Chapter 5

The lace expansion

5.1 Inclusion-exclusion

So far the lace expansion is the only method which has led to ri
results proving existence of critical exponents for the self-avoidin
All results obtained so far are for dimensions greater than the conj
upper critical dimension four.

The lace expansion was first introduced by Brydges and Spencer
who used it to study *weakly* self-avoiding walk above four dime
Weakly self-avoiding walk will be defined precisely in the next s
roughly speaking it is a model of a random walk where walks wh
tersect themselves do have a nonzero weight, but this weight is
than for a walk which does not intersect itself. The size of the prob
penalty imposed for a self-intersection (i.e. the weakness of the inter
is the small parameter which provided convergence for the lace exp

Now consider a model of self-avoiding walk on a lattice with la
ordination number $\mathcal{Z}$, or in other words a walk for which there are
number of steps available at each site. The probability that the con
tion of a randomly chosen single step to a given walk produces an imn
reversal is $\mathcal{Z}^{-1}$, which is small. For example, consider the usual n
neighbour model in high dimensions, with coordination number $2d$.
saw in Section 1.2, in high dimensions the main effect of the self-avo
constraint is in some sense to rule out immediate reversals, and now
pears that immediate reversals are uncommon even in the absence
constraint. This suggests that we can regard the interaction as bein
in high dimensions. Alternatively, we may consider a walk on a
dimensional lattice with large coordination number. Our basic exan

this type is a walk on the hypercubic lattice $\mathbf{Z}^d$, in which we allow all steps whose largest component has absolute value less than or equal to L, for some large parameter L. Then again the effect of ruling out immediate reversals will be small, but we must also worry about long-range effects. It will turn out that these are also small for $d > 4$, so that in this situation the self-avoidance is again weak. We shall see in Chapter 6 that in these two situations it is possible to obtain convergence of the lace expansion. In the former the small parameter responsible for convergence of the expansion is $(2d)^{-1}$, while in the latter it is L^{-1}. Remarkably, for the nearest-neighbour model in five dimensions the small parameter 1/10 is sufficiently small to prove convergence.

The lace expansion was first derived by an expansion and resummation procedure reminiscent of the cluster expansions of statistical mechanics and constructive quantum field theory. Viewed differently, however, the lace expansion can be seen as resulting from repeated application of the inclusion-exclusion relation. In this section the derivation of the lace expansion via the inclusion-exclusion relation will be discussed, and in the next section the resummation approach will be described. The inclusion-exclusion approach is geometric and is useful for providing intuition as to the origin of the expansion, while the resummation procedure has the advantage of generating terms in the expansion in a systematic and algebraic manner. In the inclusion-exclusion approach, the name lace expansion may seem somewhat inappropriate, but the laces will appear in the next section. Both derivations lead to precisely the same expansion.

At the heart of the lace expansion method is a convolution equation for the two-point function which is a multi-dimensional analogue of the renewal equations of Section 4.2. Indeed the expansion amounts to identifying an "irreducible" two-point function, which will be denoted $\Pi_z(0,x)$, such that the two-point function is essentially given by its convolution with the irreducible two-point function. Fourier transform techniques will play a key role in the analysis of the convolution equation.

We now turn to the derivation of the expansion for the case of a walk which may take more general steps than just to a nearest neighbour. We fix a finite set $\Omega \not\ni 0$ in $\mathbf{Z}^d$ which is symmetric with respect to reflections in the coordinate hyperplanes and rotations about coordinate axes by $\pi/2$. The cardinality of Ω will also be denoted by Ω. We consider the ordinary random walk taking steps in Ω. More precisely, we consider $\omega = (\omega(0), \omega(1), \ldots, \omega(N))$, where each $\omega(i)$ is an element of $\mathbf{Z}^d$ and $\omega(i+1) - \omega(i) \in \Omega$. (If Ω is the set of nearest neighbours of the origin then this is just the simple random walk.) It is shown in (A.6) that for $z < \Omega^{-1}$ the

Fourier transform of the two-point function for this walk is given

$$\hat{C}_z(k) = \frac{1}{1 - z\Omega\hat{D}(k)},$$

where

$$\hat{D}(k) = \frac{1}{\Omega}\sum_{x\in\Omega} e^{ik\cdot x}.$$

To simplify the notation we will not use a label Ω to keep track of that we are not necessarily dealing with the nearest-neighbour walks in this section will take steps in Ω. Self-avoiding walks tak in Ω will satisfy the same subadditivity inequality as the nearest-n self-avoiding walk, and hence will have a critical point $z_c = z_c($ define $\hat{\Pi}_z(k)$, for $z < z_c$, implicitly by the equation

$$\hat{G}_z(k) = \frac{1}{1 - z\Omega\hat{D}(k) - \hat{\Pi}_z(k)}.$$

Then $\hat{\Pi}_z$ can be thought of as a measure of the difference betw avoiding and simple random walk. The lace expansion is an expa $\hat{\Pi}_z$.

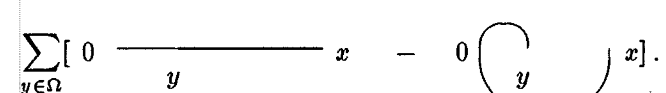
We denote by $\mathcal{C}_N(x,y)$ the set of all N-step self-avoiding walks f y (taking steps in Ω), and denote its cardinality by $c_N(x,y)$. The i in deriving the expansion is to extract the term in $G_z(0,x)$ corres to $N = 0$:

$$G_z(0,x) = \delta_{0,x} + \sum_{N=1}^{\infty} c_N(0,x) z^N.$$

We shall now argue that for $N \geq 1$,

$$c_N(0,x) = \sum_{y\in\Omega}\left[c_1(0,y)c_{N-1}(y,x) - \sum_{\omega^{(1)}\in\mathcal{C}_{N-1}(y,x)} I[0 \in \omega^{(1)}]\right].$$

Diagrammatically the right side of (5.1.3) can be represented by



In the first term on the right side the bold line is unconstraine from the fact that it should be self-avoiding. The thin line in the fi represents a single step. Equation (5.1.3) is just the inclusion-exclusion relation: the first term on the right side counts all walks from 0 to x which are self-avoiding *after* the first step, and the second subtracts the contribution due to those which are not self-avoiding from the beginning, i.e., walks that return to the origin. Since $c_1(0,y) = 1$ for $y \in \Omega$, substitution of (5.1.3) into (5.1.2) gives

$$G_z(0,x) = \delta_{0,x} + z\sum_{y\in\Omega} G_z(y,x) - \sum_{y\in\Omega}\sum_{N=0}^{\infty} z^{N+1} \sum_{\omega^{(1)}\in\mathcal{C}_N(y,x)} I[0\in\omega^{(1)}]. \tag{5.1.4}$$

The inclusion-exclusion relation can now be applied to the last term on the right side of (5.1.4), as follows. Let S be the first (and only) time that $\omega^{(1)}(S) = 0$. Then

$$\sum_{\omega^{(1)}\in\mathcal{C}_N(y,x)} I[0\in\omega^{(1)}] = \sum_{S=1}^{N} \sum_{\substack{\omega^{(2)}\in\mathcal{C}_S(y,0)\\ \omega^{(3)}\in\mathcal{C}_{N-S}(0,x)}} I[\omega^{(2)}\cap\omega^{(3)} = \{0\}]$$

$$= \sum_{S=1}^{N}\left[c_S(y,0)c_{N-S}(0,x) - \sum_{\substack{\omega^{(2)}\in\mathcal{C}_S(y,0)\\ \omega^{(3)}\in\mathcal{C}_{N-S}(0,x)}} I[\omega^{(2)}\cap\omega^{(3)} \neq \{0\}]\right].$$

We can interpret $c_S(y,0)$ as the number of $(S+1)$-step walks which step from the origin directly to y, then return to the origin in S steps, which have distinct vertices apart from the fact that they return to their starting point. Let $\mathcal{U}_S$ denote the set of all S-step self-avoiding loops at the origin (S-step walks which begin and end at the origin but which otherwise have distinct vertices), and let u_S be the cardinality of $\mathcal{U}_S$. Then

$$\sum_{y\in\Omega}\sum_{N=0}^{\infty} z^{N+1} \sum_{\omega^{(1)}\in\mathcal{C}_N(y,x)} I[0\in\omega^{(1)}]$$

$$= \sum_{S=2}^{\infty} z^S u_S \cdot G_z(0,x) - \sum_{\substack{S=2\\N=0}}^{\infty} \sum_{\substack{\omega^{(2)}\in\mathcal{U}_S\\ \omega^{(3)}\in\mathcal{C}_N(0,x)}} z^{S+N} I[\omega^{(2)}\cap\omega^{(3)} \neq \{0\}].$$

Continuing in this fashion, in the last term on the right side of the above equation let $T_1 \geq 1$ be the first time along $\omega^{(3)}$ that $\omega^{(3)}(T_1) \in \omega^{(2)}$, and let $v = \omega^{(3)}(T_1)$. Then the inclusion-exclusion relation can be applied

again to remove the avoidance between the portions of $\omega^{(3)}$ before
T_1, and correct for this removal by the subtraction of a term in
further intersection. For $z < z_c$ repetition of this procedure lea
convolution equation

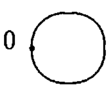
$$G_z(0,x) = \delta_{0,x} + z\sum_{y\in\Omega} G_z(y,x) + \sum_{v} \Pi_z(0,v)G_z(v,x),$$

where the "irreducible" two-point function $\Pi_z(0,x)$ is given by

$$\Pi_z(0,v) = \sum_{N=1}^{\infty}(-1)^N \Pi_z^{(N)}(0,v),$$

with the terms on the right side defined as follows. The $N =$
given by

$$\Pi_z^{(1)}(0,v) = \delta_{0,v}\sum_{S=2}^{\infty} z^S u_S \equiv \delta_{0,v}\ 0$$

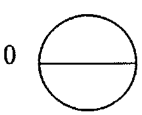


The $N = 2$ term is

$$\Pi_z^{(2)}(0,v) = \prod_{i=1}^{3}\left[\sum_{T_i=1}^{\infty} z^{T_i} \sum_{\omega_i\in\mathcal{C}_{T_i}(0,v)}\right] I(\omega_1,\omega_2,\omega_3),$$

where $I(\omega_1,\omega_2,\omega_3)$ is equal to 1 if the ω_i are pairwise mutually
apart from their common endpoints, and otherwise equals 0. Diag
ically this can be represented by

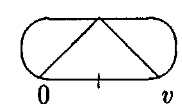
$$\Pi_z^{(2)}(0,v) = \ 0$$



v ,

where each line represents a sum over self-avoiding walks between
points of the line, weighted by z^T, with mutual avoidance between
pairs of lines in the diagram. Similarly

$$\Pi_z^{(3)}(0,v) \ =$$



,

where now there is mutual avoidance between some but not all
lines in the diagram; we defer any discussion of the details of thi
avoidance until later in the chapter. The unlabelled vertex is sum

$\mathbf{Z}^d$. A slashed propagator is used to indicate a walk which may have zero length, i.e., be a single site, whereas propagators without a slash correspond to walks of at least one step. All the higher order terms can be expressed as diagrams in this way, and with some care it is possible to keep track of the pattern of mutual avoidance between subwalks (individual lines in the diagram) which emerges. The algebraic derivation of the expansion, which will be given in the next section, keeps track of this mutual avoidance automatically. Equation (5.1.6) is the lace expansion.

In the above we have tacitly assumed that the lace expansion converges. To be more careful, we should have truncated the above procedure after some large finite number of terms had been generated, and then taken a limit as the number of terms grows to infinity. This convergence assumption will be made more explicit in the next section.

The Fourier transform of a function on $\mathbf{Z}^d$ was defined in (1.4.10). Using translation invariance, and the fact that the Fourier transform of a convolution is the product of Fourier transforms, taking the Fourier transform of (5.1.5) and solving for $\hat{G}_z(k)$ gives

$$\hat{G}_z(k) = \frac{1}{1 - z\Omega\hat{D}(k) - \hat{\Pi}_z(k)}. \tag{5.1.7}$$

Here

$$\hat{\Pi}_z(k) = \sum_{N=1}^{\infty} (-1)^N \hat{\Pi}_z^{(N)}(k). \tag{5.1.8}$$

In Section 5.4 we will show how $\hat{\Pi}_z^{(N)}(k)$ can by bounded using the diagrammatic representation of $\Pi_z^{(N)}(0,x)$ described above. But first we turn to the algebraic derivation of the lace expansion, in the next section.

5.2 Algebraic derivation of the lace expansion

In this section we give an algebraic derivation of the lace expansion for walks taking steps in a fixed set $\Omega \subset \mathbf{Z}^d$ which respects the symmetries of the lattice and does not contain the origin. As in the previous section we simplify the notation by omitting the label Ω. The number of sites in Ω is also denoted Ω.

Given a walk $\omega = (\omega(0), \omega(1), \ldots, \omega(n))$ and two "times" s and t in $\{0, 1, \ldots, n\}$, we define

$$\mathcal{U}_{st}(\omega) = \begin{cases} -1 & \text{if } \omega(s) = \omega(t) \\ 0 & \text{if } \omega(s) \neq \omega(t). \end{cases} \tag{5.2.1}$$

Then the two point function can be written

$$G_z(0,x) = \sum_{\omega:0\to x} z^{|\omega|} \prod_{0\le s<t\le|\omega|} (1 + \mathcal{U}_{st}(\omega)).$$

Here the activity z is any *complex* number for which $|z| < z_c$
The sum over ω is the sum over all ordinary (possibly self-int
walks from 0 to x, although walks which do have self-intersections
contribution to (5.2.2). The walk ω takes steps (u,v) with $v - u$
as usual $|\omega|$ denotes the number of steps in ω. The product in
equal to one if ω is self-avoiding, and is equal to zero if not. Th
self-avoiding walk, also known as the Domb–Joyce model, is d
replacing the factor $1 + \mathcal{U}_{st}$ by $1 + \lambda\mathcal{U}_{st}$, with $\lambda \in (0,1)$. Taking λ
the ordinary unconstrained random walk, while $\lambda \in (0,1)$ gives a
which self-intersections are suppressed but not prohibited. We ta

We will have need of the self-avoiding walk with a memory τ
point function is defined by

$$G_z(0,x;\tau) = \sum_{\omega:0\to x} z^{|\omega|} \prod_{\substack{0 \le s < t \le |\omega| \\ t-s\le\tau}} (1 + \mathcal{U}_{st}(\omega)).$$

If the memory τ is equal to zero, then (5.2.3) is just the two-poin
of ordinary random walk. The case $\tau = \infty$ corresponds to the sel
walk. Unless explicitly stated otherwise, the memory may take
$0 \le \tau \le \infty$.

