entries of the matrix A. What other matrix properties can be characterized or inferred from algebraic inequalities? For example: Is there a polynomial $p(X)$ in the entries of X such that A is similar to a diagonal matrix if and only if $p(A) \neq 0$?

R. Merris used the principle of the irrelevance of algebraic inequalities to show that the answer is no. Also it can be shown that there is no polynomial $q(X)$ such that $q(A)$ is the ith coefficient of the minimal polynomial of A, for all matrices A.

References

1. M. Marcus and W. Watkins, Elementary divisors of derivations, Journal of Linear and Multilinear Algebra, 2, June 1974, 65–80.
2. S. Perlis, Theory of Matrices, Reading, Mass., 1958.
3. O. Taussky, The factorization of the adjugate of a finite matrix, Linear Algebra and its Applications, 1 (1968) 39–41.
4. O. Taussky, A generalization of matrix commutativity, Linear Algebra and its Applications, 2 (1969) 349–353.
5. O. Taussky and H. Wielandt, Linear relations between higher additive commutators, Proc. AMS, 13 (1962) 732–735.
6. B. L. van der Waerden, Modern Algebra, Vol. I, Ungar, New York, 1949, English translation.
7. H. Weyl, The Classical Groups, 2nd edition, Princeton, N.J., 1946.

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AN OPTIMAL STRATEGY FOR POT-LIMIT POKER

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1. Introduction. Poker games yield a wealth of problems for the game theorist. John von Neumann and Oskar Morgenstern analyzed two simple poker games in *Theory of Games and Economic Behavior* and showed that bluffing is a very important part of the game. Later papers generalize some of the results and analyze different games. For a good bibliography, see [1]. The papers may be divided into two groups, the theoretical and the practical. The theoretical deal with extremely simplified games, almost exclusively having only two players (an exception is [4]). This is because the theory of the finite two-person zero-sum game is quite handily taken care of by the Minimax Theorem ([5], p. 153). This tells us that each player has an optimal strategy, in the sense that if both players play these strategies, then neither player can improve his lot, assuming that the other player does not change his strategy. Of course, if one player plays a different, inferior strategy, the other player must usually also change to a non-optimal strategy to get best results. If we remove finite from the hypothesis, the Minimax Theorem does not always hold (see [8]). Many of the poker games analyzed are not finite, but for all of them the Minimax Theorem does hold. If we allow more than two players, the situation becomes very different, and the theory is much more complicated.

The practical papers deal with some popular poker games, such as draw poker, and deal with many players. Computers are used to analyze the games, and the results are not optimal strategies, but approximations.

This paper is theoretical and involves a game of two players. We shall begin by discovering optimal strategies for several simple games, and shall then solve a game which is complicated enough to be of some interest. We shall conclude by looking at a poker game which can be played with cards, and might be called "deal 'em and bet 'em". Rather than looking at all possible strategies (of which there will be uncountably many), we will search for pairs of optimal strategies. This will be done by making various assumptions about these strategies which will lead to simultaneous equations. Solution of these equations then results in strategies meeting all of the conditions.

For those unfamiliar with poker games, we shall define some terms. At the start, each player puts a certain fixed number of chips into the pot. This is called the **ante.** The cards are then dealt. The first player, hereafter denoted A, can then either **check** (pass the opportunity to bet) or **bet,** whereby he adds additional chips to the pot. It is now the second player's (player B) turn. If A checks, then B can either check, in which case there is a showdown, or bet. If A bets, then B can either **fold** (forfeit his chance at the pot), **call** (in which case he adds the same number of chips to the pot as A bet, and there is a showdown), or **raise,** in which case he adds as many chips to the pot as A bet and some more in addition. If B has bet or raised, then the action reverts to A, who must either fold, call, or raise. If A checks, B bets, and A raises, this is called a **check-raise.** The betting continues until either one player folds or there is a showdown. If a showdown occurs, high hand wins the pot.

