Von Neumann and Newman poker with a flip of hand values

Nicla Bernasconi Julian Lorenz* Reto Spöhel†

Institute of Theoretical Computer Science
ETH Zürich, 8092 Zürich, Switzerland
{nicla|jlorenz|rspoehel}@inf.ethz.ch

Abstract. The von Neumann and Newman poker models are simplified two-person poker models in which hands are modeled by real values drawn uniformly at random from the unit interval. We analyze a simple extension of both models that introduces an element of uncertainty about the final strength of each player's own hand, as is present in real poker games. Whenever a showdown occurs, an unfair coin with fixed bias q is tossed, $0 \leq q \leq 1/2$. With probability $1-q$, the higher hand value wins as usual, but with the remaining probability q, the lower hand wins. Both models favour the first player for $q = 0$ and are fair for $q = 1/2$. Our somewhat surprising result is that the first player's expected payoff increases with q as long as q is not too large. That is, the first player can exploit the additional uncertainty introduced by the coin toss and extract even more value from his opponent.

1. Introduction

Simplified models of poker were first studied in a game theoretic framework by Borel [3] and von Neumann [12] in the 1920s. Von Neumann's results were published later in the seminal book *Theory of Games and Economic Behavior* with Morgenstern [13]. He considered the following $(0, 1)$ two-player game: There is an initial pot of size P (consisting of player's antes). Each player is dealt a 'hand', i.e., a real number x, respectively y, drawn independently and uniformly at random from the unit interval. After X has inspected x, he can make a bet of a chips or check. If he bets, Y can either call (i.e., match X's bet of a chips) or fold (i.e., concede the pot to X), where of course he can base this decision on his own hand y. If X checks or if his bet is called by Y, a showdown occurs, i.e., both hands are revealed and the player with the higher hand value wins the pot and all bets made. Note that only X is allowed to make a bet. Thus he has an advantage over Y, who can only react.

In the above model, one can either prescribe the amount a that X bets if he opts to bet, or let X choose the size of his bet freely as a function of his hand x. The former case is the game originally studied by von Neumann. Here we assume $a = P$ (pot size bet) for simplicity. Von Neumann showed that in optimal play X has an expected payoff of $5/9\,P$. A key insight of his solution is that *bluffing* is a game-theoretic necessity: X will not only bet with his good hands, but also with very low-ranked hands. These bluff-bets aim at inducing his opponent to make more calls, even with only mediocre hands, which in turn enables X to extract more value from Y with his good hands. The case of variable bet size $a(x)$ was studied by Newman [9] in the 1950s. Certainly, in this setup player X has an even greater advantage – Newman proved an expected payoff of $4/7\,P$ in optimal play.

Several extensions and variants of these poker models have been studied in the literature. In particular, there are two 'intermediate' models in which X can choose his bet more freely than

*The author was supported by UBS AG.
†The author was supported by Swiss National Science Foundation, grant 200020-119918.

in von Neumann's model, but not completely unrestricted as in Newman's model: Karlin and Restrepo [8] (see also [10]) analyzed the case where X may choose his bet a from a finite number of prescribed bet sizes $a_1, \ldots, a_n$, and Sakai [10] investigated the case where X is allowed to choose his bet freely in an interval $[0, a_{\max}]$. Much research was also devoted to analyzing more complex bet structures, in which both X and Y are allowed to bet, raise, or re-raise [2, 5, 6, 7, 8].

1.1. **The von Neumann and Newman model with flip.** We consider the following extension of the above model: whenever a showdown occurs, an unfair coin with bias q is tossed, where $0 \leq q \leq 1/2$ is a fixed parameter known to both players. With probability $1-q$, the higher hand value wins as usual, but with the remaining probability q, the *lower* hand wins. We shall refer to this as a *flip* of the hand values: initially strong hands become weak, and vice versa. This extension is motivated by the fact that in most real world poker variants, players are not only ignorant about the opponent's hand, but also about their own final hand value. In Texas Hold'em for example, a player's final hand strength is a function of both his private hand cards and so-called 'community cards' that are revealed over the course of the game; a weak starting hand thus may turn into a very strong hand at showdown if the right community cards appear.

Obviously, setting $q = 0$ yields the original von Neumann and Newman models, and the expected payoffs of X in optimal play are as given above. In the other extreme $q = 1/2$, each player has probability $1/2$ of having the better hand at showdown, regardless of the realizations of x and y. Thus, for $q = 1/2$ the game reduces to a fair coin toss and neither player will have an advantage: Y will call every bet, and regardless of X's strategy each player has expected payoff $P/2$ in both models.

In light of these two extreme cases, one might be tempted to conclude that the uncertainty about player's own hands introduced by the flip is always bad for the bettor X, who is in control of the game (recall that Y can only react). More specifically, one might think that player X's expected payoff as a function of q is decreasing on the entire interval $[0, 1/2]$. As we shall see, something altogether different is true.

1.2. **Our results.** An equilibrium of a two-player game is a pair of (possibly randomized) strategies such that no player can improve his expected payoff by unilaterally changing his strategy. In other words, each strategy is a best response to the other one. As it is well-known, in two-player constant-sum games all equilibria have the same expected payoffs, and thus we may define the *value* of such a game as the expected payoff of the first player X in any equilibrium pair of strategies (if one exists). Clearly, the poker games introduced above are constant-sum with sum P, and we shall present explicit equilibrium pairs of strategies for both games. Let $V_I(q)$ and $V_{II}(q)$ be the values of the von Neumann game (with pot size bet) and the Newman game with flip probability q, respectively. Certainly, we have $V_{II}(q) \geq V_I(q)$ for all values of q, as in Newman's model player X is allowed more flexibility in his betting strategy than in von Neumann's.