The Fourier transform of (5.2.3) is given by

$$\hat{G}_z(k;\tau) = \sum_{\omega} z^{|\omega|} e^{ik\cdot\omega(|\omega|)} \prod_{\substack{0 \le s < t \le |\omega| \\ t-s\le\tau}} (1 + \mathcal{U}_{st}(\omega))$$

for $k \in [-\pi,\pi]^d$; the sum over ω is the sum over all ordinary
arbitrary length beginning at the origin. The right hand side i
series in z. We denote its radius of convergence by $z_c(k;\tau)$. For

$$z_c(k;\tau) \ge z_c(0;\tau) = \mu_\tau^{-1},$$

where μ_τ was defined in (1.2.12). The self-avoiding walk critica
then $z_c = z_c(0;\infty)$.

To obtain a formula for the inverse of $\hat{G}_z(k)$, we first introd
terminology.

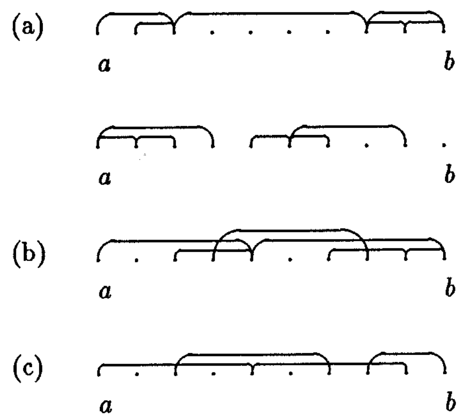


Figure 5.1: Graphs in which an edge st is represented by an arc joining s and t. (a) Examples of graphs which are not connected. (b) An example of a connected graph. (c) An example of a lace.

Definition 5.2.1 *Given an interval $I = [a,b]$ of positive integers, we refer to a pair $\{s,t\}$ $(s < t)$ of elements of I as an* edge. *To abbreviate the notation, we usually write st for $\{s,t\}$. The* length *of an edge st is $t - s$. A set of edges is called a* graph. *A graph Γ is said to be* connected *if both a and b are endpoints of edges in Γ, and if in addition, for any $c \in (a,b)$, there are $s,t \in [a,b]$ such that $s < c < t$ and $st \in \Gamma$. The set of all graphs on $[a,b]$ consisting of edges of length τ or less is denoted $\mathcal{B}_\tau[a,b]$, and the subset consisting of all connected graphs is denoted $\mathcal{G}_\tau[a,b]$. A* lace *is a minimally connected graph, i.e., a connected graph for which the removal of any edge would result in a disconnected graph. The set of laces on $[a,b]$ consisting of edges of length τ or less is denoted by $\mathcal{L}_\tau[a,b]$, and the set of laces on $[a,b]$ which consist of exactly N edges is denoted $\mathcal{L}_{\tau,N}[a,b]$.*

A convenient graphical representation of graphs and laces is illustrated in Figure 5.1.

Given a connected graph Γ, the following prescription associates to Γ a unique lace $\mathcal{L}_\Gamma$: The lace $\mathcal{L}_\Gamma$ consists of edges $s_1t_1, s_2t_2, \ldots$ where

$$s_1 = a, \quad t_1 = \max\{t : at \in \Gamma\}$$

$$t_{i+1} = \max\{t : st \in \Gamma, s < t_i\}$$

$$s_i = \min\{s : st_i \in \Gamma\}.$$

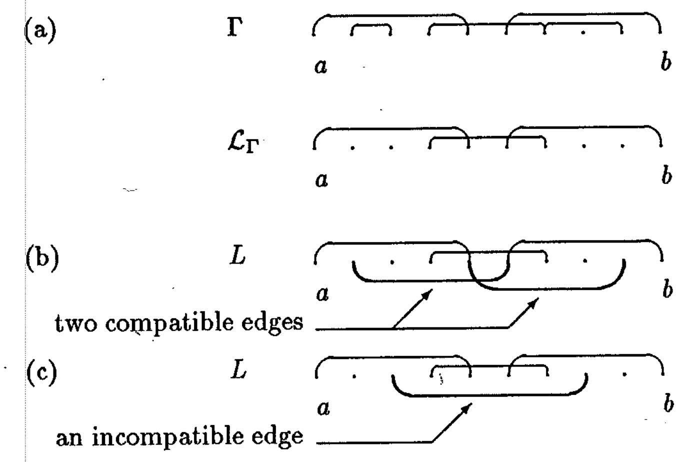


Figure 5.2: (a) An example of a connected graph Γ and its associ
$L = \mathcal{L}_\Gamma$. (b) Examples of edges compatible with the lace L. (c) An
of an edge which is not compatible with the lace L.

Given a lace L, the set of all edges $st \notin L$ (of length τ or less) s
$\mathcal{L}_{L \cup \{st\}} = L$ is denoted $\mathcal{C}_\tau(L)$. Edges in $\mathcal{C}_\tau(L)$ are said to be c
with L. Figure 5.2 illustrates these definitions.

For $a < b$ we define

$$K_\tau[a,b] = \prod_{\substack{s,t \in [a,b] \\ 0 < t - s \le \tau}} (1 + \mathcal{U}_{st}).$$

We set $K_\tau[a,a] = 1$, and if $a > b$ then we set $K_\tau[a,b] = 0$. By e
the product in (5.2.6) we obtain

$$K_\tau[a,b] = \sum_{\Gamma \in \mathcal{B}_\tau[a,b]} \prod_{st \in \Gamma} \mathcal{U}_{st}.$$

For $a < b$ we define an analogous quantity, in which the sum ov
is restricted to connected graphs:

$$J_\tau[a,b] = \sum_{\Gamma \in \mathcal{G}_\tau[a,b]} \prod_{st \in \Gamma} \mathcal{U}_{st}.$$

We set $J_\tau[a,a] = 1$, and if $a > b$ then we set $J_\tau[a,b] = 0$.

resumming the right side of (5.2.8), we obtain

$$J_\tau[a,b] = \sum_{L\in\mathcal{L}_\tau[a,b]} \sum_{\Gamma:\mathcal{L}_\Gamma=L} \prod_{st\in L} \mathcal{U}_{st} \prod_{s't'\in\Gamma\setminus L} \mathcal{U}_{s't'}$$
$$= \sum_{L\in\mathcal{L}_\tau[a,b]} \prod_{st\in L} \mathcal{U}_{st} \prod_{s't'\in\mathcal{C}_\tau(L)} (1+\mathcal{U}_{s't'}). \qquad (5.2.9)$$

For $a < b$ we define $J_{\tau,N}[a,b]$ to be the contribution to (5.2.9) from laces consisting of exactly N bonds:

$$J_{\tau,N}[a,b] = \sum_{L\in\mathcal{L}_{\tau,N}[a,b]} \prod_{st\in L} \mathcal{U}_{st} \prod_{s't'\in\mathcal{C}_\tau(L)} (1+\mathcal{U}_{s't'}). \qquad (5.2.10)$$

Each term in the above sum is either 0 or $(-1)^N$. By (5.2.9) and (5.2.10),

$$J_\tau[a,b] = \sum_{N=1}^{\infty} J_{\tau,N}[a,b]. \qquad (5.2.11)$$

The sum over N in (5.2.11) is a finite sum, since the sum in (5.2.10) is empty for $N > b-a$ and hence $J_{\tau,N}[a,b] = 0$ if $N > b-a$. By definition $J_\tau[a,a+1] = 0$, since the only lace on $[a,a+1]$ consists of the single edge $\{a,a+1\}$, and $\mathcal{U}_{a,a+1}(\omega) = 0$ for all ω, because a walk cannot be at the same place at consecutive times.

Lemma 5.2.2 *For any $a < b$,*

$$K_\tau[a,b] = K_\tau[a+1,b] + \sum_{j=a+2}^{b} J_\tau[a,j]K_\tau[j,b]. \qquad (5.2.12)$$

Proof. The contribution to the sum on the right side of (5.2.7) due to all graphs Γ for which a is not in an edge is exactly $K_\tau[a+1,b]$. To resum the contribution due to the remaining graphs we proceed as follows. If Γ does contain an edge ending at a, let $j(\Gamma)$ be the largest value of j such that the set of edges in Γ with at least one end in the interval $[a,j]$ forms a connected graph on $[a,j]$. We lose nothing by taking $j \geq a+2$, since as argued above the statement of the lemma, $\mathcal{U}_{a,a+1} = 0$. Then resummation over graphs on $[j,b]$ gives

$$K_\tau[a,b] = K_\tau[a+1,b] + \sum_{j=a+2}^{b} \sum_{\Gamma\in\mathcal{G}_\tau[a,j]} \prod_{st\in\Gamma} \mathcal{U}_{st}\, K_\tau[j,b], \qquad (5.2.13)$$

which with (5.2.8) proves the lemma. □

Now we define

$$\Pi_z^{(N)}(0,x;\tau) = (-1)^N \sum_{\substack{\omega:0\to x\\ |\omega|\geq 1}} z^{|\omega|} J_{\tau,N}[0,|\omega|]$$

and

$$\Pi_z(0,x;\tau) = \sum_{N=1}^{\infty}(-1)^N \Pi_z^{(N)}(0,x;\tau) = \sum_{\substack{\omega:0\to x\\ |\omega|\geq 1}} z^{|\omega|} J_\tau[0,|\omega|]$$

for any z for which the right side converges. The factor $(-1)^N$ or side of (5.2.14) ensures that

$$\Pi_z^{(N)}(0,x;\tau) \geq 0 \quad \text{for nonnegative } z.$$

Since $J_{\tau,N}[0,1] = 0$, the sum on the right side of (5.2.14) or (5.2 equally well be over walks ω of length greater than or equal to the notation

$$\hat{D}(k) = \frac{1}{\Omega}\sum_{x\in\Omega} e^{ik\cdot x}$$

Theorem 5.2.3 *For any value of z for which $\sum_{|\omega|\geq 2} z^{|\omega|} J_\tau[$ $\sum_{|\omega|\geq 0} z^{|\omega|} K_\tau[0,|\omega|]$ converge absolutely,*

$$G_z(0,x;\tau) = \delta_{0,x} + z\sum_{u\in\Omega} G_z(u,x;\tau) + \sum_v \Pi_z(0,v;\tau)G_z(v,x;\tau$$

and

$$\hat{G}_z(k;\tau) = \frac{1}{1 - z\Omega\hat{D}(k) - \hat{\Pi}_z(k;\tau)}.$$

Proof. It suffices to obtain (5.2.17), since then (5.2.18) follo diately upon taking the Fourier transform of (5.2.17) and usin that the Fourier transform of a convolution is the product of Fou forms. Existence of the Fourier transforms of $G_z(0,\cdot;\tau)$ and Π_z guaranteed by the hypotheses of the theorem.

To prove (5.2.17), we first extract the contribution to (5.2.2) zero step walk:

$$G_z(0,x;\tau) = \delta_{0,x} + \sum_{\substack{\omega:0\to x\\ |\omega|\geq 1}} z^{|\omega|} K_\tau[0,|\omega|].$$

Substitution of (5.2.12) into this equation results in

$$G_z(0,x;\tau) = \delta_{0,x} + \sum_{\substack{\omega:0\to x \\ |\omega|\geq 1}} z^{|\omega|} K_\tau[1,|\omega|] + \sum_{\substack{\omega:0\to x \\ |\omega|\geq 1}} z^{|\omega|} \sum_{j=2}^{|\omega|} J_\tau[0,j]K_\tau[j,|\omega|]. \quad (5.2.19)$$

In the second term on the right side, the factor $K_\tau[1,|\omega|]$ is nonzero only if the walk is self-avoiding after the first step. Explicitly summing over the endpoint of the first step in ω, the second term is equal to

$$z\sum_{y\in\Omega} G_z(y,x;\tau).$$

The third term on the right side of (5.2.19) is equal to

$$\sum_{N=1}^{\infty} z^N \sum_{\substack{\omega:0\to x \\ |\omega|=N}} \sum_{j=2}^{N} J_\tau[0,j]K_\tau[j,N].$$

Interchanging the order of summation gives the following expression for this quantity:

$$\sum_{j=2}^{\infty} z^j \sum_{N=j}^{\infty} z^{N-j} \sum_{\substack{\omega:0\to x \\ |\omega|=N}} J_\tau[0,j]K_\tau[j,N].$$

In the sum over ω, there is no interaction between the initial j-step portion and final $(N-j)$-step portion of the walk. Factorizing the walk into these two pieces gives the desired result.

In the last step we interchanged two infinite sums. This is justified by the hypothesis that these two sums converge absolutely. □

Next, we prove an identity which will be used to study the finite-dimensional distributions of the self-avoiding walk. It expresses the difference between a self-avoiding walk and two independent or decoupled self-avoiding walks, in the spirit of the inclusion-exclusion relation. First we need a definition.

Definition 5.2.4 *Any graph $B \in \mathcal{B}_\tau[0,b]$ breaks up into connected components in a natural way. Given an integer m in the open interval $(0,b)$, we let $C_m(B) = \{m\}$ if B does not contain a bond st with $s < m < t$. If B*

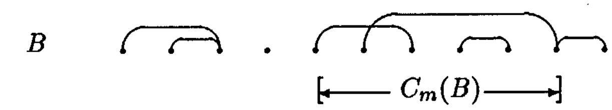


Figure 5.3: An example of a graph B and interval $C_m(B)$

does contain a bond st with $s < m < t$, then there is a connected co Γ of B with bonds having endpoints less than and greater than m case we let $C_m(B) = [i,j]$, where i is the smallest endpoint of all Γ, and j is the largest.

An example illustrating the definition is depicted in Figure 5.3.

Lemma 5.2.5 *For any integers $0 \leq m \leq b$,*

$$K_\tau[0,b] = \sum_{I\ni m} K_\tau[0,I_1]J_\tau[I_1,I_2]K_\tau[I_2,b],$$

where the sum over I is a sum over intervals $[I_1,I_2]$ of integers w $0 \leq I_1 < m < I_2 \leq b$ or $I_1 = I_2 = m$.

Proof. By definition, $C_m(B)$ is an interval of the type being over in the statement of the lemma. Therefore a partial resumn (5.2.7) gives

$$K_\tau[0,b] = \sum_{I\ni m} \sum_{B:C_m(B)=I} \prod_{st\in B} U_{st}.$$

Factoring the sum over B into three independent sums over graphs connected graphs on $I=[I_1,I_2]$, and graphs on $[I_2,b]$, gives

$$K_\tau[0,b] = \sum_{I\ni m} K_\tau[0,I_1]\left(\sum_{\Gamma\in\mathcal{G}_\tau(I)} \prod_{st\in\Gamma} U_{st}\right)K_\tau[I_2,b].$$

The lemma then follows from (5.2.8).

