Multiplayer CHOMP

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Abstract: This article expands the famous problem CHOMP to multiplayer version. Since it is no longer a combinatorial game, its characteristics should be interesting and fruitful to discover. This article analyzes two specific kind of multiplayer Chomp game and yields satisfying results.

Key Words: Chomp, multiplayer, positions, scoring

Generic Multiplayer Chomp

We know that Chomp is a standard two-player combinatorial game. When it is extended to more players, it is natural to make those players to pick chocolate pieces in order. Different from win or lose, multiplayer game can value each player by scores. Players always seek for higher scores. For two-player game, winner gets 1 point and loser gets 0 points so valuing players by scores is reasonable.

Clockwise-Scoring Multiplayer Chomp

Assume that there are N players $(N \geq 3)$ numbered 1 through N play the Chomp game. The scale of the chocolate bar is $m \times n$ $(m, n \geq 1)$. The one who eats the last piece of chocolate scores 0 points, while his proceeding player scores 1 point, and the player after the proceeding player scores 2 points, and so on. The player before him scores $(N-1)$ points, which is the highest. All players want to maximize their scores. Who can get the most score in this multiplayer chomp game?

Start from the Simplest

We start from $N = 3$ and run some easy examples. **All the positions and graphs in this article are aligned to the top left.**

X X

X

Fig.1 The simplest example
(X means a piece of chocolate)

