

A COMBINATORIAL APPROACH TO SOME POSITIVITY PROBLEMS*

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Abstract. We give a purely combinatorial proof of the nonnegativity of the integrals

$$\int_0^\infty e^{-x} x^\alpha L_n^\alpha(\lambda x) L_m^\alpha((1-\lambda)x) L_k^\alpha(x) dx \quad \text{and} \quad \int_0^\infty e^{-2x} x^\alpha L_n^\alpha(x) L_m^\alpha(x) L_k^\alpha(x) dx,$$

where α, k, m, n are nonnegative integers and $\lambda \in [0, 1]$. We also state the combinatorial equivalent of a conjecture of H. Lewy. This is done by applying the Master theorem of MacMahon.

1. Introduction. In their study of discretization of the time dependent wave equation in two dimensions, K. O. Friedrichs and H. Lewy needed the positivity of the coefficients $A(k, m, n)$ in

$$(1.1) \quad \{(1-r)(1-s) + (1-r)(1-t) + (1-s)(1-t)\}^{-1} = \sum_{k,m,n=0}^\infty A(k, m, n) r^k s^m t^n$$

in order to show that the finite difference approximations to the solution do converge to a solution of the wave equation. The question of the positivity of A 's is very simple to state and surprisingly turned out to be difficult to prove. G. Szegő [13] solved this problem using Sonine type integrals for Bessel functions. Szegő was also able to generalize this problem in several directions. He also observed that, Askey and Gasper [2], the coefficients $A(k, m, n)$ can be expressed in terms of the simple Laguerre polynomials $\{L_n(x)\}_{n=0}^\infty$ as

$$(1.2) \quad A(k, m, n) = \int_0^\infty e^{-3x} L_k(x) L_m(x) L_n(x) dx.$$

The generalized Laguerre polynomials $\{L_n^\alpha(x)\}_{n=0}^\infty$ have the generating function, Szegő [14]

$$(1.3) \quad \sum_{n=0}^\infty L_n^\alpha(x) t^n = (1-t)^{-\alpha-1} \exp\left(\frac{-xt}{1-t}\right),$$

and satisfy the orthogonality relation, [11],

$$(1.4) \quad \int_0^\infty e^{-x} x^\alpha L_m^\alpha(x) L_n^\alpha(x) dx = \frac{\Gamma(\alpha+1+n)}{n!} \delta_{mn}, \quad \alpha > -1.$$

The simple Laguerre polynomial $L_n(x)$ is $L_n^0(x)$. The Laguerre polynomials are often called Rook polynomials; see Riordan [12]. Askey and Gasper [2] observed that, due to the orthogonality relation (1.4), the coefficients $A(k, m, n)$ are linearization coefficients in

$$e^{-2x} L_k(x) e^{-2x} L_m(x) = \sum_{n=0}^\infty A(k, m, n) e^{-2x} L_n(x),$$

and proceeded to study the numbers

$$(1.5) \quad A^\alpha(\mu; k, m, n) = \int_0^\infty e^{-\mu x} x^\alpha L_k^\alpha(x) L_m^\alpha(x) L_n^\alpha(x) dx, \quad \alpha > -1, \quad \mu > 0.$$

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Because of the well-known formula, Rainville [11, p. 209],

$$(1.6) \quad L_n^\alpha(xy) = \sum_{k=0}^n \frac{(1+\alpha)_n (1-y)^{n-k} y^k}{(n-k)!(1+\alpha)_k} L_k^\alpha(x)$$

the nonnegativity of $A^\alpha(\mu; k, m, n)$ for some μ implies the nonnegativity of $A^\alpha(\nu; k, m, n)$ for all $\nu \geq \mu > 0$. The following are three results on the sign behavior of these coefficients.

