New results on D-optimal matrices

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Introduction

Let v be an odd positive integer and consider $(-1,1)$-matrices H of order $2v$.

Ehlich's bound $\quad \det(H) \leq 2^v(2v-1)(v-1)^{v-1}$.

D-optimal matrices are $2v \times 2v$ $(-1,1)$-matrices that attain Ehlich's bound, maxdet

Ehlich also proved that if A and B are circulant $(-1,1)$-matrices of order v such that $AA^T + BB^T = 2(v-1)I_v + 2J_v$ then the matrix

$$H = \begin{pmatrix} A & B \\ -B^T & A^T \end{pmatrix} \qquad \text{has maximal determinant}$$

Such A and B can be constructed by cyclic $SDS(v; r, s; r + s - \frac{v-1}{2})$.

D-optimal matrices of circulant type, D-optimal SDS.

Diophantine constraint: $a^2 + b^2 = 4v - 2$, a and b are row sums of A and B, resp.

Obtain feasible parameters r and s for the required SDS: $a = v - 2r$ & $b = v - 2s$.

Normalization: we may assume that $0 < a \leq b$ which implies that $r \geq s$.

Supplementary Difference Sets, SDS

Seberry, 1971

Let G be an additive abelian group of order v and $S_1, \ldots, S_n$ be subsets of G containing $k_1, \ldots, k_n$ elements respectively.

For every $d \in G - \{0\}$ we define the numbers

$$\lambda_i(d) = \#\{(r,s) : d = r - s, r, s, \in S_i\},\ 1 \leq i \leq n.$$

If $\lambda(d) = \lambda_1(d) + \cdots + \lambda_n(d)$ has a constant value λ, then $S_1, \ldots, S_n$ are called $n - \{v; k_1, \ldots, k_n; \lambda\}$ supplementary difference sets. $SDS(v; k_1, \ldots, k_n; \lambda)$

Crucial Property :

SDS can be constructed by taking unions of cyclotomic classes.

Yamada, 1992

Supplementary difference sets and Jacobi sums

Discrete Mathematics 103 (1992) pp. 75–90

Known Orders, Infinite Series

- Comprehensive table of all odd $v < 100$ for which D-optimal SDSs are known: Djokovic 1997
- Updated table of all $v < 100$ for which D-optimal matrices of order $2v$ are known: Kharaghani, Orrick 2007
- `http://www.indiana.edu/~maxdet/` Orrick

Two Infinite Series

- $v = q^2 + q + 1$ where q is a prime power, Seberry et al. 1991
- $v = 2q^2 + 2q + 1$ where q is an odd prime power, see Whiteman, 1990

Djokovic, Gysin, Seberry, 1997, 1998: the only odd values of $v > 100$ for which D-optimal matrices of order $2v$ are currently known:

- $v = 113$
- $v = 145$
- $v = 157$
- $v = 181$

Recent Progress

IS THIS NOW THE LIMIT OF WHAT WE CAN DO? IT MAY BE, BUT IT IS UNLIKELY AN ADVANCE WILL BE MADE BY PEOPLE WHO THINK THEY CANNOT SUCCEED.

- - Carl Pomerance (1994)

▷ New theoretical results:
Diophantine constraints for **divisors** of v + PSD constancy over orbits

▷ New computational results: $v = 63, 93, 103, 121, 131, 241$

Power Spectral Density (PSD) criterion

Let v be an odd integer such that the Diophantine equation $x^2 + y^2 = 4v - 2$ has solutions (α, β).

Two sequences $[a_1, \ldots, a_v]$, $[b_1, \ldots, b_v]$ can be used to form a D-optimal matrix of circulant type if and only if

$$PSD([a_1, \ldots, a_v], i) + PSD([b_1, \ldots, b_v], i) = 2v - 2, \ \forall \ i = 1, \ldots, \frac{v-1}{2}$$

where $PSD([a_1, \ldots, a_v], k)$ denotes the k-th element of the power spectral density sequence, i.e. the square magnitude of the k-th element of the DFT

$$DFT_{[a_1,\ldots,a_v]} = [\, \mu_0, \ldots, \mu_{v-1} \,], \ \text{ with } \ \mu_k = \sum_{i=0}^{v-1} a_{i+1}\, \omega^{ik}, \ k = 0, \ldots, v-1, \text{ where}$$

$\omega = e^{\frac{2\pi i}{v}} = \cos\left(\frac{2\pi}{v}\right) + i \sin\left(\frac{2\pi}{v}\right)$ is a primitive v-th root of unity (also called the principal v-th root of unity).

