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RECIPROCAL SUMS OVER PARTITIONS AND COMPOSITIONS*

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Abstract. The authors obtain precise asymptotic estimates for certain combinatorial sums over products of reciprocals of the summands in the partition or composition of a natural number n. These estimates are applied to determine the mean value of a certain arithmetical function over the polynomial ring $\mathbb{F}_q[X]$, where $\mathbb{F}_q$ is the finite field with q elements.

Key words. generating functions, infinite products, finite fields

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1. Introduction. In considering a problem concerning the factorization of polynomials over a large finite field, Greene and Knuth [3, p. 52] were led to consider the infinite product generating function

$$h(z) = \prod_{n=1}^{\infty} \left(1 + \frac{z^n}{n}\right). \tag{1.1}$$

If we regard this generating function purely from a combinatorial point of view, the coefficient of z^n in $h(z)$ corresponds to the sum

$$h(n) = \sum_{\substack{i_1+i_2+\cdots+i_k=n \\ 1 \leqq i_1 < i_2 < \cdots < i_k \leqq n \\ k \geqq 1}} \frac{1}{i_1 i_2 \cdots i_k}, \tag{1.2}$$

summed over all *distinct partitions* of the positive integer n. It seems natural to investigate the asymptotic behaviour of the analogous combinatorial sums

$$f(n) = \sum_{\substack{i_1+i_2+\cdots+i_k=n \\ 1 \leqq i_1 \leqq i_2 \leqq \cdots \leqq i_k \leqq n \\ k \geqq 1}} \frac{1}{i_1 i_2 \cdots i_k}, \tag{1.3}$$

summed over all *partitions* of n, and

$$g(n) = \sum_{\substack{i_1+i_2+\cdots+i_k=n \\ i_j \geqq 1, 1 \leqq j \leqq k, \\ k \geqq 1}} \frac{1}{i_1 i_2 \cdots i_k}, \tag{1.4}$$

summed over all *compositions* (ordered partitions) of n.

Corresponding to the asymptotic estimate

$$h(n) = e^{-\gamma} + \frac{e^{-\gamma}}{n} + O\left(\frac{\log n}{n^2}\right), \qquad n \to \infty,$$

obtained by Greene and Knuth [3], we show the following theorem by a related but somewhat different approach.

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THEOREM 1. *Let $f(n)$ be defined by* (1.3); *then, as $n \to \infty$,*

$$f(n) = e^{-\gamma}(n - \log n + 1 + \log 2 - \gamma) + O\left(\frac{\log n}{n}\right).$$

Here and henceforth, γ denotes Euler's constant. In addition, by applying techniques of complex integration we show the following theorem.

THEOREM 2. *Let $g(n)$ be defined by* (1.4); *then, as $n \to \infty$,*

$$g(n) = \frac{1}{e-1}\left(\frac{e}{e-1}\right)^n + O(1).$$

As a consequence of Theorem 2, we can deduce an asymptotic estimate for the following sum over the unsigned Stirling numbers of the first kind.

COROLLARY 3. *It holds that*

$$\sum_{k=1}^{n} k!s(n,k) = n!\left\{\frac{1}{(e-1)}\left(\frac{e}{e-1}\right)^n + O(1)\right\}, \qquad n \to \infty.$$

In § 3 we apply Theorem 2 to obtain an estimate for the mean value of a certain arithmetical function over $\mathbb{F}_q[X]$, where $\mathbb{F}_q[X]$ denotes the polynomial ring in one indeterminate X, over a finite field $\mathbb{F}_q$. Finally, in § 4 we interpret Theorems 1 and 2 in terms of cycles of permutations.

2. Analysis. Our approach to Theorems 1 and 2 is based on an analysis of the respective (ordinary) generating functions whose coefficients are $f(n)$ and $g(n)$. In the first case, it is easily verified that $f(n)$ has generating function

$$f(z) = \prod_{n=1}^{\infty}\left(1 - \frac{z^n}{n}\right)^{-1} \tag{2.1}$$

(compare (1.1)). In the second case, we use the result (see, e.g., Jordan [4, p. 146]) that

$$g_k(n) = \sum_{\substack{i_1+i_2+\cdots+i_k=n \\ i_j \geq 1, 1 \leq j \leq k}} \frac{1}{i_1 i_2 \cdots i_k}$$

has generating function $(-\log(1-z))^k$ for fixed k. Since $g(n) = \sum_{k=1}^{n} g_k(n)$, it follows that $g(n)$ has generating function

$$\sum_{k=1}^{\infty} (-\log(1-z))^k = \frac{-\log(1-z)}{1+\log(1-z)}. \tag{2.2}$$