Finally we prove a lemma which will be used in Section 6.7 existence of the infinite self-avoiding walk above four dimensions. the lemma we first introduce some notation. Since we will not nee memory in Section 6.7, we consider only the fully self-avoiding w drop subscripts τ. Given $n \geq m \geq 0$ and an n-step self-avoiding w ω_m denote the first m steps of ω. For $m \geq 1$, we write $\mathbf{k} = (k^{(1)},$.

$k^{(i)} \in [-\pi,\pi]^d$, and $\mathbf{k}\cdot\omega_m = \sum_{j=1}^m k^{(j)}\cdot\omega(j)$. We define a quantity similar to the Fourier transform $\hat{G}_z(k)$ of the two-point function by

$$\Gamma_z(\mathbf{k},m) = \sum_{n=m}^{\infty}\sum_{|\omega|=n} e^{i\mathbf{k}\cdot\omega_m} K[0,n]z^n. \qquad (5.2.20)$$

Since $|\Gamma_z(\mathbf{k},m)| \leq \chi(|z|)$, this power series converges for $|z| < z_c$. We define a quantity similar to $\hat{\Pi}_z(k)$, again for $m \geq 0$, by

$$\Psi_z(\mathbf{k},m) = \sum_{s=m}^{\infty} z^s \sum_{|\omega|=s} e^{i\mathbf{k}\cdot\omega_m} J[0,s]. \qquad (5.2.21)$$

For $j < m$ we define $\bar{\mathbf{k}}_j = (k^{(j+1)},\ldots,k^{(m)})$.

Lemma 5.2.6 *For $m \geq 1$ and for any z for which both sides make sense,*

$$\begin{aligned}\Gamma_z(\mathbf{k},m) &= z\Omega\hat{D}(\sum_{j=1}^m k^{(j)})\Gamma_z(\bar{\mathbf{k}}_1,m-1)\\ &+ \sum_{s=2}^{m-1} z^s \sum_{|\omega|=s}\exp[i\sum_{j=1}^m k^{(j)}\cdot\omega(\min\{j,s\})]J[0,s]\Gamma_z(\bar{\mathbf{k}}_s,m-s)\\ &+ \Psi_z(\mathbf{k},m)\chi(z).\end{aligned}$$

Proof. The proof is similar to the proof of (5.2.17), but is complicated by the presence of the phase factor. We begin by replacing the factor $K[0,n]$ on the right side of (5.2.20), using Lemma 5.2.2, by

$$K[0,n] = K[1,n] + \sum_{s=2}^{m-1} J[0,s]K[s,n] + \sum_{s=m}^{n} J[0,s]K[s,n]. \qquad (5.2.22)$$

The three terms in the statement of the lemma correspond to the three terms on the right side of the above equation.

For example, the $K[1,n]$ term can be written

$$\begin{aligned} &z\sum_{y\in\Omega} e^{ik^{(1)}\cdot y}\sum_{n'=m-1}^{\infty}\sum_{|\omega'|=n'}\exp\left[i\sum_{j=2}^m k^{(j)}\cdot(\omega'(j)+y)\right]K[0,n']z^{n'}\\ &= z\Omega\hat{D}(\sum_{j=1}^m k^{(j)})\Gamma_z(\bar{\mathbf{k}}_1,m-1).\end{aligned}$$

Similarly, in the second and third terms the product $J[0,s]K[s,n]$ allows the sum over ω to be replaced by sums over independent walks of lengths s and $n-s$. For $s < m$ the phase factor makes a contribution to the second of these walks, while for $s \geq m$ it does not. □

5.3 Example: the memory-two walk

As an example of a calculation using the lace expansion, we now s self-avoiding walk with memory equal to two by finding an explicit for $\hat{G}_z(k;2)$. This allows for an explicit calculation of the mean-sq placement. Although the calculation of the mean-square displace the memory-two walk is much simpler than the corresponding ca for the fully self-avoiding walk, the memory-two calculation does i some of the basic features which will occur in Chapter 6.

We begin with a formula for $\hat{\Pi}_z(k;2)$. The derivation of the requires $|z| < 1$, but the resulting expression has an analytic cont to a meromorphic function, which in turn provides a meromorphic e for $\hat{G}_z(k;2)$.

Theorem 5.3.1 *For the self-avoiding walk with memory equal to*

$$\hat{\Pi}_z(k;2) = \frac{z^2\Omega}{z^2-1}[1 - z\hat{D}(k)].$$

Proof. We use (5.2.15), (5.2.11) and (5.2.10), with $\tau = 2$, to $\hat{\Pi}_z(k;2)$. When the memory is equal to two, all contributing la all edges of length exactly equal to two. There is a unique N-e L_N contributing to $J_{2,N}[0,|\omega|]$ if $|\omega| = N+1$, illustrated in Figu $|\omega| \neq N+1$, then $J_{2,N}[0,|\omega|] = 0$. There is therefore exactly one the sum (5.2.10) defining $J_{2,N}[0,|\omega|]$ if $|\omega| = N+1$, and there are otherwise.

Thus we can write the Fourier transform of (5.2.14) as

$$\begin{aligned}\hat{\Pi}_z^{(N)}(k;2) &= \sum_x \Pi_z^{(N)}(0,x;2)e^{ik\cdot x}\\ &= (-1)^N z^{N+1}\sum_{|\omega|=N+1} J_{2,N}[0,N+1]e^{ik\cdot\omega(N+}\\ &= (-1)^N z^{N+1}\sum_{|\omega|=N+1}\prod_{st\in L_N} \mathcal{U}_{st}(\omega)e^{ik\cdot\omega(N+1)}\end{aligned}$$

where L_N is the unique memory-two N-edge lace. The product o patible edges in (5.2.10) is equal to 1 here, since all bonds compat L_N have length one. The product $\prod_{st\in L_N} U_{st}(\omega)$ is equal to $(-1$ nonzero, and it will be nonzero if and only if ω has the topology i in Figure 5.4. These walks end at the origin if N is odd, and end in Ω if N is even. In either case they simply step back and forth re between the origin and a particular site in Ω.

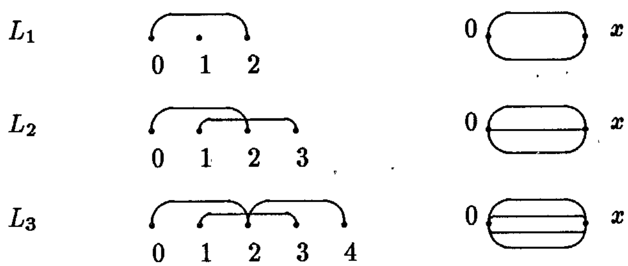


Figure 5.4: The laces for the memory-two walk, together with the corresponding walk topologies for $\prod_{st\in L_N} \mathcal{U}_{st} \neq 0$. Each line in the diagrams on the right represents a single step of the walk, and x is a site in Ω.

Summing (5.3.1) times $(-1)^N$ over N, with the odd and even values of N summed separately, gives

$$\begin{aligned}\hat{\Pi}_z(k;2) &= -\sum_{n=1}^{\infty} z^{2n}\Omega + \sum_{n=1}^{\infty} z^{2n+1} \sum_{x\in\Omega} e^{ik\cdot x} \\ &= \frac{z^2}{z^2-1}\Omega[1 - z\hat{D}(k)]. \qquad (5.3.2)\end{aligned}$$

□

Corollary 5.3.2 *For all $z \in \mathbf{C}$,*

$$\hat{G}_z(k;2) = \frac{1-z^2}{1+(\Omega-1)z^2 - z\Omega\hat{D}(k)}. \qquad (5.3.3)$$

Proof. Combining Theorem 5.3.1 with Theorem 5.2.3, we obtain (5.3.3) for all $|z| < \mu_2^{-1} = (\Omega-1)^{-1}$. But since the right side defines a function meromorphic in the plane, we have a meromorphic extension of $\hat{G}_z(k;2)$. □

The denominator of (5.3.3) has zeroes

$$z_\pm(k) = \frac{\Omega\hat{D}(k) \pm \left[\Omega^2\hat{D}(k)^2 - 4\Omega + 4\right]^{1/2}}{2(\Omega-1)}. \qquad (5.3.4)$$

At $k = 0$ these reduce to

$$z_-(0) = \frac{1}{\Omega-1}, \quad z_+(0) = 1. \qquad (5.3.5)$$

For $k = 0$ the singularity at $z_+(0) = 1$ in (5.3.3) is removable, a[...] the susceptibility $\hat{G}_z(0;2)$ has a unique singularity at the simple [...] $(\Omega-1)^{-1}$. For k near but not equal to 0, $\hat{G}_z(k;2)$ has two simp[...] The location of the closest singularity of $\hat{G}_z(0;2)$ to the origin co[...] been anticipated from the fact that $z_c(0;2)$, given in (5.2.5), is [...] μ_2^{-1}.

Let $c_{T,2}(0,x)$ denote the number of T-step memory-two walk[...] at x, and denote its Fourier transform by $\hat{c}_{T,2}(k)$. Then the mea[...] displacement of the memory-two walk is given by

$$\langle|\omega(T)|^2\rangle_{T=2} = -\frac{\nabla_k^2\hat{c}_{T,2}(0)}{\Omega(\Omega-1)^{T-1}},$$

where ∇_k denotes the gradient in k-space.

Let C be a small circle centred at the origin of the complex pla[...]

$$\hat{G}_z(k;2) = \sum_{T=0}^{\infty} \hat{c}_{T,2}(k)z^T,$$

we have

$$-\nabla_k^2\hat{c}_{T,2}(0) = \frac{1}{2\pi i}\oint_C \frac{-\nabla_k^2\hat{G}_z(0;2)}{z^{T+1}}dz.$$

By (5.3.3),

$$-\nabla_k^2\hat{G}_z(0;2) = \frac{(1+z)z\sum_{y\in\Omega}|y|^2}{[1-(\Omega-1)z]^2(1-z)}.$$

The integral on the right side of (5.3.8) can be evaluated exactly [...] residue theorem. We deform the contour of integration past the sin[...] of the integrand at $(\Omega-1)^{-1}$ and 1 to a large circle of radius R, [...] let R go to infinity. The integral over the large circle vanishes in [...] leaving the contributions from the residues. Denoting the residu[...] at z_0 by $\mathrm{Res}(f(z), z_0)$, we then have

$$\begin{aligned}&\frac{1}{2\pi i}\oint_C \frac{-\nabla_k^2\hat{G}_z(0;2)}{z^{T+1}}dz \\ &= -\mathrm{Res}\left(\frac{-\nabla_k^2\hat{G}_z(0;2)}{z^{T+1}}, \frac{1}{\Omega-1}\right) - \mathrm{Res}\left(\frac{-\nabla_k^2\hat{G}_z(0;2)}{z^{T+1}} \right.\end{aligned}$$

To abbreviate the notation, let $b^2 = \Omega^{-1}\sum_{y\in\Omega}|y|^2$ and $\delta = ($[...] Computing the residues and using (5.3.6) and (5.3.8) then gives

$$\langle|\omega(T)|^2\rangle_{T=2} = b^2\left[\left(\frac{1+\delta}{1-\delta}\right)T - \frac{2\delta(1-\delta^T)}{(1-\delta)^2}\right]$$

$$\sim \quad b^2\left(\frac{1+\delta}{1-\delta}\right)T. \qquad (5.3.11)$$

Thus the mean-square displacement is as expected asymptotically linear in the number of steps.

5.4 Bounds on the lace expansion

Equation (5.2.15) can be used to provide a diagrammatic representation for $\Pi_z(0,x;\tau)$ like that obtained using the inclusion-exclusion relation in Section 5.1. In this section we describe this diagrammatic representation, and then use it to obtain upper bounds on $\hat{\Pi}_z(k;\tau)$ and some of its derivatives. The bounds we obtain will be in terms of norms of the two-point function. Although we drop the memory τ from the notation, the results of this section are valid for any memory $2 \le \tau \le \infty$. The activity z is a complex parameter.

By (5.2.15), (5.2.14) and (5.2.10),

$$\begin{aligned} \Pi_z(0,x) &= \sum_{N=1}^{\infty}(-1)^N \Pi_z^{(N)}(0,x) \qquad (5.4.1)\\ &= \sum_{N=1}^{\infty} \sum_{\substack{\omega:0\to x\\ |\omega|\ge 1}} z^{|\omega|} \sum_{L\in\mathcal{L}_N[0,|\omega|]} \prod_{st\in L}\mathcal{U}_{st} \prod_{s't'\in\mathcal{C}(L)} (1+\mathcal{U}_{s't'}). \end{aligned}$$

In this section we do not concern ourselves with the convergence or divergence of the above series, or of other related series which will occur. Rather, we treat these series as formal power series in z, and obtain upper bounds which are valid in this context. The question of whether or not these bounds suffice to determine convergence is postponed until Section 6.2.

A lace $L \in \mathcal{L}_N[0,|\omega|]$ consists of exactly N edges, and hence the factor $\prod_{st\in L}\mathcal{U}_{st}$ occurring in the sum over ω takes on either the value $(-1)^N$ or 0. The nonzero value will be attained for those walks ω such that $\omega(s)=\omega(t)$ for each $st \in L$.

Consider the case $N=1$. There is a unique lace in $\mathcal{L}_1[0,|\omega|]$, namely $L_1 = \{0,|\omega|\}$. The first product in (5.4.1) is nonzero, in fact -1, for precisely those walks which end at the origin. Thus $\Pi_z^{(1)}(0,x)=0$ if $x \ne 0$. All edges other than $\{0,|\omega|\}$ are compatible with L_1, and hence the second product in (5.4.1) is nonzero only for those walks which have no other self-intersections. The sum over walks in $\hat{\Pi}_z^{(1)}(0,x)$ is thus the sum over all self-avoiding loops (walks which begin and end at the origin and otherwise are self-avoiding, consisting of at least two steps). Let $\mathcal{U}$ denote the set of

all self-avoiding loops. Then, introducing a diagrammatic notation, write

$$\Pi_z^{(1)}(0,x) = \delta_{0,x}\sum_{\omega\in\mathcal{U}} z^{|\omega|} \equiv \delta_{0,x}\ 0\bigcirc\ .$$

We can turn the above into a bound on $\hat{\Pi}_z^{(1)}(k)$ by simply noti

$$\hat{\Pi}_z^{(1)}(k) = \sum_{\omega\in\mathcal{U}} z^{|\omega|} = \sum_{x\in\Omega} zG_z(0,x)$$

and hence

$$|\hat{\Pi}_z^{(1)}(k)| \le |z|\,\Omega \sup_{x\ne 0} G_{|z|}(0,x).$$

Writing $\|\cdot\|_\infty$ for the x-space supremum norm

$$\|f\|_\infty = \sup_{x\in Z^d} |f(x)|,$$

and introducing

$$\begin{aligned} H_z(x,y) &= G_z(x,y) - \delta_{x,y}\\ &= \begin{cases} G_z(x,y) & x\ne y\\ 0 & x=y, \end{cases} \end{aligned}$$

(5.4.2) can be rewritten as

$$|\hat{\Pi}_z^{(1)}(k)| \le |z|\,\Omega\,\|H_{|z|}\|_\infty.$$

(Actually H_z is a function of two variables, and in writing norms c mean norms of the function $H_z(0,\cdot)$ of a single variable.)