In our game, dealing the cards will consist of each player randomly drawing a real number in $[0, 1]$. This game will later be used to approximate optimal strategies in the case where cards are dealt. In [3] the game is analyzed where k raises are allowed, each of exactly n chips. This is quite similar to our game. We shall set the ante at 1 chip, and each bet or raise must be the size of the pot at that time. That is, after the ante, there are 2 chips in the pot, hence the first bet is 2 chips. If the initial bet is made and called, there are then 6 chips in the pot, so the first raise is 6 chips. The second raise is 18, the third 54, etc. This limitation on betting size is rather severe, but is the size of bet recommended by some experts in a no-limit or table stakes game. See [2], p. 59. It will become apparent later that under these conditons the game is easier to analyze than the so-called limit games, where bets and raises are restricted to a certain fixed number of chips.

2. Some simple cases. First of all, let us look at the game analyzed on pp. 211–219 of [5]. In this game, the ante is 1 chip, and player A can either check or bet 2 chips. If A checks, there is a showdown. If A bets, then player B may either call or fold. This game is obviously in A's favor. Player A has the following strategies: For each $x \in [0,1]$ he may either bet or check. Hence his pure strategies are characteristic functions on $[0,1]$ and his mixed strategies are functions $b: [0,1] \to [0,1]$ where if A is dealt x, he bets with probability $b(x)$. Similarly, B's mixed strategies are functions $c: [0,1] \to [0,1]$, where if B is dealt y and A bets, then B calls with probability $c(y)$.

We shall see that there are pure strategies for both players that are optimal strategies. Let us start by analyzing the game from B's point of view. Suppose he has discovered A's strategy. Then, to decide if he should call A is quite simple. First of all, to call the bet he must risk 2 chips in an attempt to win 4 chips. Hence the pot is giving him 2 to 1 odds. Therefore he should call if he has a $\frac{1}{3}$ or better chance of winning. Specifically, he should call with hand y if

$$u = \frac{\int_0^y b(x)dx}{\int_0^1 b(x)dx} > \frac{1}{3},$$

fold if $u < \frac{1}{3}$, and it doesn't matter if $u = \frac{1}{3}$. Since $f(y) = \int_0^y b(x)dx$ is an increasing function, we note that there is either a single point or an interval where $u = \frac{1}{3}$. Let us suppose, for simplicity, that B chooses the following strategy: There is a fixed number $c \in (0,1)$, and B calls if and only if $y \geqq c$.

Let us now return to player A. Assuming B plays the previously mentioned strategy, then what are A's expected returns with hand x?

If he checks, his return is

$$E_{\text{check}} = 1(x) + (-1)(1-x) = 2x - 1. \tag{1}$$

If he bets, his return is

$$\begin{aligned} E_{\text{bet}} &= 1(c) - 3(1-c) = 4c - 3 && \text{if } c > x, \\ &= 1(c) + 3(x-c) - 3(1-x) = 6x - 2c - 3 && \text{if } c < x. \end{aligned} \tag{2}$$

Whether A should bet or check depends on which expected return is larger. If $c > \frac{1}{2}$, we get two intervals of betting. The first is $[0, a]$. These are A's **bluffs** (he really has a poor hand, and hoping B folds is the only way he can win). The second is A's legitimate high hand bets, in the interval $[b, 1]$. Since a and b are cut-off points between betting and checking, they must be places where $E_{\text{check}} = E_{\text{bet}}$. Hence, assuming $a < c < b$, we get $2a - 1 = 4c - 3$ or

$$2c - a = 1 \tag{3}$$

and

$$2b - 1 = 6b - 2c - 3 \text{ or}$$

$$2b - c = 1. \tag{4}$$

Finally, since for $y = c$ the probability of B's winning must be $\frac{1}{3}$, we get that the high bets and bluffs must be in the ratio of 2 to 1 or

$$2a + b = 1. \tag{5}$$

Solving (3), (4), and (5) simultaneously we get $a = \frac{1}{9}$, $b = \frac{7}{9}$, and $c = \frac{5}{9}$. Note that all assumed inequalities hold, and hence we have optimal strategies. Hence A must bet if $x < \frac{1}{9}$ or $x > \frac{7}{9}$ and check elsewhere. B must call if $y > \frac{5}{9}$ and fold otherwise.

