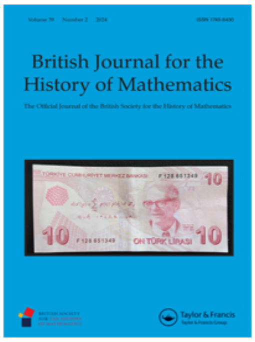

British Journal for the History of Mathematics

ISSN: (Print) (Online) Journal homepage: www.tandfonline.com/journals/tbsh21

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British Journal for the History of Mathematics, 2024
https://doi.org/10.1080/26375451.2024.2400707

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The Bauer–Ramanujan formula: historical analyses and perspectives

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The formula

$$\frac{2}{\pi} = 1 - 5 \cdot \left(\frac{1}{2}\right)^3 + 9 \cdot \left(\frac{1 \cdot 3}{2 \cdot 4}\right)^3 - \cdots$$

was famously included as a discovery in Ramanujan's first letter to Hardy in 1913, and has been referred to as the *Bauer–Ramanujan formula*, in view of Bauer's 1859 proof of the above formula. There is a rich history associated with this formula and its many and dramatically different proofs, including a computer-based proof due to Zeilberger that may be seen as groundbreaking in the history of computer-assisted proofs. In addition to a complete survey we provide of all known proofs of the Bauer–Ramanujan formula, we introduce historical analyses based on these proofs, by arguing that the history of the Bauer–Ramanujan formula and our account of this history may be seen as being representative of much broader trends in the history of mathematics. In this regard, the earlier proofs tend to rely on one of the oldest and most basic tools in classical analysis, namely, interchanging the order of limiting operations. In contrast, the more modern proofs tend to rely on computer-related approaches toward summation problems, as in with Zeilberger-type and Gosper-type telescoping arguments.

1. Introduction

The importance of Ramanujan's series for $\frac{1}{\pi}$ within number theory, special functions theory, and other areas motivates a scholarly study concerning what is considered as the most basic out of Ramanujan's series for $\frac{1}{\pi}$ and how it has gone on, over the decades, to be explored in different ways. In this regard, the now

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famous formula

$$\frac{2}{\pi} = 1 - 5 \cdot \left(\frac{1}{2}\right)^3 + 9 \cdot \left(\frac{1 \cdot 3}{2 \cdot 4}\right)^3 - \cdots \tag{1}$$

was included in Ramanujan's first letter to Hardy (Berndt and Rankin 1995, 25; Berndt 1989, 23–24), and, with an absolute convergence rate of 1, may be seen as the simplest out of Ramanujan's series for $\frac{1}{\pi}$, by comparison with the very fast-converging series now referred to as Ramanujan's 17 series for $\frac{1}{\pi}$, with reference to the survey article concerning such series by Baruah et al. (2009). As expressed in this survey, the formula in (1) was introduced by Bauer (1859), relative to Ramanujan's 1913 letter to Hardy (Berndt and Rankin 1995, 21). It is thus appropriate to refer to the formula in (1) as the *Bauer–Ramanujan* formula (Zudilin 2009a), or as the *Bauer–Ramanujan series* formula (Levrie and Campbell 2022). Apart from Bauer's 1859 discovery of (1) and Ramanujan's rediscovery of (1) and Zeilberger's famous computer proof of (1) (Ekhad and Zeilberger 1994) having been mentioned in the above survey, the Bauer–Ramanujan series was not otherwise considered there. This motivates our survey in Section 2 based on the many remarkably different proofs of (1) that have emerged over many decades. We introduce historical analyses based on our survey, as in Section 3, to argue that the history of the Bauer–Ramanujan formula may be seen as being representative of much broader trends in the history of mathematics.

The known proofs of the Bauer–Ramanujan formula were introduced in the following chronological order, referring to Section 2 below on the attributions listed below:

- Bauer (1859)
- Glaisher (1905)
- Dougall (1906)
- Hardy (1924)
- Borwein and Borwein (1987)
- Ekhad and Zeilberger (1994)
- Baranov (2006)
- Baruah and Berndt (2010)
- Aycock (2013)
- Cooper (2017)
- Levrie and Nimbran (2018)
- Ojanguren (2018)
- Campbell (to appear)

The older proofs of the Bauer–Ramanujan formula, including that by Bauer and that by Hardy, tend to rely on one of the 'oldest tricks in the book' in classical analysis: namely, interchanging the order of limiting operations. In contrast, the more modern proofs, including that by Zeilberger and by Ojanguren, tend to rely on techniques related to computer-based summation tools, as in the telescoping-based approach required according to the computer proof certificates generated by the Wilf–Zeilberger method. We argue, as in Section 3 below, that this is representative of a broader trend in the history of mathematics. In a related way, we further argue that: out of all of the proofs referenced above, Zeilberger's computer proof may be seen as especially groundbreaking and representative of a paradigm shift.

