Simplified Von Neumann Poker: Risk-Averse Players, When to Fold, When to Call?

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Picture yourself at the poker table, every decision a crucial step toward victory or defeat. Poker is not just a game of luck; it is a battlefield where strategy and probability rule. It was analyzed by the Hungarian-American mathematician John von Neumann, who believed that real life mirrors poker, involving bluffing and strategic thinking. Together with Oskar Morgenstern, he analyzed poker, resulting in their 1944 book *Theory of Games and Economic Behavior*, which laid the foundation for groundbreaking mathematical theory of economic and social organization.

Simplified Von Neumann Poker

In this paper, we consider a simplified von Neumann poker game, the payoff of which is summarized in Figure 1. This model is the simplest because raising does not play a role, and Player 2 does not play.

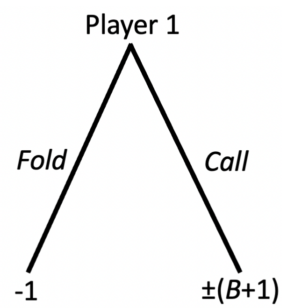

Figure 1: Payoff of the simplified von Neumann poker

Player 1 antes \$1 into the pot. Player 2 antes $\$(B+1)$ into the pot. Each player is then dealt a random card between 1 and n. After seeing their own card, Player 1 can choose to "fold," in which case Player 2 wins the pot. Player 1 can alternately "call," which means to match the $B+1$ dollars to the pot. Then the two cards are compared, and the player with the higher number wins.

Objectives

Our goal is to find Player 1's optimal strategy and conduct mathematical experiments through Maple to understand its implications on concepts such as moments, coefficient of variation, and probability generating functions. We conclude the paper with a discussion of strategies for risk-averse players who prefer lower payoff uncertainty over higher uncertainty.

Let's shuffle up and deal as we explore the mathematics of poker, pioneered by the founder of Game Theory.

Step-by-step procedure

First step: Assume that the given card to Player 1 is x. The initial question to ask is "When to call?"

Player 1 will call (rather than fold) if

$$\begin{aligned} \text{Payoff}_{\text{call}} > \text{Payoff}_{\text{fold}} &\iff (B+1)\frac{x-1}{n-1} - (B+1)\frac{n-x}{n-1} > -1 \\ &\iff x > \frac{Bn+B+2}{2(B+1)}. \end{aligned}$$

Thus, the strategy for Player 1 is to fold if the given card $x \leq \left\lfloor \dfrac{Bn+B+2}{2(B+1)} \right\rfloor$, and to call otherwise.

Notation: From here on, for simplicity, we assume $\dfrac{Bn+B+2}{2(B+1)}$ is an integer, and we denote this cutoff integer value as C.

Step 2: "What is the expected payoff?"

Let P_x be the random variable representing payoff of Player 1 under this strategy. Then,

$$P_x = \begin{cases} -1 & \text{if } x \leq C \text{ (fold)} \\ \frac{(B+1)(2x-n-1)}{n-1} & \text{if } x > C \text{ (call).} \end{cases}$$

It follows that the expected payoff according to this strategy is

$$\mu_C := E[P_x] = \frac{1}{n}\left[\sum_{x=1}^{C}(-1) + \sum_{x=C+1}^{n} \frac{(B+1)(2x-n-1)}{n-1}\right].$$

Moreover, we obtain the following formulas:

- Since C is an integer, μ_C simplifies to
$$\mu_C = \frac{B(Bn+B+2)}{4(B+1)n}.$$

- When $B=2$,
$$\mu_C = \frac{n+2}{3n}.$$

Step 3: "What is the uncertainty?"

The variance of this play, if Player 1 follows this strategy, would be:
$$\sigma_C^2 = \frac{1}{n}\left[\sum_{x=1}^{C}(-1-\mu_C)^2 + \sum_{x=C+1}^{n}\left(\frac{(B+1)(2x-n-1)}{n-1}-\mu_C\right)^2\right].$$

- Again, since C is assumed to be integer, by letting $B=2$, we have that
$$\sigma_C^2 = \frac{4(2n+1)(2n^2+1)}{9n^2(n-1)}.$$

Step 4: "Higher moments?"