For $N=2$, we proceed as follows. Laces in $\mathcal{L}_2[0,|\omega|]$ are in a correspondence with pairs of times s_2,t_1 with $0 < s_2 < t_1 < |\omega|$ be seen from Figure 5.5. For such a lace L, a walk with $\prod_{st\in L}\mathcal{U}_s$ breaks up in a natural way into three subwalks. Letting x de endpoint of ω, these three subwalks are of the form $\omega^{(1)}:0\to x$, ω 0, $\omega^{(3)}:0\to x$, of respective lengths s_2, t_1-s_2, $|\omega|-t_1$. We can factor $z^{|\omega|}$ into three corresponding factors

$$z^{|\omega|} = \prod_{i=1}^{3} z^{|\omega^{(i)}|}.$$

Each of the subwalks consists of at least one step. The factor $\prod_{s't'\in}$ $\mathcal{U}_{s't'})$ ensures that each of these three subwalks is self-avoiding, s

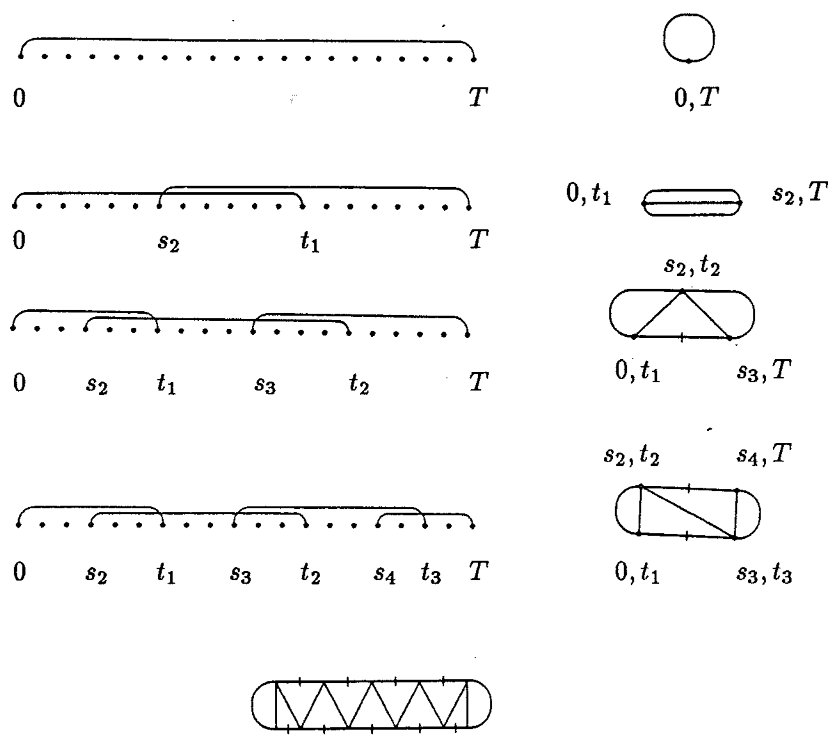


Figure 5.5: The left column shows the general form of laces consisting of $1, 2, 3$ or 4 edges. The lace edges are denoted s_i, t_i, $1 \leq i \leq N$, with $s_1 = 0$ and $t_N = T$. The right column shows the self-intersections required for a walk ω with $\prod_{st \in L} \mathcal{U}_{st}(\omega) \neq 0$. For $\prod_{s't' \in \mathcal{C}(L)}(1 + \mathcal{U}_{s't'}) \neq 0$, each of the $2N-1$ subwalks must be a self-avoiding walk, and in addition there must be mutual avoidance between some (but not all) of the subwalks. The number of loops in a diagram is equal to the number of edges in the corresponding lace. The lines which are slashed correspond to subwalks which may consist of zero steps, but the others correspond to subwalks consisting of at least one step. The eleven-loop diagram is depicted at the bottom.

bond lying entirely in one of the intervals $[0, s_2]$, $[s_2, t_1]$, $[t_1, |\omega|]$ is
patible with L. (In particular, $x \neq 0$.) This same factor also provi
avoidance interaction between the three subwalks, since there are
in $\mathcal{C}(L)$ which link distinct subwalks. This interaction between di
subwalks would be awkward to take into account exactly, but in an
bound it can be disregarded since $1 + \mathcal{U}_{s't'} \leq 1$. Therefore

$$|\Pi_z^{(2)}(0, x)| \leq H_{|z|}(0, x)^3. \quad ($$

We now show how (5.4.8) can be used to estimate $\hat{\Pi}_z^{(2)}(k)$ and its s
derivative with respect to a component k_μ of k. Writing ∂_μ^u for th
order partial derivative with respect to k_μ, and using the definition
Fourier transform,

$$\left|\partial_\mu^u \hat{\Pi}_z^{(2)}(k)\right| \leq \sum_x |x_\mu|^u \Pi_{|z|}^{(2)}(0, x) \leq \sum_x |x_\mu|^u H_{|z|}(0, x)^3$$
$$\leq \| |x_\mu|^u H_{|z|}(0, x)\|_\infty \|H_{|z|}\|_2^2. \quad ($$

Here $\|\cdot\|_2$ denotes the x-space L^2-norm

$$\|f\|_2 = \left[\sum_{x \in \mathbf{Z}^d} |f(x)|^2\right]^{1/2} \quad (5$$

In terms of the bubble diagram (1.5.4), the right side of (5.4.9) is
to

$$\| |x_\mu|^u H_{|z|}(0, x)\|_\infty [\mathsf{B}(|z|) - 1]. \quad ($$

This bound provides an indication of the critical nature of $d = 4$,
following way. Assuming that the infrared bound $\eta \geq 0$ is indeed
then $\mathsf{B}(z_c)$ is finite for $d > 4$. The infrared bound also implies th
critical two-point function decays at least as fast as $|x|^{2-d}$, so th
$d > 4$ (5.4.11) will be finite at the critical point for $u \leq 2$, and he
will $\partial_\mu^u \hat{\Pi}_{z_c}^{(2)}(k)$. For models with a suitable weak interaction, such
nearest-neighbour model in sufficiently high dimensions, or a suffi
"spread-out" model above four dimensions (see Chapter 6), the qu
$\mathsf{B}(z_c) - 1$ will be not only finite, but arbitrarily small, and will be the
parameter responsible for convergence of the expansion. For this reas
distinction between H_z and G_z will be crucial.

We wish to obtain bounds on $\hat{\Pi}_z^{(N)}(k)$ and some of its derivative
analogous way for higher values of N. To state these bounds we firs
to introduce the following definition.

Definition 5.4.1 *We define a multiplication operator $\mathcal{M}$ and convolution operators $\mathcal{H}^{(0)}$ and $\mathcal{H}^{(1)}$ by*

$$(\mathcal{M}f)(x) = H_{|z|}(0,x)f(x) \tag{5.4.12}$$

$$(\mathcal{H}^{(0)}f)(x) = \sum_y G_{|z|}(x,y)f(y) \tag{5.4.13}$$

$$(\mathcal{H}^{(1)}f)(x) = \sum_y H_{|z|}(x,y)f(y). \tag{5.4.14}$$

These operators depend on $|z|$, but to simplify the notation we do not make this dependence explicit.

Then we have the following bounds on $\hat{\Pi}_z^{(N)}(k)$, for $N \geq 2$. (The case $N = 1$ is special and the bound is given in (5.4.3).)

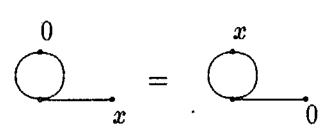
Theorem 5.4.2 *For $N \geq 2$,*

$$|\hat{\Pi}_z^{(N)}(k)| \leq \left((\mathcal{H}^{(1)}\mathcal{M})(\mathcal{H}^{(0)}\mathcal{M})^{N-2}H_{|z|}\right)(0). \tag{5.4.15}$$

Proof. The Fourier transform on the left side of (5.4.15) is obtained by multiplying the N-th term of (5.4.1) by $e^{ik\cdot x}$ and summing over x. For an upper bound, we take absolute values inside all sums in the resulting expression. This removes the k dependence, and replaces z by $|z|$. Then we bound by 1 all factors $1+\mathcal{U}_{st}$ in the product over compatible bonds in (5.4.1) for which s and t are times corresponding to distinct subwalks. The remaining factors ensure that all subwalks remain self-avoiding, but there is no longer any interaction between distinct subwalks. We claim that the resulting expression is then exactly equal to the right side of (5.4.15).

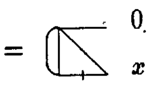
To see this, we note that for $N = 2,3,4$ we have the following diagrammatic representations, in which slashed lines correspond to $G_{|z|}$ and unslashed lines to $H_{|z|}$ (i.e. slashed lines correspond to subwalks which may consist of no steps, while unslashed lines correspond to subwalks consisting of at least one step). To begin,

$$\mathcal{H}^{(1)}\mathcal{M}H_{|z|}(x) =$$

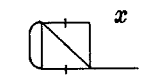


where symmetry was used in the last step. Proceeding from the analogue of the right side of the above equation, and then again using symmetry, we obtain

$$\mathcal{H}^{(1)}\mathcal{M}\mathcal{H}^{(0)}\mathcal{M}H_{|z|}(x) =$$



$$\mathcal{H}^{(1)}\mathcal{M}(\mathcal{H}^{(0)}\mathcal{M})^2H_{|z|}(x) =$$



The pattern continues for larger N, and reproduces the diagrams rep ing $\hat{\Pi}_z^{(N)}$ when $x = 0$ (with no mutual avoidance between distinct sub

Remark. We will require modifications of Theorem 5.4.2 of two t
1. The first, and most common, arises when we wish to bound derivatives of $\hat{\Pi}_z^{(N)}$, with respect to z and/or a component of k. Fo ple, in bounding the derivative of $\hat{\Pi}_z^{(N)}$ with respect to k_μ, the diff tion of the exponential brings down a factor ix_μ. The factor x_μ can written as a sum of displacements along subwalks, and when absol ues are taken and interactions between subwalks neglected, the res sum of diagrams of the same topology as those representing $\hat{\Pi}_z^{(N)}$ its with one of the subwalks in each diagram weighted by the absolu of the μ-th component of its displacement. (Taken together, the w lines give a path from 0 to x.) Such a diagram is equal to the anal the right side of (5.4.15) in which the factor corresponding to the w subwalk is replaced by the corresponding multiplication or convolu $|x_\mu|H_{|z|}(0,x)$. [Note that by (5.4.5), $x_\mu H_z(0,x) = x_\mu G_z(0,x)$.]
2. The second modification concerns an improvement to (5.4.15) f memory. For finite memory, the laces occurring in the lace expans sist of bonds of length τ or less. Consequently all subwalks in d representing $\hat{\Pi}_z$ consist of at most τ steps, so we may replace the o on the right side of (5.4.15) by multiplication and convolution by responding generating functions truncated at order z^τ. This impr will be relevant in this book only in Section 6.8, which is the o where we will not work directly with the fully self-avoiding walk.

The right side of (5.4.15) will be bounded using the following l

Lemma 5.4.3 *Given functions $f_0, f_1, \ldots, f_{2M}$ on $\mathbf{Z}^d$, define $\mathcal{H}_{2j}$ c to be respectively the operations of convolution with f_{2j} and multi by f_{2j-1}, for $j = 1, \ldots, M$. Then for any k,*

$$||\prod_{j=1}^{M}(\mathcal{H}_{2j}\mathcal{M}_{2j})f_0||_\infty \leq ||f_k||_\infty \prod_{\substack{0\leq j\leq 2M \\ j\neq k}} ||f_j||_2.$$

In the product over j on the left side, factors are to be taken with decreasing index from left to right.

Proof. Fix $k \in \{0,1,2,\ldots,2M\}$. Using $||\mathcal{A}||_{p,r}$ to denote the norm of an operator $\mathcal{A}:\ell^p \to \ell^r$, the left side of (5.4.16) can be bounded above by

$$\prod_{j>k} ||\mathcal{H}_{2j}\mathcal{M}_{2j}||_{\infty,\infty} ||\mathcal{H}_{2k}\mathcal{M}_{2k}||_{2,\infty} \prod_{i<k} ||\mathcal{H}_{2i}\mathcal{M}_{2i}||_{2,2}||f_0||_p \qquad (5.4.17)$$

where $p=2$ if $k>0$ and $p=\infty$ if $k=0$. (Also the norm of $\mathcal{H}_0\mathcal{M}_0$ should be omitted if $k=0$.) The desired result then follows from the inequalities

$$\begin{array}{lllll} ||\mathcal{H}_{2j}\mathcal{M}_{2j}||_{\infty,\infty} & \le & ||\mathcal{H}_{2j}||_{2,\infty}||\mathcal{M}_{2j}||_{\infty,2} & \le & ||f_{2j}||_2||f_{2j-1}||_2 \\ ||\mathcal{H}_{2j}\mathcal{M}_{2j}||_{2,2} & \le & ||\mathcal{H}_{2j}||_{1,2}||\mathcal{M}_{2j}||_{2,1} & \le & ||f_{2j}||_2||f_{2j-1}||_2 \\ ||\mathcal{H}_{2k}\mathcal{M}_{2k}||_{2,\infty} & \le & ||\mathcal{H}_{2k}||_{2,\infty}||\mathcal{M}_{2j}||_{2,2} & \le & ||f_{2k}||_2||f_{2k-1}||_\infty \\ ||\mathcal{H}_{2k}\mathcal{M}_{2k}||_{2,\infty} & \le & ||\mathcal{H}_{2k}||_{1,\infty}||\mathcal{M}_{2j}||_{2,1} & \le & ||f_{2k}||_\infty||f_{2k-1}||_2, \end{array}$$

where the right hand inequalities follow from the Hölder and Young inequalities. (The latter states that $||g*h||_s \le ||g||_r||h||_p$ for $1 \le p,r,s \le \infty$ satisfying $p^{-1}+r^{-1}=1+s^{-1}$.) □

In the next theorem, Lemma 5.4.3 (in conjunction with Remark 1 below Theorem 5.4.2) is used to bound various derivatives of $\hat{\Pi}_z(k)$. For finite memory, the bounds can be improved as in Remark 2 below Theorem 5.4.2.