2. Survey

Each of the below subsections is devoted to a succinct description of a different proof of the Bauer–Ramanujan formula, and these subsections are more-or-less arranged in an appropriately chronological order.

2.1. *Bauer's Legendre polynomial-based proof*

One of the most basic families of orthogonal polynomials is given by the Legendre polynomials, which may be defined according to the generating function expansion such that

$$\frac{1}{\sqrt{1-2xt+t^2}} = \sum_{n=0}^{\infty} P_n(x)t^n. \tag{2}$$

The key to Bauer's 1859 proof (Bauer 1859) of (1) is given by the Fourier–Legendre expansion such that

$$\frac{1}{\sqrt{1-x^2}} = \frac{\pi}{2}\sum_{n=0}^{\infty}(4n+1)\frac{\binom{2n}{n}^2}{16^n}P_{2n}(x). \tag{3}$$

In particular, the desired result follows from (3) by setting $x=0$, according to the identity

$$P_{2n}(0) = (-1)^n\frac{\binom{2n}{n}}{4^n}. \tag{4}$$

There are many different ways of proving both (3) and (4), which nicely reflect the diversity of techniques involved in the subsequent proofs of the Bauer–Ramanujan formula, with reference to the below subsections. For example, Almkvist, in 2013, applied a different approach relative to Bauer to prove (3) (Almkvist 2013).

Mimicking Bauer's notation, we write

$$\frac{1}{\sqrt{1-a^2x^2}} = A_0P_0(x) + A_2P_2(x) + \cdots \tag{5}$$

for $a^2 \leq 1$ and for undetermined coefficients. As a consequence of the usual recurrence relation satisfied by Legendre polynomials, referring to Rainville's classic text (Rainville 1960, §10) for details, we have that

$$\frac{x}{\sqrt{1-a^2x^2}} = \sum_{n=0}^{\infty}\left(\frac{2n+1}{4n+1}A_{2n} + \frac{2n+2}{4n+5}A_{2n+2}\right)P_{2n+1}(x),$$

being consistent with the notation in (5). The desired evaluations for the scalar coefficients in (5) were then obtained by Bauer by comparing the coefficients in Fourier–

Legendre expansions of $\int_1^x tf(t)\,dt$ and of $(1-a^2x^2)f(x)$ for $f(x)=\frac{1}{\sqrt{1-a^2x^2}}$, referring to Bauer's original work and to the second author's generalization based on Bauer's proof for details (Levrie 2010).

As noted by Almkvist, the determination of the scalar coefficients in (5) for the $a=1$ case required to prove the Bauer–Ramanujan formula is equivalent to the evaluation whereby

$$\int_{-1}^{1}\frac{P_n(x)}{\sqrt{1-x^2}}\,dx=\begin{cases}\pi\frac{\binom{2m}{m}^2}{16^m} & \text{if} \quad n \text{ is even}, \quad \text{with } n=2m,\\ 0 & \text{if} \quad n \text{ is odd},\end{cases} \tag{6}$$

according to the orthogonality relations for Legendre polynomials (Almkvist 2013). Almkvist provided an alternative proof of (6) using a manipulation of the generating function in (2). Yet another way of proving (6) would be given by rewriting $P_n(x)$ within the integrand of (6) according to classically known finite binomial sum expansions for Legendre polynomials (Rainville 1960, §10), so as to obtain a finite hypergeometric identity that is routinely verifiable with the machinery of *Wilf–Zeilberger* (WZ) *proof certificates* (Petkovšek et al. 1996). This nicely connects with Zeilberger's famous computer proof of the Bauer–Ramanujan formula, which we review in Section 2.4 below. For further research related to Bauer's 1859 proof of (1), see Chan and Cooper (2012), Chu and Zhang (2014), Guo (2018) and Zudilin (2009b), for example.

Note that the series (1) appeared again in 1869 as Problem 929 in the *Nouvelles Annales de Mathématiques*, as a problem posed by Eugène Catalan, with the words: Prove this formula. In 1898 the same journal published Solutions to problems posed, and for Problem 929 no solution was given, only a reference to Bauer's paper (Catalan 1869/1898). The series can also be found in the classical treatise by Todhunter dating from 1875 (Todhunter 1875), with reference to Bauer. See also the work of Glaisher (1905) on series for $\frac{1}{\pi}$ and $\frac{1}{\pi^2}$. Glaisher mentions the series and refers to Todhunter. Bauer's series (1) is also listed in the book of mathematical formulas published by the Smithsonian Institute in 1922 (Adams and Hippisley 1922).

