and

$$(6.7.2)\quad p(5)+p(12)x+p(19)x^2+\ldots = 7\frac{\{(1-x^7)(1-x^{14})(1-x^{21})\ldots\}^3}{\{(1-x)(1-x^2)(1-x^3)\ldots\}^4} + 49x\frac{\{(1-x^7)(1-x^{14})(1-x^{21})\ldots\}^7}{\{(1-x)(1-x^2)(1-x^3)\ldots\}^8}.$$

These make (6.4.1) and (6.4.2) intuitive, and also provide proofs of the congruences to moduli 5^2 and 7^2. Thus, if we assume (6.7.1), we have

$$\frac{p(4)x+p(9)x^2+\ldots}{5\{(1-x^5)(1-x^{10})\ldots\}^4} = \frac{x}{(1-x)(1-x^2)\ldots}\,\frac{(1-x^5)(1-x^{10})\ldots}{\{(1-x)(1-x^2)\ldots\}^5}$$

$$\equiv \frac{x}{(1-x)(1-x^2)\ldots} \pmod 5.$$

Hence (after what we have proved already) the coefficient of x^{5m+5} on the left-hand side is a multiple of 5; and from this it follows that

$$p(25m+24) \equiv 0 \pmod{5^2}.$$

Similarly (6.7.2) leads to

$$p(49m+47) \equiv 0 \pmod{7^2}.$$

Ramanujan never published a complete proof of (6.7.1) or (6.7.2); but proofs have been found by Darling and Mordell.

The Rogers-Ramanujan identities

6.8. I come next to two formulae, the "Rogers-Ramanujan identities", in which Ramanujan had been anticipated by a much less famous mathematician, but which are certainly as remarkable as any which even he ever wrote down.

The Rogers-Ramanujan identities are

$$(6.8.1)\quad 1+\frac{x}{1-x}+\frac{x^4}{(1-x)(1-x^2)}+\ldots+\frac{x^{m^2}}{(1-x)(1-x^2)\ldots(1-x^m)}+\ldots = \frac{1}{(1-x)(1-x^6)\ldots(1-x^4)(1-x^9)\ldots}$$

and

$$(6.8.2)\quad 1+\frac{x^2}{1-x}+\frac{x^6}{(1-x)(1-x^2)}+\ldots+\frac{x^{m(m+1)}}{(1-x)(1-x^2)\ldots(1-x^m)}+\ldots = \frac{1}{(1-x^2)(1-x^7)\ldots(1-x^3)(1-x^8)\ldots}.$$

The exponents in the denominators on the right form in each case two arithmetical progressions with the difference 5. This is the surprise of the formulae; the "basic series" on the left are of a comparatively familiar type.

The formulae have a very curious history. They were found first in 1894 by Rogers, a mathematician of great talent but comparatively little reputation, now remembered mainly from Ramanujan's rediscovery of his work. Rogers was a fine analyst, whose gifts were, on a smaller scale, not unlike Ramanujan's; but no one paid much attention to anything he did, and the particular paper in which he proved the formulae was quite neglected.

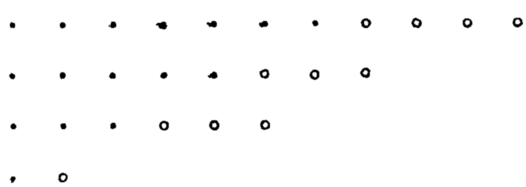
Ramanujan rediscovered the formulae sometime before 1913. He had then no proof (and knew that he had none), and none of the mathematicians to whom I communicated the formulae could find one. They are therefore stated without proof in the second volume of MacMahon's *Combinatory analysis*.

The mystery was solved, trebly, in 1917. In that year Ramanujan, looking through old volumes of the *Proceedings of the London Mathematical Society,* came accidentally across Rogers's paper. I can remember very well his surprise, and the admiration which he expressed for Rogers's work. A correspondence followed in the course of which Rogers was led to a considerable simplification of his original proof. About the same time I. Schur, who was then cut off from England by the war, rediscovered the identities again. Schur published two proofs, one of which is "combinatorial" and quite unlike any other proof known. There are now seven published proofs, the four referred to already, the two much simpler proofs found later by Rogers and Ramanujan and published in the *Papers,* and a much later proof by Watson based on quite different ideas. None of these proofs can be called both "simple" and "straightforward", since the simplest are essentially verifications; and no doubt it would be unreasonable to expect a really easy proof.