Theorem 5.4.4 *For any $v \ge 0$,*

$$|\partial_z^v \hat{\Pi}_z^{(1)}(k)| \le \Omega\, ||\partial_z^v|_{z=|z|}[zH_z]||_\infty. \qquad (5.4.18)$$

For any integer $N \ge 2$,

$$|\partial_z \hat{\Pi}_z^{(N)}(k)| \le (2N-1)||\partial_z|_{z=|z|}H_z||_\infty ||H_{|z|}||_2^N ||G_{|z|}||_2^{N-2}. \qquad (5.4.19)$$

For any integers $v=0,1$, $u \ge 0$ and $N \ge 2$,

$$\begin{aligned} |\partial_z^v\partial_\mu^u \hat{\Pi}_z^{(N)}(k)| \;&\le\; (2N-1)^v\left[\frac{N+1}{2}\right]^u ||x_\mu^u H_{|z|}||_\infty ||\partial_z^v|_{z=|z|}H_z||_2 \\ &\quad \times ||H_{|z|}||_2^{N-1}||G_{|z|}||_2^{N-2}. \end{aligned} \qquad (5.4.20)$$

For any $N \ge 2$ and for any positive p,

$$\begin{aligned} 0 \;&\le\; \hat{\Pi}_p^{(N)}(0) - \hat{\Pi}_p^{(N)}(k) \qquad (5.4.21) \\ &\le\; \frac{1}{d}\sum_{\mu=1}^d (1-\cos k_\mu)\left[\frac{N+1}{2}\right]^2 |||x|^2H_p||_\infty ||H_p||_2^N ||G_p||_2^{N-2}. \end{aligned}$$

For $N=1$, $\hat{\Pi}_z^{(1)}(0) - \hat{\Pi}_z^{(1)}(k) = 0$, and hence $\partial_\mu^u \hat{\Pi}_z^{(1)}(k) = 0$ for all $u \ge 1$.

Proof. The bound (5.4.18) follows immediately from (5.4.2). For (
applying ∂_z to $z^{|\omega|}$ brings down a factor $|\omega|$. Considering ω to co
$2N-1$ subwalks ω_j, we have

$$|\omega| = \sum_{j=0}^{2N-1} |\omega_j|.$$

Now we bound the diagram in which the j-th subwalk is weighte
factor $|\omega_j|$ using Lemma 5.4.3, with the infinity norm on the j-th s
This gives (5.4.19).

For (5.4.20) the k-derivative gives rise to an additional factor |
the factor $|\omega|$ we proceed as for (5.4.19), obtaining a sum of terms i
one subwalk is weighted by a factor $|\omega_j|$. Then for the factor $|x_\mu^u|$ we
sequence of subwalks (depending on j) connecting 0 and x spatially,
using the already weighted j-th subwalk. In general no more than (
subwalks will be needed. We then write x_μ as the sum of displa
along the subwalks, and use the inequality

$$|a_1+\ldots+a_m|^u \le m^{u-1}(|a_1|^u+\ldots+|a_m|^u)$$

to obtain a sum of diagrams where in each term one subwalk is
by $|\omega_j|$ and another by $|x_\mu^u|$. Then Lemma 5.4.3 is applied.

To prove (5.4.21), we note that the first inequality follows from
that by symmetry and (5.2.16),

$$\hat{\Pi}_p^{(N)}(0) - \hat{\Pi}_p^{(N)}(k) = \sum_x \Pi_p^{(N)}(0,x)[1-\cos k\cdot x] \ge 0.$$

For the upper bound we write

$$\hat{\Pi}_p^{(N)}(0) - \hat{\Pi}_p^{(N)}(k) = \sum_x \Pi_p^{(N)}(0,x)(1-e^{ik\cdot x}),$$

and write the last factor on the right as a telescoping sum

$$1-e^{ik\cdot x} = \sum_{\mu=1}^d (1-e^{ik_\mu x_\mu})\prod_{\nu<\mu} e^{ik_\nu x_\nu}.$$

Inserting (5.4.23) into (5.4.22) and using symmetry gives

$$\hat{\Pi}_p^{(N)}(0) - \hat{\Pi}_p^{(N)}(k) = \sum_x \Pi_p^{(N)}(0,x)\sum_{\mu=1}^d (1-\cos k_\mu x_\mu)\prod_{\nu<\mu} \cos k_\nu x_\nu.$$

Using the fact that for any real y and positive integer m,

$$0 \le 1 - \cos my \le m^2(1 - \cos y), \tag{5.4.25}$$

and again using symmetry, the right side of (5.4.24) can be bounded above by

$$\frac{1}{d}\sum_{\mu=1}^{d}(1 - \cos k_\mu)\, d \sum_x x_\mu^2 \Pi_p^{(N)}(0,x). \tag{5.4.26}$$

The bound (5.4.21) then follows, since it was shown in the proof of (5.4.20) (with $v = 0$) that the sum over x on the right side of the above equation is bounded above by the right side of (5.4.20).

Finally, the last statement of the theorem follows immediately from the first equality of (5.4.2). □

We end this section with a lemma which will be used in Section 6.8 to bound $c_n(0,x)$. In the statement of the lemma, $c_{n,\sigma}(0,x)$ denotes the number of n-step memory-σ walks from 0 to x.

Lemma 5.4.5 *For memories $\sigma < \tau \le \infty$,*

$$|\hat{\Pi}_z(k;\sigma) - \hat{\Pi}_z(k;\tau)| \le 2\|\sum_{n=\sigma/6}^{\infty} c_{n,\sigma}(0,x)|z|^n\|_\infty$$
$$\times \left[|z|\,\Omega + \sum_{N=2}^{\infty}(2N-1)\|H_{|z|}(0,\cdot;\sigma)\|_2^N \|G_{|z|}(0,\cdot;\sigma)\|_2^{N-2}\right]. \tag{5.4.27}$$

Proof. By Definition 5.2.1, $\mathcal{L}_\sigma[a,b] \subset \mathcal{L}_\tau[a,b]$, and for $L \in \mathcal{L}_\sigma[a,b]$, we have $\mathcal{C}_\sigma(L) \subset \mathcal{C}_\tau(L)$. The latter inclusion is often strict, but if L has no bond whose length exceeds $\sigma/2$ then $\mathcal{C}_\tau(L)$ can contain no bond whose length exceeds σ, and hence $\mathcal{C}_\sigma(L) = \mathcal{C}_\tau(L)$. Therefore in the difference on the left side of (5.4.27) there will be an exact cancellation of terms arising from laces having all bonds of length less than or equal to $\sigma/2$. Temporarily writing $\tilde{\Pi}_m$ for the contribution to $\hat{\Pi}_z(k;m)$ due to laces having at least one bond of length exceeding $\sigma/2$, we have

$$|\hat{\Pi}_z(k;\sigma) - \hat{\Pi}_z(k;\tau)| \le |\tilde{\Pi}_\sigma| + |\tilde{\Pi}_\tau|. \tag{5.4.28}$$

A typical bond in a lace spans either two or three subwalks; see Figure 5.5. For a lace bond of length at least $\sigma/2$, at least one of the subwalks must consist of at least $\sigma/6$ steps. Now we bound the right side of (5.4.28) as in (5.4.20) (with $u = v = 0$) and (5.4.3), using the L^∞ norm on a subwalk consisting of at least $\sigma/6$ steps. The factor $2N - 1$ arises from the

fact that any one of the $2N - 1$ subwalks may be the only subwal
least $\sigma/6$ steps. Also, the bound on $\tilde{\Pi}_\tau$ can be written in terms of n
σ rather than τ, since $c_{n,\tau}(0,x) \le c_{n,\sigma}(0,x)$.

Finally we observe that in the L^2 norms on the right side, the gen
functions H and G can be truncated at the term of order z^σ by the
Remark under Theorem 5.4.2.

5.5 Other models

The lace expansion has been adapted to study lattice trees and a
and percolation, and provides a way of proving mean-field critical be
above eight and six dimensions respectively. For these models it is
finiteness of the bubble diagram which leads to mean-field behavic
rather the finiteness of the square diagram for lattice trees and anim
of the triangle diagram for percolation. The lace expansion can be
prove that these diagrams are finite.

In this section we discuss the lace expansion for lattice trees and
and for percolation. The material developed here is not required el
in the book. The proof of convergence of the expansion in these
is very similar to the proof of convergence of the expansion for
avoiding walk (see Section 6.2), and will not be discussed here. Ins
just derive the expansions, and briefly indicate how they can be b
In the bounds the square or triangle diagrams play a key role.

5.5.1 Lattice trees and animals

Let Ω be a finite set of sites in $\mathbf{Z}^d$, not containing the origin,
symmetric with respect to the symmetries of $\mathbf{Z}^d$. We refer to a pa
of sites with $y - x \in \Omega$ as a bond. A *lattice tree* is a connected set
which has no closed loops. Although a tree T is defined as a set o
we write $x \in T$ if x is an endpoint of some bond of T. We de
number of bonds in a tree T by $|T|$. A *lattice animal* is a connecte
bonds which may contain closed loops. We denote a typical lattic
by A and the number of bonds in A by $|A|$.

Let t_n denote the number of n-bond trees containing the origin
the number of n-bond animals containing the origin. It can be sho
subadditivity arguments that both $t_n^{1/n}$ and $a_n^{1/n}$ converge to finite
limits λ and λ_a as n goes to infinity. The asymptotic behaviour
quantities is believed to be of the form

$$t_n \sim \text{const.}\lambda^n n^{-\theta+1}, \quad a_n \sim \text{const.}\lambda_a^n n^{-\theta+1}$$

Chapter 6

Above four dimensions

6.1 Overview of the results

The lace expansion has been used to resolve many of the issues co[illegible]
the self-avoiding walk in five or more dimensions. Proving conver[illegible]
the lace expansion for $d = 5$ involves a myriad of major technical dif[illegible]
due to the fact that the best bound on the small parameter respon[illegible]
convergence of the expansion, namely $||H_{z_c}||_2^2 = \mathsf{B}(z_c)-1$, is 0.493. [illegible]
many of these technical difficulties are not present if the small pa[illegible]
can be taken to be arbitrarily small, and it is in the context of an ar[illegible]
small parameter that the proof becomes most transparent. For this
in this chapter we give the proof of convergence of the lace expans[illegible]
its consequences for the critical behaviour in two contexts: for the [illegible]
neighbour model with large d, and for the "spread-out" self-avoidi[illegible]
with steps (x, y) satisfying $0 < ||x - y||_\infty \leq L$, for $d > 4$ and large [illegible]

For each of these two models we will use Ω to denote the coor[illegible]
number, i.e. $\Omega = 2d$ for the nearest-neighbour model and $\Omega = (2L$[illegible]
for the spread-out model. It will be shown that in either case the be[illegible]
of $||H_{z_c}||_2^2$ is governed by the contribution to the corresponding [illegible]
random walk critical ($z = \Omega^{-1}$) bubble diagram due to the Ω [illegible]
which two single step walks end at the same site, i.e. $\Omega/\Omega^2 = \Omega^{-1}$[illegible]
the small parameter can be made arbitrarily small by increasing Ω[illegible]

In the remainder of this section we summarize the results that
obtained in this chapter. We discuss both the nearest-neighbour m[illegible]
the spread-out model simultaneously, combining the statements t[illegible]
sufficiently large for the nearest-neighbour model, and L is sufficien[illegible]
for the spread-out model, into the single statement that Ω is suf[illegible]

large. We emphasize that all of the results stated in this section, with the exception of Theorem 6.1.3, have been proven in Hara and Slade (1992a,b) for the nearest-neighbour model for $d \geq 5$.

Asymptotic formulas for c_n and the mean-square displacement are given in the following theorem, whose proof can be found in Section 6.4.2.

Theorem 6.1.1 *There is an Ω_0 such that for $\Omega \geq \Omega_0$ there are positive A, D such that the following hold (assuming $d > 4$ for the spread-out model).*

(a) $c_n = A\mu^n[1 + O(n^{-\epsilon})]$ *as $n \to \infty$, for any $\epsilon < \min\{(d-4)/2, 1\}$.*

(b) $\langle|\omega(n)|^2\rangle = Dn[1 + O(n^{-\epsilon})]$ *as $n \to \infty$, for any $\epsilon < \min\{(d-4)/4, 1\}$.*

Remark. Bounds on the constants A and D will be given in Section 6.2.3. In particular, for the nearest-neighbour model in high dimensions D is strictly greater than one, indicating that the self-avoiding walk does move away from the origin more quickly than ordinary random walk, although only at the level of the diffusion constant. For the nearest-neighbour model in five dimensions the current best bounds are given in Hara and Slade (1992b) to be $1 \leq A \leq 1.493$ and $1.098 \leq D \leq 1.803$.

A corollary of (*a*) is that $\lim_{n\to\infty} c_{n+1}/c_n = \mu$ [cf. Equation (7.1.4)]. This is believed to be true in all dimensions, but remains unproved for $d = 2, 3, 4$. Theorem 6.1.1 is proven via a Tauberian-type theorem, after first controlling the susceptibility and correlation length of order two. The results for χ and ξ_2 are stated in the next theorem, which is proved in Section 6.2.3. [The notation $f(z) \sim g(z)$ means $\lim_{z\nearrow z_c} f(z)/g(z) = 1$.]

Theorem 6.1.2 *There is an Ω_0 such that for $\Omega \geq \Omega_0$ (assuming $d > 4$ for the spread-out model)*

$$\chi(z) \sim \frac{Az_c}{z_c - z}$$

and

$$\xi_2(z) \sim \left(\frac{Dz_c}{z_c - z}\right)^{1/2},$$

where the constants A, D are the same as in Theorem 6.1.1.

For $c_n(0, x)$ we will prove the following theorem, which gives the hyperscaling inequality $\alpha_{sing} - 2 \leq -d/2$. In fact this inequality is believed to be an equality; see Section 2.1. Theorem 6.1.3 is the only result stated in this section which has not been proved for the nearest-neighbour model for all $d \geq 5$.

Theorem 6.1.3 *There is an Ω_0 such that for $\Omega \geq \Omega_0$ (assuming d the spread-out model) there is a constant B such that*

$$\sup_{x\in Z^d} c_n(0, x) \leq B\mu^n n^{-d/2}.$$

This theorem is proved in Section 6.8. An immediate consequence o rem 6.1.3 is the following result, which is a weaker version of the sta that $\alpha_{sing} - 2 \leq -d/2$. This weaker statement has been proven nearest-neighbour model for all $d \geq 5$; we comment briefly on the r of proof in the Notes for this chapter.

Corollary 6.1.4 *There is an Ω_0 such that for $\Omega \geq \Omega_0$ (assuming for the spread-out model)*

$$\sup_{x\in Z^d} \sum_{n=0}^{\infty} n^a c_n(0, x)\mu^{-n} < \infty$$

for any $a < (d-2)/2$.

