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The Computer Solves the Three Tower Problem

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Consider the following probability problem: *We have three piles with a, b, c chips, respectively. Each second a pile X is selected at random, then another pile Y is chosen at random and a chip is moved from X to Y. Find the expected waiting time $f(a, b, c)$ until one pile is empty. (Three Tower Problem, or TTP.)*

This problem is due to Lennart Råde from Gothenburg University, Sweden. During the last 20 years he posed it to numerous people, but nobody could solve it [**5**]. I heard of it during a Statistics Conference in New Zealand 1990. It became a simulation exercise in a book I was writing at the time [**3**]. The simulation problem gives numerical answers to specific inputs a, b, c. On January 17, 1992 I loaded the program again and started to experiment. In 15 minutes I guessed the formula

$$f(a,b,c) = \frac{3abc}{a+b+c}. \tag{1}$$

Once you have guessed the formula, the proof is a routine matter. Start in state (a, b, c). In one step you are in one of the neighboring states $(a, b+1, c-1)$, $(a, b-1, c+1)$, $(a+1, b, c-1)$, $(a-1, b, c+1)$, $(a+1, b-1, c)$, $(a-1, b+1, c)$ with the same probability $1/6$. So we have

$$f(a,b,c) = 1 + \frac{1}{6}\sum f(x,y,z) \tag{2}$$

over all neighbors (x, y, z) of (a, b, c) with boundary conditions

$$f(a,b,0) = f(a,0,c) = f(0,b,c) = 0. \tag{3}$$

It looks pretty hopeless to solve the functional equation (2) with the boundary conditions (3). But thanks to the PC we have the guess (1). It obviously satisfies (3) and a short calculation shows that (2) is also satisfied. So we have a solution to our problem. Its uniqueness can be proved by a standard argument, which we reproduce to make the paper self contained. See [**1**] or [**2**].

Suppose that $g(a, b, c)$ is another solution. Consider $h(a, b, c) = f(a, b, c) - g(a, b, c)$. Then

$$h(a,b,c) = \frac{1}{6} \sum h(x, y, z) \tag{4}$$

over all neighbors of (a, b, c).

The function h is defined for finitely many points. At some of these points h assumes its maximum M. Because of (4) $h(x, y, z) = M$ for all six neighbors of (a, b, c). And their neighbors have also the same h-value M, and so on, until we reach the boundary, at which h has value 0. Thus $h(a, b, c) \leq 0$ everywhere. Similarly we can show that $-h \leq 0$. Thus $h = 0$, and $f(a, b, c) = g(a, b, c)$ everywhere. So f is unique.

Lecturing in Norway, Råde mentioned that the TTP has recently been solved by me. A listener asked about the expected duration $g(a, b, c)$ of the following *modification* of the TTP: *Start with three towers. As soon as one tower is empty continue playing with two players until just one is left*. At the lecture it was agreed that this would be a harder problem to solve. Råde challenged me to find $g(a, b, c)$. He added that he also would like to know the probability p_a that the a-tower first becomes empty.

With my PC I started to work empirically on $g(a, b, c)$. Instead of 15 minutes it took me several hours of hard work. The trouble was that I was looking for a more complicated formula. At the end I found the much simpler correct formula

$$g(a, b, c) = ab + bc + ca. \tag{5}$$

It is easy to show that (2) is satisfied when $a, b, c > 0$. The new boundary conditions are

$$g(a, b, 0) = ab, \quad g(a, 0, c) = ac, \quad g(0, b, c) = bc. \tag{6}$$

(6) is the expected duration for the Two Tower Problem. This is a classic result, which is equivalent to the Gamblers Ruin Problem. See **[4]**. Had I looked at (6) first, they would have immediately suggested (5).

By analogy I was able to write down the solution of the modified *n-Tower-Problem*:

$$g(x_1, \ldots, x_n) = \sum_{i<k} x_i x_k. \tag{7}$$

It was also easy to guess the following version of Råde's second problem: The *ith* tower is the winner (in the game which continues until one pile is left) with probability

$$p_i = \frac{x_i}{x_1 + \cdots + x_n}. \tag{8}$$

Both formulas (7) and (8) satisfy the appropriate recurrencies and boundary conditions. The proof of (7) involves induction on n. The boundary conditions for a given value of n are determined by the solution of the problem for $n - 1$. Uniqueness is proved as in the case of the Three-Tower-Problem.

A related result could also be found with my PC searching for several hours: *Players* 1, 2, 3 *start with* a, b, c *chips, respectively. In one round each player stakes one chip. Then a* 3*-sided symmetric die labeled* 1, 2, 3 *is rolled and the winner gets all the chips staked. If a player is broke the game continues with two players until one player has accumulated all the chips*. The expected number of rounds is

$$h(a, b, c) = ab + bc + ca - \frac{2abc}{a + b + c - 2}. \tag{9}$$

If the game stops as soon as one tower is empty the expected duration is

$$h(a,b,c) = \frac{abc}{a+b+c-2}. \tag{10}$$

This result was communicated to me by a former IMO contestant Michael Stoll. It was found ten years ago during a summer academy for gifted high school students. Despite huge efforts they were unable to handle four players.

The original Four Tower Problem is still unsolved. I experimented extensively for many hours, but all my guesses turned out to be wrong. $f(a,b,c,d)$ seems to be a very complicated function, as can be seen from the exact value $f(3,2,2,2) = 350612/69969$. No simple formula can give such a complicated result for so small values of a, b, c, d. The only thing I could do was to guess a good approximation

$$f(a,b,c,d) \approx \frac{6abcd}{ab+ac+ad+bc+bd+cd}. \tag{11}$$

It is easy to see that $f(a,b,c,d)$ has the form $p(a,b,c,d)/q(a,b,c,d)$ with polynomials which are symmetric in a, b, c, d. In addition q seems to be constant, depending only on $a+b+c+d$. The use of *Mathematica* may bring more success. I worked numerically with Turbo Pascal as in **[3]**.

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A Linear Algebra Approach to Cyclic Extensions in Galois Theory

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A beginning course in Galois theory often includes a discussion of cyclic extensions, that is, Galois extensions whose Galois groups are cyclic. The usual approach (see, e.g., **[1]** and **[2]**) is to derive the results on cyclic extensions as