2.2. *Dougall's generalization and Hardy's proof*

Dougall's 1906 research paper concerning Vandermonde's Theorem (Dougall 1906) introduced an identity that provides an infinite family of generalizations of Bauer's 1859 formula, but it seems that Dougall was not aware of this. In particular, as noted by Ojanguren, with regard to Ojanguren's 2018 proof of the Bauer–Ramanujan formula reviewed below (Ojanguren 2018), Dougall introduced and proved the identity such that

$$\sum_{n=0}^{\infty}(-1)^n(2n+s)\frac{(s)_n^3}{(1)_n^3}=\frac{\sin(\pi s)}{\pi}, \tag{7}$$

with the $s=\frac{1}{2}$ case producing an equivalent form of (1).

We let the *Pochhammer symbol* be such that $(x)_0=1$ and such that $(x)_n=x(x+1)\cdots(x+n-1)$ for $n\in\mathbb{N}$, and we adopt the notational convention such that

$$\begin{bmatrix} \alpha, \beta, \dots, \gamma \\ A, B, \dots, C \end{bmatrix}_n = \frac{(\alpha)_n (\beta)_n \cdots (\gamma)_n}{(A)_n (B)_n \cdots (C)_n}. \tag{8}$$

By analogy with (8), we make use of the notational shorthand such that

$$\Gamma\begin{bmatrix} \alpha, \beta, \dots, \gamma \\ A, B, \dots, C \end{bmatrix} = \frac{\Gamma(\alpha)\Gamma(\beta)\cdots\Gamma(\gamma)}{\Gamma(A)\Gamma(B)\cdots\Gamma(C)},$$

recalling the definition of the Γ-function whereby $\Gamma(x) = \int_0^\infty u^{x-1} e^{-u}\, du$ for $\Re(x) > 0$. Dougall's formula reproduced in (7) was derived by setting $z \to \infty$ and by setting $y = -c$ in the identity (Dougall 1906) such that

$$\begin{aligned} \sum_{n=0}^{\infty} (-1)^n \frac{c+2n}{c} \frac{\prod_{i=0}^{n-1}(c+i)}{n!} \frac{\prod_{i=0}^{n-1}(x-i)}{\prod_{i=0}^{n-1}(x+c+1+i)} \\ \times \frac{\prod_{i=0}^{n-1}(y-i)}{\prod_{i=0}^{n-1}(y+c+1+i)} \frac{\prod_{i=0}^{n-1}(z-i)}{\prod_{i=0}^{n-1}(z+c+1+i)} \\ = \Gamma\begin{bmatrix} x+c+1, y+c+1, z+c+1, x+y+z+c+1 \\ y+z+c+1, z+x+c+1, x+y+c+1, c+1 \end{bmatrix}. \end{aligned} \tag{9}$$

An equivalent approach toward proving (1) via Dougall's identity was given by Hardy. In this regard, what we have referred to as the Bauer–Ramanujan formula is highlighted as Example 14 in Part II of Berndt's texts on Ramanujan's Notebooks (Berndt 1989, 23). As noted by (Berndt 1989, 24), Hardy introduced a proof of (1) as a direct consequence of a hypergeometric identity that may be referred to as *Dougall's theorem*, with regard to the relevant references given in Berndt (1989, 24), including Hardy's 1924 paper concerning Ramanujan's discoveries (Hardy 1924).

Generalized hypergeometric series (Bailey 1935) may be seen as being of fundamental importance in the application of special functions and are defined so that

$${}_pF_q\left[\begin{matrix} a_1, a_2, \dots, a_p \\ b_1, b_2, \dots, b_q \end{matrix} \,\middle|\, x\right] = \sum_{k=0}^{\infty} \begin{bmatrix} a_1, a_2, \dots, a_p \\ b_1, b_2, \dots, b_q \end{bmatrix}_k \frac{x^k}{k!}.$$

