A SIMPLE THREE-PERSON POKER GAME[1]*

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§1. INTRODUCTION

In the study of games Poker, in its varied forms, has become a popular source of models for mathematical analysis. Various simple Pokers have been investigated by von Neumann,[3] Bellman and Blackwell,[4] and Kuhn.[5,6] Our paper is the first to consider a three-person model. This version has just two kinds of hands, no drawing or raising, and only one size of bet. We suppose that the game is non-cooperative and solve for "equilibrium points." The game turns out to have a well-defined value if the ante does not exceed the amount of the bet, or is more than four times the bet; but no value for at least two transition cases in between.

To cut down the magnitude of the computational task we use "behavior coefficients" in place of mixed strategies. This is an effective technique for a large class of games in extensive form.

§2. THE SOLUTION OF AN n-PERSON GAME

A definition for the solution of an n-person game, $n > 2$, based on the principle of coalition, has been developed by von Neumann and Morgenstern.[7] It is unfortunately weak in its ability to predict the actions of the players, or to ascribe a value to the game. It is most naturally applicable to games (or economic situations) in which the players are free to offer or accept side payments (outside the mechanism of the game itself) in

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[3]J. von Neumann and O. Morgenstern, "Theory of Games and Economic Behavior," 2nd ed., Princeton, 1947; pp. 186-219.

[4]Proc. N. A. S. 35 (1949), pp. 600-605.

[5]In this volume.

[6]E. Borel considers some simple two-person betting models in the course of an analysis of the probabilities of actual Poker games, in his "Traité du Calcul des Probabilités, Paris, 1938; IV, 2, pp. 91-97.

[7]Op. cit., Chapter VI.

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return for cooperation during the play. The customary ethics of Poker suggest that a non-cooperative solution concept, not recognizing side payments and pre-play agreements, would be better for our present purpose. We therefore define:

An *equilibrium point* (or *EP*) is a set of strategy choices, pure or mixed, of the n players, with the property that no player can improve his expectation by changing his own choice, the others being held fixed.

If it happens that each player's expectation is the same in all equilibrium points, then we call the n-tuple of these expectations the *value* of the game.

In a two-person, zero-sum game the equilibrium points are just the minimax points, and describe the solution in the usual sense. It has been shown that finite games always possess EP.[8] They do not necessarily have values; nor are the strategies used in different EP of the same game necessarily interchangeable.

§3. THE RULES OF THE GAME

The deck contains just two kinds of cards, "High" and "Low," in equal numbers. One card is dealt at random to each of the three players. The deck is so large that the eight possible deals occur with equal probability. Each player antes an amount a. The first player has the option of *opening* the bidding with a bet b, or of *passing*. If he passes, the second player has the same opportunity; then the third. When any player has opened, the other two, in rotation, have the choice of *calling* with a bet b, or of *folding* (dropping out), thereby forfeiting the ante money. The payoff rule: If no one opened (three consecutive passes), the players retrieve their antes. Otherwise, the players betting compare cards, and the one with the highest wins the entire accumulation of bets and antes (the *pot*). In case of a tie, the winners divide the pot equally.

There are 13 possible sequences of bids. We may represent them:

BBB BPB PBBB PBPB PPBBB PPBPB PPP
BBP BPP PBBP PBPP PPBBP PPBPP

letting "B" stand for "open" and "call," "P" for "pass" or "fold." With the eight possible deals, there are 104 different plays of the game that can occur. The possible payoffs are seen to be:

$$(2a, -a, -a) \quad (a/2, a/2, -a)$$
$$(2a + b, -a, -a - b) \quad (a/2 + b/2, a/2 + b/2, -a - b)$$
$$(2a + 2b, -a - b, -a - b) \quad (0, 0, 0)$$

[8] J. Nash, Proc. N. A. S. 36 (1950), pp. 48-49.

and their permutations. Clearly it is only the ratio of a to b that is significant, but we shall retain the separate symbols in the hope that mental verifications by the reader of such statements as the above will be made easier.

§4. BEHAVIOR COEFFICIENTS

By a calculation which we do not detail, we find that the three players have respectively 81, 100, and 256 pure strategies. If we forbid those that involve folding on a high card (see below, §7), these numbers reduce to 18, 20, and 32. The equilibrium points are then to be sought in a space of 17 + 19 + 31 = 67 dimensions, the product of the three mixed-strategy simplexes.