We calculate the higher moments about the mean:
$$E[(P_x-\mu_C)^k] = \frac{1}{n}\sum_{x=1}^{n}(P_x-\mu_C)^k = \frac{1}{n}\left[\sum_{x=1}^{C}(-1-\mu_C)^k + \sum_{x=C+1}^{n}\left(\frac{(B+1)(2x-n-1)}{n-1}-\mu_C\right)^k\right].$$

- We can use Maple to compute these quantities for each k when $B=2$.
$$\begin{aligned} E[(P_x-\mu_C)^3] &= \frac{8(2n+1)^2(n+2)}{27n^3}, \\ E[(P_x-\mu_C)^4] &= \frac{16(2n+1)(24n^6-8n^5-32n^4-20n^3-55n^2+5n+5)}{135\,n^4(n-1)^3}, \\ E[(P_x-\mu_C)^5] &= \frac{32(n+2)(4n-1)(4n^3-n^2-10n-2)(2n+1)^2}{243\,n^5(n-1)^2}, \\ &\ldots \end{aligned}$$

Step 5: "Scaled moments as $n\to\infty$?"

Let's look at the scaled moments: $\dfrac{E[(P_x-\mu_C)^k]}{E[(P_x-\mu_C)^2]^{k/2}}$. The sequence, starting at $k=1$, as $n\to\infty$, is
$$0, 1, \frac{1}{2}, \frac{9}{5}, 2, \frac{31}{7}, \frac{27}{4}, 13, \frac{112}{5}, \frac{459}{11}, \frac{151}{2}, \frac{1825}{13}, \frac{1818}{7}, \frac{2429}{5}, \frac{7279}{8}, \ldots.$$

This distribution is right-skewed as the third scaled moment, skewness, is positive.

Generating Functions

Next we use the probability generating function to double-check our results and see if there is any closed-form solution.

$$f(t) := E[t^{P_x}] = \frac{1}{n}\sum_{x=1}^{n} t^{P_x} = \frac{1}{n}\left(\sum_{x=1}^{C} t^{-1} + \sum_{x=C+1}^{n} t^{\frac{(B+1)(2x-n-1)}{n-1}}\right).$$

Remark: Unfortunately, we can't find a nice closed-form for $f(t)$.

Again, let $B = 2$ and assume C is an integer. We can verify some moments, i.e.

$$E[P_x] = f'(t)|_{t=1}, \quad E[P_x^2] = [t \cdot f'(t)]'|_{t=1}, \quad \text{etc.}$$

We actually checked a couple of moments, and they agree with the formula we had earlier. I learned that we can use command *limit*, i.e. `limit(diff(A,t),t=1);`

Risk-averse

Here, we have the freedom to vary the cutoff point C, that is, we will fold if $x \leq C$ and call otherwise.

The average payoff as a function of C (for $B = 2$) is

$$\begin{aligned}\mu_C &= \frac{1}{n}\left[\sum_{x=1}^{C}(-1) + \sum_{x=C+1}^{n} \frac{(B+1)(2x-n-1)}{n-1}\right] \\ &= \frac{C(Bn - BC - C + 1)}{n(n-1)} \\ &= \frac{C(2n - 3C + 1)}{n(n-1)}.\end{aligned}$$

The variance in terms of C (for $B = 2$) is

$$\begin{aligned}\sigma_C^2 &= \frac{1}{n}\left[\sum_{x=1}^{C}(-1-\mu_C)^2 + \sum_{x=C+1}^{n} \left(\frac{(B+1)(2x-n-1)}{n-1} - \mu_C\right)^2\right] \\ &= \frac{(n-C)(9C^3 - 6C^2 + 9C^2 n - 2nC + C - 5Cn^2 + 3n^3 - 3n)}{n^2(n-1)^2}.\end{aligned}$$

We now look at the coefficient of variation $CV = \dfrac{\sigma_C}{\mu_C}$.

Discussion for $B = 2, n = 100$:

By Maple's calculation, we have that CV is at its minimum when $C = n = 100$ (always fold), which gives $CV = 0$. This aligns with our expectations. On the other hand, when extending the CV formula to the continuous space, there is a local minimum at $C = 33.49962007$, which is close to the originally optimal cutoff of $\frac{Bn+B+2}{2(B+1)} = 34$. Note that in the discrete space, both $C = 33$ and $C = 34$ yield the same CV value.