Following Berndt's text (Berndt 1989, 23–24), the Bauer–Ramanujan formula follows in a direct way from the identity such that

$${}_4F_3\left[\begin{matrix} \frac{1}{2}n+1, n, -x, -y \\ \frac{1}{2}n, x+n+1, y+n+1 \end{matrix} \,\middle|\, -1\right] = \Gamma\begin{bmatrix} x+n+1, y+n+1 \\ n+1, x+y+n+1 \end{bmatrix} \tag{10}$$

for $\Re(2x+2y+n+2) > 0$ (Berndt 1989, 16). The hypergeometric identity in (10), in turn, is obtained by setting $z \to \infty$ in the following ${}_5F_4$-identity due to Dougall. The following ${}_5F_4$-identity is equivalent to the hypergeometric identity in (9):

$$\begin{aligned} {}_5F_4&\left[\begin{matrix} \frac{1}{2}n+1,\, n,\, -x,\, -y,\, -z \\ \frac{1}{2}n,\, x+n+1,\, y+n+1,\, z+n+1 \end{matrix} \,\middle|\, 1 \right] \\ &= \Gamma\left[\begin{matrix} x+n+1,\, y+n+1,\, z+n+1,\, x+y+z+n+1 \\ n+1,\, x+y+n+1,\, y+z+n+1,\, x+z+n+1 \end{matrix}\right]. \end{aligned} \tag{11}$$

2.3. *Pi and the AGM*

Let the complete elliptic integrals of the first and second kinds be respectively defined so that

$$\mathbf{K}(k) := \int_0^{\pi/2} \frac{\mathrm{d}\theta}{\sqrt{1-k^2\sin^2\theta}} \quad \text{and} \quad \mathbf{E}(k) := \int_0^{\pi/2} \sqrt{1-k^2\sin^2\theta}\,\mathrm{d}\theta.$$

Ramanujan's series for $\frac{1}{\pi}$ are intimately associated with *elliptic integral singular values*, i.e. expressions of the forms $\mathbf{K}(k)$ and $\mathbf{E}(k)$ that admit explicit evaluations in terms of the Γ-function for special arguments k. In this regard, the Clausen hypergeometric product identity allows us to explicitly evaluate, in terms of $\mathbf{K}$ and $\mathbf{E}$, the generating function corresponding to the summand in (1). More explicitly, Clausen's identity gives us that

$$\sum_{n=0}^{\infty} \left[\begin{matrix} \frac{1}{2},\, \frac{1}{2},\, \frac{1}{2} \\ 1,\, 1,\, 1 \end{matrix}\right]_n x^n = \frac{4}{\pi^2}\mathbf{K}^2\left(\sqrt{\frac{1-\sqrt{1-x}}{2}}\right), \tag{12}$$

with the term-by-term derivatives of the summation in (12) yielding a combination of expressions involving $\mathbf{K}$ and $\mathbf{E}$.

A standard textbook reference concerning Ramanujan's series for $\frac{1}{\pi}$ is due to the Borwein brothers, as in their 1987 monograph on π and the AGM (Borwein and Borwein 1987). There may seem to be something of a 'gap' when it comes to the time period between Dougall's/Hardy's proof covered in Section 2.2 and the Borwein brothers' text; this seems to reflect how all of Ramanujan's series for $\frac{1}{\pi}$ were, famously, only formally and rigorously proved decades later by none other than the same Borwein brothers.

The Bauer–Ramanujan formula appears in (Borwein and Borwein 1987, 184) as a special case of an infinite family

$$\frac{1}{\pi} = \sum_{n=0}^{\infty} (-1)^n \left[\begin{matrix} \frac{1}{2},\, \frac{1}{2},\, \frac{1}{2} \\ 1,\, 1,\, 1 \end{matrix}\right]_n b_n(N)\left(g_N^{-12}\right)^{2n}, \tag{13}$$

writing

$$b_n(N) := \alpha(N)(k_N')^{-2} + n\sqrt{N}\left(\frac{1+k_N^2}{1-k_N^2}\right),$$

referring to the Borwein brothers' text for details. While the derivation of (13) requires modular forms associated with Ramanujan's g-invariant, the $N = 2$ case of (13) can be proved in a much more self-contained way, and provides a remarkably different proof compared to Bauer's proof reviewed in Section 2.1 and compared to Hardy's proof reviewed in Section 2.2.

Let

$$\mathbf{K}_s(k) := \frac{\pi}{2}\,{}_2F_1\left[\begin{matrix}\frac{1}{2}-s, \frac{1}{2}+s \\ 1\end{matrix} \,\middle|\, k^2\right]$$

denote the *generalized complete elliptic integral of the first kind* (Borwein and Borwein 1987, 178). A consequence of the hypergeometric product identity due to Clausen then gives us that

$$\left(\frac{2}{\pi}\mathbf{K}_s(h)\right)^2 = {}_3F_2\left[\begin{matrix}\frac{1}{2}-s, \frac{1}{2}+s, \frac{1}{2} \\ 1, 1\end{matrix} \,\middle|\, \left(2hh'\right)^2\right], \tag{14}$$