6.9. MacMahon and Schur showed that the theorems have a simple combinatorial interpretation. I take the first. We can exhibit a square m^2 as

$$1+3+5+\ldots+(2m-1),$$

or in the manner shown by the black dots of (G_7). If we now take any partition of $n-m^2$ into m parts at most, with the parts in descending order,



(G_7)

and add it to the graph, as shown by the circles of (G_7), where $m = 4$ and $n = 4^2+11 = 27$, we obtain a partition of n (here

$$27 = 11+8+6+2)$$

into parts without repetitions or sequences,[1] or *parts whose minimal difference is* 2. The partitions of n of this type, associated with a particular m, are enumerated by

$$\frac{x^{m^2}}{(1-x)(1-x^2)\dots(1-x^m)},$$

the general term of the series on the left in (6.8.1); and the whole series enumerates all such partitions of n.

On the other hand the right-hand side enumerates partitions into numbers $5m+1$ and $5m+4$. Hence (6.8.1) may be restated as a "combinatorial" theorem: *the number of partitions of* n *with minimal difference* 2 *is equal to the number of partitions into parts* $5m+1$ *and* $5m+4$. Thus when $n = 9$ there are 5 partitions of each type;

$$9,\ 8+1,\ 7+2,\ 6+3,\ 5+3+1$$

of the first kind, and

$$9,\ 6+1+1+1,\ 4+4+1,\ 4+1+1+1+1+1,$$
$$1+1+1+1+1+1+1+1+1$$

of the second. There is a similar combinatorial interpretation of (6.8.2).

These forms of the theorems are MacMahon's (or Schur's); neither Rogers nor Ramanujan ever considered their combinatorial aspect. It is natural to ask for a proof in which we set up, by "combinatorial" arguments, a direct correspondence between the two sets of partitions, but no such proof is known. Schur's "combinatorial" proof is based, not on (6.8.1) itself, but on a transformation of the formula which I will mention in a moment.[2] It is not unlike Franklin's proof of (6.2.1), but a good deal more complicated.

6.10. The proofs given ultimately by Rogers and Ramanujan are much the same, but Rogers's form is a little easier to follow.

We can write the right-hand side of (6.8.1) as

$$\frac{1}{\Pi\{(1-x^{5m+1})(1-x^{5m+4})\}} = \frac{\Pi\{(1-x^{5m})(1-x^{5m+2})(1-x^{5m+3})\}}{(1-x)(1-x^2)(1-x^3)\dots};$$

and the numerator on the right can be transformed, by a standard formula from the theory of the theta-functions, into

$$1-x^2-x^3+x^9+x^{11}-\dots,$$

where the indices are the numbers

$$\tfrac{1}{2}(5n^2 \pm n) \qquad (n = 0, 1, 2, \dots).$$

[1] Parts differing by 1.

[2] (6.10.1), with each side multiplied by $(1-x)(1-x^2)\dots$.

We have therefore to prove that

$$(6.10.1)\quad 1+\frac{x}{1-x}+\frac{x^4}{(1-x)(1-x^2)}+\ldots=\frac{1-x^2-x^3+x^9+x^{11}-\ldots}{(1-x)(1-x^2)(1-x^3)\ldots}.$$

Similarly (6.8.2) is equivalent to

$$(6.10.2)\quad 1+\frac{x^2}{1-x}+\frac{x^6}{(1-x)(1-x^2)}+\ldots=\frac{1-x-x^4+x^7+x^{13}-\ldots}{(1-x)(1-x^2)(1-x^3)\ldots},$$

the indices in the numerator on the right being the numbers $\frac{1}{2}(5n^2 \pm 3n)$.