Periodic Autocorrelation Function (PAF)

Let v be an odd integer such that the Diophantine equation $x^2 + y^2 = 4v - 2$ has solutions (α, β).

Two sequences $[a_1, \ldots, a_v]$, $[b_1, \ldots, b_v]$ can be used to form a D-optimal matrix of circulant type if and only if

$$PAF([a_1, \ldots, a_v], s) + PAF([b_1, \ldots, b_v], s) = 2, \ \forall \ s = 1, \ldots, \frac{v-1}{2}$$

where

$$PAF([a_1, \ldots, a_v], s) = \sum_{i=1}^{v} a_i a_{i+s}$$

where $i + s$ is taken *mod* v when needed.

note for ambitious researchers

$2 \to -2 \rightsquigarrow$ **Hadamard matrices with two circulant cores**

bibl. ref. [210] in K. Horadam's celebrated PUP book

"Hadamard Matrices and Their Applications"

Optimization formalism

The search for D-optimal matrices can be formulated as an optimization problem, via the concept of the PAF.

There are optimization algorithms that deal with problems with $20K$ (discrete) variables.

We need **symmetric matrices** and certain vector/matrix products

$$\min_{x \in \{0,1\}^n} x^T A x$$

Let $a = [a_1, a_2, \ldots, a_n]^T$ be a column $n \times 1$ vector, where $a_1, a_2, \ldots, a_n \in \{-1, +1\}$ and consider the elements of the PAF vector $P_A(1), \ldots, P_A(m)$. Define the following $m = [n/2]$ symmetric matrices (which are independent of the sequence a)

$$M_i = (m_{jk}), \text{ s.t. } \begin{cases} m_{jk} = m_{kj} = \frac{1}{2}, & \text{when } a_j a_k \in P_A(i),\ j, k \in \{1, \ldots, n\} \\ 0, & \text{otherwise} \end{cases}, i = 1, \ldots, m$$

Lemma

The matrices M_i can be used to write the PAF equations in a matrix form:

- for n odd:

$$a^T M_i a = P_A(i), \;\; i = 1, \ldots, m.$$

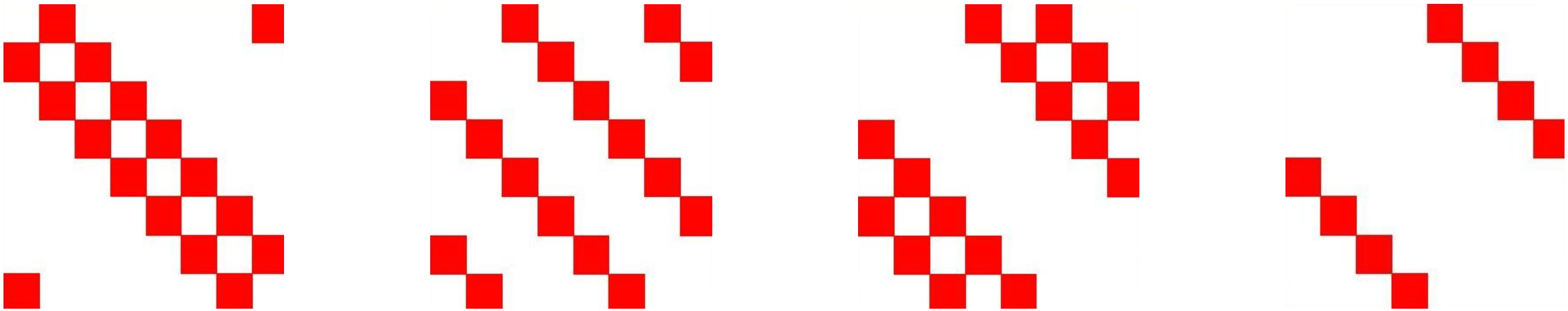
- for n even:

$$a^T M_i a = P_A(i), \;\; i = 1, \ldots, m-1 \text{ and } a^T M_m a = \frac{1}{2} P_A(m).$$

Example

Let $n = 8$, $a = [a_1, \ldots, a_8]$. Then we have that $m = 4$ and

$$a^T M_i a = P_A(i), \;\; i = 1, 2, 3 \text{ and } a^T M_4 a = \frac{1}{2} P_A(4)$$