writing $h' := \sqrt{1-h^2}$ (Borwein and Borwein 1987, 178–179). The $s = 0$ case of (14) then gives us an equivalent version of the power series identity in (12). By applying the classical differential relation

$$\frac{\mathrm{d}\mathbf{K}}{\mathrm{d}k} = \frac{\mathbf{E} - (k')^2\mathbf{K}}{k(k')^2}$$

together with term-by-term differentiation applied to (12), we obtain

$$\begin{aligned}\sum_{n=0}^{\infty}\left[\begin{matrix}\frac{1}{2}, \frac{1}{2}, \frac{1}{2} \\ 1, 1, 1\end{matrix}\right]_n n\,x^n = {} & \frac{4\mathbf{E}\left(\sqrt{\frac{1-\sqrt{1-x}}{2}}\right)\mathbf{K}\left(\sqrt{\frac{1-\sqrt{1-x}}{2}}\right)}{\pi^2\sqrt{1-x}} \\ & - \frac{2\left(\sqrt{1-x}+1\right)\mathbf{K}^2\left(\sqrt{\frac{1-\sqrt{1-x}}{2}}\right)}{\pi^2\sqrt{1-x}}.\end{aligned} \tag{15}$$

By setting $x = -1$ in both (12) and (15), and by then applying the classically known modular relations

$$\mathbf{K}\left(i\frac{k}{k'}\right) = k'\mathbf{K}(k) \quad \text{and} \quad \mathbf{E}\left(i\frac{k}{k'}\right) = \frac{1}{k'}\mathbf{E}(k),$$

along with a known singular value for $\mathbf{K}(\sqrt{2}-1)$ (Borwein and Borwein 1987, 139, 298) together with a known relationship between singular values for $\mathbf{K}$ and singular values for $\mathbf{E}$ given by the elliptic alpha function (Borwein and Borwein 1987), this provides another proof of the Bauer–Ramanujan formula.

2.4. *Zeilberger's computer proof*

WZ theory is of great importance in terms of what is meant by a *computer proof* or a *computer-assisted proof*, with the development of WZ theory having been of a groundbreaking nature in the history of automated reasoning, in the history of experimental mathematics, and with respect to many other fields. For a comprehensive treatment as to background material concerning WZ theory and its applications, we refer the interested reader to the classic $A = B$ text (Petkovšek et al. 1996), and we proceed to consider how WZ theory has been applied in relation to Ramanujan's series for $\frac{1}{\pi}$. Notably, Zeilberger applied his EKHAD computer system to prove (1) via the WZ method in 1994 (Ekhad and Zeilberger 1994), and we review this WZ proof below.

A *Ramanujan-type series* is of the form

$$\frac{1}{\pi} = \sum_{n=0}^{\infty} \frac{\left(\frac{1}{2}\right)_n \left(\frac{1}{s}\right)_n \left(1 - \frac{1}{s}\right)_n}{(1)_n^3} z^n (a + bn), \tag{16}$$

for $s \in \{2, 3, 4, 6\}$ and for parameters a, b, and z that are real and algebraic. Especially notable developments in the study of Ramanujan-type series and related series via WZ theory are due to Guillera, again with reference to the survey on Ramanujan's series for $\frac{1}{\pi}$ given by Baruah et al. (2009). In this regard, many of Guillera's research contributions (Guillera 2006, 2016, 2010, 2013) may be seen as influenced by or otherwise closely related to Zeilberger's WZ proof of (1). This is representative of the importance of Zeilberger's WZ proof in the history of computer proofs, and motivates our review of it below.

Zeilberger proved, via the WZ method, the following infinite family of generalizations of the Bauer–Ramanujan formula, with the $n = -\frac{1}{2}$ case yielding the Bauer–Ramanujan formula:

$$\Gamma\left[\begin{matrix} n + \frac{3}{2} \\ \frac{3}{2}, n + 1 \end{matrix}\right] = \sum_{k=0}^{\infty} (-1)^k (4k + 1) \left[\begin{matrix} \frac{1}{2}, \frac{1}{2}, -n \\ 1, 1, n + \frac{3}{2} \end{matrix}\right]_k. \tag{17}$$

