$$
$$= \sum_m \binom{n+j}{2m} \sum_k \frac{2k+1}{k+j+1} \binom{m-1-k}{j-k} \binom{m+k}{j+k} t_{kk}$$
$$= \sum_{j=0}^{m} \binom{n+j}{2m} t_{mj}.$$

Note that $T_0(n, m) = r_{nm}$ and $r_{nn} = \delta_{n0}$, so that r_{nm} is the solution when $t_{kk} = \delta_{k0}$. Also, since $q_{nn} = 1$,

$$\binom{n}{m}^2 = \sum_{k=0}^{m} \frac{2k+1}{m+k+1} \binom{n-1-k}{m-k} \binom{n+k}{m+k}.$$

Further results appear in the problems. ▶

1.5 ABEL'S GENERALIZATION OF THE BINOMIAL FORMULA

Abel's celebrated generalization of the binomial formula (given in his *Oeuvres Complètes*, Christiania, C. Groendahl, 1839) in one form, preferred by Hurwitz (1902), reads

(13) $$x^{-1}(x + y + na)^n = \sum_{k=0}^{n} \binom{n}{k} (x + ka)^{k-1} (y + (n-k)a)^{n-k}.$$

If x is replaced by ax and y by ay, this is the same as

(13a) $$x^{-1}(x + y + n)^n = \sum_{k=0}^{n} \binom{n}{k} (x + k)^{k-1} (y + n - k)^{n-k},$$

so the parameter a is disposable.

Abel's formula (13a) is the instance $p = -1$, $q = 0$, of a class of sums defined by

(14) $$A_n(x, y; p, q) = \sum_{k=0}^{n} \binom{n}{k} (x + k)^{k+p} (y + n - k)^{n-k+q},$$

whose study, a little surprisingly, simplifies the proof of (13a). Note first that replacing k with $n - k$ on the right of (14) yields the relation

(15) $$A_n(x, y; p, q) = A_n(y, x; q, p).$$

Next, by the basic recurrence (i),

(16) $$A_n(x, y; p, q) = A_{n-1}(x, y + 1; p, q + 1) + A_{n-1}(x + 1, y; p + 1, q)$$

Also

(17) $$A_n(x, y; p, q) = \sum \binom{n}{k} (x + k)(x + k)^{k-1+p} (y + n - k)^{n-k+q}$$
$$= xA_n(x, y; p - 1, q)$$
$$+ n \sum \binom{n-1}{k-1} (x + k)^{k-1+p} (y + n - k)^{n-k+q}$$
$$= xA_n(x, y; p - 1, q) + nA_{n-1}(x + 1, y; p, q).$$

Similarly,

$$A_n(x, y; p, q) = \sum \binom{n}{k} (y + n - k)(x + k)^{k+p} (y + n - k)^{n-k+q-1}$$

or

(17a) $$A_n(x, y; p, q) = yA_n(x, y; p, q - 1) + nA_{n-1}(x, y + 1; p, q).$$

This is an alternate to (17) because it may be obtained by interchanging both x, y and p, q and using (15). Use of (16) in (17) leads to the two relations

(18)
$$A_n(x, y; p, q) = xA_{n-1}(x, y + 1; p - 1, q + 1) + (x + n)A_{n-1}(x + 1, y; p, q)$$
$$= (x + n)A_n(x, y; p - 1, q) - nA_{n-1}(x, y + 1; p - 1, q + 1).$$

The first line of (18) with $p = 0$, $q = -1$, leads to an immediate proof of Abel's formula (13a): first

$$A_n(x, y; 0, -1) = xA_{n-1}(x, y + 1; -1, 0) + (x + n)A_{n-1}(x + 1, y; 0, -1),$$

which by (15) is the same as

$$A_n(y, x; -1, 0) = xA_{n-1}(x, y + 1; -1, 0) + (x + n)A_{n-1}(y, x + 1; -1, 0),$$

or, omitting the constant parameters -1 and 0 and interchanging x and y,

(19) $$A_n(x, y) = yA_{n-1}(y, x + 1) + (y + n)A_{n-1}(x, y + 1).$$

By (14), $A_0(x, y) \equiv A_0(x, y; -1, 0) = x^{-1}$, $A_1(x, y) = x^{-1}(x + y + 1)$, and, if $A_k(x, y) = x^{-1}(x + y + k)^k$, $k = 0(1)\overline{n-1}$, it follows from (19) that

$$A_n(x, y) = yy^{-1}(x + y + n)^{n-1} + (y + n)x^{-1}(x + y + n)^{n-1}$$
$$= x^{-1}(x + y + n)^n,$$

which is the left side of (13a), as required.