THEOREM 1.1 (Askey and Gasper [3]). *The numbers $A^\alpha(2, k, m, n)$ are nonnegative for all $k, m, n = 0, 1, \dots$, if and only if $\alpha > (-5 + \sqrt{17})/2$.*

THEOREM 1.2 (Debbi and Gillis [7]). *The numbers $(-1)^k A(\frac{3}{2}, k, n, n)$ are nonnegative for $n = 0, 1, \dots$ and $k = 0, 1, \dots, n$.*

Quite recently T. Koornwinder [9] proved

THEOREM 1.3. *Let*

$$B^\alpha(n, m, k) = \int_0^\infty e^{-x} x^\alpha L_n^\alpha(\lambda x) L_m^\alpha((1-\lambda)x) L_k^\alpha(x) dx.$$

Then $B^\alpha(n, m, k) \geq 0$ for $\alpha \geq 0$ and $0 \leq \lambda \leq 1$. Furthermore if $0 < \lambda < 1$ then $B^\alpha(0, m, n) > 0$.

Theorem 1.3 was iterated by Askey, Ismail and Koornwinder [5, Thm. 1]. Using the orthogonality relation (1.4) one can easily see that

$$L_n^\alpha(\lambda x) L_m^\alpha((1-\lambda)x) = \sum_{k=0}^{m+n} \frac{k! B^\alpha(n, m, k)}{\Gamma(\alpha+1+k)} L_k^\alpha(x).$$

Going back to the Friedrichs and Lewy conjecture, immediately after Szegő [13] settled it, Kaluza [8] published an “elementary” proof of that conjecture and obtained some monotonicity properties for the coefficients $A(k, m, n)$. Although Kaluza’s proof is elementary, in the sense that it uses only elementary series manipulations, it depends on a very tricky way of combining various powers of r, s and t . It is probably fair to say that the impression one forms by reading Kaluza’s paper is that Kaluza’s method is too tricky to work in any other problem. One purpose of the present work is to give Kaluza’s paper another chance by systematizing his method. While doing so we shall give combinatorial proofs of Theorems 1.1 and 1.3 when α is a nonnegative integer. Our proofs use a powerful tool given by MacMahon, which he calls the Master theorem; see MacMahon [10, pp. 93–98].

THEOREM 1.4 (the Master theorem). *Set*

$$V_n = \det(I_n - A_n X_n),$$

where I_n is the $n \times n$ identity matrix, $A_n = (a_{ij})_{i,j=1}^n$ and $X_n = (x_{ij})_{i,j=1}^n$, $x_{ij} = x_i \delta_{ij}$. Then the coefficient of $x_1^{k_1} \cdots x_n^{k_n}$ in the power series $1/V_n$ is the same as the coefficient of the same term in the expansion of

$$(a_{11}x_1 + \cdots + a_{1n}x_n)^{k_1} \cdots (a_{n1}x_1 + \cdots + a_{nn}x_n)^{k_n}.$$

In § 2 we start by proving Koornwinder’s theorem, Theorem 1.3 for $\alpha = 0, 1, 2, \dots$. The proof we give is purely combinatorial and uses only the Master theorem and the binomial theorem. This is followed by a similar proof of Theorem 1.1 for $\alpha = 0, 1, 2, \dots$. In § 3 we mention an interpretation of the numbers $A(2; k, m, n)$ and $B^0(k, m, n)$. In § 4 we discuss an open problem of Lewy and point out its combinatorial equivalent. Although we do not offer a solution to this open problem, we hope to eventually get a solution by this approach. The present work is a continuation of the work started in Askey and Ismail [4], and in Askey, Ismail and Koornwinder [5].

2. Combinatorial results and proofs. We first prove Theorem 1.3 then proceed to the proof of Theorem 1.1. In both proofs, we first obtain a generating function for the triple sequence under consideration by using the generating function (1.3). Then we apply the Master theorem to transform the problem to a combinatorial one. Finally using elementary manipulations we handle the combinatorial problem.