We first present our results for the von Neumann game. As mentioned above, we have $V_I(0) = 5/9\,P \approx 0.556\,P$ in the classical case without flip, and trivially $V_I(1/2) = P/2$.

Theorem 1. *The value of von Neumann poker with pot size bet and flip probability q is*

$$V_I(q) = \begin{cases} \frac{27q^2-32q+10}{6(9q^2-10q+3)}\,P, & q \in [0, 1/3] \\ \frac{3-2q}{4}\,P, & q \in (1/3, 1/2] \end{cases} . \tag{1}$$

We give a proof of Theorem 1 in Section 2. Figure 1 shows $V_I(q)$ graphically. $V_I(q)$ first *increases* as we increase q, and attains its unique maximum for $q = 1/3$ with a value of $7/12\,P \approx 0.583\,P$. In $[1/3, 1/2]$, $V_I(q)$ decreases linearly. In other words, as we increase the parameter q, player X can exploit this additional uncertainty about the outcome of the game and extract even more value from his opponent at first! Only beyond $q = 1/3$, X gradually loses his advantage.

This behaviour of the value function is related to von Neumann's observation about bluffing and can be explained as follows: As q increases, X has to make less and less bluff-bets, since gradually the flip q takes over and induces Y into calling enough of X's bets. At $q = 1/3$ something important happens – as we shall show, at this point the optimal strategies change abruptly (cf. Figure 2 in Section 2): For $q > 1/3$, Y calls every bet since his expected payoff is positive regardless of the hands x and y. Thus the need for X to bluff vanishes completely, and he can concentrate on extracting value from Y with his good hands. Due to the abrupt change of strategy, $V_I(q)$ is continuous but not smooth at $q = 1/3$. The value 1/3 stems from our assumption that $a = P$; in general, this change of strategies occurs at q^* satisfying $q^*/(1 - q*) = a/(P + a)$, i.e., at the point where the odds of a flip occurring equal Y's 'pot odds'.

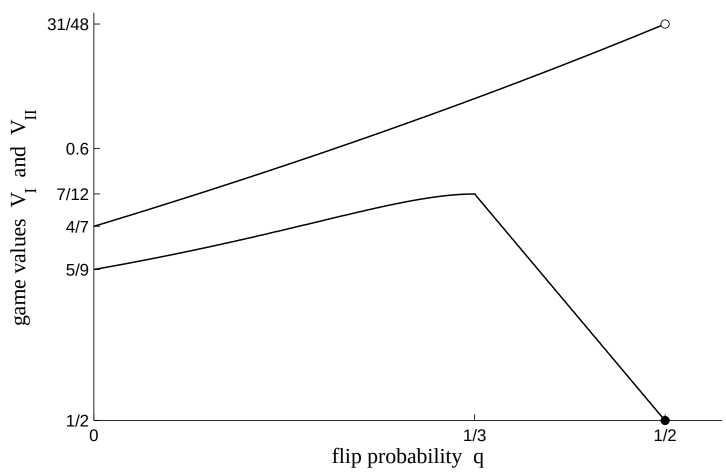

FIGURE 1. Value $V_I(q)$ of von Neumann poker (lower curve) and value $V_{II}(q)$ of Newman poker (upper curve) as function of the flip probability q. We set $P = 1$ for simplicity. $V_{II}(q)$ is discontinuous at $q = 1/2$.

We now present our results for the Newman model. We find it quite remarkable that $V_{II}(q)$ turns out to have the simple closed form below, as the calculations involved are far more complicated than in the von Neumann case.

Theorem 2. *The value of Newman poker with flip probability q is*

$$V_{II}(q) = \begin{cases} \frac{16-q}{4(7-2q)}\,P, & q \in [0, 1/2) \\ P/2, & q = 1/2 \end{cases} . \tag{2}$$

We give a proof of Theorem 2 in Section 3. As shown in Figure 1, $V_{II}(q)$ increases from $4/7\,P \approx 0.571\ P$ to $31/48\,P \approx 0.646\,P$. Unlike in the von Neumann game, the value function increases in the entire interval $[0, 1/2)$: As the outcome of the game becomes more and more uncertain, X's advantage gets larger and larger. We attribute the discontinuity at $q = 1/2$ to the fact that X's bet size a is unlimited. (In fact, in the optimal strategies we present $a(x)$ diverges both at $x = 0$ and $x = 1$ for all values of q, cf. Figure 3 in Section 3.) We believe that if a limit on the bet size is imposed as in [10], this discontinuity disappears and $V_{II}(q)$ attains a maximum at some $q_0 < 1/2$ instead, similar to the behaviour of $V_I(q)$.

An interesting feature of the equilibrium pair of strategies we exhibit is that they are discontinuous at a^* satisfying $q/(1 - q) = a^*/(P + a^*)$ (cf. Figure 3 in Section 3): Y will call all bets of at most

a^* since for such bets his expected payoff is nonnegative regardless of the hands x and y. Knowing this, X never bets an amount between 0 and a^*. Thus, again the equation $q/(1-q) = a/(P+a)$ determining the point where the odds of a flip equal Y's pot odds plays a vital role.