6.11. We use the auxiliary function

$$(6.11.1)\quad G_k = G_k(a,x) = \sum_{n=0}^{\infty} (-1)^n a^{2n} x^{\frac{1}{2}n(5n+1)-kn}(1-a^k x^{2kn})\, C_n,$$

where k is 0, 1, or 2 and

$$C_0 = 1, \quad C_n = \frac{(1-a)(1-ax)\ldots(1-ax^{n-1})}{(1-x)(1-x^2)\ldots(1-x^n)}.$$

Thus

$$(6.11.2)\quad G_k = (1-a^k)\,C_0 - a^2x^{3-k}(1-a^kx^{2k})\,C_1 + a^4x^{11-2k}(1-a^kx^{4k})\,C_2 - \ldots.$$

If $a \neq 0$ then $G_0 = 0$ for all x. Also

$$(6.11.3)\quad G_1(x,x) = 1-x-x^4+x^7+x^{13}-\ldots$$

and

$$(6.11.4)\quad G_2(x,x) = 1-x^2-x^3+x^9+x^{11}-\ldots$$

are the series which occur in (6.10.2) and (6.10.1).

If the operator η is defined by

$$\eta f(a,x) = f(ax,x),$$

then

$$(6.11.5)\quad \eta C_n = \frac{(1-ax)\ldots(1-ax^n)}{(1-x)\ldots(1-x^n)} = \frac{1-ax^n}{1-a}C_n$$

and

$$(6.11.6)\quad \eta C_{n-1} = \frac{(1-ax)\ldots(1-ax^{n-1})}{(1-x)\ldots(1-x^{n-1})} = \frac{1-x^n}{1-a}C_n.$$

Hence

$$(6.11.7)\quad (1-x^n)\,C_n = (1-a)\,\eta C_{n-1}, \quad (1-ax^n)\,C_n = (1-a)\,\eta C_n,$$

and in particular $\quad \eta C_0 = C_0 = 1.$

If k is 1 or 2, then

$$\begin{aligned} G_k - G_{k-1} &= \sum_{n=0}^{\infty} (-1)^n a^{2n} x^{\frac{1}{2}n(5n+1)-kn}\{1-a^kx^{2kn}-x^k(1-a^{k-1}x^{2(k-1)n})\}\,C_n \\ &= a^{k-1}(1-a) + \sum_{n=1}^{\infty} (-1)^n a^{2n} x^{\frac{1}{2}n(5n+1)-kn}\{(1-x^k)+a^{k-1}x^{(2k-1)n}(1-ax^n)\}\,C_n \\ &= a^{k-1}(1-a)\eta C_0 + (1-a)\sum_{n=1}^{\infty} (-1)^n a^{2n} x^{\frac{1}{2}n(5n+1)-kn}\{\eta C_{n-1} + a^{k-1}x^{(2k-1)n}\eta C_n\}, \end{aligned}$$

by (6.11.7). When we rearrange this series in terms of ηC_0, ηC_1, ..., the coefficient of ηC_n is

$$(-1)^n(1-a)\{a^{2n+k-1}x^{\frac{1}{2}n(5n+1)+(k-1)n} - a^{2n+2}x^{\frac{1}{2}(n+1)(5n+6)-k(n+1)}\}$$
$$= (-1)^n(1-a)a^{2n+k-1}x^{\frac{1}{2}n(5n+1)+(k-1)n}\{1-a^{3-k}x^{(3-k)(2n+1)}\}.$$

Hence

$$G_k - G_{k-1} = (1-a)a^{k-1}\sum_{n=0}^{\infty}(-1)^n a^{2n}x^{\frac{1}{2}n(5n+1)+(k-1)n}\{1-a^{3-k}x^{(3-k)(2n+1)}\}\eta C_n.$$

But $$G_{3-k} = \sum_{n=0}^{\infty}(-1)^n a^{2n}x^{\frac{1}{2}n(5n+1)-(3-k)n}\{1-a^{3-k}x^{(3-k)2n}\}C_n,$$

and so $$\eta G_{3-k} = \sum_{n=0}^{\infty}(-1)^n a^{2n}x^{\frac{1}{2}n(5n+1)+(k-1)n}\{1-a^{3-k}x^{(3-k)(2n+1)}\}C_n;$$