For the time being, we let $n \in \mathbb{N}_0$. For such values, the purported infinite series identity in (17) is equivalent to the corresponding identity obtained by restricting the indices of the series in (17) so that $k \in \{0, 1, \ldots, n\}$. By letting $F(n, k)$ denote the summand of (17) divided by the left-hand side of (17), Zeilberger, using the EKHAD computer system, produced the computer-generated function

$$G(n, k) := \frac{(2k + 1)^2}{(2n + 2k + 3)(4k + 1)} F(n, k)$$

according to the WZ method, so as to form the difference equation

$$F(n + 1, k) - F(n, k) = G(n, k) - G(n, k - 1). \tag{18}$$

We find that the right-hand side of (18) telescopes under the application of summation operators with respect to k, so that $\sum_k F(n, k)$ is constant, with $\sum_k F(n, k) = 1$ from the $n = 0$ case. By rewriting $\sum_k F(n, k) = 1$ as $\sum_{k=0}^{n} F(n, k) = 1$, and by rewriting this latter equality as $\sum_{k=0}^{\infty} F(n, k) = 1$, an application of Carlson's theorem (Almkvist

2013, 20–21) gives us that this final equality also holds for real values. So, by setting $n = -\frac{1}{2}$, we obtain the desired result.

2.5. *An FL-based method due to Baranov*

What appears to be an overlooked or forgotten FL-based proof of the Bauer–Ramanujan formula was given in a 2006 article by Baranov (2006). An incorrect version of Baranov's formula

$$\sum_{n=0}^{\infty} \frac{(-1)^n(4n+1)((2n-1)!!)^3}{2^{3n}(n!)^3} P_{2n}(\cos\theta) = \frac{4\mathbf{K}(\sin\theta)}{\pi^2} \tag{19}$$

was given in Gradshtein and Ryzhik's classic text (Gradshtein and Ryzhik 1994, 1043), referring via (Adams and Hippisley 1922) to a paper by Hargreaves (1898, 91). We find that the FL expansion in (19) provides an infinite family of generalizations of the Bauer–Ramanujan formula, by considering the $\theta = 0$ case.

As described by Baranov, the FL expansion in (19) may be derived by integrating both sides of the equality

$$\begin{aligned} &\pi \sum_{k=0}^{\infty} (2k+1) P_k(x) P_k(y) P_k(z) \\ &\quad = 2\big(\cos(\varphi - \psi) - \cos\lambda\big)^{-1/2}\big(\cos\lambda - \cos(\varphi+\psi)\big)^{-1/2} \end{aligned} \tag{20}$$

for $\varphi - \psi < \lambda < \min\{\varphi+\psi, 2\pi - \varphi - \psi\}$, writing $x = \cos\lambda$ and $y = \cos\varphi$ and $z = \cos\psi$. As noted by Baranov, the $x = y = 0$ case of (20) yields an equivalent version of the classical FL expansion in (3), recalling that (3) was the key to Bauer's original proof of what we refer to as the Bauer–Ramanujan formula. This is representative of 'bookends' that are formed in the course of the history of proofs of the Bauer–Ramanujan formula, and seems to touch upon an interdisciplinarity suggested by the dramatically different proofs of this formula that we review.

The triple Legendre polynomial identity in (20) was obtained by Baranov by setting $\mu = \lambda$ in

$$T(\mu) = \frac{2}{\pi} \frac{\sin((\lambda+\mu)/2)}{\sqrt{(\cos(\varphi-\psi) - \cos\mu)(\cos\mu - \cos(\varphi+\psi))}} \sqrt{\frac{\sin\mu}{\sin^3\lambda}},$$

where the operator T satisfies

$$\int_{\varphi-\psi}^{\varphi+\psi} \frac{\sin((n+1)(\mu-\lambda))}{\mu-\lambda} T(\mu)\,\mathrm{d}\mu = T(\lambda) \int_{\varphi-\psi}^{\varphi+\psi} \frac{\sin((n+1)(\mu-\lambda))}{\mu-\lambda}\,\mathrm{d}\mu,$$

referring to Baranov's work (Baranov 2006) for details.

2.6. *An Eisenstein series-based proof due to Baruah and Berndt*

Baruah and Berndt, in 2010, introduced and proved a remarkable variety of Ramanujan-type series for $\frac{1}{\pi}$ (Baruah and Berndt 2010). Baruah and Berndt's Eisenstein-based techniques (Baruah and Berndt 2010) may be seen as providing an historically notable advancement in the study of Ramanujan-type series. The foregoing considerations motivate the historical interest surrounding the Eisenstein series-based techniques applied (Baruah and Berndt 2010) to prove the Bauer–Ramanujan formula.