To prove Theorem 1, we require a number of preliminary results, see below:

$$\begin{aligned} &\text{Define } r(z) = (1-z)^2 f(z), \qquad z \neq 1 \\ &\text{and let } r(z) = \sum_{n=0}^{\infty} r_n z^n,\ r_0 = 1. \end{aligned} \tag{2.3}$$

Also, let $c(z)$ denote the formal power series

$$c(z) = \sum_{n=1}^{\infty} c_n z^n = z\frac{r'(z)}{r(z)}. \tag{2.4}$$

Formal integration and exponentiation of (2.4) leads to

$$r(z) = \exp\left(\sum_{n=1}^{\infty} \frac{c_n}{n} z^n\right). \tag{2.5}$$

PROPOSITION 4. *For $n \geqq 1$,*

$$\text{(i)} \qquad n r_n = \sum_{k=1}^{n} c_k r_{n-k}, \tag{2.6}$$

$$\text{(ii)} \qquad c_1 = -1 \quad \text{and} \quad c_n = \sum_{\substack{d \mid n \\ 1<d<n}} \left(\frac{d}{n}\right)^{d-1} \text{ for } n \geqq 2. \tag{2.7}$$

Proof. (i) By (2.4) and (2.3), we have the identity

$$\left(\sum_{n=1}^{\infty} c_n z^n\right)\left(\sum_{n=0}^{\infty} r_n z^n\right) = \sum_{n=1}^{\infty} n r_n z^n.$$

By equating coefficients on each side, (2.6) follows.

(ii) We have

$$\sum_{n=1}^{\infty} (2 + c_n) z^n = \frac{2z}{1-z} + \sum_{n=1}^{\infty} c_n z^n = z \frac{f'(z)}{f(z)} \quad \text{by (2.3).}$$

By dividing by z and then integrating, we obtain

$$\sum_{n=1}^{\infty} \frac{(2+c_n) z^n}{n} = \log f(z) = -\sum_{n=1}^{\infty} \log\left(1 - \frac{z^n}{n}\right)$$
$$= \sum_{n=1}^{\infty} \sum_{m=1}^{\infty} \frac{z^{nm}}{mn^m} = \sum_{n=1}^{\infty} \left(\sum_{d \mid n} d^{-1}\left(\frac{d}{n}\right)^d\right) z^n.$$

It follows that

$$2 + c_n = \sum_{d \mid n} \left(\frac{d}{n}\right)^{d-1} = 2 + \sum_{\substack{d \mid n \\ 2 \leqq d < n}} \left(\frac{d}{n}\right)^{d-1},$$

as required.

PROPOSITION 5. *We have that*

$$\text{(i)} \qquad c_n = \frac{1 + (-1)^n}{n} + O\left(\frac{1}{n^2}\right); \tag{2.8}$$

$$\text{(ii)} \qquad r_n = O\left(\frac{1}{n^2}\right); \tag{2.9}$$

$$\text{(iii)} \qquad r(-1) = 4 \quad \text{and} \quad r(1) = e^{-\gamma}. \tag{2.10}$$

Proof. (i) If n is even, then, by separating out the term for $d = 2$ from (2.7), we obtain

$$c_n = \frac{2}{n} + \sum_{\substack{d \mid n \\ 3 \leqq d < n}} \left(\frac{d}{n}\right)^{d-1}$$
$$= \frac{2}{n} + \frac{1}{n^2} \sum_{\substack{d \mid n \\ 3 \leqq d < n}} d^2 \left(\frac{d}{n}\right)^{d-3}$$

$$= \frac{2}{n} + \frac{1}{n^2} O\left(\sum_{d=3}^{\infty} d^2 \left(\frac{1}{2}\right)^{d-3} \right)$$

$$= \frac{2}{n} + O\left(\frac{1}{n^2}\right).$$

If n is odd, then there is no term for $d = 2$, so $c_n = O(1/n^2)$, and (2.8) follows.