However, as commonly occurs in games with several moves, there is great redundancy in this representation. Distinct mixed strategies exist which prescribe identical behaviors for the player in question. This equivalence induces a natural projection of his simplex into a convex polytope of much lower dimension, with each pure strategy going into a distinct extreme point of the polytope. We shall discover that the players have only eight essential dimensions apiece; or five apiece if we outlaw folding on a high card.

The natural way to achieve this economy is by avoiding the description of behavior as a probability mixture of the pure strategies, and instead considering on-the-spot randomizations during the course of play.[9] There are eight situations which require a decision that can face each player; and the decision is always to make or not to make a bet. We therefore introduce as the <u>behavior</u> <u>coefficients</u> the probabilities of betting in the different situations:

Player 1

Faced with:	He holds High — he bets with probability:	Low
--	α	β
PBB	o	π
PBP	ρ	σ
PPB	τ	υ

Player 2

Faced with:	He holds High — he bets with probability:	Low
B	γ	δ
P	ε	ζ
PPBB	φ	χ
PPBP	ψ	ω

Player 3

Faced with:	He holds High — he bets with probability:	Low
BB	η	θ
BP	ι	κ
PB	λ	μ
PP	ν	ξ

BEHAVIOR COEFFICIENTS

[9] This procedure would not be legitimate if each player's information were not monotone increasing.

Every mixed strategy can be completely represented, as far as its effect in the game is concerned, by an octet of values of the appropriate player's behavior coefficients.

§5. IRRELEVANCE

It can happen that if certain of the coefficients take on extreme values (i.e., 0 or 1), then the situations to which other coefficients apply can never arise. For example, if $\alpha = 1$ and $\beta = 1$ then the values of ε and ζ tell nothing about the behavior of Player 2. To keep the representation of behavior unique, we assign the conventional value 1 to an irrelevant coefficient if it refers to a high card, 0 if it refers to a low card This amounts to identifying certain vertices of the 24-dimensional cube of behavior probabilities, and does not lower dimension.

§6. DISCRIMINANTS

The expected payoffs to the three players will be certain multilinear functions[10]

$$P_1(\alpha, \beta, \ldots, \omega), \; P_2(\alpha, \beta, \ldots, \omega), \; P_3(\alpha, \beta, \ldots, \omega)$$

of the behavior coefficients, with terms of degree as high as five. It would be tedious, as well as unnecessary, to attempt to give the explicit functions here. In their place we shall work with the discriminants, $\Delta_\alpha, \Delta_\beta, \ldots, \Delta_\omega$, defined by:

$$\Delta_u = 16 \frac{\partial}{\partial u} P_{k(u)}(\alpha, \beta, \ldots, \omega), \quad u = \alpha, \beta, \ldots, \omega ;$$

where it is the $k(u)$-th player who controls u. (The factor of 16 clears out the fractions arising from the random deal and the divided pots.) Directly from the definition of equilibrium, we have, at any EP:

$$\text{(C)} \qquad \begin{cases} u = 0 \Longrightarrow \Delta_u \leq 0 \\ 0 < u < 1 \Longrightarrow \Delta_u = 0 \\ u = 1 \Longrightarrow \Delta_u \geq 0 \end{cases} \qquad u = \alpha, \beta, \ldots, \omega .$$

A set of coefficient-values satisfying (C) does not necessarily constitute an EP, since the possibility that a player would increase his expectation by varying two or more of his coefficients simultaneously is not excluded by (C). Our method of solution will be to show that among the possible EP's

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[10] In the mixed strategies they would be trilinear functions.

only one satisfies (C). By the existence theorem[11] this one must be the unique EP.

§ 7. DOMINATIONS

The values in the solution of 9 of the 24 coefficients can be fixed immediately by observing that to drop out with a high card is always definitely injurious to a player's expectation. Thus we have ...

$$\ldots \boxed{\gamma = \eta = \iota = \lambda = o = \rho = \tau = \varphi = \psi = 1}.$$

The value of ν can not quite be determined in the same way, for Player 3 might conceivably find opening on High no more profitable than passing out the hand. However, it is easily seen that if an equilibrium point exists with $\nu < 1$, then $\nu = 1$ together with the same set of other values is also an EP. Therefore we may assume ... $\boxed{\nu = 1}$. It will turn out that the unusual circumstances (i.e., $\beta = \zeta = 1$; $\alpha, \varepsilon < 1$) which permit $\nu < 1$ do not occur in any EP.