For the correlation length $\xi(z) = 1/m(z)$ [see (1.3.15)] we ha following result, which is proved in Section 6.5.1.

Theorem 6.1.5 *There is an Ω_0 such that for $\Omega \geq \Omega_0$ (assuming d the spread-out model)*

$$\xi(z) \sim \sqrt{\frac{D}{2d}}\left(\frac{z_c}{z_c - z}\right)^{1/2},$$

with the same constant D as in Theorem 6.1.1.

By Theorems 6.1.1, 6.1.2 and 6.1.5, the length scales defined by the square displacement, the correlation length of order two, and the corr length are as expected all governed by the same critical exponent ν

Using Theorem 6.1.5 it can be shown that the renormalized co constant $g(z)$ of (1.4.22) obeys

$$g(z) \simeq (z_c - z)^{(d-4)/2} \quad \text{as } z \nearrow z_c,$$

for the spread-out model with Ω sufficiently large and for the n neighbour model for $d \geq 6$. Unfortunately (6.1.1) remains unpro $d = 5$. Further details are given in the Remark under Theorem 1.5.

The results for the critical two-point function are stronger in k than in x-space, and are summarized in the following theorem, whos can be found in Section 6.5.2. The upper bound on $G_{z_c}(0,x)$ in the theorem, for $p < (d-2)/2$, follows immediately from Corollary 6.1.4 and the fact that $|x|^p c_n(0,x) \le n^p c_n(0,x)$. The k-space result provides a strong infrared bound.

Theorem 6.1.6 *There is an Ω_0 such that for $\Omega \ge \Omega_0$ (assuming $d > 4$ for the spread-out model) the following hold. For any p satisfying $p < (d-2)/2$ or $p \le 2$, there is a constant $C(p)$ such that for all x, $G_{z_c}(0,x) \le C(p)|x|^{-p}$. There is a positive constant such that the Fourier transform satisfies $\hat{G}_{z_c}(k) = const.[k^2 + O(k^{2+\epsilon})]^{-1}$ as $k \to 0$, for any $\epsilon < \min\{(d-4)/2, 1\}$. In addition, there is a positive constant such that $0 \le \hat{G}_{z_c}(k) \le const.k^{-2}$ for all $k \in [-\pi,\pi]^d$.*

Corollary 6.1.7 *There is an Ω_0 such that for $\Omega \ge \Omega_0$ (assuming $d > 4$ for the spread-out model)*

$$m(z_c) = 0.$$

Proof. The bound on $\hat{G}_{z_c}(k)$ of Theorem 6.1.6 implies that the critical bubble diagram $\mathsf{B}(z_c) = (2\pi)^{-d}\int_{[-\pi,\pi]^d} \hat{G}_{z_c}(k)^2 d^dk$ is finite (see Section 1.5). It then follows from Theorem 4.1.6 that $m(z_c) = 0$. □

To discuss the scaling limit, we first introduce some notation. Let $C_d[0,1]$ denote the continuous $\mathbf{R}^d$-valued functions on $[0,1]$, equipped with the supremum norm. Given an n-step self-avoiding walk ω, we define $X_n \in C_d[0,1]$ by setting $X_n(k/n) = (Dn)^{-1/2}\omega(k)$ for $k = 0,1,2,\ldots,n$, and taking $X_n(t)$ to be the linear interpolation of this. We denote by dW the Wiener measure on $C_d[0,1]$. Expectation with respect to the uniform measure on the n-step self-avoiding walks is denoted by $\langle\cdot\rangle_n$. The following theorem is proved in Section 6.6.

Theorem 6.1.8 *There is an Ω_0 such that for $\Omega \ge \Omega_0$ (assuming $d > 4$ for the spread-out model), the scaled self-avoiding walk converges in distribution to Brownian motion. In other words for any bounded continuous function f on $C_d[0,1]$,*

$$\lim_{n\to\infty} \langle f(X_n)\rangle_n = \int fdW.$$

The next result concerns the existence of a measure on infinitely long self-avoiding walks. We defer the precise definition of this measure until Section 6.7, where the following theorem will be proved.

Theorem 6.1.9 *There is an Ω_0 such that for $\Omega \ge \Omega_0$ (assuming d*
the spread-out model) the infinite self-avoiding walk exists.

The key ingredient in the proofs of the above theorems is the
gence of the lace expansion, which is proved in the next section.

6.2 Convergence of the lace expansion

This section is divided into three parts. The first part proves a
which encapsulates the basic structure of the proof of convergence
lace expansion, and also gives a number of properties of simple r
walk which will be needed in the convergence proof. The second par
the proof of convergence of the lace expansion, and states a nur
consequences. The last part gives the proof of Theorem 6.1.2, i.e. ex
of and mean-field values for the critical exponents for the susceptibili
the correlation length of order two.

6.2.1 Preliminaries

The following elementary lemma will be used to prove convergence
lace expansion. It states that under an appropriate continuity assun
if a set of inequalities implies a stronger set of inequalities, then in f
stronger inequalities must hold.

Lemma 6.2.1 *Let $f_1,\ldots,f_n$ be nonnegative functions defined on*
terval $[0,p_1)$, and let $p_0 \in [0,p_1)$ and $a < 1$ be given. Suppose that

1. *f_i is continuous on the interval $[0,p_1)$, for $i = 1,\ldots,n$,*
2. *$f_i(p) \le a$ for $0 \le p \le p_0$, for $i = 1,\ldots,n$,*
3. *for each $p \in [p_0,p_1)$, if $f_i(p) \le 1$ for all $i = 1,\ldots,n$, then $f_i(p) \le a$ for all $i = 1,\ldots,n$. (In other words a set of ineq*
implies a stronger set of inequalities.)

Then $f_i(p) \le a$ for all $p \in [0,p_1)$ and all $i = 1,\ldots,n$.

Proof. Define $f_{max}(p) = \max_{1\le i\le n} f_i(p)$. By the second assump
suffices to show that $f_{max}(p) \le a$ for $p \in [p_0,p_1)$. By the third a
tion $f_{max}(p) \notin (a,1]$ for all $p \in [p_0,p_1)$. By the first assumption f_{ma}
continuous in $p \in [0,p_1)$. Since $f_{max}(p_0) \le a$ by the second assun
the above two facts imply that $f_{max}(p)$ cannot enter the forbidden i
$(a,1]$ when $p \in [p_0,p_1)$ and hence $f_{max}(p) \le a$ for all $p \in [0,p_1)$.

Before defining the functions f_i that we will use, we need to introduce two models of ordinary random walk corresponding to the two models of self-avoiding walk discussed in the previous section. For the usual nearest-neighbour simple random walk we denote the coordination number by $\Omega = 2d$, and also use Ω to denote the set of sites which are nearest neighbours of the origin. The critical $(z = \Omega^{-1})$ two-point function for this model is shown in (A.8) to be given by

$$C^{(0)}(0,x) = \int_{[-\pi,\pi]^d} \frac{e^{-ik\cdot x}}{1-\hat{D}_0(k)} \frac{d^dk}{(2\pi)^d}, \tag{6.2.1}$$

where

$$\hat{D}_0(k) = \frac{1}{\Omega}\sum_{x\in\Omega} e^{ik\cdot x} = d^{-1}\sum_{\mu=1}^{d} \cos k_\mu. \tag{6.2.2}$$

Let $L \geq 1$ be an integer. For the ordinary "spread-out" random walk in $\mathbf{Z}^d$ whose steps (x,y) satisfy $0 < ||x-y||_\infty \leq L$, we will use Ω to denote the set of $x \in \mathbf{Z}^d$ with $0 < ||x||_\infty \leq L$, and also write Ω for the cardinality of this set, i.e. $\Omega = (2L+1)^d - 1$. For $x \in \mathbf{Z}^d$, let $C^{(L)}(0,x)$ denote the critical spread-out ordinary random walk two-point function. This is given in (A.8) by

$$C^{(L)}(0,x) = \int_{[-\pi,\pi]^d} \frac{e^{-ik\cdot x}}{1-\hat{D}_L(k)} \frac{d^dk}{(2\pi)^d}, \tag{6.2.3}$$

where

$$\hat{D}_L(k) = \frac{1}{\Omega}\sum_{x\in\Omega} e^{ik\cdot x} = \frac{1}{\Omega}\sum_{x\in\Omega} \cos(k\cdot x). \tag{6.2.4}$$

We write simply $C(0,x)$ and $\hat{D}(k)$ when we wish to discuss both the spread-out and nearest-neighbour models simultaneously. The following lemma is a combination of the statements of Lemmas A.3 and A.5, in which some bounds have been degraded for a unified statement.

Lemma 6.2.2 *For any $d \geq 1$ there is an Ω_0 such that for any $k \in [-\pi,\pi]^d$ and $\Omega \geq \Omega_0$,*

$$1 - \hat{D}(k) \geq \frac{k^2}{2\pi^2 d}. \tag{6.2.5}$$

For any $d \geq 1$ there is an Ω_0 such that for all $\Omega \geq \Omega_0$

$$\sup_{n\geq 0} n^{d/2} ||\hat{D}^n||_1 < \infty. \tag{6.2.6}$$

Let s denote a fixed small positive number for the spread-out model, and let $s = 0$ for the nearest-neighbour model. There is a K such that for all

Ω (assuming $d > 4$ for the spread-out model and $d \geq 5$ for th neighbour model)

$$||\hat{C}||_2^2 - 1 = \left|\left|\frac{1}{1-\hat{D}}\right|\right|_2^2 - 1 \leq K\Omega^{-1+s},$$

and

$$\left|\left|\frac{\partial_\mu^2 \hat{D}}{[1-\hat{D}]^2}\right|\right|_1 + 2\left|\left|\frac{(\partial_\mu \hat{D})^2}{[1-\hat{D}]^3}\right|\right|_1 \leq K\Omega^{-1+s+2/d}$$

(the $2/d$ in the exponent can be omitted for the nearest-neighbo The above norms are all L^p norms on $[-\pi,\pi]^d$ with measure (The constant K depends on the dimension (but not on L) for t out model, and is a universal constant for the nearest-neighbour

In the following we will maintain the convention that K and on the dimension when a statement is applied to the spread-out r are universal constants when the same statement is applied to th neighbour model.

6.2.2 The convergence proof

To prove convergence of the lace expansion, we will use Lemma $n = 2$, $p_0 = \Omega^{-1}$, $p_1 = z_c$, $a = 2/3$,

$$f_1(p) = \frac{||H_p||_2^2}{3K\Omega^{-1+s}} \text{ and } f_2(p) = \frac{||\, x_\mu^2 G_p ||_\infty}{3K\Omega^{-1+s+2/d}},$$

with K the constant of Lemma 6.2.2. Here s is as in the sta Lemma 6.2.2, and the $2/d$ can be omitted from the exponent in tion of f_2 for the nearest-neighbour model.

The following three results confirm that the hypotheses of Le are satisfied, either for the nearest-neighbour model in sufficient mensions, or for the spread-out model in more than four dimen Ω sufficiently large. It will then follow from the lemma that || $||\, x_\mu^2 G_p||_\infty$ are both small (for large Ω) uniformly in $p \in [0,$ will give good bounds on the lace expansion, when combined orem 5.4.4.

For simplicity we deal explicitly only with the strictly self-avoi although the results of this section also hold for all finite memori ∞, subject to the replacement of z_c by the finite memory cri $z_c(0;\tau)$. In particular, the constants of Corollaries 6.2.6 and Theorem 6.2.9 are independent of τ. Finite memory is used onl the bound on $c_n(0,x)$ of Theorem 6.1.3.

Lemma 6.2.3 *The above functions f_1 and f_2 are continuous on the interval $[0, z_c)$.*

Proof. We begin with f_1. Since the subcritical two-point function decays exponentially by (1.3.14), $||H_p||_2^2$ is finite for $p < z_c$. This norm can be rewritten as a power series in p with positive coefficients, which therefore must have radius of convergence at least z_c. Hence it is continuous in $p \in [0, z_c)$.

For f_2, we fix $r \in [0, z_c)$. Arguing as in the derivation of (1.3.14), there is a constant M, depending on r but not on x, such that for any $p \in [0, r]$ and any x,

$$\frac{d}{dp} x_\mu^2 G_p(0, x) \leq M. \tag{6.2.10}$$

Hence for $p_1 < p_2 \leq r$ we have

$$\begin{aligned} 0 &\leq f_2(p_2) - f_2(p_1) \\ &\leq (3K)^{-1}\Omega^{1-s-2/d} \sup_x x_\mu^2 [G_{p_2}(0,x) - G_{p_1}(0,x)] \\ &\leq (3K)^{-1}\Omega^{1-s-2/d} M(p_2 - p_1). \end{aligned}$$

This implies continuity of f_2 for $p < r$, and hence for $p < z_c$ since r is arbitrary. □

Lemma 6.2.4 *For $p \in [0, \Omega^{-1}]$, $f_i(p) \leq 1/3$ for $i = 1, 2$.*

Proof. For $p \in [0, \Omega^{-1}]$, $G_p(0, x) \leq G_{1/\Omega}(0, x)$. Since in general the self-avoiding walk two-point function is bounded above by the ordinary random walk two-point function having the same activity, $G_p(0, x) \leq G_{1/\Omega}(0, x) \leq C(0, x)$. Now $H_p(0, x) = G_p(0, x) - \delta_{0,x}$, so $||H_p||_2^2 = ||G_p||_2^2 - 1 \leq ||C||_2^2 - 1$. Hence by the Parseval relation $||H_p||_2^2 \leq ||\hat{C}||_2^2 - 1$, and the desired bound on f_1 follows from (6.2.7). For f_2 we use the Fourier transform to write

$$\begin{aligned} x_\mu^2 G_p(0, x) &\leq x_\mu^2 C(0, x) = -\int_{[-\pi,\pi]^d} \partial_\mu^2 \hat{C}(k) e^{-ik\cdot x} \frac{d^d k}{(2\pi)^d} \\ &= -\int_{[-\pi,\pi]^d} \left[\frac{\partial_\mu^2 \hat{D}}{(1-\hat{D})^2} + \frac{2(\partial_\mu \hat{D})^2}{(1-\hat{D})^3} \right] e^{-ik\cdot x} \frac{d^d k}{(2\pi)^d}. \end{aligned} \tag{6.2.11}$$

The desired bound then follows from (6.2.8). □

This leaves the last and most substantial assumption of Lemma 6.2.1 to be shown. The following result confirms that the final hypothesis of

Lemma 6.2.1 is satisfied for f_1 and f_2 of (6.2.9), i.e. that for eac
$[\Omega^{-1}, z_c)$, if $f_i(p) \leq 1$ $(i = 1, 2)$ then in fact $f_i(p) \leq 2/3$ $(i = 1, 2)$.