(ii) We use induction on n. By (2.6),

$$n r_n = \sum_{k=1}^{n-1} c_k r_{n-k} + c_n$$

$$= O\left(\sum_{k=1}^{n-1} \frac{1}{k} \frac{1}{(n-k)^2} \right) + O\left(\frac{1}{n}\right),$$

using (2.8) and the inductive hypothesis. This gives

$$n r_n = O\left(\frac{1}{n^2} \sum_{k=1}^{n-1} \left(\frac{1}{k} + \frac{1}{n-k} \right) \right) + O\left(\frac{1}{n} \sum_{k=1}^{n-1} \frac{1}{(n-k)^2} \right) + O\left(\frac{1}{n}\right)$$

$$= O\left(\frac{1}{n^2} \log n\right) + O\left(\frac{1}{n}\right) = O\left(\frac{1}{n}\right),$$

which gives (2.9).

(iii) It follows from (2.9) that the power series (2.3) for $r(z)$ converges absolutely and uniformly on and inside the unit circle, so $r(z)$ is continuous for $|z| \leqq 1$. At $z = -1$, we may use the product expression (2.1) to give

$$r(-1) = (1 - (-1))^2 \prod_{n=1}^{\infty} \left(1 - \frac{(-1)^n}{n} \right)^{-1}$$

$$= 4 \cdot \left(\frac{2}{1} \cdot \frac{1}{2} \cdot \frac{4}{3} \cdot \frac{3}{4} \cdot \frac{6}{5} \cdot \frac{5}{6} \cdots \right)^{-1} = 4.$$

From (2.3) and (2.1), we have for $|z| < 1$ that

$$\log r(z) = 2 \log (1 - z) - \sum_{n=1}^{\infty} \log \left(1 - \frac{z^n}{n} \right)$$

$$= \log (1 - z) - \sum_{n=2}^{\infty} \log \left(1 - \frac{z^n}{n} \right)$$

$$= - \sum_{n=1}^{\infty} \frac{z^n}{n} + \sum_{n=2}^{\infty} \left\{ \frac{z^n}{n} + \sum_{m=2}^{\infty} \frac{z^{mn}}{n^m m} \right\}$$

$$= -1 + \sum_{n=2}^{\infty} \sum_{m=2}^{\infty} \frac{z^{mn}}{n^m m}$$

$$= -1 + \lim_{M \to \infty} \sum_{m=2}^{M} \sum_{n=2}^{\infty} \frac{z^{mn}}{n^m m}.$$

Thus, since $r(z)$ is continuous for $|z| \leqq 1$,

$$\log r(1) = -1 + \lim_{M \to \infty} \sum_{m=2}^{M} \left\{ -\frac{1}{m} - \log \left(1 - \frac{1}{m} \right) \right\}$$

$$= \lim_{M \to \infty} \left\{ - \sum_{m=1}^{M} \frac{1}{m} + \log \left(\frac{2}{1} \cdot \frac{3}{2} \cdot \frac{4}{3} \cdots \frac{M}{M-1} \right) \right\}$$

$$= -\gamma.$$

Remark. If $\zeta(s) = \sum_{n=1}^{\infty}(1/n^s)$ denotes the Riemann zeta function, we can deduce from the proof of Proposition 5 the identity

$$\sum_{s=2}^{\infty} \frac{\zeta(s)-1}{s} = 1-\gamma.$$

By contrast, it is easily seen that $\sum_{s=2}^{\infty}(\zeta(s)-1) = 1$. These elementary identities do not appear in standard textbooks on the Riemann zeta function such as [10].

PROPOSITION 6. *As* $n \to \infty$,

$$r_n = \frac{(e^{-\gamma}+4(-1)^n)}{n^2} + O\left(\frac{\log n}{n^3}\right). \tag{2.11}$$

Proof. From (2.6), (2.8), and (2.9),

$$\begin{aligned}
nr_n &= \sum_{k=1}^{n} c_k r_{n-k} = \sum_{m=0}^{n-1} c_{n-m} r_m \\
&= \sum_{m=0}^{n-1} \frac{(1+(-1)^{n-m})}{n-m} r_m + \left\{O\left(\frac{1}{n^2}\right) + O\left(\sum_{m=1}^{n-1} \frac{1}{m^2}\frac{1}{(n-m)^2}\right)\right\} \\
&= \left\{\frac{1}{n}\sum_{m=0}^{n-1}(1+(-1)^{n-m})r_m + O\left(\sum_{m=1}^{n-1}\left(\frac{1}{n-m}-\frac{1}{n}\right)\frac{1}{m^2}\right)\right\} + O\left(\frac{1}{n^2}\right) \\
&= \left\{\frac{1}{n}(r(1)+(-1)^n r(-1)) + O\left(\frac{1}{n^2}\right)\right\} + O\left(\frac{1}{n^2}\sum_{m=1}^{n-1}\left(\frac{1}{n-m}+\frac{1}{m}\right)\right) + O\left(\frac{1}{n^2}\right) \\
&= \frac{1}{n}(e^{-\gamma}+4(-1)^n) + O\left(\frac{\log n}{n^2}\right)
\end{aligned}$$

by (2.10).