2. The von Neumann Model

In this section we prove Theorem 1. We give an explicit equilibrium pair of deterministic strategies, and compute the value of the game from there.

Recall that in the von Neumann poker model player X can decide whether to check or to bet $a = P$ after inspecting his hand x. If X bets, Y either calls or folds depending on his hand y.

Formally, a deterministic strategy for X is given by an arbitrary (measurable) subset $\mathcal{B} \subseteq (0,1)$ determining for which hand values x he will make a bet. Similarly, a deterministic strategy for Y is given by an arbitrary (measurable) subset $\mathcal{C} \subseteq (0,1)$ determining with which hands y he will call if X bets.

To make the formulas more understandable we write a for the bet and P for the pot despite the assumption that $a = P$. Throughout, we neglect the case $x = y$, which occurs with probability 0.

2.1. The case $\mathbf{1/3 < q < 1/2}$.

We show that for any fixed $q \in (1/3, 1/2)$, an equilibrium pair of strategies is given by

$$\mathcal{B} = (1/2, 1) \ , \tag{3}$$

$$\mathcal{C} = (0, 1) \ . \tag{4}$$

In words, X bets the better half of his hands, and Y calls every bet regardless of his hand.

Claim 3. *Y's calling strategy* (4) *is a best response to any betting strategy* $\tilde{\mathcal{B}}$ *of player X.*

Proof. Note that if Y folds, his payoff is 0, and if he calls with a hand of y, his expected payoff *conditional on X having a hand* $x > y$ is

$$(P+a)q - a(1-q) \ , \tag{5}$$

while his expected payoff *conditional on X having a hand* $x < y$ is

$$(P+a)(1-q) - aq \ . \tag{6}$$

As for $1/2 > q > 1/3 = a/(P+2a)$ both (5) and (6) are positive, calling all bets of X indeed corresponds to Y's pointwise optimal behaviour, regardless of X's strategy. □

Claim 4. *X's betting strategy* (3) *is a best response to Y's calling strategy* (4).

Proof. As Y always calls, X's expected payoff if he bets with a hand of x is given by

$$\begin{aligned} & x\big((P+a)(1-q) - aq\big) + (1-x)\big((P+a)q - a(1-q)\big) \\ & \quad = P\left((1-2q)x + q\right) + a\,(1-2q)(2x-1) \ , \end{aligned} \tag{7}$$

and if he checks, by

$$xP(1-q) + (1-x)Pq = P\left((1-2q)x + q\right) \ . \tag{8}$$

Here x and $1-x$ are the probabilities that Y's hand y is lower, resp. higher than x. Since (7) is larger than (8) for $x > 1/2$ and smaller for $x < 1/2$, (3) indeed is a pointwise best response to Y's calling strategy (4). □

Let E_x denote X's expected payoff when holding a hand of x, with X and Y playing strategies (3, 4). Combining (7, 8) and using that $a = P$, we obtain

$$E_x = \begin{cases} P\left((1-2q)x + q\right), & x \in (0, 1/2] \\ P\left(3(1-2q)x + 3q - 1\right), & x \in (1/2, 1) \end{cases} .$$

Integrating E_x over $(0,1)$ we obtain $V_I(q)$ as stated in Theorem 1 for the case $1/3 < q \leq 1/2$.

2.2. **The case $\mathbf{0 \leq q \leq 1/3}$.** Before we state and verify an explicit equilibrium pair of strategies, we briefly outline how such strategies can be derived heuristically. We use the well-known principle of indifference, a more detailed description of which can be found e.g. in [4].

We assume that player X bets if and only if his hand x is in the interval $(0, x_0)$ (bluff bets) or in $(x_1, 1)$ (value bets) for some fixed numbers $x_0, x_1 \in (0,1)$. Similarly, we assume that Y calls a bet of X if and only if his hand y is in $[y_1, 1)$ for some fixed y_1, $x_0 \leq y_1 \leq x_1$.

Assume that Y's calling threshold y_1 is fixed and known to X. If X bets with a hand $x \leq y_1$, his payoff is given by $y_1 P + (1-y_1)\big((P+a)q - a(1-q)\big)$. Here the first term corresponds to the case where Y folds, and the second one to the case where Y calls. Clearly, X should bet if and only if this exceeds (8), which is his expected payoff if he checks. For continuity reasons, at his bluff threshold x_0 the two options should have the same expected payoff ('at x_0, X is indifferent between betting or checking'), which yields the indifference equation

$$x_0 P(1-q) + (1-x_0)Pq \stackrel{!}{=} y_1 P + (1-y_1)\big((P+a)q - a(1-q)\big) \ . \tag{9}$$

Similarly, X's indifference between checking and betting at x_1 yields the equation

$$\begin{aligned} x_1 P(1-q) + (1-x_1)Pq \stackrel{!}{=} y_1 P &+ (x_1 - y_1)\big((P+a)(1-q) - aq\big) \\ &+ (1-x_1)\big((P+a)q - a(1-q)\big) \ , \end{aligned} \tag{10}$$

and from Y's indifference at y_1 between calling and folding we obtain

$$x_0\big((P+a)(1-q) - aq\big) + (1-x_1)\big((P+a)q - a(1-q)\big) \stackrel{!}{=} 0 \ . \tag{11}$$

Solving the equation system (9, 10, 11) observing that $a = P$ yields

$$x_0 = (1-3q)/D \ , \tag{12}$$

$$x_1 = 1 - (2-3q)/D \ , \tag{13}$$

$$y_1 = (1-2q)(5-9q)/D \ , \tag{14}$$

where $D = 3\,(9q^2 - 10q + 3)$. Note that $x_0 < y_1 < x_1$ for all values of q. Figure 2 illustrates these thresholds as q varies from 0 to 1/3.

