ACKNOWLEDGMENTS. The author gratefully acknowledges valuable suggestions from the anonymous referees.

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A Simple Solution to a Multiple Player Gambler's Ruin Problem

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1. PROBLEM. Consider a gambler's ruin problem involving r players, with player i initially having n_i units, $n_i > 0$, $i = 1, \dots, r$. At each stage, two of the players are chosen to play a game, with the winner of the game receiving 1 unit from the loser. Any player whose fortune drops to 0 is eliminated, and this continues until a single player has all $n \equiv \sum_{i=1}^{r} n_i$ units, with that player designated as the victor. Assuming that the results of successive games are independent and that each game is equally likely to be won by either of its two players, among other results we find

(a) the probability that player i is the victor;
(b) the expected number of stages until one of the players has all the money;
(c) the expected number of games played between two specified players.

Moreover we show that none of the preceding quantities depend on the rule for choosing the players in each stage.

2. SOLUTION. We first argue that the expected number of games played is finite. Let X_i denote the number of games that involve player i, and let

$$X = \frac{1}{2}\sum_{i=1}^{r} X_i$$

denote the total number of games played.

Lemma 1. *No matter how the choices of the players in each stage are made,* $E[X] < \infty$.

Proof. Fix i, and let L_j equal 1 either if player i loses the jth game she plays or if she plays less than j games, and let it equal 0 otherwise. Also, for $k \geq 1$, let A_k be the event that $L_{(k-1)n+1} = L_{(k-1)n+2} = \cdots = L_{kn} = 1$. Then, with

$$F = \min(k : A_k \text{ occurs})$$

we have that $X_i \leq nF$, implying that

$$\begin{aligned} E[X_i] &\leq nE[F] \\ &= n\sum_{j\geq 1} P(F \geq j) \\ &= n\sum_{j\geq 1} P(A_1^c A_2^c \cdots A_{j-1}^c) \\ &\leq n\sum_{j\geq 1}(1-(1/2)^n)^{j-1} \\ &< \infty. \end{aligned}$$

Consequently,

$$E[X] = \sum_{i=1}^{r} E[X_i] < \infty. \qquad \blacksquare$$

Proposition 2. *No matter how the choices of the players in each stage are made, the probability that player i is the victor is* n_i/n.

Proof. To begin, suppose that there are n players, with each player initially having 1 unit. Consider player i. She starts with 1 and each stage she plays will be equally likely to result in her either winning or losing 1 unit, with the results from each stage being independent. In addition, she will continue to play stages until her fortune becomes either 0 or n. Because this is the same for all players, it follows that each player has the same chance of being the victor. Consequently, no matter how the choices of the players in each stage are made, each player has probability $1/n$ of being the victor. Now, suppose these n players are partitioned into r teams, with team i containing n_i players, $i = 1, \ldots, r$. Then the probability that the victor is a member of team i is n_i/n. But because team i initially has a total fortune of n_i units, $i = 1, \ldots, r$, and each game played by members of different teams results in the fortune of the winning team increasing by 1 and that of the losing team decreasing by 1, it follows that the probability that the victor is from team i is exactly the probability asked for. Consequently, no matter how the choices of the players in each stage are made,

$$P(i \text{ is victor}) = n_i/n. \quad \blacksquare$$

To find $E[X_i]$, we will make use of the following lemma.

Lemma 3. *Let m_j denote the expected number of games needed when there are only 2 players with initial fortunes j and $n-j$. Then*

$$m_j = j(n-j).$$

Proof. Conditioning on the outcome of the first game yields that

$$m_j = 1 + \frac{1}{2}m_{j+1} + \frac{1}{2}m_{j-1}, \quad j = 1, \ldots, n-1. \tag{1}$$

Consequently,

$$m_{j+1} = 2m_j - m_{j-1} - 2, \quad j = 1, \ldots, n-1. \tag{2}$$

Using that $m_0 = 0$, the preceding yields that

$$\begin{aligned} m_2 &= 2m_1 - 2 \\ m_3 &= 2m_2 - m_1 - 2 = 3m_1 - 6 = 3(m_1 - 2) \\ m_4 &= 2m_3 - m_2 - 2 = 4m_1 - 12 = 4(m_1 - 3) \end{aligned}$$

suggesting that

$$m_i = i(m_1 - i + 1), \qquad i = 1, \ldots, n. \tag{3}$$

Using (2), the preceding is easily shown by mathematical induction. Letting $i = n$ in (3), and using that $m_n = 0$, gives that $m_1 = n - 1$, and yields the result

$$m_i = i(n-i). \quad \blacksquare$$

Proposition 4. *X_i, the number of games played by player i, has the same distribution no matter how the choices of the players in each stage are made. Also,*

$$E[X_i] = n_i(n - n_i).$$

Proof. From the perspective of player i, starting with n_i he will continue to play stages, independently being equally likely to win or lose each one, until his fortune is either n or 0. Thus, the number of stages he plays is exactly the same as when he has a single opponent with an initial fortune of $n - n_i$, and the result follows from Lemma 3. ■