Proof of Theorem 1. By equating coefficients in the defining identity $f(z) = (1-z)^{-2}r(z)$ and using (2.10), we obtain

$$\begin{aligned}
f(n) &= \sum_{k=0}^{n}(n+1-k)r_k = (n+1)\sum_{k=0}^{n} r_k - \sum_{k=0}^{n} kr_k \\
&= (n+1)e^{-\gamma} - (n+1)\sum_{k=n+1}^{\infty} r_k - \sum_{k=0}^{n} kr_k.
\end{aligned} \tag{2.12}$$

First, by (2.11),

$$\begin{aligned}
\sum_{k=n+1}^{\infty} r_k &= e^{-\gamma}\sum_{k=n+1}^{\infty}\frac{1}{k^2} + 4\sum_{k=n+1}^{\infty}\frac{(-1)^k}{k^2} + O\left(\sum_{k=n+1}^{\infty}\frac{\log k}{k^3}\right) \\
&= \frac{e^{-\gamma}}{n} + O\left(\frac{1}{n^2}\right) + O\left(\frac{\log n}{n^2}\right).
\end{aligned} \tag{2.13}$$

Hence

$$(n+1)\sum_{k=n+1}^{\infty} r_k = e^{-\gamma} + O\left(\frac{\log n}{n}\right). \tag{2.14}$$

Also by (2.11),

$$\sum_{k=1}^{n} kr_k = e^{-\gamma} \sum_{k=1}^{n} \frac{1}{k} + 4 \sum_{k=1}^{n} \frac{(-1)^k}{k} + \sum_{k=1}^{n} O\left(\frac{\log k}{k^2}\right)$$
$$= e^{-\gamma} \log n + O(1),$$

from which we can immediately deduce the estimate

$$f(n) = e^{-\gamma}(n - \log n) + O(1).$$

The sharper estimate of Theorem 1 requires a more detailed analysis, which is set forth in Lemma 7, below.

LEMMA 7. *As* $n \to \infty$,

$$\sum_{k=1}^{n} kr_k = e^{-\gamma}\{\log n + \gamma - 1 - \log 2\} + O\left(\frac{\log n}{n}\right). \tag{2.15}$$

Proof. By (2.6),

$$\sum_{k=1}^{n} kr_k = \sum_{k=1}^{n} \sum_{j=1}^{k} c_j r_{k-j} = \sum_{j=1}^{n} c_j \sum_{k=j}^{n} r_{k-j}$$
$$= \sum_{j=1}^{n} c_j \sum_{k=0}^{n-j} r_k = \sum_{j=1}^{n} c_j \left\{ \sum_{k=0}^{\infty} r_k - \sum_{k=n-j+1}^{\infty} r_k \right\} \tag{2.16}$$
$$= e^{-\gamma} \sum_{j=1}^{n} c_j - \sum_{j=1}^{n} c_j \sum_{k=n-j+1}^{\infty} r_k.$$

Now, by (2.7),

$$\sum_{j=1}^{n} c_j = -1 + \sum_{j=4}^{n} \sum_{\substack{d|j \\ 2 \le d \le j/2}} \left(\frac{d}{j}\right)^{d-1}$$
$$= -1 + \sum_{\substack{j=4 \\ 2|j}}^{n} \frac{2}{j} + \sum_{j=6}^{n} \sum_{\substack{d|j \\ 3 \le d \le j/2}} \left(\frac{d}{j}\right)^{d-1}. \tag{2.17}$$

Now

$$\sum_{\substack{j=4 \\ 2|j}}^{n} \frac{2}{j} = \sum_{k=2}^{[n/2]} \frac{1}{k} = \log\left(\frac{n}{2}\right) + \gamma - 1 + O\left(\frac{1}{n}\right). \tag{2.18}$$