and therefore

(6.11.8) $$G_k - G_{k-1} = (1-a)a^{k-1}\eta G_{3-k} \qquad (k=1,2).$$

6.12. If now

$$H_k = H_k(a,x) = \frac{G_k(a,x)}{(1-a)(1-ax)(1-ax^2)\dots}$$

(so that $H_0 = 0$), then (6.11.8) becomes

$$H_k - H_{k-1} = a^{k-1}\eta H_{3-k}.$$

In particular

(6.12.1) $$H_1 = \eta H_2, \quad H_2 - H_1 = a\eta H_1$$

and so

(6.12.2) $$H_2 = \eta H_2 + a\eta^2 H_2.$$

Suppose now that $H_2 = 1 + c_1 a + c_2 a^2 + \dots$,

where the coefficients depend on x only. Substituting into (6.12.2), we obtain

$$1 + c_1 a + c_2 a^2 + \dots = 1 + c_1 ax + c_2 a^2x^2 + \dots + a(1 + c_1 ax^2 + c_2 a^2 x^4 + \dots).$$

Hence, equating coefficients,

$$c_1 = \frac{1}{1-x}, \quad c_2 = \frac{x^2}{1-x^2}c_1, \quad c_3 = \frac{x^4}{1-x^3}c_2, \dots$$

and $$c_n = \frac{x^{n(n-1)}}{(1-x)(1-x^2)\dots(1-x^n)};$$

and hence

$$\frac{G_2(a,x)}{(1-a)(1-ax)\dots} = H_2(a,x) = 1 + \frac{a}{1-x} + \frac{a^2x^2}{(1-x)(1-x^2)} + \frac{a^3x^6}{(1-x)(1-x^2)(1-x^3)} + \dots.$$

Also

$$\frac{G_1(a,x)}{(1-a)(1-ax)\dots} = H_1(a,x) = \eta H_2(a,x) = 1+\frac{ax}{1-x}$$
$$+\frac{a^2x^4}{(1-x)(1-x^2)}+\frac{a^3x^9}{(1-x)(1-x^2)(1-x^3)}+\dots.$$

Finally, putting $a = x$ in these two formulae, and using (6.11.4) and (6.11.3), we obtain (6.10.1) and (6.10.2).

This proof is elementary, and reasonably simple; but it is undeniably rather artificial. It is a "verification"; we verify that the series (6.11.1) satisfies a functional equation, and the argument gives no explanation of our choice of this particular series.

6.13. There is another proof by Rogers which seems to assume a little more but is really more illuminating.[1]

We shorten our formulae by writing

$$x_n = 1-x^n, \quad x_n! = x_1x_2\dots x_n,$$

and begin by expanding the function

$$f(a) = \prod_1^\infty (1+ax^n)$$

in powers of a. The function satisfies

$$f(a) = (1+ax)f(ax).$$

Substituting a power series in a for $f(a)$, and equating coefficients, we find without difficulty that

$$f(a) = 1+\frac{x}{x_1!}a+\frac{x^3}{x_2!}a^2+\dots+\frac{x^{\frac{1}{2}n(n+1)}}{x_n!}a^n+\dots.$$

Replacing a by $ae^{i\theta}$ and $ae^{-i\theta}$, and multiplying the resulting series, we find that

$$(6.13.1)\quad \Phi(x,\theta,a) = \prod_1^\infty(1+2ax^n\cos\theta+a^2x^{2n})$$
$$= \left(1+\frac{x}{x_1!}ae^{i\theta}+\frac{x^3}{x_2!}a^2e^{2i\theta}+\dots\right)\left(1+\frac{x}{x_1!}ae^{-i\theta}+\frac{x^3}{x_2!}a^2e^{-2i\theta}+\dots\right)$$
$$= 1+\sum_1^\infty\frac{B_n(\theta)}{x_n!}a^n,$$

[1] The argument of §§ 10–12, if regarded as a proof of (6.10.1), assumes *nothing*, though some knowledge of theta-functions is required to identify (6.8.1) and (6.10.1). In the proof here we use formulae from the theory of theta-functions in the proof.

