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Combinatorial correspondences for Young tableaux, balanced tableaux, and maximal chains in the weak Bruhat order of S_n

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1. INTRODUCTION. R. Stanley conjectured in [8] that the number of maximal chains in the (weak) Bruhat order of S_n is equal to the number f of Young tableaux of "staircase" shape $\lambda = \{n-1,n-2,\ldots,2,1\}$. (The weak Bruhat order on S_n is defined by letting $\sigma \leq \theta$ iff $\theta = \sigma\tau_1\tau_2\ldots\tau_k$, where each τ_i is a transposition of adjacent increasing elements). Stanley subsequently proved his conjecture [9], by algebraic methods, but his proof does not give a direct combinatorial correspondence and leaves a number of interesting combinatorial questions unanswered. The purpose of this note is to describe three bijections, which prove Stanley's conjecture and contain a wealth of information relating tableaux to chains in the weak Bruhat order. In the course of this work we were led to introduce and study a new class of tableaux (equinumerous with standard Young tableaux), called <u>balanced tableaux</u>. These tableaux have a rich structure, which has only begun to be explored. In what follows, we give a sketch of the main ideas, with an indication of the major results obtained to date. Complete proofs and other details will be published elsewhere [1].

2. BALANCED TABLEAUX. Let $\lambda = \{\lambda_1 \geq \lambda_2 \geq \ldots \geq \lambda_m\}$ be a partition of n, with conjugate partition $\lambda^* = \{\lambda_1^* \geq \ldots \geq \lambda_m^*\}$. A <u>tableau</u> of shape λ is a doubly indexed array T of integers t_{ij}, $1 \leq i \leq m$, $1 \leq j \leq \lambda_i$. For each cell (i,j) define the <u>hook</u> H_{ij} to be the multiset $\{t_{ik}\}_{k \geq j} \cup \{t_{kj}\}_{k > i}$. A tableaux T is said to be <u>balanced</u> if (1) its entries are a permutation of 1,2,...,n and (2) each t_{ij} is the r_{ij}th largest element of its hook H_{ij}, where $r_{ij} = \lambda_j^* - i + 1$. For example,

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5 7 4 8 9
6 10
2 1
3

is a balanced tableau of shape $\lambda = \{5,2,2,1\}$.

Our first result is that maximal chains in the weak Bruhat order of S_n correspond bijectively to balanced tableaux of shape $\{n-1,n-2,\dots,1\}$. The proof is not difficult.

THEOREM 1. Let Γ denote a maximal chain in the weak Bruhat order of S_n. Let B_Γ denote the tableau of shape $\{n-1,n-2,\dots,2,1\}$ defined by setting $b_{(n+1-i)j} = k$ if the kth step in Γ transposes $i > j$. Then B_Γ is a balanced tableau, and the correspondence $\Gamma \longrightarrow B_\Gamma$ between maximal chains and balanced tableaux is bijective.

EXAMPLE 1. If Γ is the chain in S_4 whose successive elements are (1234), (1324), (3124), (3142), (3412), (4312), (4321), then the corresponding balanced tableau B_Γ is

4 3 5
2 1
6

Let λ be an arbitrary partition, and let b_λ denote the number of balanced tableaux of shape λ. Then Stanley's conjecture is equivalent to proving that $b_\lambda = f_\lambda$ when λ is of staircase type, that is $\lambda = \{n-1,n-2,\dots,1\}$ for some n. Our principal result is the following:

THEOREM 2. $b_\lambda = f_\lambda$ for any partition λ.

The proof is nontrivial. One might hope to show, for example, that b_λ obeys the well-known recursion

$$f_\lambda = \sum f_{\lambda^-} \tag{1}$$

summed over all partitions λ^- obtained by removing a cell from the Ferrers diagram of λ. This recursion is elementary, and underlies many basic properties of the f_λ's. We can show that formula (1) holds for b_λ, but only as a consequence Theorem 2, not as a step in its proof.