Graphical representations of the four symmetric matrices M_1, M_2, M_3, M_4

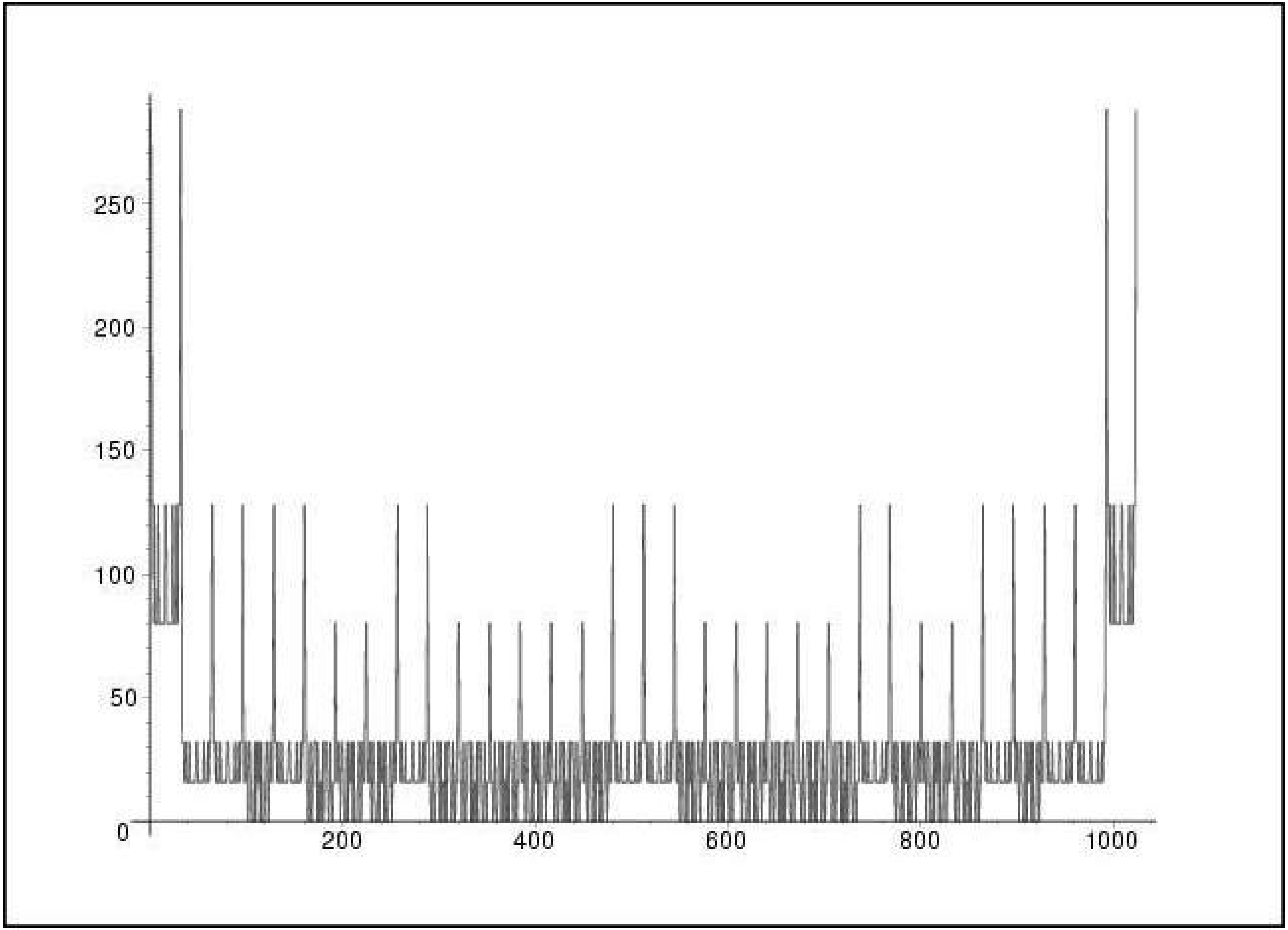
Problem I Now suppose that we are looking for two $\{-1, +1\}$ sequences A and B of lengths n, such that

$$P_A(i) + P_B(i) = 2, \;\; i = 1, \ldots, m.$$

Via the previous lemma we can reformulate this problem as follows:

Problem II Find two binary sequences a, b, (viewed as $n \times 1$ column vectors) such that

$$a^T M_i a + b^T M_i b = 2, \;\; i = 1, \ldots, m.$$


250
200
150
100
50
0
200
400
600
800
1000

Explicit DFT/PSD evaluations

The elements of the DFT/PSD vectors associated to a $\{-1,+1\}$-sequence are usually complex numbers with floating point real and imaginary parts.