We prove that for any fixed $q \in [0, 1/3]$, an equilibrium is given by the following pair of strategies. Player X bets if and only if his hand x is in

$$\mathcal{B} = (0, x_0) \cup (x_1, 1) \ , \tag{15}$$

and player Y calls a bet of X if and only if his hand y is in

$$\mathcal{C} = [y_1, 1) \ , \tag{16}$$

where x_0, x_1 and y_1 given by (12, 13, 14).

The fact that the strategies (15, 16) are in equilibrium is proved by the two following claims.

Claim 5. *Any calling strategy $\tilde{\mathcal{C}}$ of player Y satisfying*

$$y > x_1 \Longrightarrow y \in \tilde{\mathcal{C}} \tag{17}$$

$$y < x_0 \Longrightarrow y \notin \tilde{\mathcal{C}} \tag{18}$$

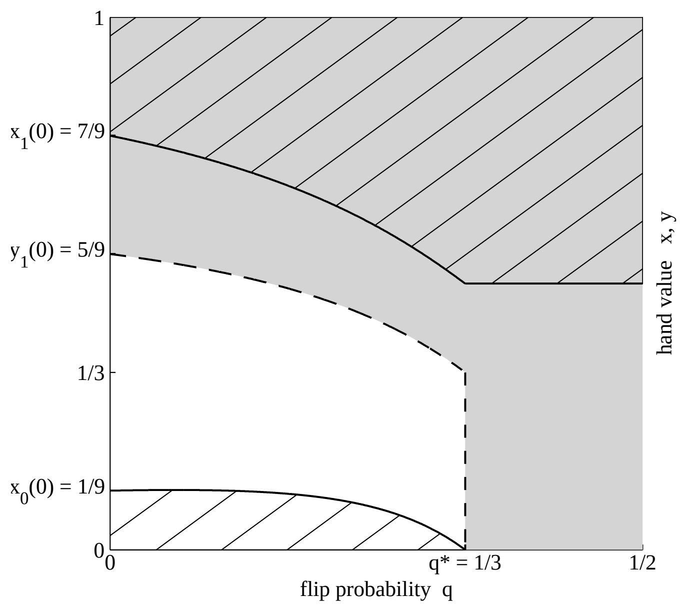

FIGURE 2. Equilibrium strategy pairs for players X and Y as function of parameter q. The hatched regions indicate with which hands x player X makes a (bluff or value) bet, and the shaded region indicates with which hands y player Y calls X's bets.

is a best response to X's betting strategy (15)*.*

Proof. Assume that Y has a hand of y and is faced with a bet. Clearly, if he folds, his payoff is 0. If he calls, his expected payoff is the weighted average of (5, 6), where the weights are the probabilities that $x > y$, resp. $x < y$, *conditional on* $x \in \mathcal{B}$ (i.e., X making a bet). If Y's hand y is between x_0 and x_1, these conditional probabilities are proportional to $1 - x_1$ resp. x_0, and Y's expected payoff is some constant times the left hand side of (11), which vanishes for x_0 and x_1 as defined in (12, 13). Thus for hands between x_0 and x_1, calling and folding have the same expected payoff.

For $y < x_0$, the conditional probabilities in question are proportional to $(1 - x_1) + (x_0 - y)$ and y respectively. Thus the smaller term (5) gets strictly larger weight, and Y's expected payoff when calling is negative. Analogously it follows that his expected payoff when calling with a hand $y > x_1$ is positive. Together this shows that any strategy satisfying (17, 18) corresponds to pointwise optimal behaviour of Y, which proves the claim. □

Claim 6. *X's betting strategy* (15) *is a best response to Y's calling strategy* (16)*.*

Proof. The expected payoff for X if he checks with a hand of x is given by (8). On the other hand, if he bets with a hand $x \le y_1$, his expected payoff is

$$y_1 \, P + (1 - y_1)\big((P + a)q - a(1 - q)\big) = P\,(27q^3 - 24q^2 + 4q + 1)/D \qquad (19)$$

independently of x, and if he bets with a hand with a hand $x > y_1$,

$$\begin{aligned} &y_1 \, P + (x - y_1)\big((P + a)(1 - q) - aq\big) + (1 - x)\big((P + a)q - a(1 - q)\big) \\ &\quad = P\,\big((-162q^3 + 261q^2 - 144q + 27)\,x/D\big) + (135q^3 - 192q^2 + 91q - 14)/D\big) \ , \qquad (20) \end{aligned}$$

where we used that $a = P$ and y_1 as defined in (14). Straightforward calculation yields that (19) is larger than (8) for $x < x_0$ and that (20) is larger than (8) for $x > x_1$, reflecting equalities (9) and (10). Thus (15) is a best response to Y's calling strategy (16). □

Let E_x denote X's expected payoff when holding a hand of x, with X and Y playing strategies (15, 16). Combining (8, 19, 20), we obtain

$$E_x = \begin{cases} P\,\frac{27q^3-24q^2+4q+1}{D}, & x \in (0, x_0] \\ P\left((1-2q)x+q\right), & x \in (x_0, x_1) \\ P\left(\frac{-162q^3+261q^2-144q+27}{D}\,x + \frac{135q^3-192q^2+91q-14}{D}\right), & x \in [x_1, 1) \end{cases}.$$

Integrating E_x over $[0,1]$ we obtain $V_I(q)$ as stated in Theorem 1 for the case $0 \leq q \leq 1/3$.