From this result it follows at once from (16) and (15) that

$$(20)\quad \begin{aligned} A_n(x, y; -1, -1) &= A_{n-1}(x, y+1; -1, 0) + A_{n-1}(x+1, y; 0, -1) \\ &= A_{n-1}(x, y+1; -1, 0) + A_{n-1}(y, x+1; -1, 0) \\ &= (x^{-1} + y^{-1})(x+y+n)^{n-1}, \end{aligned}$$

which is a well-known companion to (13). On the other hand, by (17)

$$\begin{aligned} xA_n(x, y; -2, 0) &= A_n(x, y; -1, 0) - nA_{n-1}(x+1, y; -1, 0) \\ &= x^{-1}(x+y+n)^n - n(x+1)^{-1}(x+y+n)^{n-1} \end{aligned}$$

or

(21)

$$A_n(x, y; -2, 0) = x^{-2}(x+1)^{-1}[(x+1)(x+y+n)^n - nx(x+y+n)^{n-1}],$$

which is less well known.

Iteration of a form of (17), namely

$$xA_n(x, y; p-1, q) = A_n(x, y; p, q) - nA_{n-1}(x+1, y; p, q);$$

gives in the first place

$$\begin{aligned} x^2(x+1)A_n(x, y; p-2, q) = (x+1)&A_n(x, y; p, q) \\ &- n(2x+1)A_{n-1}(x+1, y; p, q) \\ &+ n(n-1)xA_{n-2}(x+2, y; p, q). \end{aligned}$$

Hence

$$\begin{aligned} x^2(x+1)A_n(x, y; -3, 0) = x^{-1}&(x+1)(x+y+n)^n \\ &- n(x+1)^{-1}(2x+1)(x+y+n)^{n-1} \\ &+ n(n-1)(x+2)^{-1}x(x+y+n)^{n-2} \end{aligned}$$

or

$$(22)\quad \begin{aligned} A_n(x, y; -3, 0) = x^{-3}(x+1)^{-2}(x+2)^{-1}[&(x+1)^2(x+2)(x+y+n)^n \\ &- nx(x+2)(2x+1)(x+y+n)^{n-1} \\ &+ n(n-1)x^2(x+1)(x+y+n)^{n-2}]. \end{aligned}$$

Further development of this form of iteration, which is somewhat intricate, is found in Problem 18.

The iteration of (17) as written is immediate; the result is

$$(23)\quad A_n(x, y; p, q) = \sum_{k=0} \binom{n}{k} k!(x+k)A_{n-k}(x+k, y; p-1, q).$$

Hence in the first place

$$(24)\quad \begin{aligned} A_n(x, y; 0, 0) &= \sum_{k=0} \binom{n}{k} k!(x+k)(x+k)^{-1}(x+y+n)^{n-k} \\ &= (x+y+n+\alpha)^n, \qquad \alpha^k \equiv \alpha_k = k!; \end{aligned}$$

that is,

$$\sum_{k=0} \binom{n}{k}(x+k)^k(y+n-k)^{n-k} = \sum_{k=0} \binom{n}{k} k!(x+y+k)^{n-k}.$$

This is usually called Cauchy's formula. Note that by (17)

$$xA_n(x, y; -1, 0) = A_n(x, y; 0, 0) - nA_{n-1}(x+1, y; 0, 0);$$

that is,

$$(x+y+n)^n = (x+y+n+\alpha)^n - n(x+y+n+\alpha)^{n-1},$$

which is readily verified.