However, for $n \equiv 0\,(mod\ 3)$

LEMMA

v odd integer, $v \equiv 0\,(mod\ 3)$, $m = \frac{v}{3}$, $[a_1,\ldots,a_v]$ $\{-1,+1\}$-sequence. Then we have the explicit evaluations:

$$DFT([a_1,\ldots,a_v],m) = \left(A_1 - \frac{1}{2}A_2 - \frac{1}{2}A_3\right) + \left(\frac{\sqrt{3}}{2}A_2 - \frac{\sqrt{3}}{2}A_3\right) i$$

$$PSD([a_1,\ldots,a_v],m) = A_1^2 + A_2^2 + A_3^2 - A_1A_2 - A_1A_3 - A_2A_3$$

where

$$A_1 = \sum_{i=0}^{m-1} a_{3i+1},\ \ A_2 = \sum_{i=0}^{m-1} a_{3i+2},\ \ A_3 = \sum_{i=0}^{m-1} a_{3i+3}.$$

COROLLARY $PSD([a_1,\ldots,a_n],m)$ is a non-negative integer.

Vertical Constraint

If $v \equiv 0\,(mod\ 3)$ and the two sequences $[a_1, \ldots, a_v]$, $[b_1, \ldots, b_v]$ can be used to form a D-optimal matrix of circulant type then

$$A_1^2 + A_2^2 + A_3^2 + B_1^2 + B_2^2 + B_3^2 = 8m - 2$$

where $m = \dfrac{n}{3}$ and

$$A_1 = \sum_{i=0}^{m-1} a_{3i+1}, \;\; A_2 = \sum_{i=0}^{m-1} a_{3i+2}, \;\; A_3 = \sum_{i=0}^{m-1} a_{3i+3},$$

$$B_1 = \sum_{i=0}^{m-1} b_{3i+1}, \;\; B_2 = \sum_{i=0}^{m-1} b_{3i+2}, \;\; B_3 = \sum_{i=0}^{m-1} b_{3i+3}.$$

Horizontal Constraint

If $v \equiv 0\,(mod\ 3)$ and the two sequences $[a_1, \ldots, a_v]$, $[b_1, \ldots, b_v]$ can be used to form a D-optimal matrix of circulant type then

Following Cohn, 1992

let $C_i = a_i + a_{i+m} + a_{i+2m}$, $D_i = b_i + b_{i+m} + b_{i+2m}$, for $i = 1, \ldots, m$.

Note that $a_i, b_i \in \{-1, +1\}$ for $i = 1, \ldots, n$, implies
$C_i, D_i \in \{-3, -1, +1, +3\}$, for $i = 1, \ldots, m$.