Proof of Theorem 1.3 for $\alpha = 0, 1, 2, \dots$. A generating function for $B^\alpha(k, m, n)$ can be obtained as follows

$$\begin{aligned} \sum_{k,m,n=0}^{\infty} B^\alpha(k, m, n) r^k s^m t^n &= \int_0^\infty e^{-x} x^\alpha \left\{ \sum_{k,m,n=0}^{\infty} r^k L_k^\alpha(\lambda x) s^m L_m^\alpha((1-\lambda)x) t^n L_n^\alpha(x) \right\} dx \\ &= \int_0^\infty e^{-x} x^\alpha \left\{ \sum_{k=0}^{\infty} L_k^\alpha(\lambda x) r^k \right\} \\ &\quad \cdot \left\{ \sum_{m=0}^{\infty} L_m^\alpha((1-\lambda)x) s^m \right\} \left\{ \sum_{n=0}^{\infty} L_n^\alpha(x) t^n \right\} dx. \end{aligned}$$

Using the generating function (1.3) we get

$$\begin{aligned} \sum_{k,m,n=0}^{\infty} B^\alpha(k, m, n) r^k s^m t^n &= \{(1-r)(1-s) \\ &\quad \cdot (1-t)\}^{\alpha-1} \int_0^\infty x^\alpha \exp \left\{ -x \left(1 + \frac{\lambda r}{1-r} + \frac{(1-\lambda)s}{1-s} + \frac{t}{1-t} \right) \right\} dx \\ &= \Gamma(\alpha+1) \left\{ (1-r)(1-s)(1-t) \left(1 + \frac{\lambda r}{1-r} + \frac{(1-\lambda)s}{1-s} + \frac{t}{1-t} \right) \right\}^{-\alpha-1} \\ &= \Gamma(\alpha+1) \{1 - (1-\lambda)r - \lambda s - \lambda r t - (1-\lambda)st + rst\}^{-\alpha-1}. \end{aligned}$$

Because α is a nonnegative integer, it then suffices to show that the rational function $\{1 - (1-\lambda)r - \lambda s - \lambda r t - (1-\lambda)st + rst\}^{-1}$ has nonnegative power series coefficients. It is not too difficult to see that

$$V_3 = 1 - (1-\lambda)x_1 - \lambda x_2 - \lambda x_1 x_3 - (1-\lambda)x_2 x_3 + x_1 x_2 x_3$$

if we choose the corresponding matrix A_3 of the Master theorem as

$$A_3 = \begin{pmatrix} 1-\lambda & -\sqrt{\lambda(1-\lambda)} & -\sqrt{\lambda} \\ -\sqrt{\lambda(1-\lambda)} & \lambda & -\sqrt{1-\lambda} \\ -\sqrt{\lambda} & -\sqrt{1-\lambda} & 0 \end{pmatrix}.$$

So, it remains to show that the coefficient of $r^k s^m t^n$ in $[(1-\lambda)r - \sqrt{\lambda(1-\lambda)}s - \sqrt{\lambda}t]^k [-\sqrt{\lambda(1-\lambda)}r + \lambda s - \sqrt{1-\lambda}t]^m [-\sqrt{\lambda}r - \sqrt{1-\lambda}s]^n$ is nonnegative. This is a combinatorial problem and can be proved as follows:

$$\begin{aligned} &[(1-\lambda)r - \sqrt{\lambda(1-\lambda)}s - \sqrt{\lambda}t]^k [-\sqrt{\lambda(1-\lambda)}r + \lambda s - \sqrt{1-\lambda}t]^m [-\sqrt{\lambda}r - \sqrt{1-\lambda}s]^n \\ &= (-1)^n \sum_{i,j} \binom{k}{i,j} [(1-\lambda)r - \sqrt{\lambda(1-\lambda)}s]^i (-\sqrt{\lambda}t)^{k-i} \binom{m}{j} \\ &\quad \cdot [-\sqrt{\lambda(1-\lambda)}r + \lambda s]^j (-\sqrt{1-\lambda}t)^{m-j} [\sqrt{\lambda}r + \sqrt{1-\lambda}s]^n \\ &= (-1)^{k+m+n} \sum_{i,j} (-1)^i \binom{k}{i} \\ &\quad \cdot \binom{m}{j} [\sqrt{1-\lambda}r - \sqrt{\lambda}s]^{i+j} [\sqrt{\lambda}r + \sqrt{1-\lambda}s]^n t^{k+m-i-j} \lambda^{(k+j-i)/2} (1-\lambda)^{(m+i-j)/2} \\ &= (-1)^{k+m+n} \sum_{i,j,p,q} (-1)^{j+p} \binom{k}{i} \binom{m}{j} \binom{i+j}{p} \\ &\quad \cdot \binom{n}{q} r^{p+q} s^{i+j+n-p-q} t^{k+m-i-j} \lambda^{(q+2j+k-p)/2} (1-\lambda)^{(p+n-q+m+i-j)/2}. \end{aligned}$$

Here we follow the standard convention that the binomial coefficient $\binom{a}{b}$ is zero if either b or $a-b$ is a negative integer. From the above computations we see that the coefficient of $r^k s^m t^n$ can be obtained by letting $p+q=k$, $i+j+n-p-q=m$ and $k+m-i-j=n$. Therefore for $k+m \geq n$

$$\begin{aligned} B^0(k, m, n) &= \sum_{i,p} (-1)^{i+p} \binom{k}{i} \binom{m}{k+m-n-i} \binom{k+m-n}{p} \\ &\quad \cdot \binom{n}{k-p} \lambda^{2k+m-n-i-p} (1-\lambda)^{n-k+p+i} \\ &= \lambda^{2k+m-n} (1-\lambda)^{n-k} \frac{(k+m-n)! n!}{k! m!} \left\{ \sum_i (-1)^i [(1-\lambda)/\lambda]^i \binom{k}{i} \binom{m}{n-k+i} \right\}^2 \\ &\geq 0. \end{aligned}$$

Clearly $B^0(0, m, n) > 0$ if $0 < \lambda < 1$. This completes the proof.

Note that in the process of proving Theorem 1.3 we actually proved that the coefficient of $r^k s^m t^n$ in $[(1-\lambda)r - \sqrt{\lambda(1-\lambda)}s - \sqrt{\lambda t}]^k [-\sqrt{\lambda(1-\lambda)}r + \lambda s - \sqrt{1-\lambda}t]^m \cdot [-\sqrt{\lambda}r - \sqrt{1-\lambda}s]^n$ is $B^0(k, m, n)$. Now we prove Theorem 1.1.

Proof of Theorem 1.1 for $\alpha = 0, 1, \dots$ First we compute a generating function for $A^\alpha(2, k, m, n)$. Straightforward manipulations similar to the first steps in the proof of Theorem 1.3 yield, see also Askey and Ismail [4],

$$\begin{aligned} \sum_{k,m,n=0}^{\infty} A^\alpha(2; k, m, n) r^k s^m t^n &= \int_0^\infty e^{-2x} x^\alpha \left\{ \sum_{k=0}^{\infty} L_k^\alpha(x) r^k \right\} \left\{ \sum_{m=0}^{\infty} L_m^\alpha(x) s^m \right\} \\ &\quad \cdot \left\{ \sum_{n=0}^{\infty} L_n^\alpha(x) t^n \right\} dx \\ &= \left\{ (1-r)(1-s)(1-t) \left(2 + \frac{r}{1-r} + \frac{s}{1-s} + \frac{t}{1-t} \right) \right\}^{-\alpha-1}; \end{aligned}$$

hence it suffices to consider the case $\alpha = 0$. Thus

$$2 \sum_{k,m,n=0}^{\infty} A^0(2, k, m, n) r^k s^m t^n = \{1 - \frac{1}{2}(r+s+t) + \frac{1}{2}rst\}^{-1}.$$