Next, by (23) and (20)

$$\begin{aligned} A_n(x, y; 0, -1) &= \sum \binom{n}{k} k!(x+k)A_{n-k}(x+k, y; -1, -1) \\ &= \sum \binom{n}{k} k!\, y^{-1}(x+y+k)(x+y+n)^{n-k-1}, \end{aligned}$$

but $A_n(x, y; 0, -1) = A_n(y, x; -1, 0) = y^{-1}(x+y+n)^n$, so that

$$(25)\quad (x+y+n)^{n+1} = \sum \binom{n}{k} k!(x+y+k)(x+y+n)^{n-k}.$$

Again, using (23)

(26)

$$\begin{aligned} A_n(x, y; 1, 0) &= \sum \binom{n}{k} k!(x+k)A_{n-k}(x+k, y; 0, 0) \\ &= \sum \binom{n}{k} k!(x+k)(x+y+n+\alpha)^{n-k}, \qquad \alpha^k \equiv \alpha_k = k! \\ &= [x+y+n+\alpha+\beta(x)]^n, \qquad \beta^k(x) \equiv \beta_k(x) = k!(x+k). \end{aligned}$$

Noting that

$$\exp t\alpha = \sum \alpha_n \frac{t^n}{n!} = (1-t)^{-1}$$

$$\exp t\beta(x) = \sum \beta_n(x)\frac{t^n}{n!} = (1-t)^{-2}[x + t(1-x)]$$

it follows that

$$\exp t[\alpha + \beta(x)] = (1-t)^{-3}[x + t(1-x)] = \sum \left[\binom{n+1}{2} + x(n+1)\right] t^n.$$

Hence (26) is the same as

$$(26a) \qquad \sum \binom{n}{k}(x+k)^{k+1}(y+n-k)^{n-k}$$
$$= \sum \binom{n}{k} k! \left[\binom{k+1}{2} + x(k+1)\right](x+y+n)^{n-k}.$$

Continuing the use of (23),

$$(27) \qquad A_n(x, y; 1, -1) = \sum \binom{n}{k}\beta_k(x)A_{n-k}(x+k, y; 0, -1)$$
$$= y^{-1}\sum \binom{n}{k}\beta_k(x)(x+y+n)^{n-k}$$
$$= y^{-1}[x+y+n+\beta(x)]^n$$

and

$$(28) \qquad A_n(x, y; 1, 1) = \sum \binom{m}{k}\beta_k(x)A_{n-k}(x+k, y; 0, 1)$$
$$= \sum \binom{n}{k}\beta_k(x)[x+y+n+\alpha+\beta(y)]^{n-k}$$
$$= [x+y+n+\alpha+\beta(x)+\beta(y)]^n.$$

Note that

$$\exp t[\alpha + \beta(x) + \beta(y)] = (1-t)^{-5}[t + x(1-t)][t + y(1-t)]$$

so that

$$[\alpha + \beta(x) + \beta(y)]^k = k!\left[\binom{k+2}{4} + (x+y)\binom{k+2}{3} + xy\binom{k+2}{2}\right].$$

The sequel to (26) is worth examining because of a new complication. This is derived as follows; first

$$A_n(x, y; 2, 0) = \sum \binom{n}{k}\beta_k(x)A_{n-k}(x+k, y; 1, 0).$$

Then by (26)

$$A_{n-k}(x+k, y; 1, 0) = [x+y+n+\alpha+\beta(x+k)]^{n-k}$$
$$= \sum \binom{n-k}{j}(x+y+n+\alpha)^{n-k-j}j!(x+j+k)$$
$$= [x+y+n+\alpha+\beta(x)]^{n-k} + k(x+y+n+\alpha+\alpha)^{n-k}$$

and, finally, by writing $\alpha^k(2) \equiv \alpha_k(2) = (\alpha+\alpha)^k$, $\gamma_k(x) = k\beta_k(x)$, $\beta_k(x; 2) = [\beta(x)+\beta(x)]^k$,

$$(29) \qquad A_n(x, y; 2, 0) = [x+y+n+\alpha+\beta(x; 2)]^n + [x+y+n+\alpha(2)+\gamma(x)]^n.$$

All of these results, and a little more, are summarized in Table 1.2. The reader is reminded that additional results appear in the problems.