$$\sum_{i=1}^{m} (C_i^2 + D_i^2) = 2v + 4$$

Properties of D-optimal matrices of circulant type

1. n is a on odd integer, s.t. $x^2 + y^2 = 4n - 2$ has solutions
2. $\alpha^2 + \beta^2 = 4n - 2$
3. $a_1 + \ldots + a_n = \pm\alpha$
4. $b_1 + \ldots + b_n = \pm\beta$
5. $PAF([a_1, \ldots, a_n], i) + PAF([b_1, \ldots, b_n], i) = 2, \forall\ i = 1, \ldots, \frac{n-1}{2}$
6. $PSD([a_1, \ldots, a_n], i) + PSD([b_1, \ldots, b_n], i) = 2n - 2,\ \forall\ i = 1, \ldots, \frac{n-1}{2}$
7. $m = n/3$
   (a) $PSD([a_1, \ldots, a_n], m) = A_1^2 + A_2^2 + A_3^2 - A_1A_2 - A_1A_3 - A_2A_3$
   (b) $PSD([b_1, \ldots, b_n], m) = B_1^2 + B_2^2 + B_3^2 - B_1B_2 - B_1B_3 - B_2B_3$
   (c)
   $$\begin{cases} pam := PSD([a_1, \ldots, a_n], m) = \frac{3}{2}\left(A_1^2 + A_2^2 + A_3^2\right) - \frac{\alpha^2}{2} \\ pbm := PSD([b_1, \ldots, b_n], m) = \frac{3}{2}\left(B_1^2 + B_2^2 + B_3^2\right) - \frac{\beta^2}{2} \end{cases}$$
   (d) Vertical Constraint $A_1^2 + A_2^2 + A_3^2 + B_1^2 + B_2^2 + B_3^2 = 8m - 2$ where
   $$A_1 = \sum_{i=0}^{m-1} a_{3i+1},\ \ A_2 = \sum_{i=0}^{m-1} a_{3i+2},\ \ A_3 = \sum_{i=0}^{m-1} a_{3i+3},\ \ B_1 = \sum_{i=0}^{m-1} b_{3i+1},\ \ B_2 = \sum_{i=0}^{m-1} b_{3i+2},\ \ B_3 = \sum_{i=0}^{m-1} b_{3i+3}.$$
   (e) $A_1 + A_2 + A_3 = \pm\alpha$
   (f) $B_1 + B_2 + B_3 = \pm\beta$
   (g) $pam + pbm = 2n - 2$
   (h) $A_1^2 + A_2^2 + A_3^2 = \frac{2pam + \alpha^2}{3} \qquad B_1^2 + B_2^2 + B_3^2 = \frac{2pbm + \beta^2}{3}$
   (i) Horizontal Constraint $\sum_{i=1}^{m}(C_i^2 + D_i^2) = 2n + 4$ where
   $$C_i = a_i + a_{i+m} + a_{i+2m}, D_i = b_i + b_{i+m} + b_{i+2m}, i = 1, \ldots, m.$$
   $a_i, b_i \in \{-1, +1\}, i = 1, \ldots, n, C_i, D_i \in \{-3, -1, +1, +3\}, i = 1, \ldots, m.$
   (j)
   $$\sum_{i=1}^{m} C_i = \pm\alpha \text{ and } \sum_{i=1}^{m} D_i = \pm\beta$$
   (k) Let $3_C, 1_C$ denote the number of $C_i's$ s.t. $\mid C_i \mid= 3, \mid C_i \mid= 1$ resp.
   Let $3_D, 1_C$ denote the number of $D_i's$ s.t. $\mid D_i \mid= 3, \mid D_i \mid= 1$ resp.
   Then we have:
   $$3_C + 3_D = \frac{m+1}{2} \qquad 1_C + 1_D = \frac{3m-1}{2}$$

Generalized horizontal and vertical constraint

It turns out we can generalize the previous constraints for any divisor of v.

THEOREM

If the two sequences $[a_0, \ldots, a_{v-1}]$ and $[b_0, \ldots, b_{v-1}]$ can be used to form a D-optimal matrix of circulant type and $v = dm$ and we set

$$A_j = a_j + a_{j+d} + \cdots + a_{j+(m-1)d}$$

$$B_j = b_j + b_{j+d} + \cdots + b_{j+(m-1)d}$$

for $j = 0, \ldots, d-1$,

then

$$\sum_{j=0}^{d-1} (A_j^2 + B_j^2) = 2(v + m - 1)$$

and

$$\sum_{k<l} (A_k A_l + B_k B_l) = v - m.$$

Power spectral density constancy over orbits

Let Z_v be the ring of integers $\mod v$, i.e $Z_v = \{0, 1, \ldots, v-1\}$. Let $Z_v^\star$ be the group of invertible elements of Z_v, i.e. $Z_v^\star = \{k \in Z_v : \gcd(k, v) = 1\}$.

The order of $Z_v^\star$ is equal to $\phi(v)$.

Let $H \leqslant Z_v^\star$ be a subgroup of $Z_v^\star$. Then H acts on Z_v and we denote the orbits of this action by

$$\mathcal{O}_1 = \{0\}, \mathcal{O}_2, \ldots \mathcal{O}_m.$$

Thus we have the disjoint union relationship $Z_v = \mathcal{O}_1 \cup \mathcal{O}_2 \cup \cdots \cup \mathcal{O}_m$.

Djokovic, Gysin, Seberry, 1991,1997,1998 constructed solutions for circulant type D-optimal matrices by expressing the corresponding SDSs as unions of certain orbits associated to a suitable subgroup of $Z_v^\star$.

The special structure of these solutions implies certain constraints on the possible range of values of the power spectral densities of the two sequences associated to the SDS.

The power spectral densities remain constant over the orbits.