Next, if $j = qd$, then

$$\sum_{j=6}^{n} \sum_{\substack{d|j \\ 3 \le d \le j/2}} \left(\frac{d}{j}\right)^{d-1} = \sum_{q=2}^{[n/3]} \sum_{d=3}^{[n/q]} \left(\frac{1}{q}\right)^{d-1}$$
$$= \sum_{q=2}^{[n/3]} \frac{1}{q^2} \frac{1 - (q^{-1})^{[n/q]-2}}{1 - q^{-1}}$$
$$= \sum_{q=2}^{[n/3]} \frac{1}{q(q-1)} - \sum_{q=2}^{[n/3]} \frac{q^{-[n/q]}}{1 - q^{-1}} \tag{2.19}$$
$$= 1 + O\left(\frac{1}{n}\right) + O\left(\sum_{q=2}^{[n/3]} q^{-[n/q]}\right).$$

To bound this sum, we use

$$\sum_{q=2}^{[n/3]} q^{-[n/q]} = O\left\{\left(\left[\frac{n}{3}\right]-1\right)\max_{2\le q\le [n/3]} q^{-[n/q]}\right\}$$
$$= O\left\{\left(\left[\frac{n}{3}\right]-1\right)\left[\frac{n}{3}\right]^{-3}\right\} = O\left(\frac{1}{n^2}\right),$$

since the function x^{nx} decreases for $0 < x < e^{-1}$. Substituting this into (2.19), and then substituting (2.19) and (2.18) into (2.17), yields

$$\sum_{j=1}^{n} c_j = \log n - \log 2 + \gamma - 1 + O\left(\frac{1}{n}\right). \tag{2.20}$$

Next, consider the second sum in (2.16). Since the inner sum is $O(1/(n-j+1))$, by (2.13) and $c_j = O(1/j)$ by (2.8),

$$\begin{aligned}\sum_{j=1}^{n} c_j \sum_{k=n-j+1}^{\infty} r_k &= O\left(\sum_{j=1}^{n}\frac{1}{j}\frac{1}{n-j+1}\right)\\ &= O\left(\frac{\log n}{n}\right).\end{aligned} \tag{2.21}$$

We substitute (2.21) and (2.20) into (2.16) to obtain (2.15), which completes the proof of Lemma 7.

Theorem 1 now follows from (2.12), (2.14), and (2.15).

Remarks. (i) Equation (2.1) is a special case of a general product expansion $\prod_{n=1}^{\infty}(1-a_n z^n)^{-1}$, which can be developed for an arbitrary power series $A(z) = 1 + A_1 z + A_2 z^2 + \cdots$. These and other more general product representations for power series are investigated in [6]. In addition, the problem of finding the region of analyticity of such product expansions corresponding to a given function $A(z)$ is considered in [5].

(ii) We can further improve the error estimate in Theorem 1 from $O(\log n/n)$ to $e^{-\gamma}\log n/n + O(1/n)$, but we omit the details. This improved estimate, which we denote by $\hat{f}(n)$, can be used to obtain accurate numerical estimates for the coefficients $f(n)$, even for fairly small n, as Table 1 shows.

(iii) Since the generating function (2.1) has the unit circle as a natural boundary, we cannot apply standard asymptotic techniques such as the Darboux method (see, e.g., Bender [1]) or the transfer methods for singularity analysis of generating functions, developed recently by Flajolet and Odlyzko [2]. These methods are, however, applicable in the case of generating function (2.2) but lead to weaker error estimates than that of Theorem 2.

Proof of Theorem 2. By substituting $w = 1 - z$ into (2.2), we obtain

$$\sum_{n=1}^{\infty} g(n)(1-w)^n = \frac{-\log w}{1+\log w}.$$

We wish to expand this function as a power series about $w = 1$. It has a branch point at $w = 0$ because of $\log w$ and simple pole at $w = e^{-1}$ with residue e^{-1}. Therefore $-\log w/(1+\log w) - e^{-1}/(w-e^{-1})$ is analytic for $|w-1| < 1$. In addition, this function is bounded on the circle $|w-1| = 1$ because

$$\frac{-\log w}{1+\log w} = \frac{-1}{(\log w)^{-1}+1} \to -1 \quad \text{as } w \to 0.$$

TABLE 1

Estimates of $f(n)$ with errors of order $1/n$.