Next we look for a matrix A_3 such that the corresponding determinant V_3 of the Master theorem is

$$V_3 = 1 - \frac{1}{2}(r+s+t) + \frac{1}{2}rst.$$

The matrix A_3 is certainly not unique, see Askey and Ismail [4], and

$$A_3 = \frac{1}{2} \begin{pmatrix} 1 & -1 & -1 \\ -1 & 1 & -1 \\ -1 & -1 & 1 \end{pmatrix},$$

is one such matrix. Hence $A^0(2, k, m, n)$ is a positive multiple ($= 2^{-k-m-n}$) of the coefficient of $r^k s^m t^n$ in $(r-s-t)^k (-r+s-t)^m (-r-s+t)^n$. The last step in the proof is to establish the nonnegativity of the coefficient of $r^k s^m t^n$ in $(r-s-t)^k (-r+s-t)^m (-r-s+t)^n$. Let $C(k, m, n)$ be the coefficient of $r^k s^m t^n$ in $(r-s-t)^k (-r+s-t)^m (-r-s+t)^n$.

Clearly $(r-s-t)^k (-r+s-t)^m (-r-s+t)^n = \sum_{j=0}^n \binom{n}{j} t^{n-j} (r-s-t)^k (-r+s-t)^m (-r-s)^j$.

Therefore the nonnegativity of $C(k, m, n)$ will imply the nonnegativity of $A^0(2, k, m, n)$.

The nonnegativity of $C(k, m, n)$ can be established exactly as in the last part of Theorem 1.3. Indeed it is the special case $\lambda = \frac{1}{2}$ of Theorem 1.3 and the proof is complete.

With reference to the remark made in the Introduction about Kaluza's proof for Theorem 1.3, it should be noted that the use of the Master theorem and the manipulations in the proof of the Theorem 1.1 capture and systematize the essential idea of Kaluza's proof.

3. Weighted permutation problems. The combinatorial approach to the problems discussed in the preceding sections reveals their combinatorial nature. The numbers $A(\mu; k, m, n)$ can be interpreted as the number of certain weighted permutations. Let us consider the following derangement problem. We have three boxes containing k , m and n objects, respectively. We rearrange the objects in such a way that the number of objects in each box remains unchanged and no object remains in its own box. One can easily see that there is a one-to-one correspondence between derangements and ways of obtaining the monomial $x^k y^m z^n$ in $(y+z)^k (x+z)^m (x+y)^n$. Indeed the coefficient of $x^k y^m z^n$ in $(y+z)^k (x+z)^m (x+y)^n$ is the number of derangements. Askey, Ismail and Rashed [6] used this idea and the generating function (1.3) to show that

$$\mathcal{D}(n_1, n_2, \dots, n_k) = (-1)^k \int_0^\infty e^{-x} L_{n_1}(x) \cdots L_{n_k}(x) dx,$$

where $\mathcal{D}(n_1, n_2, \dots, n_k)$ is the number of derangements of objects in boxes with n_j objects in the j th box, $j = 1, \dots, k$. These derangements are really weighted permutations. In this case a permutation has weight zero if an object remains in its original box, otherwise the permutation has weight 1. As another example we consider the numbers of Askey and Gasper

$$A^0(2; k, m, n) = \int_0^\infty e^{-2x} L_k(x) L_m(x) L_n(x) dx.$$

We have seen, in § 2, that $A^0(2; k, m, n)$ is a positive multiple of the coefficient of $x^k y^m z^n$ in $(x-y-z)^k (-x+y-z)^m (-x-y+z)^n$. The coefficient of $x^k y^m z^n$ in $(x-y-z)^k (-x+y-z)^m (-x-y+z)^n$ has the following interpretation. Consider three boxes with k objects in the first box, m objects in the second box and n objects in the third box. Rearrange the objects so that the number of objects in each box remains unchanged. To each permutation attach the weight

$$(-1)^{\text{number of objects changing boxes}}.$$

It is easy to see that the coefficient of $x^k y^m z^n$ in $(x-y-z)^k (-x+y-z)^m (-x-y+z)^n$ equals the number of these weighted permutations. Another combinatorial interpretation of $A(2; k, m, n)$, in terms of distances in a Hamming scheme, can be found in Askey, Ismail and Koornwinder [5].