A solution for n = 63, satisfies:

(1) PSD(A) + PSD(B) = 124

(2) PAF(A) + PAF(B) = 2

(3) a^2+b^2 = 250 (i.e. a = +/-9 , b = +/-13)

```
> a :=
  [1,-1,1,1,-1,1,-1,1,1,1,1,-1,1,-1,1,-1,-1,1,1,-1,1,-1,-1,1,-1,-1,-1,1,1,1,1,-1,1,-1,1,1,1,-
  1,-1,1,1,-1,1,1,-1,1,1,1,1,1,-1,-1,-1,1,1,-1,1,1,1,1,-1,-1,1];
  b :=
  [1,-1,-1,1,-1,-1,1,1,-1,-1,-1,1,1,1,1,-1,-1,-1,-1,1,-1,1,1,1,1,1,1,1,1,1,-1,-1,-1,1,-1,1,-1
  ,1,1,-1,-1,1,1,1,1,1,1,-1,1,1,1,-1,1,1,1,-1,1,-1,1,-1,-1,-1,-1];
  checkSol(a,b,63);
  vertical(a,b,63);
  horizontal(a,b,63);
```

$a := [1, -1, 1, 1, -1, 1, -1, 1, 1, 1, 1, -1, 1, -1, 1, -1, -1, 1, 1, -1, 1, -1, -1, 1, -1, -1, -1, 1, 1, 1, 1, -1, 1, -1, 1, 1, 1, -1, -1, 1, 1, -1, 1, 1, -1, 1, 1, 1, 1, 1, -1, -1, -1, 1, 1, -1, 1, 1, 1, 1, -1, -1, 1]$

$b := [1, -1, -1, 1, -1, -1, 1, 1, -1, -1, -1, 1, 1, 1, 1, -1, -1, -1, -1, 1, -1, 1, 1, 1, 1, 1, 1, 1, 1, 1, -1, -1, -1, 1, -1, 1, -1, 1, 1, -1, -1, 1, 1, 1, 1, 1, 1, -1, 1, 1, 1, -1, 1, 1, 1, -1, 1, -1, 1, -1, -1, -1, -1]$

```
a = 13 b = 9

1, 26.8603423702914903856742 89, 97.139659900902955711927041 1351, 124.00000227119444609760133 0135
```

```
10, 79.93001509021748119622 56, 44.06997997626226096876 7744, 123.99999506647974216499 3344
```

10, 79.930015090217481196225 6, 44.069979976262260968767744, 123.999995066479742164993344
11, 97.209640151295844447917161, 26.790358980521497431344547 0017, 123.99999131817341879261708002
12, 12.000000075622660252774 20089, 112.00000001509186808260 7018245, 124.00000009071452833538 1219135
13, 97.20964194125476873772 25, 26.79035844437076003269 7881, 124.00000038562552877042 0381
14, 124.00000002757524683529 9369024, .2673123647296e-17, 124.00000002757524683797 2492671
15, 12.00000016511465886227 901521, 112.000000015697510349, 124.00000018081216921127 901521
16, 26.86034237029149038567 4289, 97.13965990090295571192 70411351, 124.00000227119444609760 1330135
17, 79.93001800188271441757 09672104, 44.06997997626226096876 7744, 123.99999797814497538633 8711210
18, 36.00000000000000000000 0001234, 87.99999999651151418770 20435389, 123.99999999651151418770 2043662
19, 97.20964194125476873772 25, 26.79035844437076003269 7881, 124.00000038562552877042 0381
20, 79.93001800188271441757 09672104, 44.06997997626226096876 7744, 123.99999797814497538633 8711210
21, 123.99999999999958774526 9097352, .10379024321310390944555 5740251e-28, 123.99999999999958774526 9097352
22, 97.20964194125476873772 25, 26.79035898052149743134 45470017, 124.00000092177626616906 7047002
23, 79.93001800188271441757 09672104, 44.06998409942462414605 21, 124.00000210130733856362 3067210
24, 12.00000016511465886227 901521, 112.00000001509186808260 7018245, 124.00000018020652694488 6033455
25, 97.20964194125476873772 25, 26.79035898052149743134 45470017, 124.00000092177626616906 7047002
26, 97.20964015129584444791 7161, 26.79035844437076003269 7881, 123.99999859566660448061 5042
27, 36.00000000000000000000 0001234, 88.00000003691863503348 7746366, 124.00000003691863503348 774760
28, 124.00000000402459801727 9351327, .2673123647296e-17, 124.00000000402459801995 2474974
29, 79.93001800188271441757 09672104, 44.06998409942462414605 21, 124.00000210130733856362 3067210
30, 12.00000007562266025277 420089, 112.000000015697510349, 124.00000009132017060177 420089
31, 26.86034237029149038567 4289, 97.13966255393785360773 46242279, 124.00000492422934399340 8913228