While the original motivation for this work was to prove Stanley's conjecture by generalizing it to non-staircase shapes, the proof of Theorem 2 is ultimately based on an explicit bijection for staircase tableaux. The ideas rest heavily on an elegant theory of tableau transformations developed by M.P. Schützenberger ([5],[6],[7]), and on a new variant of the Robinson-

Schensted-Knuth insertion algorithm (see [4] for details). We describe these ideas briefly in the next section.

3. SCHÜTZENBERGER OPERATORS. For purposes of this discussion, define a <u>standard tableau</u> to be a tableau to be a tableau whose entries form a permutation of k+1,k+2,...,k+n, with rows and columns strictly increasing. When k=0, we call T a <u>standard Young tableau</u>. If T is standard, define a transformation $T \longrightarrow T^{\partial}$ (called "promotion*") as follows: if the largest entry in T occurs in position (i,j), i.e. $t_{ij} = k+n$, define

$$t^{\partial}_{ij} = \max\{t_{(i-1)j}, t_{i(j-1)}\} \tag{2}$$

with the convention that $t_{i0} = t_{0j} = 0$ for all $i,j > 0$. If $t_{(i-1)j} \geq t_{i(j-1)}$ replace i by i-1 (otherwise replace j by j-1), and iterate (2) in this manner until i = j = 1. Finally define $t^{\partial}_{11} = k$. This process results in a new tableau T^{∂} with entries k,k+1,...,k+n-1 whose rows and columns strictly increase. Let $T^{[i]}$ denote the result of applying ∂^i to T. Thus $T^{[1]} = T^{\partial}$ and $T^{[0]} = T$. Now define operators P and S on standard tableaux T as follows*:

(a) $T^P = T^{[n]} + n$. In other words, T^P is the result of promoting T n times, then adding n to each entry.

(b) T^S is the tableau of shape λ obtained by letting $t^S_{ij} = q$ if $t^{[q]}_{ij} \leq k$ but $t^{[q-1]}_{ij} > k$. In other words, T^S records the times at which cells in T receive "new" labels, as ∂ is iterated.

EXAMPLE 2. As an illustration of the operators ∂, P, and S, consider the standard Young tableau

$$T = \begin{matrix} 1 & 3 & 4 \\ 2 & 6 & \\ 5 & & \end{matrix}$$

Then

$$T^{\partial} = \begin{matrix} 0 & 1 & 4 \\ 2 & 3 & \\ 5 & & \end{matrix} \qquad T^P = \begin{matrix} 1 & 2 & 5 \\ 3 & 6 & \\ 4 & & \end{matrix} \qquad T^S = \begin{matrix} 1 & 3 & 6 \\ 2 & 4 & \\ 5 & & \end{matrix}$$

A remarkable result of Schützenberger states that the map $T \longrightarrow T^S$ is an involution on standard tableaux: indeed an analogous result can be shown to hold more generally, when T is replaced by an arbitrary partially ordered set with a monotone labelling [7]. The situation with P is is more complicated.

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*Our notation differs slightly from the definitions of ∂ found in [5], [6], and [7], and from that of S found in [4].

When T has rectangular shape one can deduce from results in [6] and [7] that $T^P = T$, although the derivation is not obvious. The following theorem, proved in [3], is somewhat more difficult. It underlies several of our major results, although it is not used explicitly in the proofs.

THEOREM 3. If T is a standard tableau of staircase shape, then T^P is the transpose of T. Thus the map $T \longrightarrow T^P$ is an involution on standard tableaux of staircase shape.