We shall in general not try to figure out all of the optimal strategies, but for this particular case we might mention that A has only this one optimal strategy, while B has many. (He must always fold if $y < \frac{1}{9}$ and call if $y > \frac{7}{9}$, but if $\frac{1}{9} < y < \frac{7}{9}$ he may vary his strategy.)

Before we continue, it is well to note, as is always the case later, that the ratio of high bets to bluffs is 2 to 1, and the call line falls somewhere in between.

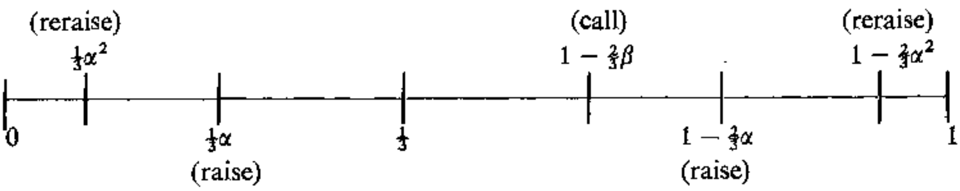
Now let us analyze the game of "force-in." In this game, A is forced to bet, regardless of his hand. B can then fold, call, or raise, and so on. We shall allow for an unlimited number of reraises. The rule that the first player is forced to bet is quite frequently used, and in our study it will help to simplify things.

First we note that since $y = \frac{1}{3}$ is the cut-off point at which B's expectation of winning is $\frac{1}{3}$, then this is the cut-off point between calling and folding. Next we note that B will want to raise on his best hands, say $a \leqq y \leqq 1$. We shall denote the ratio $(1 - a)/(\frac{2}{3})$ by α. Hence $a = 1 - \frac{2}{3}\alpha$. α is the percentage of good hands that B chooses to raise with. Since the ratio of high bets to bluffs must be 2 to 1, he must also raise with half as many bad hands. As we shall see later, it does not matter which bad hands he bluffs, as long as they are hands he would otherwise fold. Let us let him bluff the worst hands, or for $0 \leqq y \leqq \frac{1}{3}\alpha$.

Now if B raises, what does A do? We note a curious thing. A is in exactly the same position as B was before, except that the pot is 3 times as large. A knows that B holds in the range $[0, \frac{1}{3}\alpha]$ or $[1 - \frac{2}{3}\alpha, 1]$, which are in the ratio of 1 to 2. Hence the cut-off point for calling is somewhere between $\frac{1}{3}\alpha$ and $1 - \frac{2}{3}\alpha$, call it b. We denote the ratio $(1 - b)/(\frac{2}{3})$ by β. This is the percentage of good hands that A calls with.

He will also reraise on the α best hands of the collection of good hands that B raised with. Hence A reraises in $[1 - \frac{2}{3}\alpha^2, 1]$. He also bluff-reraises with hands $[0, \frac{1}{3}\alpha^2]$.

Let us graph these points:



We will suppose that the points lie in the relative positions indicated. We will now endeavor to compute α and β.

$1 - \frac{2}{3}\alpha$ is the cut-off between calling and raising (but not calling a reraise), hence the expected returns for each action must be the same.

Therefore

$$E_{\text{call}} = 3(1 - \tfrac{2}{3}\alpha) - 3(\tfrac{2}{3}\alpha) =$$

$$E_{\text{raise}} = -9(\tfrac{1}{3}\alpha^2) + 3((1 - \tfrac{2}{3}\beta) - \tfrac{1}{3}\alpha^2) + 9((1 - \tfrac{2}{3}\alpha) - (1 - \tfrac{2}{3}\beta)) - 9(\tfrac{2}{3}\alpha)$$

which reduces to

$$\alpha^2 + 2\alpha - \beta = 0. \tag{6}$$

Similarly, checking the bluff-raise cut-off point ($\frac{1}{3}\alpha$) we get the equation