126, 2, 2, 2, 2, 2, 2, 2, 2, 2, 2, 2, 2, 2, 2, 2, 2, 2, 2, 2, 2, 2, 2, 2, 2, 2, 2, 2, 2, 2, 2, 2, 2, 2, 2, 2, 2, 2, 2, 2, 2, 2, 2, 2, 2, 2, 2, 2, 2, 2, 2, 2, 2,
2, 2, 2, 2, 2, 2, 2, 2, 2, 2

[[7, -3, 9], 13, [3, 3, 3], 9, [139, 27, 166]]

[[1, -1, 1, 1, -1, 1, 1, 3, 1, 1, -1, 1, 1, -1, 3, 1, -1, 1, 1, -1, 1], 13, [3, 1, 1, 3, 1, -1, 3, 3, 1, -3, -1, 1, 3, -1, 3, -3, 1, -1, -3, -1, -1], 9,
[37, 93, 130]]

>

Let (X, Y) be an SDS of Z_v with parameters $(v; r, s; \lambda)$, with v odd and $\lambda = r + s - \frac{v-1}{2}$, corresponding to a circulant D-optimal matrix.

Assume that

$$X = \bigcup_{j \in J} \mathcal{O}_j, \quad Y = \bigcup_{k \in K} \mathcal{O}_k$$

for some subsets J, K of $\{1, 2, \ldots, m\}$.

By abuse of notation, let X also denote the sequence $x_0, x_1, \ldots, x_{v-1}$ where

$$x_i = \begin{cases} 1 & \text{if} \quad i \notin X \\ -1 & \text{if} \quad i \in X \end{cases}$$

and define similarly the sequence $Y = y_0, y_1, \ldots, y_{v-1}$.

THEOREM

If k and k' belong to the same orbit $\mathcal{O}_r \subseteq Z_v$ and the sequence X is as defined above, then

$$PSD_X(k) = PSD_X(k').$$

Supercomputing developments

http://www.top500.org/ November 2011, 800 racks, Kobe, Japan, made by Fujitsu

"the K Computer achieved an impressive 10.51 Petaflop/s on the Linpack benchmark using $705,024$ SPARC64 processing cores"

New D-optimal matrices for $v = 103$

Consider the subgroup $H = \{1, 46, 56\}$ of order 3, of $Z^{\star}_{103}$

Consider the 35 orbits of the action of H on Z_{103}.

$H \cdot 0 = \{0\}$	$H \cdot 1 = \{1, 46, 56\}$	$H \cdot 2 = \{2, 9, 92\}$
$H \cdot 3 = \{3, 35, 65\}$	$H \cdot 4 = \{4, 18, 81\}$	$H \cdot 5 = \{5, 24, 74\}$
$H \cdot 6 = \{6, 27, 70\}$	$H \cdot 7 = \{7, 13, 83\}$	$H \cdot 8 = \{8, 36, 59\}$
$H \cdot 10 = \{10, 45, 48\}$	$H \cdot 11 = \{11, 94, 101\}$	$H \cdot 12 = \{12, 37, 54\}$
$H \cdot 14 = \{14, 26, 63\}$	$H \cdot 15 = \{15, 16, 72\}$	$H \cdot 17 = \{17, 25, 61\}$
$H \cdot 19 = \{19, 34, 50\}$	$H \cdot 20 = \{20, 90, 96\}$	$H \cdot 21 = \{21, 39, 43\}$
$H \cdot 22 = \{22, 85, 99\}$	$H \cdot 23 = \{23, 28, 52\}$	$H \cdot 29 = \{29, 79, 98\}$
$H \cdot 30 = \{30, 32, 41\}$	$H \cdot 31 = \{31, 87, 88\}$	$H \cdot 33 = \{33, 76, 97\}$
$H \cdot 38 = \{38, 68, 100\}$	$H \cdot 40 = \{40, 77, 89\}$	$H \cdot 42 = \{42, 78, 86\}$
$H \cdot 44 = \{44, 67, 95\}$	$H \cdot 47 = \{47, 57, 102\}$	$H \cdot 49 = \{49, 66, 91\}$
$H \cdot 51 = \{51, 75, 80\}$	$H \cdot 53 = \{53, 69, 84\}$	$H \cdot 55 = \{55, 58, 93\}$
$H \cdot 60 = \{60, 64, 82\}$	$H \cdot 62 = \{62, 71, 73\}$	