TABLE 1.2 ABEL IDENTITIES

$$A_n(x, y; p, q) = \sum_{k=0} \binom{n}{k}(x+k)^{k+p}(y+n-k)^{n-k+q}$$

p	q	$A_n(x, y; p, q)$
-3	0	$x^{-3}(x+1)^{-2}(x+2)^{-1}[(x+1)^2(x+2)(x+y+n)^n - nx(x+2)(x+1)(x+y+n)^{n-1} + n(n-1)x^2(x+1)(x+y+n)^{n-2}]$
-2	0	$x^{-2}(x+1)^{-1}[(x+1)(x+y+n)^n - nx(x+y+n)^{n-1}]$
-1	0	$x^{-1}(x+y+n)^n$
0	0	$(x+y+n+\alpha)^n$
1	0	$[x+y+n+\alpha+\beta(x)]^n$
2	0	$[x+y+n+\alpha+\beta(x; 2)]^n + [x+y+n+\alpha(2)+\gamma(x)]^n$
-1	-1	$(x^{-1}+y^{-1})(x+y+n)^{n-1}$
-1	1	$x^{-1}[x+y+n+\beta(y)]^n$
-1	2	$x^{-1}\{[x+y+n+\beta(y; 2)]^n + [x+y+n+\alpha+\gamma(y)]^n\}$
1	1	$[x+y+n+\alpha+\beta(x)+\beta(y)]^n$
1	2	$[x+y+n+\alpha+\beta(x)+\beta(y; 2)]^n + [x+y+n+\alpha(2)+\gamma(y)]^n$
2	2	$[x+y+n+\alpha+\beta(x; 2)]^n + [x+y+n+\alpha(2)+\gamma(x)+\beta(y; 2)]^n + [x+y+n+\alpha(2)+\beta(x; 2)+\gamma(y)]^n + [x+y+n+\alpha(3)+\gamma(x)+\gamma(y)]^n$

Notation. $\alpha^k \equiv \alpha_k = k!$

$$[\alpha(j)]^k \equiv \alpha_k(j) = (\alpha + \cdots + \alpha)^k \ (j \text{ terms}) = \binom{k+j-1}{k}k!$$
$$\beta^k(x) \equiv \beta_k(x) = k!\,(x+k)$$
$$[\beta(x;j)]^k \equiv \beta_k(x;j) = [\beta(x) + \cdots + \beta(x)]^k \qquad (j \text{ terms})$$
$$\gamma^k(x) \equiv \gamma_k(x) = k \cdot k!\,(x+k)$$

1.6 MULTINOMIAL ABEL IDENTITIES

Multinomial extensions of three of the binomial Abel identities appeared in Hurwitz (1902). They are probably the most significant of the wide range of possibilities, some of which are now examined. Write

$$(30)\qquad A_n(x_1, \ldots, x_n; p_1, \ldots, p_m) = \sum (n; k_1, \ldots, k_m) \prod_{j=1}^{m} (x_j + k_j)^{k_j + p_j}$$

for the multinomial extension of (14); $(n; k_1, \ldots, k_m)$ is the multinomial coefficient $n!/k_1! \cdots !k_m!$, with $k_1 + \cdots + k_m = n$. Then, first, by the basic recurrence for multinomial coefficients, namely

$$(31)\qquad \begin{aligned}(n; k_1, \ldots, k_m) = {} & (n-1; k_1 - 1, k_2, \ldots, k_m) \\ & + (n-1; k_1, k_2 - 1, k_3, \ldots, k_m) + \cdots \\ & + (n-1; k_1, \ldots, k_j - 1, k_{j+1}, \ldots, k_m) + \cdots \\ & + (n-1; k_1, \ldots, k_{m-1}, k_m - 1),\end{aligned}$$

the correspondent to (16) is found to be

$$\begin{aligned}A_n(x_1, \ldots, x_m; p_1, \ldots, p_m) = {} & A_{n-1}(x_1 + 1, x_2, \ldots, x_m; p_1 + 1, p_2, \ldots, p_m) \\ & + A_{n-1}(x_1, x_2 + 1, x_3, \ldots, x_m; \\ & \qquad p_1, p_2 + 1, p_3, \ldots, p_m) \\ & + \cdots \\ & + A_{n-1}(x_1, \ldots, x_m + 1; p_1, \ldots, p_m + 1).\end{aligned}$$