where

$$(6.13.2)\quad \frac{B_{2n}(\theta)}{x_{2n}!} = \frac{x^{n(n+1)}}{x_n!\,x_n!}\left(1+\frac{x_n}{x_{n+1}}2x\cos 2\theta+\frac{x_n x_{n-1}}{x_{n+1}x_{n+2}}2x^4\cos 4\theta+\dots \right.$$
$$\left. +\frac{x_n x_{n-1}\dots x_1}{x_{n+1}x_{n+2}\dots x_{2n}}2x^{n^2}\cos 2n\theta\right),$$

$$(6.13.3)\quad \frac{B_{2n+1}(\theta)}{x_{2n+1}!} = \frac{x^{(n+1)^2}}{x_n!\,x_{n+1}!}\left(2\cos\theta+\frac{x_n}{x_{n+2}}2x^2\cos 3\theta \right.$$
$$\left. +\frac{x_n x_{n-1}}{x_{n+2}x_{n+3}}2x^6\cos 5\theta+\dots+\frac{x_n x_{n-1}\dots x_1}{x_{n+2}x_{n+3}\dots x_{2n+1}}2x^{n(n+1)}\cos(2n+1)\theta\right).$$

Finally, we replace a in (6.13.1) by $x^{-\frac{1}{2}}$, and use a standard formula from the theory of the theta-functions, viz.

$$\Phi(x,\theta,x^{-\frac{1}{2}}) = \Pi(1+2x^{n-\frac{1}{2}}\cos\theta+x^{2n-1})$$
$$=\frac{1+2x^{\frac{1}{2}}\cos\theta+2x^{\frac{4}{2}}\cos 2\theta+2x^{\frac{9}{2}}\cos 3\theta+\dots}{(1-x)(1-x^2)(1-x^3)\dots}.$$

We thus obtain

$$(6.13.4)\quad \frac{1+2x^{\frac{1}{2}}\cos\theta+2x^{\frac{4}{2}}\cos 2\theta+2x^{\frac{9}{2}}\cos 2\theta+\dots}{(1-x)(1-x^2)(1-x^3)\dots} = 1+\sum_1^\infty \frac{B_n(\theta)}{x_n!}x^{-\frac{1}{2}n}.$$

6.14. If we replace $B_n(\theta)$, in (6.13.4), by its explicit expression (6.13.2) or (6.13.3), and rearrange the right-hand side as a trigonometrical series, we obtain an equality between two convergent trigonometrical series. Such an equality must be an identity, the coefficients of $\cos n\theta$ in the two series being the same. It follows that *we may replace*

$$1,\ 2\cos\theta,\ 2\cos 2\theta,\ 2\cos 3\theta,\ \dots$$

by any numbers for which the series remain convergent.

If we replace

$$1,\ 2\cos 2\theta,\ 2\cos 4\theta,\ \dots,\ 2\cos 2n\theta,\ \dots$$

by $\quad 1,\ -(1+x),\ x(1+x^2),\ \dots,\ (-1)^n x^{\frac{1}{2}n(n-1)}(1+x^n),\ \dots$

and all the odd cosines by 0, then $B_{2n}(\theta)$ becomes

$$(6.14.1)\quad \beta_{2n} = \frac{x_{2n}!}{x_n!\,x_n!}x^{n(n+1)}\left\{1-\frac{x_n}{x_{n+1}}x(1+x)+\frac{x_n x_{n-1}}{x_{n+1}x_{n+2}}x^5(1+x^2)-\dots \right.$$
$$\left. +(-1)^n\frac{x_n x_{n-1}\dots x_1}{x_{n+1}x_{n+2}\dots x_{2n}}x^{\frac{1}{2}n(3n-1)}(1+x^n)\right\},$$

and we obtain

$$(6.14.2)\quad 1+\frac{\beta_2}{x_2!}x^{-1}+\frac{\beta_4}{x_4!}x^{-2}+\dots = \frac{1-x^2-x^3+x^9+x^{11}-\dots}{(1-x)(1-x^2)(1-x^3)\dots},$$

where the right-hand side is the same as in (6.10.1). On the other hand, if we replace

