# A SIMPLIFIED TWO-PERSON POKER*

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A fascinating problem for the game theoretician is posed by the common card game, Poker. While generally regarded as partaking of psychological aspects (such as bluffing) which supposedly render it inaccessible to mathematical treatment, it is evident that Poker falls within the general theory of games as elaborated by von Neumann and Morgenstern [1]. Relevant probability problems have been considered by Borel and Ville [2] and several variants are examined by von Neumann [1] and by Bellman and Blackwell [3].

As actually played, Poker is far too complex a game to permit a complete analysis at present; however, this complexity is computational and the restrictions that we will impose serve only to bring the numbers involved within a reasonable range. The only restriction that is not of this nature consists in setting the number of players at two. (The games considered in [1] and [3] also require this condition.) The simplifications, though radical, enable us to compute all optimal strategies for both players. In spite of these modifications, however, it seems that Simplified Poker retains many of the essential characteristics of the usual game.

An ante of one unit is required of each of the two players. They obtain a fixed hand at the beginning of a play by drawing one card apiece from a pack of three cards (rather than the $\binom{52}{5} = 2{,}598{,}960$ hands possible in Poker) numbered 1, 2, 3. Then the players choose alternatively either to bet one unit or pass without betting. Two successive bets or passes terminate a play, at which time the player holding the higher card wins the amount wagered previously by the other player. A player passing after a bet also ends a play and loses his ante.

Thus thirty possible plays are permitted by the rules. First of all, there are six possible deals; for each deal the action of the players may follow one of five courses which are described in the following diagram:

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| | First Round Player I | First Round Player II | Second Round Player I | Payoff |
|---|---|---|---|---|
| (1) | pass | pass | | 1 to holder of higher card |
| (2) | | bet | pass | 1 to player II |
| (3) | | | bet | 2 to holder of higher card |
| (4) | bet | pass | | 1 to player I |
| (5) | | bet | | 2 to holder of higher card |

We code the pure strategies available to the players by ordered triples $(x_1, x_2, x_3)$ and $(y_1, y_2, y_3)$ for players I and II, respectively $(x_i = 0, 1, 2;\ y_j = 0, 1, 2, 3)$. The instructions contained in $x_i$ are for card $i$ and are deciphered by expanding $x_i$ in the binary system, the first figure giving directions for the first round of betting, the second giving directions for the second, with 0 meaning pass and 1 meaning bet. For example, $(x_1, x_2, x_3) = (2, 0, 1) = (10, 00, 01)$ means player I should bet on a 1 in the first round, always pass with a 2 and wait until the second round to bet on a 3.

Similarly, to decode $y_j$, one expands in the binary system, the first figure giving directions when confronted by a pass, the second when confronted by a bet, with 0 meaning pass and 1 meaning bet. Thus $(y_1, y_2, y_3) = (2, 0, 1) = (10, 00, 01)$ means that player II should pass except when holding a 1 and confronted by a pass or holding a 3 and confronted by a bet.

In terms of this description of the pure strategies, the payoff to player I is given by the following scheme:

| $x_i$ \ $y_j$ | 0 = 00 | 1 = 01 | 2 = 10 | 3 = 11 |
|---|---|---|---|---|
| 0 = 00 | $\pm 1$ | $\pm 1$ | $-1$ | $-1$ |
| 1 = 01 | $\pm 1$ | $\pm 1$ | $\pm 2$ | $\pm 2$ |
| 2 = 10 | 1 | $\pm 2$ | 1 | $\pm 2$ |

where the ambiguous sign is + if $i > j$ and - if $i < j$.

From the coding of the pure strategies it is clear that player I has 27 pure strategies while player II has 64 pure strategies. Fortunately Poker sense indicates a method of reducing this unwieldy number of strategies.

Obviously, no player will decide either to bet on a 1 or to pass with a 3 when confronted by a bet. For player I (II) this heuristic argument recognizes the domination of certain rows (columns) of the game matrix by other rows (columns). It is well known that we may drop the dominated rows (dominating columns) without changing the value of the game and that any optimal strategy for the matrix thus reduced will be optimal for the original game. However, since the domination is not proper, these pure strategies could appear in some optimal mixed strategy. For the careful solver who may wish to find all of the optimal strategies, complementary arguments may be made to show that the pure strategies dropped are actually superfluous in this game. After we have found at least one of the optimal strategies for each player we shall give an indication of these arguments.

Now that these strategies have been eliminated new dominations appear. First, we notice that if player I holds a 2 he may as well pass in the first round, deciding to bet in the second if confronted by a bet, as bet originally. On either strategy he will lose the same amount if player II holds a 3; on the other hand, player II may bet on a 1 if confronted by a pass but certainly will not if confronted by a bet. Secondly player II may as well pass as bet, when holding a 2 and confronted by a pass since player I will now answer a bet only when he holds a 3.