Next separation of a factor $x_1 + k_1$ in A_n, as in the derivation of (17), leads to

(32)

$$\begin{aligned}A_n(x_1, \ldots, x_m; p_1, \ldots, p_m) = {} & x_1 A_n(x_1, \ldots, x_m; p_1 - 1, p_2, \ldots, p_m) \\ & + nA_{n-1}(x_1 + 1, x_2, \ldots, x_m; p_1, \ldots, p_m).\end{aligned}$$

Hence the multinomial companion to (23) is

$$(33)\quad A_n(x_1, \ldots, x_m; p_1, \ldots, p_m) = \sum_{k=0} \binom{n}{k} k!(x_1 + k) \times A_{n-k}(x_1 + k, x_2, \ldots, x_m; p_1 - 1, p_2, \ldots, p_m).$$

Turn now to the first of Hurwitz's multinomial extensions, which is an extension of (13), that is, of

$$A_n(x, y; -1, 0) = \sum \binom{n}{k} (x + k)^{k-1} (y + n - k)^{n-k} = x^{-1}(x + y + n)^n.$$

A derivation by iteration, quite different from that of Hurwitz, is as follows.

First

$$\begin{aligned}y^{-1}A_n(x, y+z; -1, 0) &= (xy)^{-1}(x + y + z + n)^n \\ &= \sum \binom{n}{k} (y + k)^{k-1} y^{-1} (y + z + n - k)^{n-k} \\ &= \sum \binom{n}{k} (x + k)^{k-1} \sum \binom{n-k}{j} (y + j)^{j-1} \\ &\qquad \times (z + n - k - j)^{n-k-j} \\ &= \sum (n; k, j, n-k-j)(x + k)^{k-1} (y + j)^{j-1} \\ &\qquad \times (z + n - k - j)^{n-k-j}\end{aligned}$$

or

$$A_n(x, y, z; -1, -1, 0) = (xy)^{-1}(x + y + z + n)^n.$$

Repetitions of the procedure lead to the Hurwitz identity:

$$(34)\qquad A_n(x_1, \ldots, x_m; -1, -1, \ldots, -1, 0) = (x_1 x_2 \cdots x_m)^{-1} x_m (x + n)^n,$$

with $x = x_1 + x_2 + \cdots + x_m$.

The second of Hurwitz's extensions is that of

$$A_n(x, y; 0, 0) = (x + y + n + \alpha)^n, \qquad \alpha^k \equiv \alpha_k = k!$$

Then, as before,

$$\begin{aligned}A_n(x, y + z + \alpha; 0, 0) &= [x + y + z + n + \alpha(2)]^n \\ &= \sum \binom{n}{k} (x + k)^k (y + z + \alpha + n - k)^{n-k} \\ &= \sum \binom{n}{k} (x + k)^k \sum \binom{n-k}{j} (y + j)(z + n - k - j)^{n-k-j} \\ &= A_n(x, y, z; 0, 0, 0).\end{aligned}$$

The general result is clearly (again $x = x_1 + x_2 + \cdots + x_m$)

$$(35)\qquad \begin{aligned}A_n(x_1, \ldots, x_m; 0, \ldots, 0) &= [x + n + \alpha(m-1)]^n \\ &= \sum_{k=0} \binom{n}{k} (x + n)^{n-k} \alpha_k(m-1),\end{aligned}$$

where, of course, as in (29) and Table 1.2

$$\exp t\alpha(m) = (\exp t\alpha)^m = (1 - t)^{-m} = \sum_{k=0} \binom{m + k - 1}{k} t^k,$$

so that $\alpha_k(m) = k!(m + k - 1)!/k!(m - 1)! = (m + k - 1)!/(m - 1)!$.