$$2\cos\theta,\ 2\cos 3\theta,\ \ldots,\ 2\cos(2n+1)\theta,\ \ldots$$

by $$(1-x),\ -(1-x^3),\ \ldots,\ (-1)^n x^{\frac{1}{2}n(n-1)}(1-x^{2n+1}),$$

and the even cosines by 0, then $B_{2n+1}(\theta)$ becomes

(6.14.3)
$$\beta_{2n+1} = \frac{x_{2n+1}!}{x_n!\,x_{n+1}!}x^{(n+1)^2}\left\{1-\frac{x_n}{x_{n+2}}x^2(1-x^3)+\frac{x_n x_{n-1}}{x_{n+2}x_{n+3}}x^7(1-x^5)-\ldots\right.$$
$$\left.+(-1)^n\frac{x_n x_{n-1}\ldots x_1}{x_{n+2}x_{n+3}\ldots x_{2n+1}}x^{\frac{1}{2}n(3n+1)}(1-x^{2n+1})\right\}.$$

Multiplying by $x^{-\frac{1}{4}}$, we obtain

(6.14.4) $$\frac{\beta_1}{x_1!}x^{-1}+\frac{\beta_3}{x_3!}x^{-2}+\ldots = \frac{1-x-x^4+x^7+x^{13}-\ldots}{(1-x)(1-x^2)(1-x^3)\ldots},$$

where the right-hand side is the same as in (6.10.2). It remains to prove that the series on the left in (6.14.2) and (6.14.4) are the same as in (6.10.1) and (6.10.2).

6.15. We can do this by evaluating β_n in an elementary manner. But before doing this I observe that the substitutions of § 6.14 correspond to linear analytical transformations. Thus if $x = e^{-\delta}$ then

$$\int_{-\infty}^{\infty} e^{-2\theta^2/\delta}\cos\theta\,.\,2\cos 2n\theta\,.\,d\theta = \int_{-\infty}^{\infty} e^{-2\theta^2/\delta}\{\cos(2n-1)\theta+\cos(2n+1)\theta\}\,d\theta$$
$$= \sqrt{(\tfrac{1}{2}\pi\delta)}\,\{e^{-\frac{1}{8}(2n-1)^2\delta}+e^{-\frac{1}{8}(2n+1)^2\delta}\} = \sqrt{(\tfrac{1}{2}\pi\delta)}\,x^{\frac{1}{8}}\,.\,x^{\frac{1}{2}n(n-1)}(1+x^n).$$

Hence the first substitution of § 6.14 corresponds to the result of (i) replacing θ by $\theta+\frac{1}{2}\pi$ and (ii) operating with

$$\sqrt{\left(\frac{2}{\pi\delta}\right)}\,x^{-\frac{1}{8}}\int_{-\infty}^{\infty} e^{-2\theta^2/\delta}\cos\theta\ldots d\theta.$$

Similarly it may be verified that the second is the result of (i) replacing θ by $\theta+\frac{1}{2}\pi$ and (ii) operating with

$$-\sqrt{\left(\frac{2}{\pi\delta}\right)}\,x^{-\frac{1}{8}}\int_{-\infty}^{\infty} e^{-2\theta^2/\delta}\sin 2\theta\ldots d\theta.$$

6.16. We now write

(6.16.1) $$\beta_{2n} = \frac{x_{2n}!}{x_n!\,x_n!}x^{n(n+1)}\gamma_{2n},\quad \beta_{2n+1} = \frac{x_{2n+1}!}{x_n!\,x_{n+1}!}x^{(n+1)^2}\gamma_{2n+1},$$

and prove that

(6.16.2) $$\gamma_{2n} = x_n!,\quad \gamma_{2n+1} = x_{n+1}!.$$

We shall then have

(6.16.3) $\beta_{2n} = x^{n(n+1)}x_{n+1}x_{n+2}\ldots x_{2n}, \quad \beta_{2n+1} = x^{(n+1)^2}x_{n+1}x_{n+2}\ldots x_{2n+1},$

and it may be verified at once that the series on the left of (6.14.2) and (6.14.4) reduce to the Rogers-Ramanujan series.