n	$f(n)$	$\hat{f}(n)$	$n(f(n) - \hat{f}(n))$
100	54.21582	54.21274	0.3077
200	109.95989	109.95853	0.2723
300	165.87349	165.87263	0.2581
400	221.85541	221.85479	0.2501
500	277.87451	277.87402	0.2449
600	333.91701	333.91661	0.2412
700	389.97562	389.97528	0.2384
800	446.04599	446.04569	0.2363
900	502.12532	502.12506	0.2345
1000	558.21172	558.21149	0.2331
1100	614.30383	614.30362	0.2319
1200	670.40065	670.40046	0.2308
1300	726.50142	726.50124	0.2299
1400	782.60556	782.60539	0.2292
1500	838.71259	838.71244	0.2285
1600	894.82214	894.82200	0.2278
1700	950.93391	950.93378	0.2273
1800	1007.04764	1007.04752	0.2268
1900	1063.16312	1063.16300	0.2263
2000	1119.28017	1119.28005	0.2259

Suppose therefore that

$$G(w) = \frac{-\log w}{1 + \log w} - \frac{e^{-1}}{w - e^{-1}} = \sum_{n=0}^{\infty} a_n (w-1)^n. \tag{2.22}$$

By Taylor's theorem,

$$a_n = \frac{1}{2\pi i} \oint_C \frac{G(w)}{(w-1)^{n+1}}\, dw,$$

where, by the above, C can be taken to be the circle $|w - 1| = 1$. Therefore

$$|a_n| \leqq \max \{ |G(w)| : |w - 1| = 1 \}, \tag{2.23}$$

since C has circumference 2π. Now, for w on C,

$$\begin{aligned} |G(w)| &\leqq \left| \frac{\log w}{1 + \log w} \right| + \left| \frac{1}{ew - 1} \right| \\ &\leqq \frac{1}{\min\limits_{|w-1|=1} \{1 + (\log w)^{-1}\}} + 1 = O(1). \end{aligned} \tag{2.24}$$

For $|1 - w| < 1 - e^{-1}$, however,

$$\begin{aligned} \sum_{n=1}^{\infty} g(n)(1-w)^n &= \sum_{n=0}^{\infty} a_n (w-1)^n + \frac{e^{-1}}{(w-1) - (e^{-1} - 1)} \\ &= \sum_{n=0}^{\infty} a_n (w-1)^n + \frac{e^{-1}}{1 - e^{-1}} \sum_{n=0}^{\infty} \left(\frac{1-w}{1-e^{-1}} \right)^n. \end{aligned}$$

It follows that

$$g(n) = \frac{e^{-1}}{(1 - e^{-1})^{n+1}} + (-1)^n a_n$$
$$= \frac{1}{(e-1)}\left(\frac{e}{e-1}\right)^n + O(1),$$

using (2.24).

Remarks. (i) Using a computer, we obtain the numerical estimate

$$\min_{|w-1|=1} \left(1 + \frac{1}{\log w}\right) = 0.72159\cdots.$$

From this, we deduce that

$$\left|g(n) - \frac{1}{(e-1)}\left(\frac{e}{e-1}\right)^n\right| < \frac{1}{0.7216} + 1 = 2.3858\cdots.$$

(ii) The generating function $g(z)$ can be used to derive the following recurrence relation for $g(n)$, which is useful for computational purposes. Since

$$g(z)(1 + \log(1-z)) = -\log(1-z),$$

we can equate coefficients on each side of this identity to obtain

$$g(n) = \sum_{k=0}^{n-1} \frac{g(k)}{n-k}, \qquad n \geqq 1, \tag{2.25}$$

where we define $g(0) = 1$.

Proof of Corollary 3. By Jordan [4, p. 146], the unsigned Stirling number of the first kind $s(n, k)$ is given by the formula

$$s(n,k) = \frac{n!}{k!} \sum_{i_1 + i_2 + \cdots + i_k = n} \frac{1}{i_1 i_2 \cdots i_k}, \tag{2.26}$$

summed over all compositions of n of length k. It follows that $(1/n!) \sum k! s(n, k) = g(n)$, from which we now deduce the asymptotic estimate using Theorem 2.

3. Ordered irreducible factorizations of polynomials. As is the case with Greene and Knuth's estimates for $h(n)$, our asymptotic estimates for $g(n)$ can be applied to investigate a problem involving polynomials over a large finite field.

Let $\mathbb{F}_q[X]$ denote a polynomial ring in one indeterminate X over a finite field $\mathbb{F}_q$ with exactly q elements. Let P_n denote the set of all monic polynomials of degree n in $\mathbb{F}_q[X]$, of cardinality q^n. Let $\pi(n) = \pi_q(n)$ denote the number of monic irreducible polynomials in P_n. It is well known (see [8, p. 93]) that

$$\pi(n) = \frac{q^n}{n}(1 + O(q^{-n/2})). \tag{3.1}$$

For $a(X) \in P_n$, we use $\Omega(a)$ and $\omega(a)$ to denote the arithmetical functions that count the total number (respectively, the distinct number) of irreducible factors of $a(X)$.

