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Source: *SIAM Journal on Applied Mathematics*, Vol. 14, No. 6 (Nov., 1966), pp. 1266-1274
Published by: Society for Industrial and Applied Mathematics
Stable URL: http://www.jstor.org/stable/2946240
Accessed: 24-05-2018 14:19 UTC

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SIAM J. Appl. Math.
Vol. 14, No. 6, November, 1966
Printed in U.S.A.

AN OCCUPANCY DISCIPLINE AND APPLICATIONS*

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1. Introduction. Most systems of filing, cataloguing or storing units of information have the following structure: each record, book or information unit has a natural name or *record identification number* associated with it. The set of all possible names, which we denote by $\{a_1, a_2, \cdots, a_m\}$, is usually very large in comparison to the actual number, r say, of records $\{a_{i_1}, a_{i_2}, \cdots, a_{i_r}\}$ that are to be stored in any one problem. The storage procedure consists of assigning to each record a_{i_k} a unique *record location number* $A_{i_k} \in \{0, 1, \cdots, n-1\}$ where n is the size of the storage (memory). Of course $r \leqq n$. Typical values of m and n are 2^{36} and 2^{10} respectively. The problem is to devise a procedure for assigning the record location numbers so that the time needed to store and recover a record, knowing only its name, is minimized in some sense.

In certain circumstances the names $\{a_{i_1}, a_{i_2}, \cdots, a_{i_r}\}$ are structured in such a manner that a simple function $g: \{a_1, a_2, \cdots, a_m\} \to \{0, 1, 2, \cdots, n-1\}$ can be found with the property $g(a_{i_j}) = g(a_{i_k})$ if and only if $j = k$. When this happens the storage and recovery is quite trivial.

In most situations however $\{a_{i_1}, a_{i_2}, \cdots, a_{i_r}\}$ lacks a definite structure and m is much larger than n. Various schemes for storage have been considered. One has been described by Peterson in [1] and proceeds as follows: one begins by "randomly" selecting a function $g: \{a_1, a_2, \cdots, a_m\} \to \{0, 1, \cdots, n-1\}$. The record location numbers $\{A_{i_1}, A_{i_2}, \cdots, A_{i_r}\}$ of the records $a_{i_1}, a_{i_2}, \cdots, a_{i_r}$ are defined inductively as follows:

$$\text{(i)} \quad A_{i_1} = g(a_{i_1}),$$

$$\text{(ii)} \quad A_{i_k} = g(a_{i_k}) + s_k \ (\text{modulo } n),$$

where s_k is the smallest nonnegative integer such that $g(a_{i_k}) + s_k$ (modulo n) $\notin \{A_{i_1}, A_{i_2}, \cdots, A_{i_{k-1}}\}$. To recover the record a_{i_k} one computes in succession the record location numbers $g(a_{i_k})$, $g(a_{i_k}) + 1$ (modulo n), $\cdots$, comparing after each computation the name of the record stored in each of these locations with a_{i_k}. It is clear that the number of comparisons needed to recover the record a_{i_k} is just $s_k + 1$.

Peterson gives experimental data for the average value of $s_k + 1$. After some combinatorial preliminaries in §2 we introduce in §3 a new occupancy discipline and apply it in §4 to obtain the probability distribution of the

* Received by the editors October 2, 1964, and in revised form April 11, 1966.

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random variable s_k for a certain model of selecting the function g. This is used in §5 to calculate the limiting value of $E\{s_k\}$[1] as $n \to \infty$ with $k/n = \mu \in (0, 1)$. We conclude by applying §3 to a curious "parking" problem in §6.