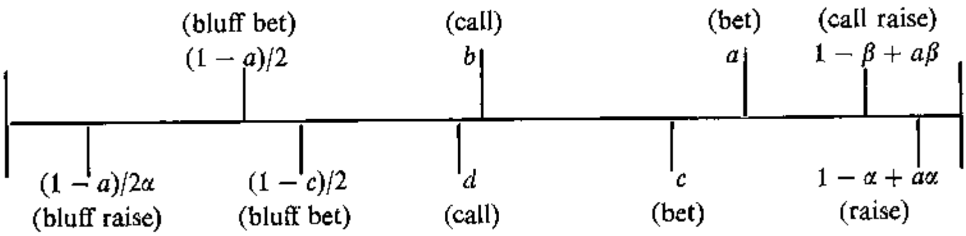
$$-1 = -9(\tfrac{1}{3}\alpha^2) + 3((1 - \tfrac{2}{3}\beta) - \tfrac{1}{3}\alpha^2) - 9(\tfrac{2}{3}\beta) \text{ or } \alpha^2 + 2\beta = 1. \tag{7}$$

Solving (6) and (7) simultaneously, we get $\alpha = \frac{1}{3}\sqrt{7} - \frac{2}{3} \cong 0.215$, and $\beta = \frac{2}{9}\sqrt{7} - \frac{1}{9} \cong 0.477$.

3. The main case. We are now ready to analyze the game with unlimited betting, restricted only by forbidding check-raises. Player A will presumably want to bet on his better hands, say for $a \leqq x \leqq 1$. He will bluff half as often, or for $0 \leqq x \leqq (1-a)/2$. If Player B is confronted by a check, this indicates that $(1-a)/2 \leqq x \leqq a$, hence he will bet if $y \geqq c$, where $c < a$. He will also bluff if $y \leqq (1-c)/2$.

Now if A checks and B bets, then A will call somewhere in $[(1-c)/2, c]$, call this cut-off point b. If A bets, then B knows that $x \leqq (1-a)/2$ or $x \geqq a$. Hence his call line lies somewhere in $[(1-a)/2, a]$, call it d. Also, B will raise with his best hands. But the situation he is in is precisely the force-in situation, so he will raise on the best α good hands, or hands in $[a, 1]$. Hence he raises if $y \geqq 1 - \alpha + a\alpha$. He also bluff-raises if $y \leqq (1-a)/2\alpha$. Finally, A will call this raise on the best β of his hands in $[a, 1]$, hence for $x \geqq 1 - \beta + a\beta$.

Graphing these points, we get the following picture:



Investigating the bet cut-off at a, we see that

$$3((1-c)/2) + 1(c - (1-c)/2) + 3(a - c) - 3(1 - a)$$

$$= -3((1-a)/2\alpha) + (d - (1-a)/2\alpha) + 3(a-d) - 3(1-a)$$

or

$$2\alpha a + 3c - 2d = 1 + 2\alpha. \tag{8}$$

Investigating the point $(1-a)/2$ we get

$$\alpha a + 2d = 1 + \alpha. \tag{9}$$

Finally, the points c and $(1-c)/2$ yield

$$a + b - 2c = 0 \quad \text{and} \tag{10}$$

$$2a - 4b - c = -1. \tag{11}$$

Solving (8), (9), (10), and (11) simultaneously, we get

$$a = 1 - 1/21\sqrt{7} \cong 0.874$$

$$b = d = 5/9 - 1/63\sqrt{7} \cong 0.514$$

$$c = 7/9 - 2/63\sqrt{7} \cong 0.694.$$

We have now completely solved the problem. Cut-off points for bets and reraises are a, $1-(1-a)\alpha$, $1-(1-a)\alpha^2$, $1-(1-a)\alpha^3, \cdots$ or 0.874, 0.973, 0.994 and 0.999. Call lines for raises are $1-(1-a)\beta$, $1-(1-a)\alpha\beta$, $1-(1-a)\alpha^2\beta$, $1-(1-a)\alpha^3\beta, \cdots$ or 0.940, 0.987, 0.997 and 0.999 The number of bluffs to be made are $(1-a)/2\alpha$, $(1-a)/2\alpha^2, \cdots$ or 0.063, 0.014, 0.003 etc.,

Now a word about the bluff-raises. If it is raised back to either player, then some bluff-raises should be made. The bluff-raises should be made with hands which otherwise would have been folded. If the opponent is playing an optimal strategy, then he will never call with a hand worse than any of these, hence it makes no difference which of these hands are bluffed. However, a poor player might call on a weak hand, hence in our strategy we will bluff-raise on the best such hands.