A key to Baruah and Berndt's derivation of the Bauer–Ramanujan formula is given by the equalities

$$\begin{aligned} P\left(e^{-2\pi\sqrt{2}}\right) &= \frac{3}{\pi\sqrt{2}} + \frac{\sqrt{2}-1}{\sqrt{2}(1-x_2)} \sum_{k=0}^{\infty} (-1)^k \frac{\left(\frac{1}{2}\right)_k^3}{(1)_k^3} Y_2^k \\ &= \frac{3}{\pi\sqrt{2}} + \frac{1}{2\sqrt{2}} \sum_{k=0}^{\infty} (-1)^k \frac{\left(\frac{1}{2}\right)_k^3}{(1)_k^3}, \end{aligned} \tag{21}$$

where Eisenstein series-related notation involved in (21) may be given as follows (Baruah and Berndt 2010). Ramanujan's Eisenstein series may be defined so that

$$P(q) := 1 - 24 \sum_{k=1}^{\infty} \frac{kq^k}{1-q^k}$$

for $|q| < 1$. *Singular moduli* are denoted so that $x_n := x(e^{-\pi\sqrt{n}})$, where $x = x(q)$ is related to q according to one of the most important relationships in the study of elliptic functions, namely:

$$\phi^2(q) = {}_2F_1\left[\begin{matrix} \frac{1}{2}, \frac{1}{2} \\ 1 \end{matrix} \,\middle|\, x\right], \tag{22}$$

where

$$\phi(q) := \sum_{n=-\infty}^{\infty} q^{n^2}.$$

With regard to the notation in (21), we also set $Y_n := \frac{4x_n}{(1-x_n)^2}$.

Baruah and Berndt's derivation of the Bauer–Ramanujan formula is given by a combination of (21) and the $n = 2$ case of

$$P\left(e^{-2\pi\sqrt{n}}\right) = \frac{1+x_n}{1-x_n} \sum_{k=0}^{\infty} (3k+1)(-1)^k \frac{\left(\frac{1}{2}\right)_k^3}{(1)_k^3} Y_n^k. \tag{23}$$

Both (21) and (23) are derived via the relation

$$z^2 = \frac{1}{1-x} \sum_{k=0}^{\infty} (-1)^k \frac{\left(\frac{1}{2}\right)_k^3}{(1)_k^3} Y^k$$

for $0 \leq x \leq 3 - 2\sqrt{2}$, writing $z := z(q) = \phi^2(q)$, $Y := \frac{4x}{(1-x)^2}$ and referring to Baruah and Berndt's work (Baruah and Berndt 2010) for details.

2.7. *A translation method due to Aycock*

In a 2013 paper, Aycock applied a so-called *translation method* via Pfaff's classical transform

$$ {}_2F_1\left[\begin{matrix}\alpha, \beta \\ \gamma\end{matrix} \,\middle|\, x\right] = (1-x)^{-\beta}\, {}_2F_1\left[\begin{matrix}\gamma-\alpha, \beta \\ \gamma\end{matrix} \,\middle|\, \frac{x}{x-1}\right] \tag{24} $$

to formulate a new proof of the Bauer–Ramanujan formula (Aycock, unpublished). The classical identity in (24) together with the above referenced Clausen hypergeometric product identity give us that

$$ \left({}_2F_1\left[\begin{matrix}\frac{1}{4}, \frac{3}{4} \\ 1\end{matrix} \,\middle|\, x\right]\right)^2 = (1-x)^{-\frac{1}{2}}\, {}_2F_1\left[\begin{matrix}\frac{1}{2}, \frac{1}{2}, \frac{1}{2} \\ 1, 1\end{matrix} \,\middle|\, \frac{x}{x-1}\right]. \tag{25} $$

By applying the operator $x\frac{x}{\mathrm{d}x}\cdot$ to both sides of the equality in (25), we obtain

$$ \begin{aligned} &\sum_{n=0}^{\infty} n \sum_{k=0}^{n} \frac{\left(\frac{1}{4}\right)_k\left(\frac{3}{4}\right)_k}{(1)_k} \frac{\left(\frac{1}{4}\right)_{n-k}\left(\frac{3}{4}\right)_{n-k}}{(1)_{n-k}} \frac{x^n}{n!(n-k)!} \\ &\quad = (1-x)^{-\frac{3}{2}}\left(\frac{x}{2}\sum_{n=0}^{\infty} a_n\left(\frac{x}{x-1}\right)^n + \sum_{n=0}^{\infty} n a_n\left(\frac{x}{x-1}\right)^2\right), \end{aligned} $$

adopting Aycock's notation whereby $a_n = \frac{\left(\frac{1}{2}\right)_n^3}{(1)_n^3}$. Setting $x = \frac{1}{2}$ and $s = \frac{1}{4}$, we obtain

$$ \frac{2\sin\frac{\pi}{4}}{\pi} = \frac{\sqrt{2}}{2}\sum_{n=0}^{\infty} \frac{\left(\frac{1}{2}\right)_n^3}{(1)_n^3}(4n+1)(-1)^n, $$