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As shown above, for this piece of L-shaped chocolate, player 1 has two options: pick all or pick one (note the symmetry!). Picking all yields 0 points while Picking one yields 1 point (player 2 can pick one too, so that he can get 2 points, see below). In order to maximize the score player 1 must choose the latter option letting player 2 get the most point.

```
X   X       X          X
X           X
Initial Position -> Player 1 (1 pt)-> Player 2 (2 pts)-> Player 3 (0 pts)
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Fig.2 Playing the simplest example

To characterize all possible options player 1 can take, we can define an **option graph**, labeling scores player 1 could get on each chocolate piece.

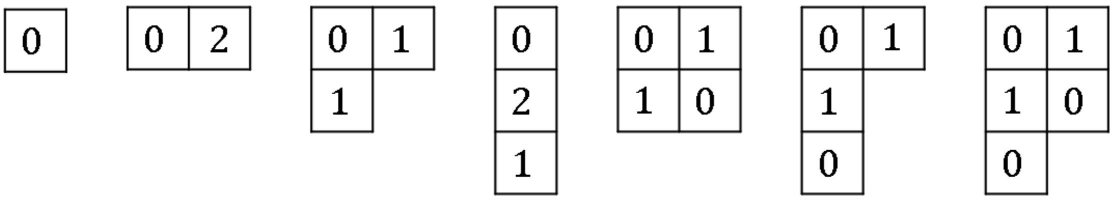

Fig.3 Option graph of several positions

Obviously, Player 1 will choose the chocolate piece with the highest score labeled on it.

Then We Go to $1 \times n$

We take a great leap to $1 \times n$ $(n \geq 1)$ with arbitrary N because it is easy to figure out.

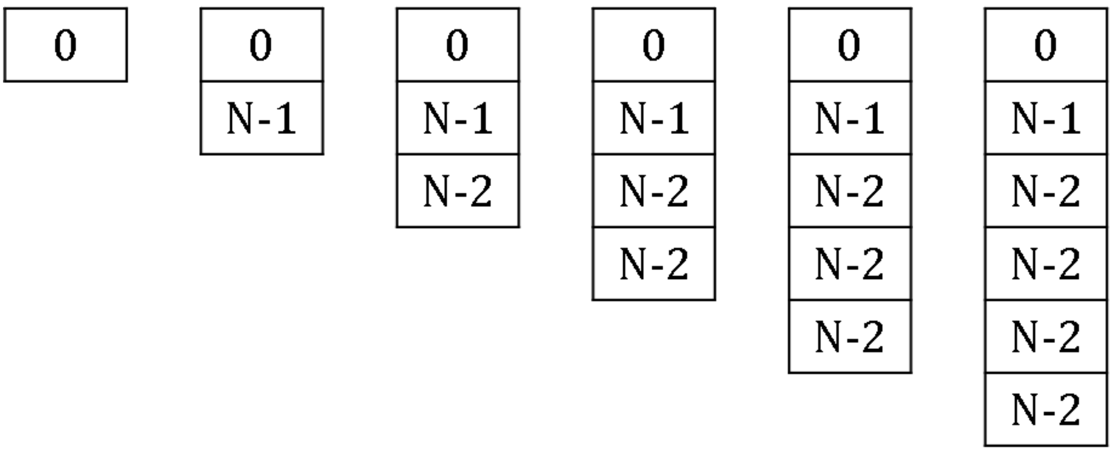

Fig.4 Option graph for $1 \times n$ ($n = 1,2,3,4,5,6$)

Here we show the reasons.

When $n = 1$, player 1 get 0 points and player N wins the highest points.
When $n = 2$, player 1 can either pick all and get 0 points or pick one and get $N - 1$ points. Obviously player 1 will choose the latter and get the highest points.
When $n = 3$, player 1 can pick all (0 pts), pick two ($N - 1$ pts), or pick one ($N - 2$ pts). Obviously player 1 will choose the second option and get the highest points.

Then we see a pattern: after picking up some chocolates, player 2 must pick the chocolate with the **highest** number labeled on **the position after player 1's move**. Then player 1's score (aka the number that should be labeled on the chocolate piece **of the original position**) should be exactly one less than player 2. But note that there is one exception: if player 2's score is 0, player 1 can get the highest score $N - 1$.
Therefore, by a simple deduction, we can easily figure out the answer for the **clockwise multiplayer chomp for** $\mathbf{1 \times n}$ **(**$\boldsymbol{n \geq 1}$**)**:

player N wins the highest points when $n = 1$,
and player 1 wins the highest points for any $n \geq 2$.

March to Our Final Goal

With this crucial conclusion in hand, we can directly solve this multiplayer chomp once and for all. All we have left is a chocolate bar with scale $m \times n$ ($m, n \geq 2$). However, to prove this situation, we should be greedy: we should get all possible positions involved. That is, the position need not to be rectangular. We call a position two-dimensional if it is NOT $1 \times n$ ($n \geq 1$). For $1 \times n$ ($n \geq 2$) position we call it one-dimensional. For 1×1 we call it zero-dimensional.
For a chocolate bar with scale $m \times n$ ($m, n \geq 2$), the initial position is surely two-dimensional. We can not only prove that player 2 scores the most in rectangular chocolate bar, but also prove that player 2 scores the most in ANY two-dimensional positions!

We execute the deduction upon the total chocolate count T of the initial position.

(1) When $T = 3$, only forming L-shape can make the position two-dimensional. Player 1 has two options: pick away one chocolate remaining 1×2 position OR pick away all. The first option forces player 2 to pick away one, making player 3 achieve 0 points, so that player 2 can have $(N - 1)$ points. Player 1 can get $(N - 2)$ points. The second option makes player 1 get 0 points. So,

player 1 must choose the first option, making player 2 get the most points.

(2) When conclusion holds for every $T \leq k$, we consider $T = k + 1$.
Player 1 has three options:
Option 1: Take away all chocolates and get 0 points;
Option 2: Reduce the position to one-dimensional. Player 2 can reduce one-dimensional position to zero-dimensional so that player 3 get 0 points. Player 1 therefore get $(N - 2)$ points;
Option 3: Remain the position on two-dimensional. Because after player 1's move, the chocolate count must be no more than k, so we can use the deduction assumption that player 3 gets the most points (because player 2 now executes the 'first' move). Player 1 therefore get $(N - 3)$ points.