We are now in a position to describe the game matrix composed of those strategies not eliminated by the previous heuristic arguments. Each entry is computed by adding the payoffs for the plays determined by the six possible deals for each pair of pure strategies and thus this matrix is six times the actual game matrix.

| $(x_1,x_2,x_3)$ \ $(y_1,y_2,y_3)$ | (0,0,3) | (0,1,3) | (2,0,3) | (2,1,3) |
|---|---|---|---|---|
| (0,0,1) | 0 | 0 | -1 | -1 |
| (0,0,2) | 0 | 1 | -2 | -1 |
| (0,1,1) | -1 | -1 | 1 | 1 |
| (0,1,2) | -1 | 0 | 0 | 1 |
| (2,0,1) | 1 | -2 | 0 | -3 |
| (2,0,2) | 1 | -1 | -1 | -3 |
| (2,1,1) | 0 | -3 | 2 | -1 |
| (2,1,2) | 0 | -2 | 1 | -1 |

One easily verifies that the following mixed strategies are optimal for this game matrix (and hence for Simplified Poker):

Player I:

| | | | | | | | | |
|---|---|---|---|---|---|---|---|---|
| (A) | 2/3 | (0,0,1) | + 1/3 | (0,1,1) | | | | |
| (B) | 1/3 | (0,0,1) | + 1/2 | (0,1,2) | + 1/6 | (2,0,1) | | |
| (C) | 5/9 | (0,0,1) | + 1/3 | (0,1,2) | + 1/9 | (2,1,1) | | |
| (D) | 1/2 | (0,0,1) | + 1/3 | (0,1,2) | + 1/6 | (2,1,2) | | |
| (E) | 2/5 | (0,0,2) | + 7/15 | (0,1,1) | + 2/15 | (2,0,1) | | |
| (F) | 1/3 | (0,0,2) | + 1/2 | (0,1,1) | + 1/6 | (2,0,2) | | |
| (G) | 1/2 | (0,0,2) | + 1/3 | (0,1,1) | + 1/6 | (2,1,1) | | |
| (H) | 4/9 | (0,0,2) | + 1/3 | (0,1,1) | + 2/9 | (2,1,2) | | |
| (I) | 1/6 | (0,0,2) | + 7/12 | (0,1,2) | + 1/4 | (2,0,1) | | |
| (J) | 5/12 | (0,0,2) | + 1/3 | (0,1,2) | + 1/4 | (2,1,1) | | |
| (K) | 1/3 | (0,0,2) | + 1/3 | (0,1,2) | + 1/3 | (2,1,2) | | |
| (L) | 2/3 | (0,1,2) | + 1/3 | (2,0,2) | | | | |

Player II:

| | | | | | |
|---|---|---|---|---|---|
| 1/3 | (0,0,3) | + 1/3 | (0,1,3) | + 1/3 | (2,0,3) |
| 2/3 | (0,0,3) | + 1/3 | (2,1,3) | | |

These strategies yield the value of Simplified Poker as $-1/18$.

As an example of the complementary arguments which assure us that no solutions are lost by discarding dominated rows and dominating columns, consider the pure strategies of the form $(1, x_2, x_3)$ which we have eliminated for player I. The verbal arguments assure us that player I will do at least as well by using $(0, x_2, x_3)$ no matter what mixed strategy II uses and irrespective of the deal. However, if I is dealt card 1 and II is dealt card 3, and if II plays either of his optimal strategies then player I loses two units when he plays $(1, x_2, x_3)$ while he loses but one unit when he plays $(0, x_2, x_3)$. Thus, since all of the pure strategies $(0, x_2, x_3)$ are essential, player I's expectation is less than the value of the game when he plays $(1, x_2, x_3)$ against II's optimal strategies and we see that the pure strategies $(1, x_2, x_3)$ are superfluous. (These terms are used as defined by Gale and Sherman in [4].)

The rank of the essential matrix is easily computed to be 3 (the sum of the first and last columns is equal to the sum of the center two columns), hence, by results of Bohnenblust, Karlin, and Shapley [5], Gale and Sherman [4], players I and II have precisely 6 and 2 linearly independent optimal strategies, respectively. A simple application of the work of Shapley and Snow [6] proves that all of the optimal strategies given above are basic. Therefore we know immediately that we have found all of the basic optimal strategies for player II. The corresponding result for player I is proved by considering the remaining kernels in the sense of [6]. This calculation is facilitated by the fact that, since the essential game was obtained by dominations, all solutions of the essential game can be extended to solutions of the full game and hence we need only consider kernels within the essential game. Moreover, the knowledge of the full set of solutions for II enables I to eliminate all 2 by 2 kernels except those involving the first and last column; there can be no 4 by 4 kernels since the essential game matrix has rank 3 and the value of the game is different from zero. Thus we verify that _all optimal strategies are convex linear combinations of the strategies given above_.