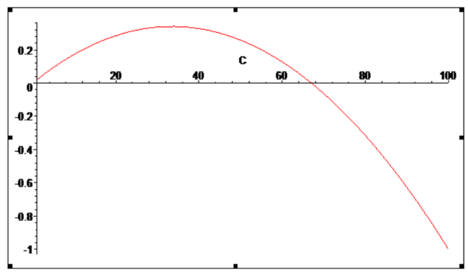

Figure 2: Plot of Player 1's payoff for the strategy with cutoff point C

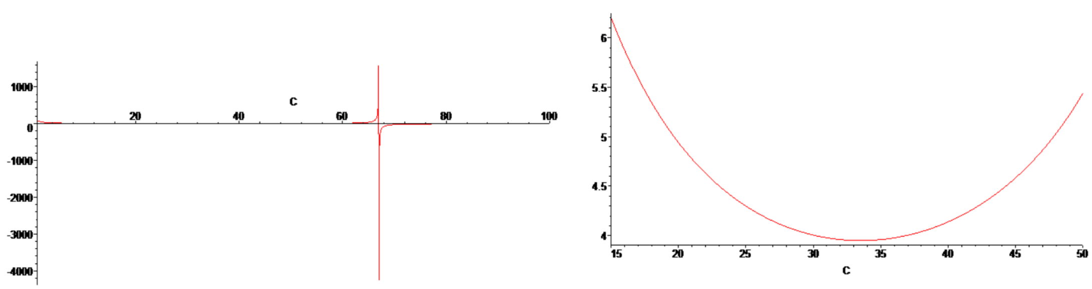

Figure 3: Plot of the CV for each cutoff point strategy from $1 \leq C \leq 100$ (left) and zoomed in on $15 \leq C \leq 50$ (right). A local minimum is at $C = 33.49962007$.

Note that at $C = 67$, $\mu = 0$ so CV blows up at this point. In general $\mu = 0$ when $C = \dfrac{Bn+1}{B+1}$.

An example of the histogram of Player 1's payoff is shown:

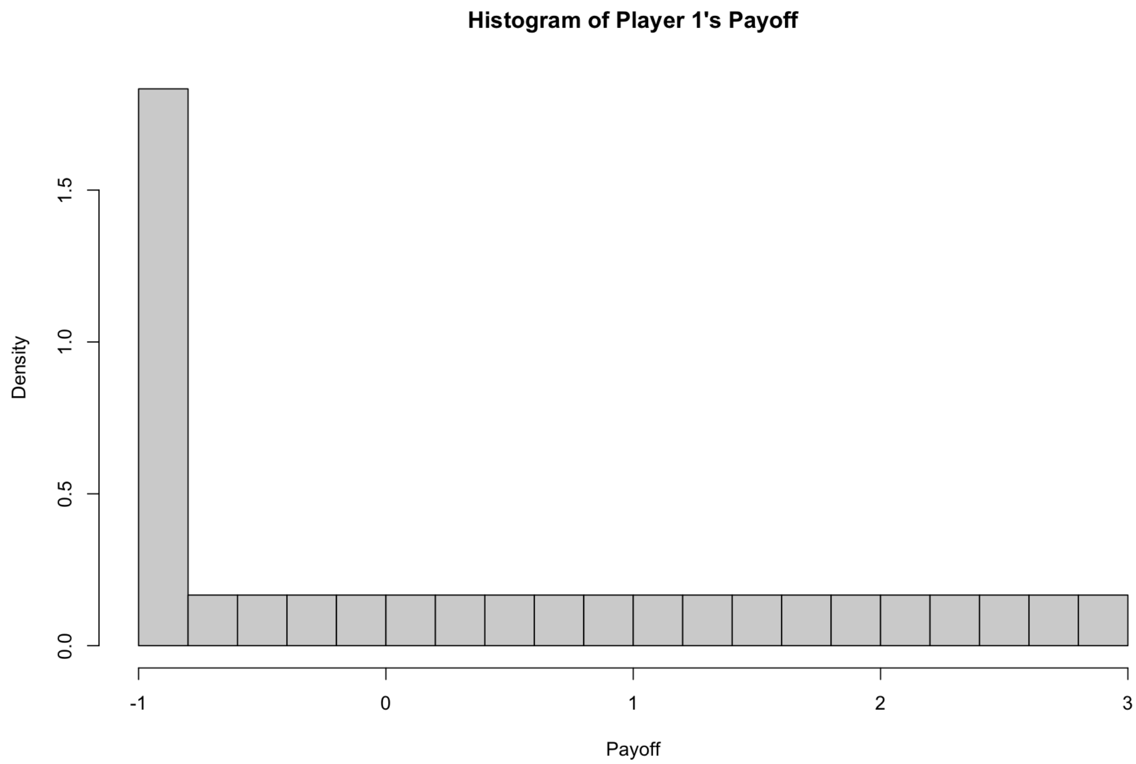

Figure 4: Histogram of Player 1's payoff when $B = 2, n = 100, C = 34$