These functions are investigated in [7] for general additive arithmetical semigroups that satisfy a certain axiom. For convenience, we confine our discussion below to the case of $\mathbb{F}_q[X]$. We introduce a new arithmetical function as follows: For any $a(X) \in P_n$, define $b(a)$ to be the number of *ordered factorizations* of $a(X)$ into irreducibles. The function b is prime independent, since, if $p(X)$ is any irreducible polynomial, then $b(p^m(X)) = 1$ for all m, which is independent of the choice of $p(X)$. However, the function b is not multiplicative, since $b(p_1p_2) = 2 \neq b(p_1)b(p_2) = 1$ for irreducible polynomials p_1, p_2. In general, if $a(X) = p_1^{\alpha_1}(X)p_2^{\alpha_2}(X)\cdots p_k^{\alpha_k}(X)$, where $k = \omega(a)$, $\alpha_1 + \alpha_2 + \cdots + \alpha_k = \Omega(a)$, we have $b(a) = \Omega(a)!/\alpha_1!\alpha_2!\cdots\alpha_k!$. We deduce that

$$\omega(a)! \leqq b(a) \leqq \Omega(a)! \tag{3.2}$$

with $b(a) = \omega(a)!$ in the case that $a(X)$ is square-free.

An arithmetical function $F : \mathbb{F}_q[X] \to \mathbb{R}$ is said to be of *normal order* $G(n)$ if, for almost all polynomials $a(X)$ in P_n (i.e., for all except $o(q^n)$ polynomials in P_n),

$$F(a) = G(n)(1 + o(1)) \quad \text{as } n \to \infty.$$

We now show that $\log b(a)$ has normal order $\log n(\log\log n - 1)$.

THEOREM 8. *For almost all $a(X) \in P_n$ and any $\delta > 0$,*

$$\log b(a) = \log n(\log\log n - 1)(1 + O(\log n)^{-1/2+\delta}) \quad \textit{as } n \to \infty.$$

Proof. By Theorem 9.7 of Knopfmacher [7, p. 90] applied in the case of $\mathbb{F}_q[X]$, we have, for almost all $a \in P_n$ and for any $\delta > 0$, that

$$\omega(a) = \log n(1 + O((\log n)^{-1/2+\delta})), \qquad \Omega(a) = \log n(1 + O((\log n)^{-1/2+\delta})).$$

Thus, for almost every $a \in P_n$,

$$\log \omega(a) = \log\log n + \log(1 + O(\log n)^{-1/2+\delta}) = \log\log n + O(\log n)^{-1/2+\delta}.$$

Now, by Stirling's formula,

$$\log n! = n(\log n - 1)\left(1 + O\left(\frac{1}{n}\right)\right).$$

Hence, for almost every $a \in P_n$,

$$\begin{aligned}\log \omega(a)! &= \log n(1 + O(\log n)^{-1/2+\delta})(\log\log n - 1)\\ &\quad\times\left(1 + O\left(\frac{(\log n)^{-1/2+\delta}}{\log\log n}\right)\right)\left(1 + O\left(\frac{1}{\log n}\right)\right)\\ &= \log n(\log\log n - 1)(1 + O(\log n)^{-1/2+\delta}).\end{aligned}$$

Since exactly the same estimate holds for $\log \Omega(a)!$, we can deduce the result from (3.2).

In addition to the normal value, it is useful to investigate the *mean value* of $b(a)$, given by $(1/q^n)\sum_{a\in P_n} b(a)$. By letting $q \to \infty$, we can apply Theorem 2 to estimate the asymptotic mean value of $b(a)$ as $n \to \infty$.

THEOREM 9. *As $q \to \infty$, the asymptotic mean value of $b(a)$ tends to*

$$\frac{1}{q^n}\sum_{a\in P_n} b(a) = \frac{1}{e-1}\left(\frac{e}{e-1}\right)^n + O(1), \quad n \to \infty.$$

Proof. By letting $q \to \infty$ in (3.1), we may take the proportion of irreducible polynomials in P_n, namely $\pi(n)/q^n$, to be $1/n$. Then

$$\begin{aligned}\frac{1}{q^n}\sum_{a\in P_n} b(a) &= \frac{1}{q^n}\sum_{\substack{i_1+i_2+\cdots+i_k=n\\ i_j\geqq 1, 1\leqq i\leqq k\\ k\geqq 1}} \pi(i_1)\pi(i_2)\cdots\pi(i_k)\\ &= g(n).\end{aligned}$$

Now we apply Theorem 2 to obtain the asymptotic estimate.