3. The Newman Model

In this section we prove Theorem 2. As in the previous section, we give an equilibrium pair of deterministic strategies for players X and Y, and compute the value of the game from there. Throughout, we assume that $0 \leq q < 1/2$. As argued in the introduction, the case $q = 1/2$ is trivial.

Recall that in the Newman poker model, X can choose the size of his bet a freely as a function of his hand x. In this formulation, if X wants to check, he does so by choosing bet size $a = 0$. Y sees the size of the bet, and can use this information together with his hand value y in deciding whether or not to call.

Formally, a deterministic strategy for X is given by an arbitrary (measurable) function $a(x) : (0,1) \to [0,\infty)$ determining his bet a as a function of his hand x. A deterministic strategy for Y is given by an arbitrary (measurable) subset $\mathcal{C} \subseteq (0,1) \times [0,\infty)$, which is interpreted as follows: Y calls a bet of a with a hand of y if and only if $(y,a) \in \mathcal{C}$.

Before we state and verify an explicit equilibrium pair of strategies, we outline the heuristics that lead to these strategies. First of all, recall that the expected payoff of Y when calling a bet of a is given by (5, 6). Since for a at most

$$a^* = \frac{q}{1-2q}\,P$$

both terms are nonnegative, Y will call *every* bet of at most a^*. Similarly to Section 2.1, it follows that X should never bet an amount $0 < a < a^*$ with hands below $1/2$, and should always bet at least a^* with hands above $1/2$.

In view of the findings in the von Neumann poker model about mixing value bets with one's *best hands* and bluff bets with one's *worst hands*, we assume that X plays as follows: He bets $a > a^*$ if and only if his hand is either $x_0(a)$ or $x_1(a)$ for some functions $x_{0,1} : [a^*,\infty) \to (0,1)$. We assume that $x_0(a)$ is strictly decreasing with $\lim_{a\to\infty} x_0(a) = 0$, and that $x_1(a)$ is strictly increasing with $\lim_{a\to\infty} x_1(a) = 1$ (cf. Figure 3 below). The function $x_0(a)$ corresponds to X's bluff bets, and $x_1(a)$ to his value bets. For Y's calling strategy, we assume that he calls a bet of $a > a^*$ if and only if his hand is greater or equal than $y_1(a)$ for some function $y_1 : [a^*,\infty) \to (0,1)$. We assume that $x_0(a) \leq y_1(a) \leq x_1(a)$ for $a \in [a^*,\infty)$. We will use the notations $x_0^* = x_0(a^*)$, $x_1^* = x_1(a^*)$, $y_1^* = y_1(a^*)$.

Assume first that Y's strategy is fixed, and that X has a hand of x. Knowing Y's strategy $y_1(a)$, if X bets he will bet the amount which maximizes his expected payoff. For bluff bets this expected payoff is $f(a) = y_1(a)\,P + (1-y_1(a))\,\big((P+a)q - a(1-q)\big)$, independently of x. First-order condition yields

$$\frac{\partial f(a)}{\partial a} = y_1'(a)\,\Big(P - \big((P+a)q - a(1-q)\big)\Big) \;-\; (1-y_1(a))\,(1-2q) \overset{!}{=} 0 \tag{21}$$

for $a \in (a^*, \infty)$. Similarly, for X's value bets the expected payoff is $g(a) = y_1(a)\, P + (x - y_1(a))\, \big((P + a)(1-q) - aq\big) + (1-x)\, \big((P+a)q - a(1-q)\big)$, and first-order condition yields

$$\frac{\partial g(a)}{\partial a} = y_1'(a) \left(P - \big((P+a)(1-q) - aq\big)\right) + \big((x - y_1(a)) - (1-x)\big)(1-2q) = 0 \ , \tag{22}$$

from which we derive the indifference equation

$$y_1'(a) \left(P - \big((P+a)(1-q) - aq\big)\right) + (2x_1(a) - y_1(a) - 1)(1-2q) \overset{!}{=} 0 \tag{23}$$

for $a \in (a^*, \infty)$ (note that we replaced x by $x_1(a)$ *after* taking the derivative). Now suppose that X's strategy is fixed. If Y is faced with a bet of $a > a^*$, clearly his payoff is 0 if he folds. From a bet of a player Y can deduce that X either bluffs with the hand $x_0(a)$ or bets for value with the hand $x_1(a)$. For a hand $x_0(a) \le y \le x_1(a)$, he will call if his expected payoff $h(a)$ is positive. To calculate $h(a)$, he needs to take into account the probability with which a bet of a is a bluff compared to a being a value bet. This ratio of bluff bets to value bets of a is given by the ratio of the slopes $-x_0'(a)$ and $x_1'(a)$, and with (5, 6) (recall that we assumed $x_0(a) \le y_1(a) \le x_1(a)$) we obtain the third equation

$$\frac{\partial h(a)}{\partial a} = -x_0'(a)\, \big((P+a)(1-q) - aq\big) + x_1'(a)\, \big((P+a)q - a(1-q)\big) \overset{!}{=} 0 \tag{24}$$

for $a \in (a^*, \infty)$.