Before proceeding, we restrict ourselves to the case $b \geq a$. Later, by a process of continuation, we shall find EP's for smaller bet sizes. But the demonstration of the existence of a value to the game requires that <u>all</u> EP's be known, and to prove completeness is too complicated for $b < a$.

We now show, in a more elaborate argument, that $b \geq a$ entails $\beta = 0$. If Player 1 opens on Low then he must expect to lose $a + b$ in .75 of the deals, and gains at most $2a$ in the remaining .25. This expectation of at most $-\frac{1}{4}(a + 3b)$ must be compared with that of at least $-a$ he can obtain by not betting at all (i.e., $\beta = \pi = \sigma = \upsilon = 0$). Since we have assumed $b \geq a$, a behavior involving opening on Low (i.e., $\beta > 0$) is possible in an EP only if conditions are most favorable to that policy. That is, b and a must be equal, and

(i) Player 1 must always win the amount 2a by opening in the low-low-low deal (i.e., $\delta = \kappa = 0$);

(ii) He must never be allowed to recover his ante when he passes (i.e., either $\xi = 1$ or $\varepsilon = \zeta = 1$).

These conditions have a decisive effect on α. We may estimate Δ_α by the following tabulation of Player 1's payoff:

[11] Nash, loc. cit.

Deal	$\alpha = 1$	$\alpha = 0$
HHH	0	0
HHL	at most (a+b)/2	at least a/2
HLH	a/2	at least a/2
HLL	2a	at least 2a + b

We conclude that $\Delta_{\alpha} < -b$ and $\alpha = 0$. It is now easily verified that $\Delta_{\kappa} = 3\beta a$. But, with κ already zero by (i), there can be no equilibrium with $\Delta_{\kappa} > 0$ (condition (C)).[12] Thus, even after assuming most favorable conditions for β, we are led to the conclusion ... $\boxed{\beta = 0}$.

§8. FURTHER REDUCTIONS

With $\beta = 0$, it is easily calculated that:

$$\Delta_{\delta} = -4\alpha b$$
$$\Delta_{\theta} = -2\alpha(1 + \delta)b$$
$$\Delta_{\kappa} = -2\alpha(1 - \delta)b \ .$$

If $\alpha > 0$ then these three discriminants are strictly negative.[13] But if $\alpha = 0$ then δ, θ, and κ become irrelevant. In any case ... $\boxed{\delta = \theta = \kappa = 0}$.

Continuing, we have:

$$\Delta_{\pi} = 2(\zeta\mu a - \zeta b - \varepsilon\mu b - \varepsilon b)$$
$$\Delta_{\chi} = 2\bar{\zeta}(\xi \upsilon a - \upsilon b - \bar{\alpha}\xi b - \bar{\alpha} b) \ .$$

(We use the bar to denote complementary probabilities: $\bar{\alpha} = 1 - \alpha$, etc.) If either of these discriminants is to be non-negative then several of the coefficients are forced to assume extreme values, and a and b must be equal. When these restrictions are applied to the other discriminants a contradiction for each case is soon reached. The process is similar in form to the proof of $\beta = 0$, as given above, and we shall not burden this account with the details. The conclusion is ... $\boxed{\pi = \chi = 0}$.

A succession of results can be established in like manner, by deducing contradictions from the alternative hypotheses We list them in the order in which we found the arguments to go through most easily.

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[12]The corresponding verbal argument: when confronted by BP, Player 3 knows that both opponents hold Low; hence a call is always profitable for him.

[13]For Δ_{κ} we argue:

$$\Delta_{\delta} < 0 \Longrightarrow \delta = 0 \Longrightarrow \Delta_{\kappa} < 0 \ .$$

$\boxed{\alpha > 0;}$ $\boxed{\xi < \frac{2}{3};}$ $\boxed{\upsilon = \omega = 0;}$ $\boxed{\varepsilon < 1;}$ $\boxed{\sigma < 1,\ \mu = 0;}$

$\boxed{\zeta < 1;}$ $\boxed{\xi > 0;}$ $\boxed{\varepsilon > 0;}$ $\boxed{\alpha < 1;}$ $\boxed{\sigma = 0.}$

Again we omit the details, which are rather lengthy and not particularly interesting.