To prove (6.16.2) it is enough to prove that

(6.16.4) $$\gamma_{2n+1} = x_{n+1}\gamma_{2n}, \quad \gamma_{2n+2} = \gamma_{2n+1}.$$

Now

$$\gamma_{2n} = 1 - \frac{x_n}{x_{n+1}}x(1+x) + \frac{x_n x_{n-1}}{x_{n+1}x_{n+2}}x^5(1+x^2) - \frac{x_n x_{n-1}x_{n-2}}{x_{n+1}x_{n+2}x_{n+3}}x^{12}(1+x^3) + \ldots$$

$$= \left(1 - x\frac{x_n}{x_{n+1}}\right) - x^2\frac{x_n}{x_{n+1}}\left(1 - x^3\frac{x_{n-1}}{x_{n+2}}\right) + x^7\frac{x_n x_{n-1}}{x_{n+1}x_{n+2}}\left(1 - x^5\frac{x_{n-2}}{x_{n+3}}\right) - \ldots$$

$$= \frac{1}{x_{n+1}}(1-x) - x^2\frac{x_n}{x_{n+1}x_{n+2}}(1-x^3) + x^7\frac{x_n x_{n-1}}{x_{n+1}x_{n+2}x_{n+3}}(1-x^5) - \ldots$$

$$= \frac{\gamma_{2n+1}}{x_{n+1}},$$

and

$$\gamma_{2n+1} = (1-x) - \frac{x_n}{x_{n+2}}x^2(1-x^3) + \frac{x_n x_{n-1}}{x_{n+2}x_{n+3}}x^7(1-x^5) - \frac{x_n x_{n-1}x_{n-2}}{x_{n+2}x_{n+3}x_{n+4}}x^{15}(1-x^7) + \ldots$$

$$= 1 - x\left(1 + x\frac{x_n}{x_{n+2}}\right) + x^5\frac{x_n}{x_{n+2}}\left(1 + x^2\frac{x_{n-1}}{x_{n+3}}\right) - x^{12}\frac{x_n x_{n-1}}{x_{n+2}x_{n+3}}\left(1 + x^3\frac{x_{n-2}}{x_{n+4}}\right) + \ldots$$

$$= 1 - \frac{x_{n+1}}{x_{n+2}}x(1+x) + \frac{x_{n+1}x_n}{x_{n+2}x_{n+3}}x^5(1+x^2) - \frac{x_{n+1}x_n x_{n-1}}{x_{n+2}x_{n+3}x_{n+4}}x^{12}(1+x^3) + \ldots$$

$$= \gamma_{2n+2}.$$

These are the relations required.

The equations (6.16.2) are

(6.16.5) $$1 - \frac{1-x^n}{1-x^{n+1}}x(1+x) + \frac{(1-x^n)(1-x^{n-1})}{(1-x^{n+1})(1-x^{n+2})}x^5(1+x^2) - \ldots = (1-x)(1-x^2)\ldots(1-x^n)$$

and

(6.16.6) $$(1-x) - \frac{1-x^n}{1-x^{n+2}}x^2(1-x^3) + \frac{(1-x^n)(1-x^{n-1})}{(1-x^{n+2})(1-x^{n+3})}x^7(1-x^5) - \ldots = (1-x)(1-x^2)\ldots(1-x^{n+1}).$$

Each of them reduces, when $n \to \infty$, to

$$1 - x - x^2 + x^5 + x^7 - \ldots = (1-x)(1-x^2)(1-x^3)\ldots,$$

Euler's identity; and the argument of this section gives a particularly simple proof of the identity. We shall be led to the formulae (6.16.5) and (6.16.6) in a different manner later.[1]

6.17. It follows from (6.12.2), or may be verified directly, that

$$F(a) = H_1(a,x) = 1+\frac{ax}{1-x}+\frac{a^2x^4}{(1-x)(1-x^2)}+\dots$$

satisfies the functional equation

$$F(a) = F(ax)+axF(ax^2).$$

From this it follows that

$$\frac{F(a)}{F(ax)} = 1+ax\frac{F(ax^2)}{F(ax)} = 1+\cfrac{ax}{1+ax^2\cfrac{F(ax^3)}{F(ax^2)}}$$

$$= 1+\frac{ax}{1+}\,\frac{ax^2}{1+}\,\frac{ax^3}{1+\dots}.$$

In particular

$$1+\frac{x}{1+}\,\frac{x^2}{1+}\,\frac{x^3}{1+\dots} = \frac{F(1)}{F(x)}$$

$$= \frac{(1-x^2)(1-x^7)\dots(1-x^3)(1-x^8)\dots}{(1-x)(1-x^6)\dots(1-x^4)(1-x^9)\dots}$$

$$= \frac{1-x^2-x^3+x^9+x^{11}-\dots}{1-x-x^4+x^7+x^{13}-\dots}$$

is a quotient of elliptic theta-functions, which may be evaluated for certain special values of x. This formula is the key to Ramanujan's evaluations of the continued fraction for special values of x, which I quoted in my first lecture.