As Askey, Ismail and Koornwinder pointed out in [5], the numbers

$$B^0(k, m, n) = \int_0^\infty e^{-x} L_k(\lambda x) L_m((1-\lambda)x) L_n(x) dx,$$

have a combinatorial interpretation. We have three boxes containing k , m and n objects, respectively. As we have seen, in § 2, $B^0(k, m, n)$ equals the coefficient of $x^k y^m z^n$ in

$$[(1-\lambda)x - \sqrt{\lambda(1-\lambda)}y - \sqrt{\lambda}z]^k [-\sqrt{\lambda(1-\lambda)}x + \lambda y - \sqrt{1-\lambda}z]^m [-\sqrt{\lambda}x - \sqrt{1-\lambda}y]^n.$$

Thus the third box is a derangement box in the sense that all the objects originally occupying it must move to other boxes. The reader can easily figure out the weights.

For more examples and related results we refer the interested reader to [4] and [5].

4. A problem of H. Lewy. The finite difference approximations to the time dependent wave equation in three (space) dimensions lead to the following four variable analogue of Friedrichs and Lewy's problem.

Conjecture 1 (H. Lewy). If

$$(4.1) \quad \begin{aligned} & [(1-r)(1-s)(1-t)(1-u)\{(1-r)(1-s)+(1-r)(1-t)+(1-r)(1-u) \\ & \quad + (1-s)(1-t) + (1-s)(1-u) + (1-t)(1-u)\}]^{-1} \\ & = \sum_{k,l,m,n=0}^{\infty} E(k, l, m, n) r^k s^l t^m u^n, \end{aligned}$$

then $E(k, l, m, n) \geq 0$.

The early coefficients in the power series expansion of

$$\{(1-r)(1-s)+(1-r)(1-t)+(1-r)(1-u)+(1-s)(1-t)+(1-s)(1-u) + (1-t)(1-u)\}^{-1}$$

are positive but the later coefficients do change sign, because Huygen's principle holds in three-space. The factor $\{(1-r)(1-s)(1-t)(1-u)\}^{-1}$ is an averaging factor that makes the early positive terms count more than the later terms. This fascinating problem of Lewy was mentioned in Askey [1] and in Askey and Gasper [2]. The following is a combinatorial equivalent of Lewy's conjecture.

Conjecture 2. Let

$$\begin{pmatrix} y_1 \\ y_2 \\ y_3 \\ y_4 \end{pmatrix} = \begin{pmatrix} \frac{1}{2} & 1 & \sqrt{12}i & -12 \\ \frac{1}{12} & \frac{1}{2} & 1 & -\sqrt{12}i \\ -\frac{i\sqrt{12}}{144} & \frac{1}{12} & \frac{1}{2} & 1 \\ -\frac{1}{144} & \frac{i\sqrt{12}}{144} & \frac{1}{12} & \frac{1}{2} \end{pmatrix} \begin{pmatrix} r \\ s \\ t \\ u \end{pmatrix}$$

and let $G(k_1, k_2, k_3, k_4)$ be the coefficient of $r^{k_1}s^{k_2}t^{k_3}u^{k_4}$ in $y_1^{k_1}y_2^{k_2}y_3^{k_3}y_4^{k_4}$. Then

$$\sum_{\substack{0 \leq k_j \leq l_j \\ 1 \leq j \leq 4}} G(k_1, k_2, k_3, k_4) \geq 0 \quad \text{for all } l_j, 0 \leq l_j, 1 \leq j \leq 4.$$