§ 9. THE SOLUTION FOR $b \geq a$

There remain at this juncture two systems of equations, differing in the way in which they involve ζ, which might possibly determine an EP. Together with the inequalities which they entail, they are:

$$(\mathrm{I}) \begin{cases} \Delta_\alpha = 0, & 0 < \alpha < 1 \\ \Delta_\varepsilon = 0, & 0 < \varepsilon < 1 \\ \zeta = 0, & \Delta_\zeta \leq 0 \\ \Delta_\xi = 0, & 0 < \xi < 2/3 \end{cases} \qquad (\mathrm{II}) \begin{cases} \Delta_\alpha = 0, & 0 < \alpha < 1 \\ \Delta_\varepsilon = 0, & 0 < \varepsilon < 1 \\ \Delta_\zeta = 0, & 0 \leq \zeta < 1 \\ \Delta_\xi = 0, & 0 < \xi < 2/3 \ . \end{cases}$$

The four discriminants in question are:

$$\Delta_\alpha = (a+b)\bar{\xi}\,\bar{\varepsilon} + (4a+2b)\bar{\xi}\,\bar{\zeta} - b\bar{\varepsilon} + b\bar{\zeta} - 3b \ ,$$

$$\Delta_\varepsilon = (a+b)\bar{\xi}\bar{\alpha} + (4a+2b)\bar{\xi} - b\bar{\alpha} - 2b \ ,$$

$$\Delta_\zeta = -2a\bar{\xi}\bar{\alpha} - 2a\bar{\xi} - 4b\bar{\alpha} + 6a - 2b \ ,$$

$$\Delta_\xi = -2(a+b)(\bar{\alpha}\,\bar{\varepsilon} + \bar{\alpha}\,\bar{\zeta} + \bar{\varepsilon}) + 4a\bar{\zeta} \ .$$

The solution of system (I) is:

$$\alpha = \varepsilon = 2 - S, \ \zeta = 0, \ \xi = 1 - \frac{b}{(a+b)S}; \ S = \sqrt{\frac{3a+b}{a+b}} \ .$$

The inequalities are satisfied in the range:

R_{I} $\qquad 0 < a/b \leq A_1 = 0.7058 \ \ldots \ .$[14]

If a/b exceeds A_1 then Δ_ζ becomes positive.

The equations of (II) give complicated expressions for α, ε, ζ, and ξ. The inequalities are satisfied in the range:

R_{II} $\qquad A_1 \leq a/b \leq 1 \ .$

If a/b is less than A_1, then ζ becomes negative. On the other hand, none of the inequalities is violated immediately if a/b is allowed to exceed 1. At the endpoints of R_{II} the numerical values are:

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[14] A_1 is the positive root of $9A^4+18A^3+3A^2-10A-3$.

at $a/b = A_1$:	α = .6482	ε = .6482	ζ = 0	ξ = .5664 ;
at $a/b = 1$:	.3084	.8257	.0441	.6354 .

As may be seen in Figure 1A, the coefficients are practically linear in R_{II}, and the connection at A_1 is continuous.