Solving the system of differential equations (21, 23, 24) requires further boundary conditions. We require that at X's bluff threshold $x_0^* = x_0(a^*)$, checking and betting a^* should have the same expected payoff, i.e.,

$$x_0^*\, P\big(1-q\big) + (1 - x_0^*)Pq \overset{!}{=} y_1^*\, P + (1 - y_1^*)\, \big((P + a^*)q - a^*(1-q)\big) \ . \tag{25}$$

The last boundary condition is given by the assumption

$$\lim_{a \to \infty} x_0(a) \overset{!}{=} 0 \ . \tag{26}$$

Solving the system of differential equations (21, 23, 24) under the boundary conditions (25, 26) yields

$$x_0(a) = \frac{1-q}{7-2q} \cdot P^2 \cdot \frac{(1-3q)P + 3(1-2q)a}{\big((1-q)P + (1-2q)a\big)^3} \ , \tag{27}$$

$$x_1(a) = \frac{1-2q}{7-2q} \cdot \frac{(1-q)(4-q)P^2 + 2(1-q)(7-2q)aP + (1-2q)(7-2q)a^2}{\big((1-q)P + (1-2q)a\big)^2} \ , \tag{28}$$

$$y_1(a) = \frac{1-2q}{7-2q} \cdot \frac{(1-q)P + (7-2q)a}{(1-q)P + (1-2q)a} \ . \tag{29}$$

Moreover, straightforward calculation gives

$$x_0^* = x_0(a^*) = \frac{1-q}{7-2q} \ , \qquad x_1^* = x_1(a^*) = \frac{4+q}{7-2q} \ , \qquad y_1^* = y_1(a^*) = \frac{1+4q}{7-2q} \ .$$

Note that for all values of q we have $x_0^* < y_1^* < x_1^*$.

We shall prove that for $0 \le q < 1/2$ an equilibrium is given by the following pair of strategies, where $x_0(a)$, $x_1(a)$, $y_1(a)$, x_0^*, x_1^*, and a^* are as defined above. As a function of his hand x, player X bets an amount of

$$a(x) = \begin{cases} a, & x = x_0(a) \in (0, x_0^*) \\ 0, & x \in [x_0^*, 1/2] \\ a^*, & x \in (1/2, x_1^*] \\ a, & x = x_1(a) \in (x_1^*, 1) \end{cases} \ . \tag{30}$$

Since both $x_0 : [a^*, \infty) \to (0, x_0^*]$ and $x_1 : [a^*, \infty) \to [x_1^*, 1)$ are bijective, $a(x)$ is indeed well-defined.[1] Player Y, holding a hand of y, calls a bet of a if and only if (y, a) is in

$$\mathcal{C} = \Big\{ (y, a) \;\Big|\; a \leq a^* \;\vee\; y \geq y_1(a) \Big\} \;. \tag{31}$$

Figure 3 illustrates these strategies for $q = 0$ and $q = 1/3$.

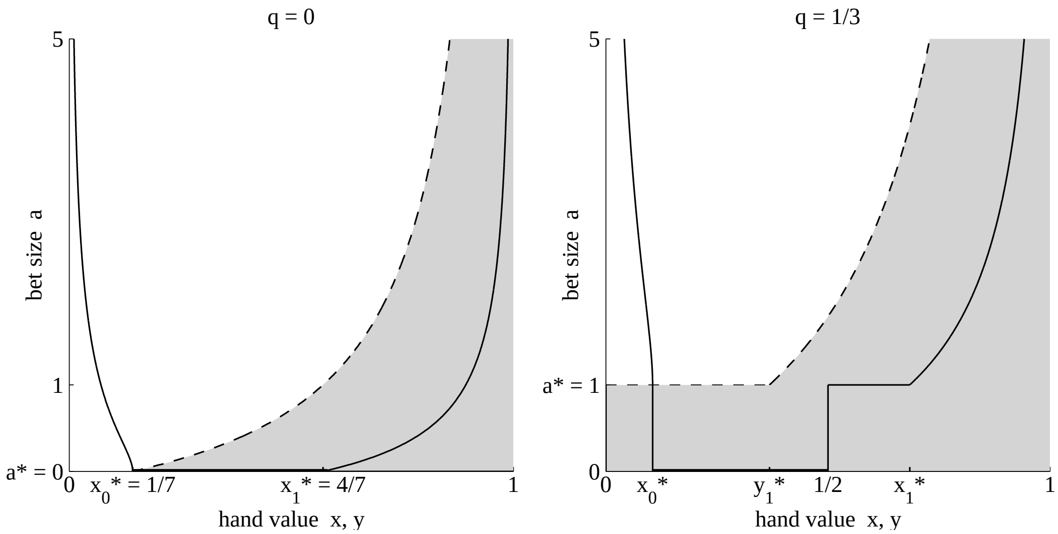

FIGURE 3. Equilibrium strategy pair for $q = 0$ (left) and $q = 1/3$ (right). Player X's bet function $a(x)$ is given by the solid line, and Y's call region $\mathcal{C}$ is the shaded region.

In order to verify that these strategies indeed form an equilibrium pair, we prove two stronger claims, showing that if one of the two strategies is fixed, there is in fact a large class of deterministic counterstrategies.