2. Combinatorial preliminaries. Let n be a positive integer and

$$\pi = (\pi_1, \pi_2, \cdots, \pi_n) \in P_n$$

a permutation of the integers 1, 2, $\cdots$, n. Define $\tau_{j,n}$, τ_n and T_n by

$$\tau_{j,n}(\pi) = \max\{k: k \leqq j, \pi_j \geqq \pi_m \quad \text{for} \quad m = j, j-1, \cdots, j-k+1\}, \tag{2.1}$$

$$\tau_n(\pi) = \prod_{j=1}^{n} \tau_{j,n}(\pi), \tag{2.2}$$

and

$$T_n = \sum_{\pi \in P_n} \tau_n(\pi). \tag{2.3}$$

LEMMA 1. $T_n = (n+1)^{n-1}$, $n = 1, 2, \cdots$.
Proof.

$$\begin{aligned} T_n &= \sum_{j=0}^{n-1} \sum_{\pi \in P_n, \pi_{n-j}=n} \tau_n(\pi) \\ &= \sum_{j=0}^{n-1} \sum_{\pi \in P_n, \pi_{n-j}=n} (n-j) \prod_{k=1}^{n-j-1} \tau_{k,n}(\pi) \prod_{k=n-j+1}^{n} \tau_{k,n}(\pi) \\ &= \sum_{j=0}^{n-1} \binom{n-1}{j} (n-j) \sum_{\pi_1 \in P_{n-j-1}, \pi_2 \in P_j} \tau_{n-j-1}(\pi_1)\tau_j(\pi_2) \\ &= \sum_{j=0}^{n-1} \binom{n-1}{j} (n-j)\, T_{n-j-1} T_j, \end{aligned} \tag{2.4}$$

where we have set $T_0 = 1$. Proceeding formally we introduce the generating function

$$\mathcal{G}(z) = \sum_{n=0}^{\infty} \frac{T_n}{n!} z^n. \tag{2.5}$$

The recurrence relation of (2.4) implies that $\mathcal{G}$ satisfies the differential equation

$$z\mathcal{G}(z) \frac{d}{dz}(z\mathcal{G}(z)) = z \frac{d}{dz} \mathcal{G}(z) \tag{2.6}$$

and hence

[1] $E\{\cdot\}$ denotes the expectation of $\cdot$.

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$$\mathcal{g}(z) = \exp(z\mathcal{g}(z)). \tag{2.7}$$

On the other hand it is well known [2] that the only solution of (2.7) analytic in $|z| < 1/e$ is

$$\mathcal{g}(z) = \sum_{n=0}^{\infty} \frac{(n+1)^{n-1}}{n!} z^n, \qquad |z| < 1/e. \tag{2.8}$$

This analytic function satisfies (2.7) and this in turn implies that $T_n = (n+1)^{n-1}$.

3. An occupancy discipline. Consider r balls $B_1, B_2, \cdots, B_r$ which are to be placed into n cells $C_0, C_1, \cdots, C_{n-1}$. We assume that $r \leqq n$. The locations of the r balls are determined according to the following occupancy discipline: suppose r "fictitious" cell numbers $(j_1, j_2, \cdots, j_k, \cdots, j_r)$ have been selected $(0 \leqq j_k < n, 1 \leqq k \leqq r)$. The "actual" location of the kth ball B_k, say l_k, is defined inductively according to the rules

(i) $l_1 = j_1$,

(ii) for $k \geqq 2$, $l_k = j_k + s_k$ (modulo n), where s_k is the smallest nonnegative integer such that

$$l_k = j_k + s_k \text{ (modulo } n) \notin \{l_1, l_2, \cdots, l_{k-1}\}.$$

We let A denote the transformation

$$A: \mathbf{j} = (j_1, j_2, \cdots, j_r) \to A\mathbf{j} = \mathbf{l} = (l_1, \cdots, l_r),$$

and set

$$\mathcal{A}_\mathbf{l} = \{\mathbf{j}: A\mathbf{j} = \mathbf{l}\}.$$

Note that

$$\mu(\mathcal{A}_\mathbf{l}) = (\text{the number of elements in } \mathcal{A}_\mathbf{l}) = \prod_{k=1}^{r} \psi_k(\mathbf{l}),$$

where

$$\psi_k(\mathbf{l}) = \max\{\sigma: \{l_k, l_k - 1 \text{ (modulo } n), \cdots, l_k - \sigma + 1 \text{ (modulo } n)\} \subseteq \{l_1, \cdots, l_k\}\}.$$

Let $f(n, r)$, $1 \leqq r < n$, $n = 2, 3, \cdots$, be the total number of ways of placing the r balls into the n cells leaving the last cell (C_{n-1}) empty, i.e.,

$$f(n, r) = \sum_{\mathbf{l}, \max_{1 \leqq i \leqq r} l_i < n-1} \mu(\mathcal{A}_\mathbf{l}).$$

Comparing the construction of $f(n, r)$ to the definition of T_n in Lemma 2 we observe that $f(n, n-1) = T_{n-1} = n^{n-2}$.