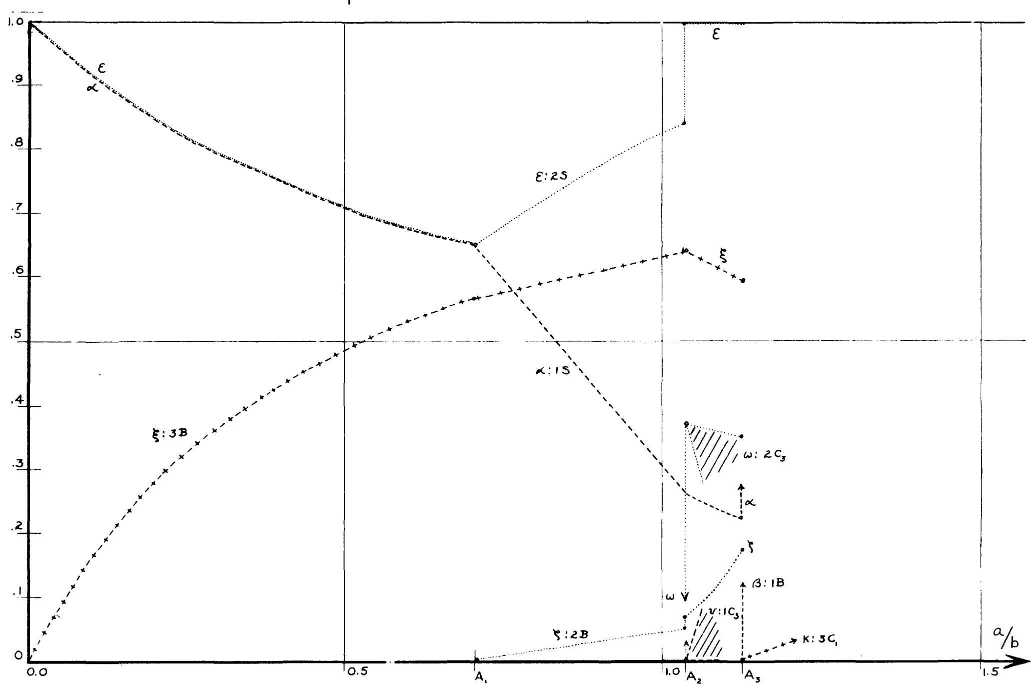

Figure 1A
Equilibrium Point Behavior Coefficients

----- Player 1	kS:	Player k	sandbags
..... Player 2	kB:	Player k	bluffs
+-+-+ Player 3	kC_h:	Player k	calls Player h's bluff

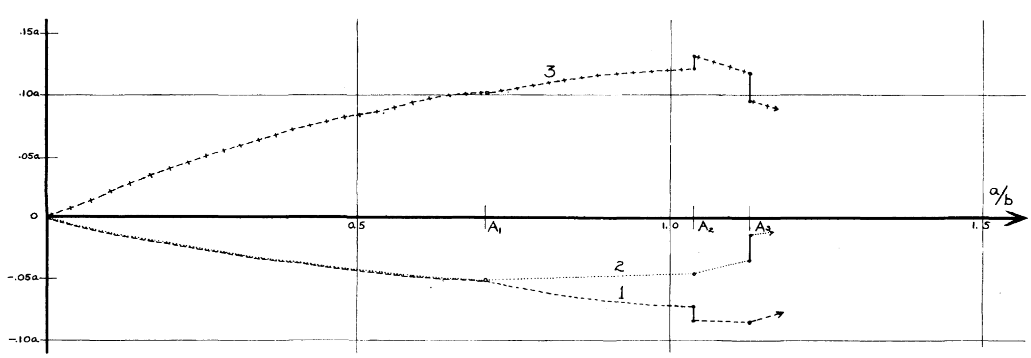

Figure 1B
Value of the Game

Since the EP is unique for each a/b, the value of the game is well-defined throughout R_I and R_{II}. In R_I it is the triple

$$V = \left\langle -\frac{a^2}{8(a+b)}, -\frac{a^2}{8(a+b)}, \frac{a^2}{4(a+b)} \right\rangle .$$

In R_{II} the results are again best given graphically (Figures 1B and 2) and numerically. We have:

$$V = \langle -.0518a, -.0518a, .1036a \rangle \quad \text{at} \quad a/b = A_1 ;$$

$$V = \langle -.0735a, -.0479a, .1214a \rangle \quad \text{at} \quad a/b = 1 .$$

The connection at A_1 between the two cases is of course continuous.

Thus, Player 3 enjoys a definite advantage. This may be ascribed to his ability to cause a pass-out.[15] He bluffs (ξ) an increasing proportion of the time as the relative size of the ante increases. His opponents meet this strategem by the manoeuvre known as "trapping" or "sandbagging" ($\bar{\alpha}$, $\bar{\varepsilon}$): passing with a high card on the first round, then calling on the second. For the small ante case (a/b in R_I), equilibrium is maintained only if the first two players follow the same strategy ($\alpha = \varepsilon$), and receive the same (negative) payoff. But for larger ante (a/b in R_{II}) it becomes possible for Player 2 also to bluff (ζ) a small fraction of the time, and his fortunes take a turn for the better. Player 1 sharply increases his sandbagging activity ($\bar{\alpha}$) (while Player 2 diminshes his), but this appears to be a defensive tactic only, for his position continues to worsen as a/b increases.