Corollary 5. *No matter how the choices of the players in each stage are made, the expected number of games played is given by*

$$E[X] = \frac{1}{2}\sum_{i=1}^{r} n_i(n - n_i) = \frac{1}{2}\left(n^2 - \sum_{i=1}^{r} n_i^2\right).$$

Proof. This follows from Proposition 4 upon using the identity $X = \frac{1}{2}\sum_{i=1}^{n} X_i$. ■

Remark. Whereas the expected number of games played does not depend on the rule used for choosing players, its distribution (as opposed to the distribution of X_i) does depend on the rule. To see this, suppose $r = 3$, $n_1 = n_2 = 1$, and $n_3 = 2$. Then if players 1 and 2 are chosen in the first stage, then it takes at least three stages to determine a victor, whereas if player 3 is in the first stage then it is possible for there to be only two stages.

Now, for any set of players $S \subset \{1, \dots, r\}$, let $X(S)$ denote the number of games involving only members of S. Also, for disjoint subsets of players A and B, let $X(A, B)$ denote the number of games in which one of the players is in A and the other is in B.

Proposition 6. *No matter how the choices of the players in each stage are made,*

$$E[X(S)] = \sum_{i<j,\ \{i,j\}\subset S} n_i n_j.$$

Proof. Let S^c be the complement of S. By regarding the players in S as constituting one team and those in S^c as constituting a second team, it follows that the changes of a team's total fortune will move exactly as if there were 2 players with initial fortunes of $\sum_{i\in S} n_i$ and $n - \sum_{i\in S} n_i$. Because there will continue to be games between members of S and S^c until the cumulative fortune of one of these teams hits 0, it follows from Lemma 3 that

$$E[X(S, S^c)] = \left(\sum_{i\in S} n_i\right)\left(n - \sum_{i\in S} n_i\right) = n\sum_{i\in S} n_i - \left(\sum_{i\in S} n_i\right)^2.$$

Now, imagine that each player earns 1 point whenever she plays in a game. Then the total number of points earned by players in team S is $\sum_{i\in S} X_i$. But since team S earns 1 point for each game between a member of S and one of S^c, and the team earns 2 points for each game between members of S, it follows that

$$\sum_{i\in S} X_i = X(S, S^c) + 2X(S).$$

Taking expectations yields the result

$$2E[X(S)] = \sum_{i\in S} n_i(n - n_i) - n\sum_{i\in S} n_i + \left(\sum_{i\in S} n_i\right)^2.$$ ■

Corollary 7. *For disjoint sets of players A and B, no matter how the choices of the players in each stage are made,*

$$E[X(A, B)] = \sum_{i\in A}\sum_{j\in B} n_i n_j.$$

Proof. For $i \neq j$, let $X(i, j)$ denote the number of games between players i and j. Proposition 6 yields that

$$E[X(i, j)] = E[X(\{i, j\})] = n_i n_j.$$

The result now follows by taking expectations of both sides of the identity

$$X(A, B) = \sum_{i \in A} \sum_{j \in B} X(i, j). \qquad \blacksquare$$

The problem of this paper had previously been considered in [**1**] and [**3**]. These papers assumed that $r = 3$ and also that each choice of the pair of players to contest a game was randomly made in the sense that each pair of remaining players had equal probability of being the contestants. These papers used, respectively, recursive equations and martingale techniques to solve for the mean time $E[X]$ (see [**2**] for an introduction to martingales). Neither recognized that $E[X]$ can be so immediately obtained using the $r = 2$ result, nor that the result is independent of the manner in which the pairs are selected. Also, neither of these papers considered questions related to the random variables $X(S)$ and $X(A, B)$.

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Mathematics Is . . .

"Mathematics is to nature as Sherlock Holmes is to evidence."

Ian Stewart, *Nature's Numbers: The Unreal Reality of Mathematical Imagination*, Basic Books, New York, 1995, p. 2.

—Submitted by Carl C. Gaither, Killeen, TX