Let us also look at those forbidden check raises. Suppose A is dealt $x = .98$. Our strategy advises him to bet and call a raise. Simple computation will show, however, that if check raises are permitted, he will come out better if he checks, raises if bet into, and folds if reraised. Hence check raises, if permitted, will be used. However, solving the problem with check raises appears to be quite difficult, as the force-in game cannot be applied.

4. Limiting the raises. In any poker game, there is either a limit on the number of raises or a limit on the amount that a given player can bet. This is obviously necessary from a practical standpoint. We shall see, however, that if the game is limited to k raises of pot size, then the game can essentially be played as the game with unlimited raises.

For example, it can easily be seen that the last raise should be made with the best $\frac{1}{4}$ of those hands that the previous raise was made with. Hence $\alpha = .250$ for this raise. Also, $\beta = .500$.

For the second last raise, $\alpha = .211$ and $\beta = .474$, and for the third last raise, $\alpha = .216$ and $\beta = .477$. Hence, these values converge rather rapidly to the .215 and .477 of the previous section.

5. Other strategies. It can be computed easily that the above mentioned strategy, when played against an optimal strategy, will yield a return of about -0.093 for A. This is the value of the game to A. Hence, as would be expected, the game favors B.

Let us suppose that B is superconservative. Suppose he calls A if $y \geqq .514$, folds otherwise, and never bets or raises. This will give a return to A of about 0.091.

On the other hand, suppose that B is a super-bluffer. That is, he always bets or raises when given the chance. The return to A for this case is also about 0.091.

Hence we see that if A plays the optimal strategy, the return is very small under any of B's possible strategies. We can notice that if B changes his call or bluff lines from the optimal strategy, he usually does not lose anything. For instance, if A has bet, and B's hand falls in the category where he will win if and only if A is bluffing, then his returns for raising, calling, or folding are all the same. Also, if B holds a hand that he is sure is beaten, then it does not matter whether he folds or raises.

Changing the line for legitimate bets or raises will result in a loss to the player, however.

Hence we can get some clue as to the nature of the set of all optimal strategies. High hand raises must be made in precisely the intervals indicated. Calls must be made in the interval between possible bluffs and high hand bets, and must be in the proper ratio with the number of high hand bets. Bluffs may be made almost anywhere, but must be in proper proportion to high hand bets.

6. The discrete game. We will now consider the discrete game, where each player is dealt a 5-card hand from a regular 52-card deck, and the betting proceeds as before. (There is no draw.) There are 2,598,960 different hands, and they are linearly ordered. Straight flushes are best, followed by four-of-a-kind, full house, flush, straight, three-of-a-kind, two pair, one pair, and bust. We shall get our approximate optimal strategies for this game in the obvious way. If the 5-card hand of a player ranks n (from the bottom) out of 2,598,960, we treat his hand as if he were dealt $n/2{,}598{,}960$ in the continuous game.

There are at least 3 objections to this approximation. First of all, some different hands are equal, as the ordering disregards suits. This is a minor objection. Second, the optimal strategies for the discrete case may not be gotten this way from the optimal strategies for the continuous case. This is serious, but according to [5], p. 207, the error is about 1/2,598,960. Finally, the hands are dealt from one deck, not two. That is, what one player holds affects what another player may hold. Milton Parnes pointed out to this author that increasing the rank of a hand does not necessarily increase its value. In fact, he conjectures that there is a discontinuity at each change of rank. For instance, if you hold a straight flush to the 5, your opponent may hold 31 higher straight flushes or 3 equal ones. However, if you hold four aces with a six kicker, your opponent may only beat you with 27 different straight flushes.