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LEMMA 2. $f(n, r) = n^{r-1}(n - r)$.

Proof. We begin by establishing the recurrence formula

$$f(n, r) = \sum_{j=0}^{r} \binom{r}{j} (j + 1)^{j-1} f(n - 1 - j, r - j), \tag{3.1}$$

where we define

$$\begin{aligned} f(n, n) &= 0, && n > 0, \\ f(n, 0) &= 1, && n \geqq 0. \end{aligned} \tag{3.2}$$

By virtue of (3.2) the recurrence (3.1) is certainly true for $r = 0$ and $r = n - 1$. Henceforth we shall assume $0 < r < n - 1$ and $n > 2$. The set

$$\Gamma = \{1: \max_{1 \leqq i \leqq r} l_i < n - 1\}$$

can be decomposed into the $r + 1$ disjoint sets

$$\Gamma_0 = \{1: \max_{l \leqq i \leqq r} l_i < n-2\},$$

$$\Gamma_j = \{1:1 \in \Gamma, \{n - 2, n - 3, \cdots, n - j - 1\} \subseteq \{l_1, l_2, \cdots, l_r\}$$
$$\text{but} \quad n - j - 2 \notin \{l_1, l_2, \cdots, l_r\}\}, \qquad j = 1, 2, \cdots, r.$$

Note that Γ_j is precisely the set of occupancy numbers for which

(i) C_{n-1} is empty,
(ii) C_ν is occupied for $n - j - 1 \leqq \nu < n - 1$, and
(iii) C_{n-j-2} is empty.

For each of the $\binom{r}{j}$ choices of j balls $B_{i_1}, B_{i_2}, \cdots, B_{i_j}, 1 \leqq i_1 < i_2 < \cdots < i_j \leqq r$, there are $f(j + 1, j) = (j + 1)^{j-1}$ sets of "fictitious" cell numbers for $B_{i_1}, B_{i_2}, \cdots, B_{i_j}$ which will result in these balls being placed into the cells $C_{n-2}, C_{n-3}, \cdots, C_{n-j-1}$, in some order, leaving cell C_{n-1} empty. For each choice of these $(j + 1)^{j-1}$ "fictitious" cell numbers for $B_{i_1}, B_{i_2}, \cdots, B_{i_j}$ there remain $f(n - 1 - j, r - j)$ sets of $r - j$ "fictitious" cell numbers for the remaining $r - j$ balls which will result in their being placed into $r - j$ of the cells $C_0, C_1, \cdots, C_{n-j-2}$, leaving cell C_{n-j-2} empty. Thus

$$f(n, r) = \sum_{j=0}^{r} \binom{r}{j}(j + 1)^{j-1} f(n - 1 - j, r - j), \quad 0 \leqq r < n, n \geqq 1.$$

Starting from (3.1) we obtain the formula

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$$\frac{f(n,r)}{r!} = \sum_{j_p \geqq 0, 1 \leqq p \leqq n-r, j_1+j_2+\cdots+j_{n-r}=r} \prod_{p=1}^{n-r} \frac{(j_p+1)^{j_p-1}}{j_p!}, \tag{3.3}$$
$$0 \leqq r < n, n \geqq 1,$$

and this is equivalent to

$$f(n,r) = \frac{d^r}{dz^r}(\mathscr{g}(z))^{n-r}\big|_{z=0}, \tag{3.4}$$

where $\mathscr{g}$ is defined by (2.8). But according to (2.7),

$$\begin{aligned}(\mathscr{g}(z))^{n-r} = \exp\,(n-r)z\mathscr{g}(z) &= \sum_{j=0}^{\infty}(n-r)^j\frac{z^j}{j!}(\mathscr{g}(z))^j \\ &= \sum_{r=0}^{\infty} z^r \sum_{j=0}^{r} \frac{(n-r)^j}{j!}\frac{f(r, r-j)}{(r-j)!},\end{aligned} \tag{3.5}$$

or

$$f(n,r) = \sum_{j=0}^{r}\binom{r}{j}(n-r)^j f(r, r-j). \tag{3.6}$$

From (3.6) we easily deduce that $f(n,r) = n^{r-1}(n-r)$, thus completing the proof of Lemma 2.