$a^2 + b^2 = 4 \cdot 103 - 2 = 410$ has two solutions $(11, 17)$ and $(7, 19)$

- $SDS(103; 48, 42; 39)$, D-optimal matrix of order $2 \times 103 = 206$.

$$X = \bigcup_{j \in J} H \cdot j, \quad Y = \bigcup_{k \in K} H \cdot k$$

$$\begin{aligned} J &= \{2, 4, 8, 10, 12, 14, 17, 19, 30, 33, 42, 44, 47, 51, 60, 62\} \\ K &= \{1, 2, 3, 14, 20, 21, 30, 33, 38, 40, 42, 44, 53, 60\} \end{aligned}$$

- $SDS(103; 46, 43; 38)$, D-optimal matrix of order $2 \times 103 = 206$.

$$X = \bigcup_{j \in J} H \cdot j, \quad Y = \bigcup_{k \in K} H \cdot k$$

$$\begin{aligned} J &= \{0, 2, 7, 10, 11, 12, 15, 17, 19, 29, 31, 33, 38, 40, 42, 47\} \\ K &= \{0, 8, 10, 15, 20, 21, 22, 23, 33, 38, 40, 47, 49, 53, 55\} \end{aligned}$$

New D-optimal matrix for $v = 241$

Consider the subgroup

$H = \{1, 15, 24, 54, 87, 91, 94, 98, 100, 119, 160, 183, 205, 225, 231\}$ of order 15, of $Z^{\star}_{241}$.

$SDS(241; 120, 105; 105)$, D-optimal matrix of order $2 \times 241 = 482$.

$$X = \bigcup_{j \in J} H \cdot j, \quad Y = \bigcup_{k \in K} H \cdot k$$

$$\begin{aligned} J &= \{3, 4, 5, 6, 7, 10, 13, 38\} \\ K &= \{3, 5, 7, 11, 19, 35, 38\} \end{aligned}$$

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Open cases for D-optimal SDSs with $v < 200$

v	r	s	λ	Existence
3	1	0	0	Yes
5	1	1	0	Yes
7	3	1	1	Yes
9	3	2	1	Yes
13	6	3	3	Yes
13	4	4	2	Yes
15	6	4	3	Yes
19	7	6	4	Yes
21	10	6	6	Yes
23	10	7	6	Yes
25	9	9	6	Yes
27	11	9	7	Yes
31	15	10	10	Yes
33	15	11	10	Yes
33	13	12	9	Yes
37	16	13	11	Yes
41	16	16	12	Yes
43	21	15	15	Yes
43	18	16	13	Yes
45	21	16	15	Yes
49	22	18	16	Yes

v	r	s	λ	Existence
51	21	20	16	Yes
55	24	21	18	Yes
57	28	21	21	Yes
59	28	22	21	Yes
61	25	25	20	Yes
63	29	24	22	Yes⋆
63	27	25	21	Yes
69	31	27	24	?
73	36	28	28	Yes
73	31	30	25	Yes
75	36	29	28	?
77	34	31	27	?
79	37	31	29	Yes
85	39	34	31	?
85	36	36	30	Yes
87	38	36	31	?
91	45	36	36	Yes
93	45	37	36	Yes⋆
93	42	38	34	Yes
97	46	39	37	Yes
99	43	42	36	?

Open cases for D-optimal SDSs with $v < 200$

v	r	s	λ	Existence
103	48	42	39	Yes ⋆
103	46	43	38	Yes ⋆
111	55	45	45	?
111	51	46	42	?
113	55	46	45	?
113	49	49	42	Yes
115	51	49	43	?
117	56	48	46	?
121	55	51	46	Yes ⋆
123	58	51	48	?
129	57	56	49	?
131	61	55	51	Yes ⋆
133	66	55	55	Yes
133	60	57	51	?
135	66	56	55	?
139	67	58	56	?
141	65	60	55	?
145	69	61	58	?
145	64	64	56	Yes
147	66	64	57	?

v	r	s	λ	Existence
153	72	65	61	?
153	70	66	60	?
157	78	66	66	Yes
159	78	67	66	?
163	79	69	67	?
163	76	70	65	?
163	73	72	64	?
167	76	73	66	?
169	81	72	69	?
175	81	76	70	?
177	84	76	72	?
181	81	81	72	Yes
183	91	78	78	Yes
183	83	81	73	?
185	91	79	78	?
187	88	81	76	?
189	92	81	79	?
189	87	83	76	?
195	94	84	81	?
199	93	87	81	?