§10. EXTENSION OF THE SOLUTION

The solutions of system (II) can be continued through the range

R_{II}', $$1 < a/b < A_2 = 1.0376 \ldots$$ [16]

without violating (C). If a/b exceeds A_2 then the value of Δ_ω becomes positive. It is easily verified that we still have an EP in this extension, although uniqueness and the existence of a well-defined value are no longer assured.

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[15] This is analogous to the advantage in von Neumann's variant "C" (op. cit., pp. 211-218) which accrues to the first player. For that player also has the power to stop the play without forfeiting his basic investment ("b", loc. cit.) and without risking an additional sum ("a - b", loc. cit.). But "initiative," in the sense of having the first move, is a distinct handicap in our game.

[16] A_2 is the positive root of $162A^4+135A^3-144A^2-150A-28$.

At $a/b = A_2$ a qualitatively new phenomenon occurs. System (II) gives the EP:

$$\alpha = .2656, \ \varepsilon = .8442, \ \zeta = .0491, \ \xi = .6423, \ \omega = 0 \ .$$

But this is just one extreme of a one-parameter family of EP's, the other extreme of which is the EP:

$$\alpha = .2656, \ \varepsilon = 1, \ \zeta = .0623, \ \xi = .6423, \ \omega = .3720 \ .$$

The coefficients ε, ζ, and ω increase together while the others remain fixed. Since just one player's behavior is variable, the family of EP's forms an <u>interchange system</u>: every triple of strategies selected individually from the EP's is itself an EP.

The value of the game at $a/b = A_2$ is not well-defined. Player 3 gains at the expense of Player 1 as ε, ζ, ω increase, while Player 2's share remains (necessarily) constant. The numerical range of values:

$$V = \langle -.0755a, -.0475a, .1230a \rangle \quad \text{at} \quad a/b = A_2, \ \omega = 0 \ ;$$

$$V = \langle -.0826a, -.0475a, .1301a \rangle \quad \text{at} \quad a/b = A_2, \ \varepsilon = 1 \ .$$

Beyond A_2 another new effect appears. If we set $\varepsilon = 1$ and $\Delta_\omega = 0$, we find that $\Delta_\upsilon = 0$ also. This suggests the system:

$$\text{(III)} \quad \begin{cases} \Delta_\alpha = 0, & 0 \le \alpha \le 1 \\ \varepsilon = 1, & \Delta_\varepsilon \ge 0 \\ \Delta_\zeta = 0, & 0 \le \zeta \le 1 \\ \Delta_\xi = 0, & 0 \le \xi \le 1 \\ \Delta_\upsilon = 0, & 0 \le \upsilon \le 1 \\ \Delta_\omega = 0, & 0 \le \omega \le 1 \ . \end{cases}$$

Solving gives unique values for α, ζ, and ξ, and for the product $\bar{\upsilon}\bar{\omega}$, and puts a restriction on υ,

$$\upsilon \le \upsilon_{max} \approx 110 \ (a/b - A_2) \ .$$

With these values (C) is satisfied throughout the range:

$$R_{III} \qquad A_2 < a/b < A_3 = 1.1262 \ \ldots \ .[17]$$

In R_{III} the EP's are not unique, as just noted, nor do they form an interchange system, since different players control υ and ω. But it is a curious fact that the game does indeed possess a well-defined value (if we

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[17] A_3 is the largest root of $12A^3 - 14A^2 - 3A + 4$. Δ_β becomes positive if a/b exceeds A_3.

assume that there are no EP's undiscovered). Thus the two properties — existence of a value and interchangeability of equilibrium strategies — which are found in the solution of all two-person, zero-sum games, are quite independent of one another in our three-person Poker.