The equivalence of conjectures 1 and 2 follows from the MacNahon's Master theorem, since

$$\begin{vmatrix} 1-\frac{1}{2}r & -s & -\sqrt{12}it & 12u \\ -\frac{1}{12}r & 1-\frac{1}{2}s & -t & \sqrt{12}iu \\ \frac{ir}{12\sqrt{12}} & -\frac{1}{12}s & 1-\frac{1}{2}t & -u \\ \frac{r}{144} & \frac{-is}{12\sqrt{12}} & -\frac{1}{12}t & 1-\frac{1}{2}u \end{vmatrix}$$

$$= 1 - \frac{1}{2}(r+s+t+u) + \frac{1}{6}(rs+rt+ru+st+su+tu)$$

$$= \frac{1}{6}[(1-r)(1-s)+(1-r)(1-t)+(1-r)(1-u)+(1-s)(1-t)+(1-s)(1-u) + (1-t)(1-u)].$$

We conclude this section by proving that the 4×4 matrix appearing in Conjecture 2 is essentially unique. This is not the case in all the previously mentioned problems; see Askey and Ismail [4]. Let

$$\begin{vmatrix} 1-ar & -es & -ht & -ju \\ -e'r & 1-bs & -ft & -ku \\ -h'r & -f's & 1-ct & -gu \\ -j'r & -k's & -g't & 1-du \end{vmatrix} = 1 - \frac{1}{2}(r+s+t+u) + \frac{1}{6}(rs+rt+ru+st+su+tu).$$

If we let $r=1, s=t=u=0$ we see that $a=\frac{1}{12}$. Similarly $b=c=d=\frac{1}{2}$. Now let $r=s=1, t=u=0$. This proves $ee'=\frac{1}{2}$ and similarly $ff'=gg'=hh'=kk'=jj'=\frac{1}{12}$. Thus

$$\begin{vmatrix} 1-\frac{1}{2}r & -es & -ht & -ju \\ -\frac{r}{12e} & 1-\frac{1}{2}s & -ft & -ku \\ -\frac{r}{12h} & -\frac{s}{12f} & 1-\frac{1}{2}t & -gu \\ -\frac{r}{12j} & -\frac{s}{12k} & -\frac{t}{12g} & 1-\frac{1}{2}u \end{vmatrix} = 1 - \frac{1}{2}(r+s+t+u) + \frac{1}{6}(rs+rt+ru+st+su+tu).$$

The substitution $r=s=t=2, u=0$ implies $12e^2f^2+h^2=0$; hence $h=\lambda ef$ where $\lambda=\pm\sqrt{12}i$. Similarly $k=\mu fg, j=\nu hg$ where $\mu=\pm\sqrt{12}i, \nu=\pm\sqrt{12}i$. Clearly, by rescaling, we may take $e=f=g=1$. If in the resulting identity we let $r=s=t=u=2$ we arrive at

$$1 = \frac{1}{9} - \frac{\lambda}{9\mu} - \frac{\lambda\nu}{108} - \frac{\mu}{9\lambda} - \frac{4}{3\lambda\nu} - \frac{(\mu-\nu)^2}{9\mu\nu}.$$

Assuming that $\mu=\delta\lambda, \nu=\varepsilon\lambda$, where $\delta=\pm 1, \varepsilon=\pm 1$, we can rewrite the above equation as

$$1 = \frac{1}{9} - \frac{\delta}{9} + \frac{\varepsilon}{9} - \frac{\delta}{9} + \frac{\varepsilon}{9} - \frac{(\delta-\varepsilon)^2}{9\delta\varepsilon},$$

that is

$$6 = 2\varepsilon - 2\delta - \frac{\delta}{\varepsilon} - \frac{\varepsilon}{\delta},$$

which holds only when $\delta=-1$ and $\varepsilon=1$, since the only admissible values for ε and δ are ± 1 . Thus there are only two matrices corresponding to $\lambda=\pm\sqrt{12}i$. The entries in these two matrices are complex conjugates, which is expected because their determinants are real. This shows that, up to a complex conjugation, the matrix appearing in Conjecture 2 is unique.

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