NOTES ON LECTURE VI

This lecture contains a good deal of the substance of Hardy and Wright, ch. 19, and there is inevitably a certain amount of repetition; but the account here is naturally less systematic. There is nothing in Hardy and Wright corresponding to §§ 6.13–16.

§ 6.2. See Hardy and Wright, § 19.11, or MacMahon, *Combinatory analysis*, ii, 21–23. Franklin's proof was first published in *Comptes rendus*, 92 (1881), 448–450.

Both Hardy and Wright and MacMahon give other examples of 'graphical' proofs.

§ 6.3. For a more rigorous proof that $F(x)$ enumerates $p(n)$, see Hardy and Wright, § 19.3.

§§ 6.4–5. Compare Hardy and Wright, § 19.12, where however there is no proof of (6.4.2). There are alternative proofs of (6.4.1) and (6.4.2), and a proof of (6.4.3), in no. 30 of the *Papers*. Darling (3) gave further proofs of (6.4.1) and (6.4.2).

[1] See Lecture VII, § 7.8.

There are interesting remarks on the parity of $p(n)$ in MacMahon's paper **2**. MacMahon does not prove any general theorem, but gives recurrence congruences (mod 2) by which it is possible to calculate the parity of $p(n)$, for quite large n, very quickly. Thus he proves, in 'about five minutes work', that $p(1000)$ is odd.

One of the standard proofs of (6.5.1) is reproduced in Hardy and Wright, § 19.9.

§§ 6.6–7. Ramanujan's proofs of the congruences for moduli 5^2, 7^2, and 11^2 are contained in an unpublished manuscript now in the possession of Prof. Watson. Darling (**2**) gave a proof of (6.7.1), and Mordell (**1**) much shorter proofs of both (6.7.1) and (6.7.2).

The references relevant to the work of Chowla, Gupta, Krečmar, Lehmer, and Watson are S. Chowla (**1**); Gupta (**1, 2, 3**); Krečmar (**1**); D. H. Lehmer (**1, 3**); and Watson (**24**).

§ 6.8. Rogers (**1**): the identities are formulae (1) and (2) of § 5. Rogers also anticipated 'Hölder's inequality' (and is quoted by Hölder), but without writing it in the standard form or recognising its fundamental importance. See Hardy, Littlewood, and Pólya, *Inequalities*, 25 and 311.

Roger's two later proofs are in his papers **2** and **4**: the latter contains the proof given in §§ 6.10–6.12, the former that given in §§ 6.13–6.16.

For Ramanujan's own proof see no. 26 of the *Papers*. Schur's two proofs appeared in *Berliner Sitzungsberichte* (1917), 301–321, and Watson's in Watson (**3**).

Ramanujan does not seem to have stated the formulae explicitly in his letters to me, but formulae IX, (4)–(7), of his first letter depend upon them. See Lecture I, formulae (1.10)–(1.12); *Papers*, xxvii; and Watson (**4**). Ramanujan proposed the formulae as a problem in *Journal Indian Math. Soc.* 6 (1914), 199; see *Papers*, 330.

§ 6.9. See Hardy and Wright, § 19.13, and MacMahon, *Combinatory analysis*, ii, 33–36.

§§ 6.10–6.12. See Hardy and Wright, § 19.14. The theta-function formulae required at the beginning of the proof are proved in §§ 19.8–19.9 (Theorems 355 and 356).

§ 6.13. The expansion of $f(a)$ in powers of a goes back to Euler. The theta-function formula is proved in Hardy and Wright, § 19.8, or in any of the standard treatises on elliptic functions.

§ 6.17. See Lecture I, formulae (1.10)–(1.12). Proofs were given by Watson (**4**).