4. Length of search in a certain associative memory. Let $X = \{a_1, a_2, \cdots, a_m\}$, $Y = \{0, 1, 2, \cdots, n-1\}$ and $\mathcal{G}(X, Y) = \{g: g: X \to Y\}$. The elements of X are record identification numbers and the elements of Y are record location numbers. Let $S = \{a_{i_1}, a_{i_2}, \cdots, a_{i_r}\}$ be a fixed (ordered) set of record identification numbers with $r \leqq n$. An element $g \in \mathcal{G}(X, Y)$ determines record location numbers for S according to the rules:

(i) the record location number for a_{i_1} is $A_{i_1} = g(a_{i_1})$,

(ii) for $k \geqq 2$, the record location number for a_{i_k} is $A_{i_k} = g(a_{i_k}) + s_k$ (modulo n), where s_k is the smallest nonnegative integer such that $g(a_{i_k}) + s_k$ (modulo n) $\notin \{A_{i_1}, A_{i_2}, \cdots, A_{i_{k-1}}\}$.

Using the terminology of §3 the set S labels the r balls, the numbers $(g(a_{i_1}), g(a_{i_2}), \cdots, g(a_{i_r}))$ are the "fictitious" cell numbers and the $(A_{i_1}, A_{i_2}, \cdots, A_{i_r})$ are the "actual" cell numbers. Within the context of the computer usage $s_k + 1$ can be interpreted as the number of steps required to search for and recover the kth record.

Let $(\Omega, \mathcal{E}, \Pr)$ be a probability space, i.e.,

(i) Ω is a set of points (the sample space),

(ii) $\mathcal{E}$ is a σ-field of subsets of Ω, and

(iii) Pr is a probability measure on Ω.

Let G be a $\mathcal{G}(X, Y)$-valued random variable on $(\Omega, \mathcal{E}, \Pr)$ with

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$$\Pr\{\omega: G(\omega) = g\} = n^{-m}, \qquad g \in \mathcal{G}(X, Y).$$

In designing a "random access associative memory" we perform a chance experiment and choose the mapping $G(\omega) = g \in \mathcal{G}(X, Y)$. In this section we shall calculate the quantities

$$\Pr\{\omega: s_k(\omega) = j\}, \qquad 0 \leqq j < k, \qquad k = 1, 2, \cdots.$$

First observe that

$$\Pr\{\omega: s_k(\omega) = j\} = \sum_{p=0}^{n-1} \Pr_{E_p}\{\omega: s_k(\omega) = j\} \Pr(E_p),$$

where

(i) $\Pr_{E_p}\{\cdot\}$ denotes the conditional probability of $\{\cdot\}$ given (conditioned by) the event E_p, and

(ii) $E_p = \{\omega: G(\omega)(a_{i_k}) = p\}$.

The event $\{\omega: s_k(\omega) = j, G(\omega)(a_{i_k}) = p\}$, $0 \leqq j \leqq k - 1$, can occur in any one of the following ways:

for $j \geqq 1$,

$$\text{(i)} \quad \{p - q \pmod n, p - q + 1 \pmod n, \cdots, p, p + 1 \pmod n, \cdots, p + j - 1 \pmod n\} \subseteq \{A_{i_1}(\omega), A_{i_2}(\omega), \cdots, A_{i_{k-1}}(\omega)\};$$

and

$$\text{(ii)} \quad \{p - q - 1 \pmod n, p + j \pmod n\} \notin \{A_{i_1}(\omega), A_{i_2}(\omega), \cdots, A_{i_{k-1}}(\omega)\}, \quad q = 0, 1, 2, \cdots, k - j - 1;$$

for $j = 0$,

$$p \notin \{A_{i_1}(\omega), A_{i_2}(\omega), \cdots, A_{i_{k-1}}(\omega)\}.$$

Thus

$$\text{(4.1)} \quad \Pr_{E_p}\{\omega: s_k(\omega) = j\} = \begin{cases} \dfrac{f(n, k-1)}{n^{k-1}} & \text{if } j = 0, \\ \displaystyle\sum_{q=0}^{k-j-1} \binom{k-1}{q+j} (j + q + 1)^{j+q+1} \cdot \frac{f(n - j - q - 1, k - j - q - 1)}{n^{k-1}} & \text{if } 1 \leqq j \leqq k - 1. \end{cases}$$

Thus we have Theorem 1.