Remark. The next theorem states that a pair of inequalities i
a stronger pair. In conjunction with Lemmas 6.2.1, 6.2.3 and 6.2.
means that in fact the stronger pair of inequalities holds. Hence the
inequalities also hold, and any consequences of the weaker inequaliti
in the course of the proof [such as the infrared bound (6.2.19)] wi
been shown to hold, once the theorem is proved.

Theorem 6.2.5 *There is an Ω_0 such that for $\Omega \geq \Omega_0$ (with $d > 4$*
spread-out model) the following implication holds. For any $p \in [\Omega^{-1}$

$$||H_p||_2^2 \leq 3K\Omega^{-1+s} \ \ and \ \ ||x_\mu^2 G_p||_\infty \leq 3K\Omega^{-1+s+2/d} \quad ($$

then in fact

$$||H_p||_2^2 \leq 2K\Omega^{-1+s} \ \ and \ \ ||x_\mu^2 G_p||_\infty \leq 2K\Omega^{-1+s+2/d}. \quad ($$

Here s is as in the statement of Lemma 6.2.2, and the $2/d$ in the ex
in the bound on $||x_\mu^2 G_p||_\infty$ can be omitted for the nearest-neighbour

Proof. We assume the weaker pair of bounds, and prove the stronge
For the proof we will work with Fourier transforms. As will be des
in more detail below [in the paragraph containing (6.2.27)], the as
bounds (6.2.12), together with Theorem 5.4.4, imply (absolute) conve
of the lace expansion. Hence by (5.2.18),

$$\hat{F}_p(k) \equiv \hat{G}_p(k)^{-1} = 1 - p\Omega\hat{D}(k) - \hat{\Pi}_p(k). \quad ($$

Since $\hat{F}_p(0) = \chi(p)^{-1} > 0$ for $p < z_c$, it follows by adding and subtr
$\hat{F}_p(0)$ to $\hat{F}_p(k)$ that for $p \geq \Omega^{-1}$

$$\begin{aligned} \hat{F}_p(k) &= \hat{F}_p(0) + p\Omega[1 - \hat{D}(k)] + \hat{\Pi}_p(0) - \hat{\Pi}_p(k) \\ &\geq [1 - \hat{D}(k)] + [\hat{\Pi}_p(0) - \hat{\Pi}_p(k)]. \quad (\end{aligned}$$

The basic idea of the proof is that the assumed bounds imply th
second term on the right side is a small perturbation of the first, w
turn implies that $G_p = 1/\hat{F}_p(k)$ is bounded above by a small pertur
of its ordinary random walk counterpart, and hence by Lemma 6.
improved bounds hold.

We now bound the difference $\hat{\Pi}_p(0) - \hat{\Pi}_p(k)$, using Theorem 5.4.4. It follows from (5.4.1), (5.2.16), symmetry, and (5.4.21) that

$$\begin{aligned} \hat{\Pi}_p(0) - \hat{\Pi}_p(k) &\geq -\sum_{j=1}^{\infty}[\hat{\Pi}_p^{(2j+1)}(0) - \hat{\Pi}_p^{(2j+1)}(k)] \\ &\geq -\sum_{\mu=1}^{d}(1-\cos k_\mu)\|x_1^2 H_p\|_\infty \\ &\quad \times \sum_{j=1}^{\infty}(j+1)^2\|H_p\|_2^{2j+1}\|G_p\|_2^{2j-1}. \end{aligned} \tag{6.2.16}$$

For the norm $\|G_p\|_2$, we note that by definition $H_p(0,x) = G_p(0,x) - \delta_{0,x}$, and hence using (6.2.12) we have

$$\|G_p\|_2^2 = \|H_p\|_2^2 + 1 \leq 2 \tag{6.2.17}$$

for sufficiently large Ω. The right side of (6.2.16) is dominated for large Ω by the $j = 1$ term, and hence by (6.2.5) and (6.2.12) we have

$$\hat{\Pi}_p(0) - \hat{\Pi}_p(k) \geq -K_1\Omega^{-u}[1 - \hat{D}(k)], \tag{6.2.18}$$

where $u = 3/2$ for the nearest-neighbour model and $u = 5/2 - 5s/2 - 2/d$ for the spread-out model, and K_1 is a constant which is independent of L for the spread-out model and independent of d for the nearest-neighbour model. We will use K_1 as a "variable constant" in what follows, to denote various constants which are independent of L or d as in (6.2.18) and whose precise values are irrelevant. Substituting (6.2.18) into (6.2.15) gives the infrared bound

$$\hat{F}_p(k) \geq [1 - K_1\Omega^{-u}][1 - \hat{D}(k)]. \tag{6.2.19}$$

We are now in a position to obtain the improved bound on $\|H_p\|_2^2$. By the Parseval relation and (6.2.17),

$$\|H_p\|_2^2 = \|\hat{G}_p\|_2^2 - 1,$$

where the norm on the right side denotes the L^2 norm on $[-\pi,\pi]^d$ with measure $(2\pi)^{-d}d^dk$. Hence by (6.2.19) we have

$$\begin{aligned} \|H_p\|_2^2 &= \left\|\frac{1}{\hat{F}_p}\right\|_2^2 - 1 \\ &\leq (1 + K_1\Omega^{-u})\left\|\frac{1}{1-\hat{D}}\right\|_2^2 - 1. \end{aligned} \tag{6.2.20}$$

Applying (6.2.7), this gives

$$\|H_p\|_2^2 \leq (1 + K_1\Omega^{-u+1-s})K\Omega^{-1+s}.$$

For the nearest-neighbour model $-u+1-s = -1/2$, while for the s[...] out model with $d > 4$, $-u+1-s = -3/2 + 3s/2 + 2/d < 0$. Thi[...] the desired result that for Ω sufficiently large

$$\|H_p\|_2^2 \leq 2K\Omega^{-1+s}.$$

We turn now to the bound on $\|x_\mu^2 G_p\|_\infty$. We give the proof w[...] $2/d$ present in the exponent, but for the nearest-neighbour model t[...] be omitted by following the same proof. [The significant difference b[...] the two models occurs in (6.2.31).]

In terms of the Fourier transform we can write

$$x_\mu^2 G_p(0,x) = -\int \partial_\mu^2 \hat{G}_p(k) e^{-ik\cdot x}\frac{d^dk}{(2\pi)^d}.$$

Explicit computation of the derivative on the right side gives the fo[...] expression, in which we have simplified the notation by dropping arg[...] and denoting partial differentiation with respect to k_μ by the subsc[...]

$$\hat{G}_{\mu,\mu} = p\Omega\frac{\hat{D}_{\mu,\mu}}{\hat{F}^2} + 2(p\Omega)^2\frac{\hat{D}_\mu^2}{\hat{F}^3} + \frac{\hat{\Pi}_{\mu,\mu}}{\hat{F}^2} + 4p\Omega\frac{\hat{D}_\mu\hat{\Pi}_\mu}{\hat{F}^3} + 2\frac{\hat{\Pi}_\mu^2}{\hat{F}^3}.$$

We insert (6.2.24) into (6.2.23), and take absolute values inside the i[...] and the sum of five terms. Applying (6.2.19) to bound $\hat{F}$ from belo[...] $\hat{F}^{-j} \leq (1+K_1\Omega^{-u})(1-\hat{D})^{-j}$ for $j \geq 1$. Applying (6.2.8) and using [...] then yields

$$\begin{aligned} x_\mu^2 G_p(0,x) &\leq (1+K_1\Omega^{-u})(p\Omega)^2 \\ &\quad \times \left[K\Omega^{-1+s+2/d} + \|\hat{\Pi}_{\mu,\mu}\|_\infty\left\|\frac{1}{1-\hat{D}}\right\|_2^2 \right. \\ &\quad \left. + 4\left\|\frac{\hat{D}_\mu\hat{\Pi}_\mu}{(1-\hat{D})^3}\right\|_1 + 2\left\|\frac{\hat{\Pi}_\mu^2}{(1-\hat{D})^3}\right\|_1\right]. \end{aligned}$$

The last three terms on the right side are error terms. Before bo[...] these, we first bound the factor $p\Omega$. By (6.2.12),

$$\|H_p\|_\infty \leq \|x_\mu^2 G_p\|_\infty \leq 3K\Omega^{-1+s+2/d};$$

this follows from the facts that $H_p(0,0) = 0$, and for $x \neq 0$, $H_p($[...] $G_p(0,x)$ and $1 \leq x_\mu^2$ for some μ. (This bound on $\|H_p\|_\infty$ is ineffici[...]

the spread-out model, for which the factor $\Omega^{2/d}$ on the right side should not be necessary, but it is adequate for our needs.) Applying (6.2.26) and (6.2.12) to (5.4.18) and (5.4.20), we see that for sufficiently large Ω the lace expansion converges and

$$|\hat{\Pi}_p(k)| \leq p\Omega K_1 \Omega^{-1+s+2/d}. \qquad (6.2.27)$$

Since $\chi(p)^{-1} = 1 - p\Omega - \hat{\Pi}_p(0) > 0$,

$$p\Omega \leq 1 - \hat{\Pi}_p(0) \leq 1 + p\Omega K_1 \Omega^{-1+s+2/d},$$

so that for Ω sufficiently large

$$p \leq \Omega^{-1}[1 + K_1\Omega^{-1+s+2/d}]. \qquad (6.2.28)$$

Since $-u \leq -1 + s + 2/d$, the factor $(1 + K_1\Omega^{-u})(p\Omega)^2$ in (6.2.25) can be replaced by $1 + K_1\Omega^{-1+s+2/d}$, for Ω large.

We next consider bounds on the derivatives of $\hat{\Pi}_p$ appearing in (6.2.25). It follows from (6.2.12) and (5.4.20) that

$$|\partial_\mu^2 \hat{\Pi}_p(k)| \leq K_1 \Omega^{-2+2s+2/d} \qquad (6.2.29)$$

(the $N = 2$ loop term dominates). We also will need a bound on $\partial_\mu \hat{\Pi}_p(k)$. Since by symmetry this derivative is zero whenever $k_\mu = 0$, it follows from Taylor's Theorem and the above bound on the second derivative that

$$|\partial_\mu \hat{\Pi}_p(k)| \leq K_1 \Omega^{-2+2s+2/d}|k_\mu|. \qquad (6.2.30)$$

Similarly,

$$|\partial_\mu \hat{D}(k)| \leq K_1 \Omega^{2/d}|k_\mu|. \qquad (6.2.31)$$

Turning now to the three error terms in (6.2.25), for the first we use (6.2.29) and (6.2.7) to bound it above by $K_1\Omega^{-2+2s+2/d}$. For the other two terms we first note that by symmetry, (6.2.5) and (6.2.7),

$$\left\|\frac{k_\mu^2}{(1-\hat{D})^3}\right\|_1 \leq K_1. \qquad (6.2.32)$$

Hence by (6.2.30) and (6.2.31) the second error term is bounded above by

$$K_1\Omega^{2/d}\Omega^{-2+2s+2/d}\left\|\frac{k_\mu^2}{(1-\hat{D})^3}\right\|_1 \leq K_1\Omega^{-2+2s+4/d}. \qquad (6.2.33)$$

Finally, the last error term can be bounded above by $K_1\Omega^{-4+4s+4}$ (6.2.30) and then (6.2.32). Taking Ω sufficiently large then gives th result

$$x_\mu^2 G_p(0,x) \leq 2K\Omega^{-1+s+2/d}.$$

The following results, which follow relatively easily from Theore will be fundamental in the rest of the chapter.

Corollary 6.2.6 *For $\Omega \geq \Omega_0$ (with $d > 4$ for the spread-out mod*

$$\begin{aligned} \|H_z\|_\infty &\leq 2K\Omega^{-1+s+2/d}, \\ \|x_\mu^2 G_z\|_\infty &\leq 2K\Omega^{-1+s+2/d}, \end{aligned}$$

and

$$\|H_z\|_2^2 \leq 2K\Omega^{-1+s}$$

for all complex z in the closed disk $|z| \leq z_c$. Here s is as in the stat Lemma 6.2.2, and for the nearest-neighbour model the $2/d$ can be from the exponent in the first two inequalities.

Proof. Since the left sides are largest at $z = z_c$, we can restrict a to this case. The left sides are monotone increasing in real po and satisfy the above bounds uniformly in $z < z_c$ by Theorem 6 (6.2.26) and the Remark preceding Theorem 6.2.5). Therefore t bounds hold at $z = z_c$ by the monotone convergence theorem.

Corollary 6.2.7 *For $\Omega \geq \Omega_0$ (with $d > 4$ for the spread-out mod is a constant K_1 such that the following bounds hold uniformly $[-\pi,\pi]^d$ and $|z| \leq z_c$:*

$$\begin{aligned} |\hat{\Pi}_z(k)| &\leq K_1\Omega^{-1+s+2/d} \\ |\partial_\mu \hat{\Pi}_z(k)| &\leq K_1\Omega^{-2+2s+2/d}|k_\mu| \\ |\partial_\mu^2 \hat{\Pi}_z(k)| &\leq K_1\Omega^{-2+2s+2/d}. \end{aligned}$$

In fact the series representations of these quantities are bounded a (absolute values inside sums over x, N) and uniformly by the rig The critical point obeys

$$\Omega^{-1} \leq z_c \leq \Omega^{-1}[1 + K_1\Omega^{-1+s+2/d}].$$

Also,

$$1 - z_c\Omega - \hat{\Pi}_{z_c}(0) = 0.$$

For any $p \in [0, z_c]$

$$\hat{\Pi}_p(0) - \hat{\Pi}_p(k) \geq -K_1\Omega^{-u}[1 - \hat{D}(k)]$$

and for any $p \in [\Omega^{-1}, z_c]$

$$\hat{F}_p(k) \geq [1 - K_1\Omega^{-u}][1 - \hat{D}(k)].$$

Here s is as in the statement of Lemma 6.2.2, and for the nearest-neighbour model the $2/d$ can be omitted from the exponents in the first four inequalities. The exponent u is equal to $3/2$ for the nearest-neighbour model, while for the spread-out model $u = 5/2 - 5s/2 - 2/d$.