4. THE STAIRCASE BIJECTION. Our second explicit bijection is obtained by studying in detail how the promotion operator ∂ acts on staircase tableaux. Let T be a standard Young tableau of shape $\lambda = \{n-1,n-2,\ldots,2,1\}$, with entries $1,2,\ldots,\binom{n}{2}$. Let $\Omega = \{X_1,X_2,\ldots,X_{n-1}\}$ be an alphabet of n-1 letters. Associate with T a word

$$W_T = W_1W_2\ldots W_{\binom{n}{2}}$$

with letters in Ω as follows: for $i = 0,1,\ldots,\binom{n}{2}-1$ define $W_{i+1} = X_j$ if the largest entry in $T^{[i]}$ occurs in cell (n-j,j). Thus the letters X_j record the columns from which the largest entries in T "exit" as ∂ is iterated. For example, if T is defined as in Example 2, then $W_T = X_2X_1X_3X_2X_1X_3$.

THEOREM 4. Let T be a standard Young tableau of (staircase) shape $\lambda = \{n-1,n-2,\ldots,2,1\}$, and let W_T be defined as above. Then W_T represents a maximal chain in the weak Bruhat order of S_n, if X_i is identified with the transposition (i,i+1). Moreover the map $T \longrightarrow W_T$ defines a bijection between standard Young tableaux (of shape λ) and maximal chains in S_n .

For example, if T is the tableau defined in Example 2, then W_T represents the chain Γ defined in Example 1.

It is not obvious that the correspondence defined by Theorem 4 is one-to-one, onto, or even well-defined for all staircase tableaux. To show the latter, one must prove that the adjacent elements transposed are always increasing. The proof of this fact relies on "Jeu-de-Taquin" arguments (see [6]), and leads to many other results concerning actions of tableaux and skew tableaux on permutations.

Combining Theorems 4 and 1, we obtain a bijection

$$T \longrightarrow W_T \longrightarrow B_{W_T} \tag{3}$$

from standard Young tableaux to balanced tableaux of the same shape. This bijection is valid for all staircase shapes, and provides a proof of Theorem 2 in this special case.

5. THE INVERSE MAPPING. The inverse to (3) is based on a variant of the Robinson-Schensted-Knuth insertion algorithm, and has a surprisingly simple form. Let X_i denote the transposition (i,i+1), and let

$$C = C_1C_2\ldots C_{\binom{n}{2}}$$

be a word in letters $X_1, X_2, \ldots, X_{n-1}$ which represents a maximal chain in the weak Bruhat order of S_n. We regard the alphabet $X_1, X_2, \ldots, X_{n-1}$ as linearly ordered in the obvious way, and define a tableau K_C of shape $\{n-1, n-2, \ldots, 2, 1\}$ by successively inserting the letters $C_1, C_2, C_3, \ldots$ according to the Robinson-Schensted-Knuth scheme (see [4] for definitions), with one exception: if the letter X_i bumps X_{i+1} from a row in which X_i is already present, the result is $\ldots X_iX_{i+1}\ldots$ rather than $\ldots X_iX_i\ldots$. This variation on the rules is derived from the standard algorithm by replacing certain of the so-called "elementary Knuth equivalence relations" (see [5]) by the "Coxeter relations $X_{i+1}X_iX_{i+1} = X_iX_{i+1}X_i$ which hold for adjacent transpositions in S_n.

As in the Robinson-Schensted-Knuth case, one can define a second tableau L_C, which records the order in which new cells appear in K_C. The construction of K_C and L_C is illustrated by the next example.

EXAMPLE 3. If C is the word $X_2X_1X_3X_2X_1X_3$ then

$$K_C = \begin{matrix} 1 & 2 & 3 \\ 2 & 3 & \\ 3 & & \end{matrix} \qquad L_C = \begin{matrix} 1 & 3 & 6 \\ 2 & 4 & \\ 5 & & \end{matrix}$$

THEOREM 5. <u>If C represents a maximal chain in S_n, then K_C is row and column strict of (staircase) shape $\{n-1, n-2, \ldots, 1\}$, and always has the form</u>

$$\begin{matrix} 1 & 2 & 3 & \ldots & n \\ 2 & 3 & \ldots & n & \\ \cdot & & & & \\ \cdot & & & & \\ n-1 & n & & & \\ n & & & & \end{matrix}$$