THEOREM 1.

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$$\Pr\{\omega: s_k(\omega) = j\} = \frac{1}{n^{k-1}} \sum_{q=j}^{k-1} \binom{k-1}{q} (q+1)^{q-1} \cdot (n-k)(n-q-1)^{k-q-2}, \tag{4.2}$$

$$E\{s_k\} = \frac{n-k}{2n^{k-1}} \sum_{q=1}^{k-1} \binom{k-1}{q} (q+1)^q q (n-q-1)^{k-2-q}. \tag{4.3}$$

5. Limiting behavior of $E\{s_k\}$.

THEOREM 2. *Let $\mu \in (0, 1)$. Then*

$$\lim_{n\to\infty} E\{s_{\mu n}\} = \frac{1}{2}\mu \frac{(2-\mu)}{(1-\mu)^2}. \tag{5.1}$$

Proof. Let

$$\alpha_q = \frac{n-k}{2n^{k-1}} \binom{k-1}{q} (q+1)^q q (n-q-1)^{k-q-2}$$

and set

$$S_1 = \sum_{q=1}^{q_0(n)-1} \alpha_q, \qquad S_2 = \sum_{q=q_0(n)}^{k-1} \alpha_q, \tag{5.2}$$

where $q_0(n)$ will be specified later. We shall prove that for an appropriate choice of $q_0(n)$

$$S_1(n) \to \frac{1-\mu}{2} e^{-\mu} \sum_{q=1}^{\infty} \frac{(q+1)^q q}{q!} (\mu e^{-\mu})^q, \tag{5.3}$$

$$S_2(n) \to 0. \tag{5.4}$$

We start with (5.4); it suffices to prove

$$\text{(i)} \quad \alpha_{q_0(n)} \to 0 \qquad (n \to \infty), \quad \text{and} \tag{5.5}$$

$$\text{(ii)} \quad \alpha_q/\alpha_{q-1} \leqq 1 - \epsilon, \qquad q \geqq q_0(n), \qquad n \geqq N, \tag{5.6}$$

where $0 < \epsilon < 1$. Now

$$\begin{aligned} \alpha_q &= \frac{1}{2} \frac{n-k}{n} \frac{(k-1)\cdots(k-q)}{n^q} \frac{q}{q!} (q+1)^q \left(1 - \frac{q+1}{n}\right)^{k-q-2} \\ &\leqq \frac{1}{2}(1-\mu)\mu^q \frac{1}{\sqrt{2\pi}} \sqrt{q}\, e^q (1+q^{-1})^q \left(1 - \frac{q+1}{n}\right)^{k-q-2} \\ &\leqq C\sqrt{q} \exp q(1 - \mu + \log \mu), \end{aligned} \tag{5.7}$$

where C is a constant. We have used in deriving (5.7) the inequalities

$$q! > \sqrt{2\pi q}\, q^q e^{-q}$$

and

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$$(1 - x)^a \leqq e^{-ax}, \qquad 0 < x < 1, \quad a > 0.$$

We shall set $q = q_0(n) = n^{1/4}$ in (5.7) and note that

$$\alpha_{q_0(n)} \leqq Cn^{1/8} \exp(-A^2 n^{1/4}) \to 0,$$

since $1 - \mu + \log \mu < 0$ for $\mu \in (0, 1)$. This proves (5.5).

$$\begin{aligned} \alpha_q/\alpha_{q-1} &= \frac{k-q}{q} \frac{(q+1)^q}{q^{q-1}} \frac{q}{q-1} \frac{(n-q-1)^{k-q-2}}{(n-q)^{k-q-1}} \\ &\leqq \frac{q}{q-1} e \frac{k-q}{n-q} \exp\left\{-\frac{k-q}{n-q} + \frac{2}{n-q}\right\} \\ &\leqq \frac{q}{q-1} \exp\left(\frac{2}{n-q}\right) \exp\left\{1 + \log\frac{k-q}{n-q} - \frac{k-q}{n-q}\right\}. \end{aligned} \tag{5.8}$$

For $q \geqq q_0(n)$ and $n \geqq N$ it follows from (5.8) that

$$\alpha_q/\alpha_{q-1} \leqq 1 - \epsilon,$$

thus proving (5.6).