Claim 7. *Any calling strategy $\tilde{\mathcal{C}}$ of player Y satisfying*

$$y > x_1(a) \Longrightarrow (y, a) \in \tilde{\mathcal{C}} \tag{32}$$

$$y < x_0(a) \Longrightarrow (y, a) \notin \tilde{\mathcal{C}} \tag{33}$$

$$a \leq a^* \Longrightarrow (y, a) \in \tilde{\mathcal{C}} \tag{34}$$

is a best response to X's betting strategy (30).

Proof. We first show that the restrictions (32, 33, 34) reflect the pointwise optimal behaviour of Y when faced with a bet of a holding a hand of y.

Recall that Y's expected payoff when calling is given by (5) or (6). If Y is faced with a bet of size $a > a^*$, there are exactly two hands X can possibly have: a value bet hand $x_1(a)$ and a bluff bet hand $x_0(a)$, where $x_0(a) < x_1(a)$. Thus with any hand $y < x_0(a)$, Y's expected payoff is given by (5) and is strictly less than $(P + a^*)q - a^*(1 - q) = 0$, since (5) is decreasing in a. Conversely, with any hand $y > x_1(a)$, Y's expected payoff is given by (6) and strictly larger than $(P + a^*)(1 - q) - a^*q = P > 0$. As moreover for $a \leq a^*$ both (5) or (6) are nonnegative, his expected payoff will be nonnegative when calling such hands, regardless of his hand y. This proves that indeed the restrictions given by (32, 33, 34) correspond to Y's pointwise optimal behaviour.

[1] An alternative equilibrium pair is obtained by replacing $x_0(a)$ with $x_0^* - x_0(a)$, as is done in [9]. Instead of betting arbitrarily large amounts a as x approaches 0, X's bet size then diverges as $x \to x_0^*$.

It remains to show that it is completely irrelevant what Y does in the remaining situations (y, a). For a fixed hand y and any call strategy $\tilde{\mathcal{C}}$ satisfying (32, 33, 34), his expected payoff against X's strategy (30) *conditional on no flip occurring* is

$$\begin{aligned}
&\int_0^1 (P + a(x))\, \mathbb{1}_{\{(y,a(x))\in\tilde{\mathcal{C}} \wedge y>x\}} - a(x)\, \mathbb{1}_{\{(y,a(x))\in\tilde{\mathcal{C}} \wedge y<x\}}\, dx \\
\overset{(30,34)}{=} &\int_\infty^{a^*} \big((P+a)\, \mathbb{1}_{\{(y,a)\in\tilde{\mathcal{C}} \wedge y>x_0(a)\}} - a\, \mathbb{1}_{\{(y,a)\in\tilde{\mathcal{C}} \wedge y<x_0(a)\}}\big) x_0'(a)\, da \\
&+ \int_{x_0^*}^{1/2} P\, \mathbb{1}_{\{y>x\}}\, dx + \int_{1/2}^{x_1^*} (P+a^*)\, \mathbb{1}_{\{y>x\}} - a^*\, \mathbb{1}_{\{y<x\}}\, dx \\
&+ \int_{a^*}^{\infty} \big((P+a)\, \mathbb{1}_{\{(y,a)\in\tilde{\mathcal{C}} \wedge y>x_1(a)\}} - a\, \mathbb{1}_{\{(y,a)\in\tilde{\mathcal{C}} \wedge y<x_1(a)\}}\big) x_1'(a)\, da \\
\overset{(32,33)}{=} &\int_\infty^{a^*} (P+a)\, \mathbb{1}_{\{(y,a)\in\tilde{\mathcal{C}}\}}\, x_0'(a)\, da \\
&+ \int_{x_0^*}^{1/2} P\, \mathbb{1}_{\{y>x\}}\, dx + \int_{1/2}^{x_1^*} (P+a^*)\, \mathbb{1}_{\{y>x\}} - a^*\, \mathbb{1}_{\{y<x\}}\, dx \\
&+ \int_{a^*}^{\infty} \big((P+a)\, \mathbb{1}_{\{y>x_1(a)\}} - a\, (\mathbb{1}_{\{(y,a)\in\tilde{\mathcal{C}}\}} - \mathbb{1}_{\{y>x_1(a)\}})\big) x_1'(a)\, da \\
= &\; C_0(y) - \int_{a^*}^{\infty} \big((P+a)\, x_0'(a) + a\, x_1'(a)\big)\, \mathbb{1}_{\{(y,a)\in\tilde{\mathcal{C}}\}}\, da
\end{aligned} \tag{35}$$

for some function $C_0(y)$ independent from $\tilde{\mathcal{C}}$. Similarly, *conditional on a flip occurring* the expected payoff of player Y is

$$C_1(y) + \int_{a^*}^{\infty} \big(a\, x_0'(a) + (P+a)\, x_1'(a)\big)\, \mathbb{1}_{\{(y,a)\in\tilde{\mathcal{C}}\}}\, da \tag{36}$$

for some other function $C_1(y)$ independent from $\tilde{\mathcal{C}}$. Combining (35, 36) and using the fact that $x_0(a)$, $x_1(a)$ as defined in (27, 28) satisfy (24) yield that the total expected payoff against X's strategy (30) for a fixed hand y is $(1-q)\, C_0(y) + q\, C_1(y)$, independently from $\tilde{\mathcal{C}}$. □

