With

$$x = \frac{b(1-u)}{b+u} \quad \text{we get that} \quad dx = -\frac{b(1+b)}{(b+u)^2}\,du.$$

We find that 0 maps to 1 and 1 to 0. Thus

$$\begin{aligned}
\int_0^1 \frac{x^n(1-x)^n}{\big((x+a)(x+b)\big)^{n+1}}\,dx &= \int_0^1 \frac{\left(\frac{b(1-u)}{b+u}\right)^n\left(1-\frac{b(1-u)}{b+u}\right)^n}{\left[\left(\frac{b(1-u)}{b+u}+a\right)\left(\frac{b(1-u)}{b+u}+b\right)\right]^{n+1}}\frac{b(1+b)}{(b+u)^2}\,du \\
&= \int_0^1 \frac{(1-u)^n u^n b^{n+1}(1+b)^{n+1}}{\big[\big(b(1-u)+a(b+u)\big)\big(b(1-u)+b(b+u)\big)\big]^{n+1}}\,du \\
&= \int_0^1 \frac{(1-u)^n u^n}{\big((a-b)u+(a+1)b\big)^{n+1}}\,du.
\end{aligned}$$