Proof. Given Corollary 6.2.6, the first four inequalities follow exactly as in the proof of Theorem 6.2.5. It then follows from the dominated convergence theorem that for $u \in \{0, 1, 2\}$, $\partial_\mu^u \hat{\Pi}_z(k)$ is continuous on the closed disk $|z| \leq z_c$. Since $\chi(p) \to \infty$ as $p \nearrow z_c$ by (1.3.6), we have

$$\chi(p)^{-1} = \hat{F}_p(0) \to 1 - z_c\Omega - \hat{\Pi}_{z_c}(0) = 0.$$

The last two bounds of the corollary follow from (6.2.18) and (6.2.19) for $p < z_c$, and then follow at z_c by taking the limit. □

By Corollary 6.2.7 and the fact that $-\nabla_k^2 \hat{D}(0) \geq 1$, there is a constant C_4 such that for Ω sufficiently large and $p \in [\Omega^{-1}, z_c]$,

$$\begin{aligned} \nabla_k^2 \hat{F}_p(0) &= -p\Omega\nabla_k^2 \hat{D}(0) - \nabla_k^2 \hat{\Pi}_p(0) \\ &\geq C_4 > 0. \end{aligned} \tag{6.2.35}$$

The following lemma will allow for bounds on $\partial_z \hat{\Pi}_z(k)$ in the closed disk $|z| \leq z_c$.

Lemma 6.2.8 *For any $p \in (0, z_c]$ and $m = 1, 2, 3, \ldots$,*

$$\partial_p^m G_p(0, x) \leq m!\, p^{-m}\, H_p * \cdots * H_p * G_p(x), \tag{6.2.36}$$

where there are m factors of H_p in the convolution.

Proof. By definition,

$$\partial_p^m G_p(0, x) = m! p^{-m} \sum_{\substack{\omega : 0 \to x \\ |\omega| \geq m}} \binom{|\omega|}{m} p^{|\omega|},$$

where the sum is over all self-avoiding walks from 0 to x. The binomial coefficient on the right side counts the number of ways to choose $0 < i_1 <$

$i_2 < \cdots < i_m \leq |\omega|$, so it is also the number of ways to brea
$m + 1$ pieces such that the first m pieces each consist of at least
The upper bound then follows by neglecting the mutual avoidance
these pieces.

Theorem 6.2.9 *For Ω sufficiently large (with $d > 4$ for the s*
model),

$$|\partial_z \hat{\Pi}_z(k)| \leq 7K\Omega^{s+2/d}$$

uniformly in $k \in [-\pi, \pi]^d$ and $|z| \leq z_c$. In fact the series represen
the left side is bounded absolutely (absolute values inside sums o
N) and uniformly by the right side. Here K is the constant of Lem
s is as in the statement of Lemma 6.2.2, and the $2/d$ can be omi
the exponent for the nearest-neighbour model. Hence for Ω sufficie
there is a positive constant C_3 such that for any $p \in (0, z_c]$

$$-\partial_z \hat{F}_p(0) = \Omega + \partial_z \hat{\Pi}_p(0) \geq C_3 > 0$$

Proof. The bound (6.2.38) clearly follows from (6.2.37), so it s
obtain (6.2.37). But by (5.4.18), (5.4.19), Corollary 6.2.6 and t
bound on $z_c\Omega$ of Corollary 6.2.7, to prove (6.2.37) it suffices to sh

$$||\partial_z|_{z=|z|} H_z||_\infty \leq 4K\Omega^{s+2/d}.$$

Since $H_p(0, x)$ is a power series with nonnegative coefficients, it s
obtain (6.2.39) at $z = z_c$. By Lemma 6.2.8 and the fact that G
$H_z(0, x) + \delta_{0,x}$,

$$\begin{aligned} \partial_z H_{z_c}(0, x) = \partial_z G_{z_c}(0, x) &\leq z_c^{-1} H_{z_c} * H_{z_c}(x) + z_c^{-1} H_{z_c}(0 \\ &\leq z_c^{-1} ||H_{z_c}||_2^2 + z_c^{-1} H_{z_c}(0, x). \end{aligned}$$

The desired result now follows from Corollary 6.2.6 and the fact
bounded below by Ω^{-1}.

We conclude this section with an upper bound on the susce
which in particular implies that it is finite in the closed disk
everywhere except at the critical point itself.

Theorem 6.2.10 *For Ω sufficiently large (with $d > 4$ for the s*
model), the inverse susceptibility $\hat{F}_z(0) = 1 - z\Omega - \hat{\Pi}_z(0)$ satisfies

$$|\hat{F}_z(0)| \geq \frac{\Omega}{2}|z_c - z|$$

for all z with $|z| \leq z_c$.

Proof. Let $|z| \le z_c$. By Corollary 6.2.7 $\hat{F}_{z_c}(0) = 0$ and hence

$$\begin{aligned} |\hat{F}_z(0)| &= \left| \int_{z_c}^{z} \partial_z \hat{F}_z(0) dz \right| \\ &= |z_c - z| \left| \Omega + \int_0^1 \partial_z \hat{\Pi}_{(1-t)z_c + tz}(0) dt \right|. \end{aligned} \tag{6.2.41}$$

The lemma then follows, using Theorem 6.2.9. □

6.2.3 Proof of Theorem 6.1.2

The critical bubble diagram $\mathsf{B}(z_c) = ||G_{z_c}||_2^2 = 1 + ||H_{z_c}||_2^2$ is finite by Corollary 6.2.6. It follows from Theorem 1.5.3 that $\bar{\gamma} = 1$, in the sense that there are positive constants c_1 and c_2 such that for all $p < z_c$,

$$c_1(z_c - p)^{-1} \le \chi(p) \le c_2(z_c - p)^{-1}. \tag{6.2.42}$$

To obtain the stronger *asymptotic* behaviour stated in Theorem 6.1.2, we observe that since $\hat{F}_{z_c}(0) = 1 - z_c\Omega - \hat{\Pi}_{z_c}(0) = 0$ by Corollary 6.2.7,

$$\begin{aligned} \chi(z) &= \frac{1}{\hat{F}_z(0) - \hat{F}_{z_c}(0)} \\ &= \left(\frac{1}{z_c - z}\right)\left(\Omega + \frac{\hat{\Pi}_{z_c}(0) - \hat{\Pi}_z(0)}{z_c - z}\right)^{-1}. \end{aligned} \tag{6.2.43}$$

It then follows from Theorem 6.2.9 that as $z \nearrow z_c$

$$\chi(z) \sim [\Omega + \partial_z \hat{\Pi}_{z_c}(0)]^{-1}(z_c - z)^{-1}. \tag{6.2.44}$$

Defining

$$A = z_c^{-1}[\Omega + \partial_z \hat{\Pi}_{z_c}(0)]^{-1} \tag{6.2.45}$$

gives the statement of Theorem 6.1.2 for the susceptibility.

For the correlation length of order 2, we note that by symmetry and direct calculation,

$$\xi_2(z)^2 = \frac{-\nabla_k^2 \hat{G}_z(0)}{\hat{G}_z(0)} = [-z\Omega\nabla_k^2 \hat{D}(0) - \nabla_k^2 \hat{\Pi}_z(0)]\chi(z). \tag{6.2.46}$$

The desired asymptotic behaviour of $\xi_2(z)$ now follows from the asymptotic behaviour of $\chi(z)$ and (6.2.35), if we define

$$D = A[-z_c\Omega\nabla_k^2 \hat{D}(0) - \nabla_k^2 \hat{\Pi}_{z_c}(0)]. \tag{6.2.47}$$

[Continuity at z_c of $\nabla_k^2 \hat{\Pi}_z(0)$ is discussed in the proof of Corolla

We end this section with bounds on the constants A and D, plicity restricting the discussion to the nearest-neighbour mode dimensions.

Proposition 6.2.11 *For the nearest-neighbour model with d su large, there are positive universal constants c_1, c_2, c_3 such that*

$$1 \le A \le 1 + c_1 d^{-1} \quad and \quad 1 + c_2 d^{-1} \le D \le 1 + c_3 d^{-1}.$$

In particular D is strictly greater than 1.

Proof. For the first bound we conclude from Theorem 1.5.3 that $\mathsf{B}(z_c)$. But by Corollary 6.2.6, $\mathsf{B}(z_c) \le 1 + c_1 d^{-1}$ for some consta

For the bound on the diffusion constant D, we have from (6.2 (6.2.45) that

$$D = \frac{1 - (2dz_c)^{-1}\nabla_k^2 \hat{\Pi}_{z_c}(0)}{1 + (2d)^{-1}\partial_z \hat{\Pi}_{z_c}(0)}.$$

It suffices to show that there are positive constants a_i such that

$$-a_1 d^{-3/2} \le -(2dz_c)^{-1}\nabla_k^2 \hat{\Pi}_{z_c}(0) \le a_2 d^{-1}$$

and

$$-a_3 d^{-1} \le (2d)^{-1}\partial_z \hat{\Pi}_{z_c}(0) \le -a_4 d^{-1}.$$

Beginning with (6.2.49), it follows from Corollary 6.2.7 and the $2dz_c \ge 1$ that

$$|(2dz_c)^{-1}\nabla_k^2 \hat{\Pi}_{z_c}(0)| \le a_2 d^{-1}.$$

This gives the upper bound of (6.2.49). For the lower bound, by s it can be concluded that for fixed μ

$$-\nabla_k^2 \hat{\Pi}_{z_c}(0) \ge -d \sum_{j=1}^{\infty} \sum_x x_\mu^2 \Pi_{z_c}^{(2j+1)}(0,x).$$

By (5.4.20) and Corollary 6.2.6 the right side is bounded below by a of $-d^{-3/2}$.

Turning now to (6.2.50), the lower bound follows immedia (6.2.37). For the upper bound, we write

$$\partial_z \hat{\Pi}_{z_c}(0) = -\partial_z \hat{\Pi}_{z_c}^{(1)}(0) + \sum_{N=2}^{\infty} (-1)^N \partial_z \hat{\Pi}_{z_c}^{(N)}(0).$$

The first term on the right side (with its minus sign) is bounded above by the contribution due to the walk which steps to a neighbour of the origin and then back to the origin, which is $-\partial_z(2dz^2) = -4dz_c \leq -2$. Thus it suffices to show that the second term on the right side is bounded in absolute value by a multiple of d^{-1}. This follows from Corollary 6.2.6 and the bound $||\partial_z H_{z_c}||_\infty \leq K_1$ of (6.2.39), together with (5.4.19). □

6.3 Fractional derivatives

In this section we describe some elementary properties of what we term fractional derivatives. This terminology is somewhat inaccurate, but is useful in a suggestive sense in the analysis of the large-n asymptotics of power series coefficients. Given a power series $f(z) = \sum_{n=0}^{\infty} a_n z^n$ and $\epsilon \geq 0$, we define the fractional derivative

$$\delta_z^\epsilon f(z) = \sum_{n=0}^{\infty} n^\epsilon a_n z^n. \tag{6.3.1}$$

Note that for ϵ equal to a positive integer, δ_z^ϵ does not give the usual derivative. We will use (6.3.1) with $\epsilon \in (0,1)$. Allowing ϵ to take on arbitrary negative values defines a relative of the antiderivative, as follows. For $\alpha > 0$ we define

$$\delta_z^{-\alpha} f(z) = \sum_{n=1}^{\infty} n^{-\alpha} a_n z^n. \tag{6.3.2}$$

Both of the above quantities will be finite at least strictly within the circle of convergence of $f(z)$.

The following lemma provides formulas which are convenient for estimating fractional derivatives.

Lemma 6.3.1 *Let $f(z) = \sum_{n=0}^{\infty} a_n z^n$ have radius of convergence R. Then for any z with $|z| < R$, and for any $\alpha > 0$,*

$$\delta_z^{-\alpha} f(z) = C_\alpha \int_0^\infty [f(ze^{-\lambda^{1/\alpha}}) - f(0)] d\lambda, \tag{6.3.3}$$

where $C_\alpha = [\alpha\Gamma(\alpha)]^{-1}$. In addition, for any z with $|z| < R$ and for any $\epsilon \in (0,1)$,

$$\delta_z^\epsilon f(z) = C_{1-\epsilon} z \int_0^\infty f'(ze^{-\lambda^{1/(1-\epsilon)}}) e^{-\lambda^{1/(1-\epsilon)}} d\lambda. \tag{6.3.4}$$

The identities (6.3.3) and (6.3.4) also hold for $z = R$, if $a_n \geq 0$.

Proof. Let $|z| < R$. We first note that for any $\alpha > 0$,

$$n^{-\alpha} = \frac{1}{\alpha\Gamma(\alpha)} \int_0^\infty e^{-n\lambda^{1/\alpha}} d\lambda,$$

as can be seen by making the substitution $y = n\lambda^{1/\alpha}$ in the integ
right side. Therefore

$$\sum_{n=1}^{\infty} n^{-\alpha} a_n z^n = C_\alpha \sum_{n=1}^{\infty} a_n \int_0^\infty (ze^{-\lambda^{1/\alpha}})^n d\lambda.$$

Since the right side converges absolutely the order of integration
mation can be interchanged to yield (6.3.3).

For (6.3.4), we write $n^\epsilon = n^{-(1-\epsilon)} n$ and use (6.3.5) with $\alpha =$
obtain

$$\sum_{n=0}^{\infty} n^\epsilon a_n z^n = C_{1-\epsilon} z \sum_{n=1}^{\infty} n a_n \int_0^\infty (ze^{-\lambda^{1/(1-\epsilon)}})^{n-1} e^{-\lambda^{1/(1-\epsilon)}} d\lambda.$$

Since the right side converges absolutely we can interchange th
summation and integration to obtain

$$\sum_{n=0}^{\infty} n^\epsilon a_n z^n = C_{1-\epsilon} z \int_0^\infty f'(ze^{-\lambda^{1/(1-\epsilon)}}) e^{-\lambda^{1/(1-\epsilon)}} d\lambda.$$

Now suppose that $a_n \geq 0$ and take $z = R$. Then the above int
of sum and integral are justified by Fubini's Theorem.

The following lemma provides an error estimate analogous to
estimate in Taylor's theorem. In applications of the lemma, R w
radius of convergence of f.

Lemma 6.3.2 *Let $\epsilon \in (0,1)$ and let $f(z) = \sum_{n=0}^{\infty} a_n z^n$. Let R*
suppose that $A_\epsilon \equiv \sum_{n=0}^{\infty} n^\epsilon |a_n| R^{n-\epsilon} < \infty$, so in particular $f(z)$
for $|z| \leq R$. Then for any z with $|z| \leq R$,

$$|f(z) - f(R)| \leq 2^{1-\epsilon} A_\epsilon |R - z|^\epsilon.$$

Suppose that $B_\epsilon \equiv \sum_{n=1}^{\infty} n^{1+\epsilon} |a_n| R^{n-1-\epsilon} < \infty$, so in particula
$\sum_{n=1}^{\infty} n a_n z^{n-1}$ converges for $|z| \leq R$. Then for any z with $|z| \leq$

$$|f(z) - f(R) - f'(R)(z - R)| \leq \frac{2^{1-\epsilon}}{1+\epsilon} B_\epsilon |R - z|^{1+\epsilon}.$$