Ignoring all of these difficulties, our approximations for the optimal strategies come out as follows. For the first player, we get:

With	*Procedure*
J9632 or worse	bet, but fold if raised
J9632 — 22J109	check and fold if a bet
22Q43 — KKJ85	check and call
KKJ86 — 9922J	bet and fold if raised
9922Q — 99778	bet, reraise, and fold
997710 — 9 high straight	bet and call a raise
9 high str. — K8743 flush	bet, reraise, and fold
K8752 flush — K10965 flush	bet, 2 reraises and fold
K10972 flush — AAAJJ	bet, reraise, and call
AAAQQ — 88887	bet, 2 reraises, and fold
88889 — 8888Q	bet, 3 reraises, and fold
8888K — 6 high str. flush	bet, 2 reraises, call
7 str. fl. — 10 str. fl.	bet, 3 reraises, fold
J str. fl. — K str. fl.	bet, 3 reraises, call
A str. fl.	bet, 4 reraises

For the second player the strategy is:

With	*Procedure*
QJ763 or worse	fold if bet into, bet otherwise
QJ764 — AKQJ6	fold — check
AKQJ7 — 22J109	Raise if bet into, fold a reraise check if checked to
22Q43 — 77AQ4	call — check
77AQ5 — 222AQ	call — bet
222AK — JJJ87	raise and fold if reraised
JJJ92 — JJJK2	raise twice and fold
JJJK3 — 555JJ	raise and call
555QQ — JJJKK	raise twice and fold
JJJAA — QQQ33	3 raises and fold
QQQ44 — QQQQ9	2 raises and call
QQQQ10 — AAAA4	3 raises and fold
AAAA5	4 raises and fold
AAAA6 — Q str. fl.	3 raises and call
K str. fl.	4 raises and fold
A str. fl.	4 raises and call

Note that if the first player is dealt an eight high straight flush, he is supposed to bet, reraise three times, and fold if raised back. You will not catch this author making that particular play, even though it is undeniably part of the optimal strategy. The reason is two-fold; first, the given strategy is very conservative, and most players play much more loosely. Second, the author, like practically all poker players, has a tendency to "fall in love with his cards." Nevertheless, inspection of the strategy shows that $\frac{1}{2}$ of all good hands should be folded when reraised.

To continue in this vein, there is a well-known story, told on pp. 65–68 of [6], where "Straights" Fowler and Dundee have been raising each other the limit of $25 for awhile. With about $700 in the pot, "Straights" raises one more time, whereby Dundee folds, convinced that his four Queens and Ace kicker will lose to four Kings or better. Irv Roddy, on pp. 165–166 of [7] is quite scornful. Our analysis goes something as follows: Since the pot is offering 28 to 1 odds, "Straights" should be bluffing about 1 time in 29. For the bluff to be a break-even proposition, Dundee should be calling about 28 times out of 29. Hence Mr. Roddy is correct in saying that Dundee should usually call; but there is that one time in 29!

For those who have decided to earn their living playing this paper's strategy, a word of warning. This strategy is very conservative, and playing against most any player, you will find that he quickly tightens up also. The return is very small, if not negligible. One two hour session with a new opponent netted 6 chips.

Finally, playing this game is about as exciting as watching paint dry; in that two hour session, there were two raises!

References

1. Nicholas V. Findler, et. al, Studies on Decision Making Using the Game of Poker, Information Processing, 71-North-Holland Publishing Company, 1972.

2. Oswald Jacoby, Oswald Jacoby on Poker, Doubleday and Co., 1947.

3. Samuel Karlin and Rodrigo Restrepo, Multistage Poker Models, Contribution to the Theory of Games, Vol. 3, Princeton University Press, 1957, pp. 337–364.

4. J. F. Nash and L. S. Shapley, A Simple Three-Person Poker Game, Contribution to the Theory of Games, Vol. 1, Princeton University Press, 1950, pp. 105–116.

5. John von Neumann and Oskar Morgenstern, Theory of Games and Economic Behavior, Princeton University Press, 1944.

6. *Poker According to Maverick.*

7. I. Roddy, Friday Night Poker, Simon and Schuster, New York, 1961.

8. Maurice Sion and Philip Wolfe, On a Game Without Value, Contribution to the Theory of Games, Vol. 3, Princeton University Press, 1957, pp. 299–306.

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