Claim 8. *For any function $\widetilde{a}(x) : (0, x_0^*) \to (a^*, \infty)$, X's betting strategy*

$$a(x) = \begin{cases} \widetilde{a}(x), & x \in (0, x_0^*) \\ 0, & x \in [x_0^*, 1/2] \\ a^*, & x \in (1/2, x_1^*] \\ a, & x = x_1(a) \in (x_1^*, 1) \end{cases}$$

is a best response to Y's calling strategy (31)*.*

Proof. The first decision X has to make is whether to bet an amount $a > a^*$ or an amount $0 \le a \le a^*$. As in the latter case Y will always call, X's expected payoff is given by (7) and thus maximized for $a = 0$ if $x < 1/2$, and for $a = a^*$ if $x > 1/2$. The resulting expected payoffs are

$$P\big((1-2q)x + q\big) \tag{37}$$

and

$$x\big((P+a^*)(1-q) - a^*q\big) + (1-x)\big((P+a^*)q - a(1-q)\big) = Px \;\;, \tag{38}$$

respectively.

We need to compare these values to the expected payoff X can achieve by betting more than a^*. Betting any amount $a > a^*$ with a hand of $x \le y_1(a)$ yields an expected payoff of

$$y_1(a)\,P + (1-y_1(a))\,\big((P+a)q - a(1-q)\big) = P\,(1+4q)/(7-2q)\ , \tag{39}$$

independently of a and x. (This independence of a is a consequence of the fact that $y_1(a)$ as defined in (29) satisfies (21).) Comparing (39) to (37) shows that for hands $x < (1-q)/(7-2q) = x_0^*$, this is indeed better than checking, reflecting the boundary condition (25).

Similarly, betting any amount $a > a^*$ with a hand of $x > y_1(a)$ yields an expected payoff of

$$y_1(a)\,P + (x-y_1(a))\,\big((P+a)(1-q) - aq\big) + (1-x)\,\big((P+a)q - a(1-q)\big)\ , \tag{40}$$

which is increasing for a near a^* and decreasing for $a \to \infty$, as can be seen from the expression in (22). Thus the expected payoff is maximized for the unique $a \in (a^*, \infty)$ satisfying (22), which is a satisfying $x_1(a) = x$ (cf. (23)). The resulting expected payoff is

$$P\,\big(10(2-q)/(7-2q) - x - 4\sqrt{3(1-x)(1-q)/(7-2q)}\big)\ , \tag{41}$$

as is found by inverting (28) and plugging the resulting function $a(x)$ into (40). Comparing this to (38) shows that for hands $x > (4+q)/(7-2q) = x_1^*$ this is indeed better than the alternative of betting a^*. □

From the equilibrium pair of deterministic strategies (30, 31), we can easily calculate the value of the game. Let E_x denote X's expected payoff when holding a hand of x, with X and Y playing strategies (30, 31). Combining (37, 38, 39, 41), we obtain

$$E_x = \begin{cases} P\,\frac{1+4q}{7-2q}, & x \in (0, x_0^*) \\ P\,\big((1-2q)x + q\big), & x \in [x_0^*, 1/2] \\ Px, & x \in (1/2, x_1^*] \\ P\Big(\frac{10(2-q)}{7-2q} - x - 4\sqrt{\frac{3(1-x)(1-q)}{7-2q}}\Big), & x \in (x_1^*, 1) \end{cases} .$$

Integrating E_x over $(0,1)$ we obtain $V_{II}(q)$ as stated in Theorem 2.

References

[1] S. Bargmann. On the theory of games of strategy. 40:13–42, 1959. English translation of [12].

[2] R. Bellman and D. Blackwell. Some two-person games involving bluffing. *Proceedings of the National Academy of Sciences of the United States of America*, 35(10):600–605, 1949.

[3] É. Borel. *Traité du Calcul des Probabilités et ses Applications Volume IV, Fascicule 2, Applications aux jeux des hazard*. Gautier-Villars, Paris, 1938.

[4] B. Chen and J. Ankenman. *The Mathematics of Poker*. ConJelCo, New Pittsburgh, PA, 2006.

[5] W. H. Cutler. An optimal strategy for pot-limit poker. *Amer. Math. Monthly*, 82:368–376, 1975.

[6] C. Ferguson, T. S. Ferguson, and C. Gawargy. U(0,1) two-person poker models. In *Game theory and application, XII*, volume 12 of *Game Theory Appl.*, pages 17–37. Nova Sci. Publ., Hauppauge, NY, 2007.

[7] A. J. Goldman and J. J. Stone. A continuous poker game. *Duke Math. J.*, 27:41–53, 1960.

[8] S. Karlin and R. Restrepo. Multistage poker models. In *Contributions to the theory of games, vol. 3*, Annals of Mathematics Studies, no. 39, pages 337–363. Princeton University Press, Princeton, N. J., 1957.

[9] D. J. Newman. A model for 'real' poker. *Operations Research*, 7(5):557–560, 1959.

[10] S. Sakai. A model for real poker with an upper bound of assets. *J. Optim. Theory Appl.*, 50(1):149–163, 1986.

[11] A. H. Taub, editor. *John von Neumann: Collected Works. Volume VI: Theory of Games, Astrophysics, Hydrodynamics and Meteorology.* 1963.

[12] J. von Neumann. Zur Theorie der Gesellschaftsspiele (in German). *Mathematische Annalen*, 100:295–320, 1928. English translation in [1]. Reprinted in [11, Paper 1].

[13] J. von Neumann and O. Morgenstern. *Theory of Games and Economic Behavior*. Princeton University Press, 1944.