The third of Hurwitz's extensions is that of

$$A_n(x, y; -1, -1) = (x^{-1} + y^{-1})(x + y + n)^{n-1}.$$

This follows from (34) and the recurrence (31) and reads

$$A_n(x_1, \ldots, x_m; -1, -1, \ldots, -1) = (x_1 x_2 \cdots x_m)^{-1} x(x + n)^{n-1}. \tag{36}$$

Now return to the first Abel binomial identity $A_n(x, y; -1, 0)$. Then

$$\begin{aligned} A_n(x, y + z + \alpha; -1, 0) &= x^{-1}(x + y + z + \alpha + n)^n \\ &= \sum \binom{n}{k} (x + k)^{k-1} (y + z + \alpha + n - k)^{n-k} \\ &= \sum \binom{n}{k} (x + k)^{k-1} \sum \binom{n-k}{j} (y + j)^j \\ &\quad \times (z + n - k - j)^{n-k-j} \\ &= A_n(x, y, z; -1, 0, 0). \end{aligned}$$

Repetitions lead to (again $x = x_1 + \cdots + x_m$)

$$A_n(x_1, \ldots, x_m; -1, 0, \ldots, 0) = x_1^{-1}[x + \alpha(m - 2) + n]^n. \tag{37}$$

Note that (37) and (33) imply (35). Also, by (33) and (35)

(38)

$$\begin{aligned} A_n(x_1, \ldots, x_m; 1, 0, \ldots, 0) &= \sum \binom{n}{k} k!(x_1 + k) A_{n-k}(x_1 + k, x_2, \ldots, x_m; \\ &\qquad\qquad 0, 0, \ldots, 0) \\ &= \sum \binom{n}{k} \beta_k(x_1)[x + n - \alpha(m - 1)]^{n-k} \\ &= [x + n + \alpha(m - 1) + \beta(x_1)]^n. \end{aligned}$$

Similarly

(39)

$$\begin{aligned} A_n(x_1, \ldots, x_m; 1, 1, 0, \ldots, 0) &= [x + n + \alpha(m - 1) + \beta(x_1) + \beta(x_2)]^n, \\ A_n(x_1, \ldots, x_m; 1, 1, 1, 0, \ldots, 0) &= [x + n + \alpha(m - 1) + \beta(x_1) \\ &\qquad + \beta(x_2) + \beta(x_3)]^n, \\ &\cdots\cdots\cdots\cdots\cdots\cdots \\ A_n(x_1, \ldots, x_m; 1, 1, \ldots, 1) &= [x + n + \alpha(m - 1) + \beta(x_1) + \cdots \\ &\qquad + \beta(x_m)]^n. \end{aligned}$$

Note that

$$\begin{aligned} \exp t[\beta(x_1) + \cdots + \beta(x_m)] &= \prod_{j=1}^{m} \exp t\beta(x_j) \\ &= (1 - t)^{-2m} \prod_{j=1}^{m} [t + (1 - t)x_j], \end{aligned}$$

$$\exp t[\alpha(m - 1) + \beta(x_1) + \cdots + \beta(x_j)] = (1 - t)^{-(2j+m-1)} \prod_{i=1}^{j} [t + (1 - t)x_i],$$

which, of course, imply expressions for the coefficients of the exponential generating functions I have not taken space to write out.

As a final example of the use of (33)

(40)

$$\begin{aligned} A_n(x_1, \ldots, x_m; 1, -1, \ldots, -1) &= \sum \binom{n}{k} \beta_k(x_1) A_{n-k}(x_1 + k, x_2, \ldots, x_m; \\ &\qquad\qquad 0, -1, -1, \ldots) \\ &= \sum \binom{n}{k} \beta_k(x_1)(x_2 \cdots x_m)^{-1}(x + n)^{n-k} \\ &= (x_2 \cdots x_m)^{-1}[x + n + \beta(x_1)]^n. \end{aligned}$$

PROBLEMS

1. Generalizing Example 3, write

$$f_k(x) = \sum_{j=1}^{k} \frac{x^j}{j}.$$

Show that

$$\begin{aligned} S_n(x) &= \sum_{k=1}^{n} (-1)^{k+1} \binom{n}{k} f_k(x) \\ &= \frac{1}{n}[1 - (1 - x)^n] \end{aligned}$$

$$f_n(x) = \sum_{k=1}^{n} (-1)^{k+1} \binom{n}{k} S_k(x);$$

that is,

$$\sum_{k=1}^{n} \frac{x^k}{k} = \sum_{k=1}^{n} (-1)^{k+1} \binom{n}{k} \frac{1}{k}[1 - (1 - x)^k].$$