A striking simplification of the solution is achieved if we return to the extensive form of the game. We do this by introducing _behavior parameters_ to describe the choices remaining available to the players after we have eliminated the superfluous strategies. We define:

Player I:

$\alpha$ = probability of bet with 1 in first round.
$\beta$ = probability of bet with 2 in second round.
$\gamma$ = probability of bet with 3 in first round.

Player II:

$\xi$ = probability of bet with 1 against a pass.
$\eta$ = probability of bet with 2 against a bet.

In terms of these parameters, player I's basic optimal strategies fall into seven sets:

| Basic Strategies | $(\alpha, \beta, \gamma)$ |
|---|---|
| A | (0 , 1/3 , 0) |
| C | (1/9 , 4/9 , 1/3) |
| E | (2/15, 7/15, 2/5) |
| B,D,F,G | (1/6 , 1/2 , 1/2) |
| H | (2/9 , 5/9 , 2/3) |
| I,J | (1/4 , 5/12, 3/4) |
| K,L | (1/3 , 2/3 , 1) |

Thus, in the space of these behavior parameters, the five dimensions of optimal mixed strategies for player I collapse onto the one parameter family of solutions:

$$\alpha = \gamma/3$$
$$\beta = \gamma/3 + 1/3$$
$$0 \leq \gamma \leq 1$$

These may be described verbally by saying that player I may pass on a 3 in the first round with arbitrary probability, but then he must bet on a 1 in the first round one third as often, while the probability with which he bets on a 2 in the second round is one third more than the probability with which he bets on a 1 in the first round.

On the other hand, we find that player II has the single solution:

$$(\xi, \eta) = (1/3,\ 1/3)\ ,$$

which instructs him to bet one third of the time when holding a 1 and confronted by a pass and to bet one third of the time when holding a 2 and confronted by a bet.

The presence of <u>bluffing</u> and <u>underbidding</u> in these solutions is noteworthy (<u>bluffing</u> means betting with a 1; <u>underbidding</u> means passing on a 3). All but the extreme strategies for player I, in terms of the behavior parameters, involve both bluffing and underbidding while player II's single optimal strategy instructs him to bluff with constant probability 1/3 (underbidding is not available to him). These results compare favorably with presence of bluffing in the von Neumann example, while bluffing is not available to player II in the continuous variant considered by Bellman and Blackwell.

The sensitive nature of bluffing and underbidding in this example is exposed by varying the ratio of the bet to the ante. Consider the games described by the same rules in which the ante is a positive real number a and the bet is a positive real number b. We will state the solutions in terms of the behavior parameters without proof. This one parameter family of games falls naturally into four intervals:

<u>Case 1: $0 < b < a$</u>

Player I: $(\alpha, \beta, \gamma) = (\frac{b}{2a + b}, \frac{2a}{2a + b}, 1)$

Player II: $(\xi, \eta) = (\frac{b}{2a + b}, \frac{2a - b}{2a + b})$

Remarks: Both players have a unique optimal mode of behavior. Player I never underbids and bluffs with probability $\frac{b}{2a + b}$; player II bluffs with probability $\frac{b}{2a + b}$. The value of this game is $-\frac{b^2}{6(2a + b)}$.

<u>Case 2: $0 < b = a$</u>

This is our original game. The value is $-\frac{b}{18}$ .

<u>Case 3: $0 < a < b < 2a$</u>

Player I: $(\alpha, \beta, \gamma) = (0, \frac{2a - b}{2a + b}, 0)$

Player II: $\xi = \frac{b}{2a + b}$

$$\frac{b}{2a + b} \leq \eta \leq \frac{a + b}{2a + b}$$

Remarks:: Player I never bluffs and always underbids; player II bluffs with probability $\frac{b}{2a + b}$. The value of this game is $-\frac{b}{6}(\frac{2a - b}{2a + b})$ .

Case 4: $0 < 2a = b$

This game has a saddle point in which player I never bluffs, always underbids and never bets on a 2 while player II bets only and always with a 3. The strategy for I is unique while II can vary his strategy considerably. The value of this game is 0.

It is remarkable that player I has a negative expectation for a play, *i.e.*, a disadvantage that is plausibly imputable to his being forced to take the initiative. (Compare von Neumann's variant (c), in which the possession of the initiative seems to be an advantage.) It is also noteworthy that Simplified Poker, which was not constructed with an eye to solutions, but rather as a modification of an actual game, has many solutions notwithstanding the fact that games with unique optimal strategies are dense in the space of all games.

In conclusion, it is hoped that these considerations will prove instructive in a qualitative way and contribute in some small measure to the casuistry of game solving.

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