To prove (5.3) we note that for each fixed q,

$$\begin{aligned} \alpha_q &= \frac{1}{2} \frac{n-k}{n} \frac{(k-1)\cdots(k-q)}{n^q} \frac{q}{q!} (q+1)^q \left(1 - \frac{q+1}{n}\right)^{k-q-2} \\ &\to \frac{1}{2}(1-\mu)\mu^q (q+1)^q \frac{q}{q!} e^{-\mu} e^{-\mu q} \end{aligned} \tag{5.9}$$

as $n \to \infty$, $k = \mu n$. The convergence in (5.9) is such that

$$\alpha_q \leqq C'(q+1)^q \frac{q}{q!} (\mu e^{-\mu})^q, \qquad q \leqq n^{1/4},$$

and hence (5.3) is established.

If we set $\Psi(z) = z\vartheta(z)$, then by (2.8)

$$S_1(n) \to \frac{(1-\mu)e^{-\mu}}{2} \Psi''(\mu e^{-\mu}) \cdot (\mu e^{-\mu}). \tag{5.10}$$

To evaluate $\Psi''(\mu e^{-\mu})$ we employ the functional equation (2.7) obtaining after straightforward calculation

$$z\Psi''(z) = \frac{\Psi(z)}{1 - \Psi(z)} \frac{1}{z} \left\{\frac{1}{(1 - \Psi(z))^2} - 1\right\}. \tag{5.11}$$

Since $\Psi(\mu e^{-\mu}) = \mu$, (5.10)–(5.11) yield Theorem 2.

Finally we note that the quantity

$$M(r, n) = \frac{1}{r} \sum_{k=1}^{r} (1 + E\{s_k\})$$

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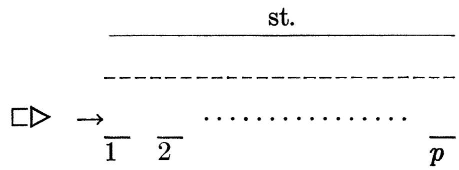
can be interpreted as the average number of steps required to recover from the memory a record of S, assuming that all records in S are requested with equal probability. A more detailed error analysis yields the following.

THEOREM 3. *If* $\mu \in (0, 1)$ *then*

$$\lim_{n\to\infty} M(\mu n, n) = 1 + \frac{1}{\mu}\int_0^{\mu} \frac{1}{2}\nu \frac{2-\nu}{(1-\nu)^2}\,d\nu = \frac{1}{2}\frac{2-\mu}{1-\mu}.$$

6. A parking problem—the case of the capricious wives. Let st. be a street with p parking places. A car



occupied by a man and his dozing wife enters st. at the left and moves towards the right. The wife awakens at a capricious moment and orders her husband to park immediately! He dutifully parks at his present location, if it is empty, and if not, continues to the right and parks at the next available space. If no space is available he leaves st.

Suppose st. to be initially empty and c cars arrive with independently capricious wives in each car. What is the probability that they all find parking places? If by "capricious" we mean that the probability of awakening in front of the ith parking place is $1/p$, $1 \leqq i \leqq p$, then the desired probability is just

$$P(c, p) = \frac{f(p+1, c)}{p^c} = \left(1 + \frac{1}{p}\right)^c \left(1 - \frac{c}{p+1}\right). \tag{6.1}$$

In particular

$$\lim_{p\to\infty} P(\mu, p, p) = (1-\mu)e^{\mu}, \qquad 0 < \mu \leqq 1.$$

The right-hand side of (6.1) is also the probability that c cars will succeed in parking in st. of length p (initially vacant) under the following more complicated parking discipline: when the ith car stops he parks if the space is free. If the space is occupied he performs a chance experiment; with probability q_i he moves backward and with probability $1 - q_i$ he moves forward, in both cases seeking the first free space. The proof is left to the reader.

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