THEOREM 6. <u>The map $C \longrightarrow L_C$ is a bijection from chains in the weak Bruhat order of S_n to standard Young tableaux of staircase shape.</u>

Although Theorem 6 defines a bijection, the actual inverse of (3) is obtained by applying the operator S to L_C. For example, the reader can check that the tableaux appearing in Examples 2 and 3 satisfy the relations

$$W_T = C \qquad L_C^S = T$$

Thus the maps

$$\begin{array}{c} T \longrightarrow W_T \longrightarrow B_{W_T} \\ \\ B \longrightarrow C_B \longrightarrow L^S_{C_B} \end{array} \qquad (4)$$

are inverses. Here, C_B denotes the maximal chain (or word) associated with balanced staircase tableau B under the bijection given by Theorem 1.

6. BALANCED TABLEAUX OF ARBITRARY SHAPE. Finally, we sketch the proof of Theorem 2 for tableaux of arbitrary shape. A bijection from standard tableaux to balanced tableaux is obtained as follows: given a standard tableau T of arbitrary shape, first imbed T in a staircase tableau $\overline{T}$ in a canonical fashion (to be described later). Then apply the staircase bijection (4) to $\overline{T}$, obtaining a balanced staircase tableau $\overline{B}$. Finally, delete the extra cells from $\overline{B}$, obtaining a balanced tableau B of the original shape. The details are best explained by example.

EXAMPLE 4. Consider the Young tableau

$$T = \begin{array}{lll} 1 & 3 & 5 \\ 2 & & \\ 4 & & \\ 6 & & \end{array}$$

of shape $\{2,1,1,1\}$. First imbed T in a staircase tableau $\overline{T}$ of shape $\{4,3,2,1\}$ by adding the missing entries 7,8,9,10 from left to right in the first row, then the second row, then the third, and so forth. Thus

$$\overline{T} = \begin{array}{llll} 1 & 3 & 5 & 7 \\ 2 & 8 & 9 & \\ 4 & 10 & & \\ 6 & & & \end{array}$$

Next apply the staircase bijection, which associates with $\overline{T}$ the chain $W_{\overline{T}} = X_2X_3X_2X_4X_1X_3X_2X_3X_1X_4$, and the balanced tableau

$$\overline{B} = B_{W_{\overline{T}}} = \begin{array}{llll} 4 & 7 & 5 & 2 \\ 6 & 9 & 8 & \\ 3 & 10 & & \\ 1 & & & \end{array}$$

Now delete the entries 10,9,8,7 in this order, each time interchanging the columns directly above and to the right of the element deleted. It is easily seen that deleting elements in this way preserves the property of being balanced. The resulting tableau is

$$B = \begin{matrix} 4 & 2 & 5 \\ 6 & & \\ 3 & & \\ 1 & & \end{matrix}$$

which (by definition) is the balanced tableau associated with T.

THEOREM 7. The correspondence described above is a bijection from standard Young tableaux to balanced tableaux of the same shape.

7. CONCLUDING REMARKS. We have not succeeded in finding a proof of Theorem 2 which is independent of (4), nor have we found any natural algebraic or geometric interpretation of balanced tableaux when the shape is not a staircase.

The bijections defined by Theorems 1, 4, and 6 contain enough information to count the number $c(\sigma)$ of maximal chains in S_n with top element equal to a fixed permutation σ. Stanley conjectured that $c(\sigma) = \sum f_\lambda$, where the sum is over a certain multiset $M(\sigma)$ of shapes. His proof [9] gives this sum but does not preclude the possibility of negative coefficients. Our correspondence describes the multiset $M(\sigma)$ in a natural way, and shows that only positive coefficients occur. For further details, see [1].

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