The situation at $a/b = A_3$ rounds out the picture, for here we have discovered a two-parameter family of EP's. The interchange systems are indexed by one parameter, while the payoffs depend only on the other parameter. The new coefficient to enter at A_3 is β, Player 1's bluff; whereas κ, Player 3's countermeasure to the bluff, comes in immediately thereafter. The value to Player 1 also finally starts to improve for $a/b > A_3$.[18]

§11. THE COALITION GAME

For the sake of making a comparison of our solution with the solutions as defined by von Neumann and Morgenstern,[19] we now calculate the

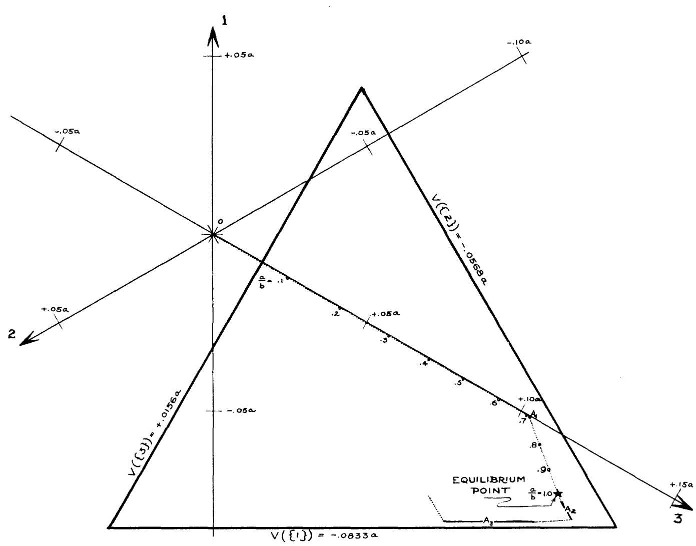

Figure 2
Characteristic Triangle of Coalition Game with $a = b$
(with values of the non-cooperative game for $0 < a/b \leq A_3$)

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[18]It is easily verified that for $a \geq 4b$ the value of the game is zero for all players, since each can guarantee himself that amount by betting on all occasions, regardless of his hand.

[19]For the actual construction of the solutions and their interpretation, see the discussion op. cit., particularly pp. 282-290.

"characteristic function" of the game for the case $a = b$ That is, for each set of players (a "coalition") we determine the maximum they can obtain for themselves as a group, irrespective of the actions of the remaining players. Essentially, therefore, we determine the values of three different two-person, zero-sum games. We do this here by exhibiting the optimal strategies.

We cannot expect to be able to represent all the strategies available to a coalition by means of our one-person behavior coefficients.[20] The correlated randomization which may be required is illustrated in the optimal strategy for $\{1, 3\}$ as given below. For the other coalitions, the one-person behavior coefficients happen to be sufficient to describe optimal play.

The optimal strategies for the several coalitions are as follows:

$\{1\}$ $\alpha = 2/3$. $\{2, 3\}$ $\xi = 2/3$.

$\{2\}$ $\varepsilon = 7/11$, $\zeta = 3/11$. $\{1, 3\}$ $\begin{cases} \alpha = 0, \ \xi = 3/16 & \text{with prob. } 8/11; \\ \alpha = 1, \ \xi = 0 & \text{with prob. } 3/11. \end{cases}$

$\{3\}$ $\mu = 1/4$, $0 \leq \xi \leq 2/3$. $\{1, 2\}$ $\alpha = 3/4$, $\varepsilon = 0$.

When not otherwise specified, always bet on High, pass on Low. The optimal strategy is not unique for $\{3\}$, but is unique for all other proper coalitions.

The characteristic function:

$$\begin{aligned} v(\{1\}) &= -v(\{2, 3\}) = -\,a/12 = -.0833a \\ v(\{2\}) &= -v(\{1, 3\}) = -5a/88 = -.0568a \\ v(\{3\}) &= -v(\{1, 2\}) = a/64 = .0156a \\ v(\{1, 2, 3\}) &= v(0) = 0 \ . \end{aligned}$$

Thus, Player 3 has a positive expectation even when the other two are allied against him.

The solutions as defined by von Neumann and Morgenstern are sets of triples lying in the triangle bounded by these values (see Figure 2). The equilibrium point is contained in precisely two of these solutions, both of them of the "discriminatory" type. From Figure 2 it is obvious that the third player has by far the most to fear from collusion between his opponents.

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[20]The information available to a two-player coalition is not monotone increasing When making its second move, the coalition has "forgotten" the contents of the hand of its first member.