So, option 2 is the best choice for player 1, then player 2 can get the most score.

Thus, player 2 scores the most in ANY two-dimensional positions.

To visualize two-dimensional positions' option graph, we take 3×3 position as an example.

0	N-2	N-3
N-2	N-3	N-3
N-3	N-3	N-3

Fig.5 Example of two-dimensional positions' option graph

Counter-Clockwise-Scoring Chomp for 3 Players

Assume that there are 3 players numbered 1 through 3 play the Chomp game. The scale of the chocolate bar is $m \times n$ $(m, n \geq 1)$. The one who eats the last piece of chocolate scores 0 points, while his proceeding player scores 2 points, and the player before him scores 1 point. All players want to maximize their scores. Who can get the most score (aka 2) in this multiplayer chomp game?

Surprisingly, changing the direction of evaluating scores makes this problem much more difficult. Here we list some option graphs.

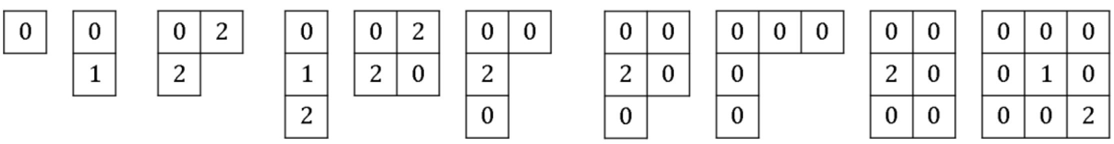

Fig.6 Some option graphs for 3-player counter-clockwise scoring chomp

Similar to clockwise version, we also have this in hand:
after picking up some chocolates, player 2 must pick the chocolate with the **highest** number labeled on **the position after player 1's move.** Then player 1's score (aka the number that should be labeled on the chocolate piece **of the original position**) should be exactly one **more** than player 2 **(not one less now)**. But note that there is one exception: if player 2's score is 2, player 1 will get the score 0.

There is a special conclusion just for counter-clockwise scoring chomp.

If there exists a 2 in an option graph, all numbers labeled on its bottom-right (edges included) should be 0.

Because take away chocolate piece labeled 2 means executing a winning move for player 1, if player 1 chooses chocolate pieces on its bottom right, player 2 can take up this winning move and get score 2. Then, player 1 will get an zero.

Walking the Tight Rope

Similar to standard two-player chomp, it is easy to analyze $1 \times n\ (n \geq 1)$ and $2 \times n\ (n \geq 2)$. Here we give the solution.

For $1 \times n\ (n \geq 1)$, if $n = 1$, then player 2 wins; if $n = 2$, then player 3 wins; if $n = 3$, then player 1 wins. What if $n \geq 4$? We can guess the option graph of it should be like this:

0	1	2	0	0	0	0	•••	0

Fig.7 Option graph for $1 \times n\ (n \geq 4)$

We can easily prove this through deduction.
Therefore, for $1 \times n\ (n \geq 1)$,

if $n = 1$, then player 2 wins;
if $n = 2$, then player 3 wins;
if $n \geq 3$, then player 1 wins.

For $2 \times n\ (n \geq 2)$, player 1 always wins because player 1 can directly chomp to 1×2 so that player 3 must take the last piece.

What about $3 \times n$?

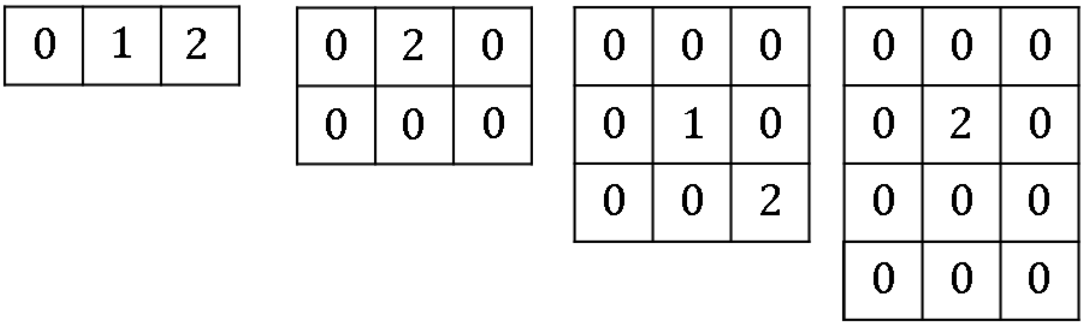

Fig.8 Option graph for $3 \times n$ ($n = 1,2,3,4$)

It seems that player 1 always wins too. For odd n it is possible to get 0, 1, or 2 points for player 1; but for even n it is possible to get 0 or 2 points, but not 1 point. The secret behind remains unrevealed.

The Rigid 'L'

We are busy finding the winner for $3 \times n$ positions, but why don't we stop to find other positions we have already covered?

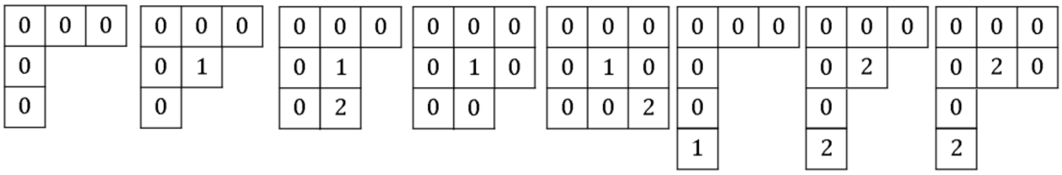

Fig.9 Discover the rigid 'L'

Chocolate pieces with coordinates (1,1), (1,2), (1,3), (2,1), (3,1) are always labeled 0. In fact, as long as this rigid 'L' exists, these five chocolate pieces must be labeled 0. The reason is as follows:

Operating (1,1) will take away all chocolate so player 1 scores 0;
Operating (1,2) or (2,1) will make the position $1 \times n$ ($n \geq 3$), player 2 definitely can find a way to win 2 points, making player 1 scores 0;
Operating (1,3) or (3,1) will make the position $2 \times n$ ($n \geq 3$), player 2 definitely can find a way to win 2 points, making player 1 scores 0;

So the conclusion holds. It is reasonable to make a guess that there are even bigger rigid shapes. However, our calculation may not achieve THAT big.

References

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