Note. (i) From the logarithm of the asymptotic mean value, we obtain $\log g(n) = n \log(e/(e-1)) + O(1)$, in contrast to the normal order of $\log b(a)$ in Theorem 8.

(ii) The mean value of the related arithmetical function $b_k(a)$, which denotes the number of ordered factorizations of $a(X)$ into products of exactly k irreducibles, is estimated as $n \to \infty$ by Knopfmacher [7, Lem. 9.12, p. 94] for fixed k. These estimates are then applied to obtain the mean value of several related arithmetical functions when k is fixed.

(iii) By letting $q \to \infty$ and using (2.28), we can show in the same way as in the proof of Theorem 9 that $b_k(a)$ has mean value $(k!/n!)s(n, k)$. By applying known asymptotics for $s(n, k)$ (see, e.g., [9]), it is possible to extend the estimates for the mean value of $b_k(a)$ in [7] to the range $1 \leqq k \leqq n$.

As noted by the referee, our estimate for $f(n)$ can also be applied to determine the asymptotic mean value of an arithmetical function $d : \mathbb{F}_q[X] \to \mathbb{R}$. For any $a(X) \in P_n$, define $d(a)$ to be the number of ways of writing $a(X) = p_1(X)p_2(X)\cdots p_s(X)$, where p_i are irreducible and $\deg p_i \geqq \deg p_{i+1}$.

THEOREM 10. *As $q \to \infty$, the asymptotic mean value of $d(a)$ tends to*

$$e^{-\gamma}(n - \log n + 1 + \log 2 - \gamma) + O\left(\frac{\log n}{n}\right), \quad n \to \infty.$$

Proof. We have that

$$\frac{1}{q^n} \sum_{a \in P_n} d(a) = \frac{1}{q^n} \sum_{\substack{i_1 + i_2 + \cdots + i_k = n \\ i_1 \leqq i_2 \leqq \cdots \leqq i_k \\ 1 \leqq i_j; k \geqq 1}} \pi(i_1)\pi(i_2)\cdots\pi(i_k)$$

$$= f(n) \quad \text{as } q \to \infty.$$

4. Cycles of permutations. Let t_n be the probability that a permutation of n letters has cycles whose lengths are all different. Wilf [11, p. 97] shows that $\{t_n\}$ has ordinary generating function $\prod_{n=1}^{\infty} (1 + z^n/n)$. It follows from the estimate for $h(n)$ obtained by Greene and Knuth [3] that

$$t_n = e^{-\gamma} + \frac{e^{-\gamma}}{n} + O\left(\frac{\log n}{n^2}\right), \quad n \to \infty.$$

Similarly, we can interpret Theorems 1 and 2 in terms of cycles of permutations.

Given a permutation $\pi \in \prod_n$, the set of permutations on n letters, let $u(\pi)$ denote the number of *ordered decompositions* of π into cycles.

THEOREM 12. *The asymptotic mean value of $u(\pi)$ is*

$$\frac{1}{n!} \sum_{\pi \in \Pi_n} u(\pi) = \left(\frac{1}{e-1}\right)\left(\frac{e}{e-1}\right)^n + O(1), \quad n \to \infty.$$

Proof. The result follows immediately from Corollary 3, since

$$\frac{1}{n!} \sum_{\pi \in \Pi_n} u(\pi) = \frac{1}{n!} \sum_{k=1}^{n} k! s(n, k).$$

Finally, given $\pi \in \prod_n$, define $v(\pi)$ to be the number of decompositions of π into cycles $\pi_1\pi_2\cdots\pi_s$, where the cycles π_i are in nondecreasing order with respect to their lengths. Then $\sum_{\pi \in \Pi_n} v(\pi)$ has exponential generating function given by (2.1), which leads to the following theorem.

THEOREM 13. *The asymptotic mean value of* $v(\pi)$ *is*

$$\frac{1}{n!}\sum_{\pi\in\Pi_n} v(\pi) = e^{-\gamma}(n - \log n + 1 + \log 2 - \gamma) + O\left(\frac{\log n}{n}\right), \quad n \to \infty.$$

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