according to the 'translation' identity

$$ \sum_{n=0}^{\infty}\sum_{k=0}^{n} \frac{(s)_k(1-s)_k(s)_{n-k}(1-s)_{n-k}}{(1)_k(1)_{n-k}} \frac{n\left(\frac{1}{2}\right)^n}{n!(n-k)!} = \frac{2\sin s\pi}{\pi} $$

proved by Aycock, where, informally, the notion of 'translation' comes from the application of $x\frac{\mathrm{d}}{\mathrm{d}x}\cdot$ to a classical hypergeometric identity, with the effect of $x\frac{\mathrm{d}}{\mathrm{d}x}\cdot$ yielding a reindexing shift.

2.8. *A translation method due to Cooper*

In Cooper's 2017 text (Cooper 2017) on Ramanujan's theta functions, Cooper provided a 'translation' method to obtain another original proof of the Bauer–Ramanujan formula. Starting with the hypergeometric transform such that

$$ {}_3F_2\left[\begin{matrix}\frac{1}{4}, \frac{1}{2}, \frac{3}{4} \\ 1, 1\end{matrix} \,\middle|\, X\right] = \sqrt{1-x}\, {}_3F_2\left[\begin{matrix}\frac{1}{2}, \frac{1}{2}, \frac{1}{2} \\ 1, 1\end{matrix} \,\middle|\, x\right] \tag{26} $$

for $X = \frac{-4x}{(1-x)^2}$, the application of the operator $X\sqrt{1-X}\frac{\mathrm{d}}{\mathrm{d}X}\cdot = x\frac{\mathrm{d}}{\mathrm{d}x}\cdot$ to both sides of

(26) yields the following (Cooper 2017, 626):

$$\sqrt{1-X}\sum_{n=0}^{\infty}\frac{\left(\frac{1}{4}\right)_n\left(\frac{1}{2}\right)_n\left(\frac{3}{4}\right)_n}{(1)_n^3}nX^n$$
$$=\sqrt{1-x}\sum_{n=0}^{\infty}\frac{\left(\frac{1}{2}\right)_n^3}{(1)_n^3}nx^n-\frac{x}{2\sqrt{1-x}}\sum_{n=0}^{\infty}\frac{\left(\frac{1}{2}\right)_n^3}{(1)_n^3}x^n.$$

Following Cooper, we then set $x \to -1^+$ so that $X \to 1^-$, so that the left-hand limit can be explicitly determined using Cooper's *translation method* given by the following result:

$$\lim_{x\to 1^-}\sqrt{1-x}\sum_{n=0}^{\infty}\frac{\left(\frac{1}{2}\right)_n(s)_n(1-s)_n}{(1)_n^3}nx^n=\frac{\sin \pi s}{\pi}.$$

The required right-hand limit can be evaluated in a direct way, yielding an equivalent version of the Bauer–Ramanujan formula (Cooper 2017, 627).

2.9. *A telescoping approach*

A 2018 research paper by Nimbran and the second author provided a new proof of the Bauer–Ramanujan formula as a main result (Levrie and Nimbran 2018). In this paper the authors show that the Bauer–Ramanujan formula is an immediate consequence of the Wallis product formula for π which can be rewritten as a series:

$$\frac{2}{\pi}=\frac{1\cdot 3}{2\cdot 2}\cdot\frac{3\cdot 5}{4\cdot 4}\cdot\frac{5\cdot 7}{6\cdot 6}\cdots=1-\frac{1}{4}\sum_{n=0}^{\infty}\frac{\left(\frac{1}{2}\right)_n\left(\frac{3}{2}\right)_n}{(2)_n^2}. \tag{27}$$

Using a telescoping technique, the Bauer–Ramanujan result follows.

We define

$$v(a,b)=\sum_{n=0}^{\infty}\frac{(a)_n(b)_n}{(a+b)_n^2}. \tag{28}$$

To use the telescoping effect, we note that the general term of the series at the RHS satisfies:

$$(a+b-1)\frac{(a)_n(b)_n}{(a+b)_n^2}=\underbrace{\frac{(a)_{n-1}(b)_n}{(a+b)_{n-1}^2}}_{=A_{n-1}}-\underbrace{\frac{(a)_n(b)_{n+1}}{(a+b)_n^2}}_{=A_n}-b^2\frac{(a)_{n-1}(b)_n}{(a+b)_n^2}$$

as can easily be verified. Using this and telescoping we can now rewrite (28) as:

$$\begin{aligned} v(a, b) &= 1 + \sum_{n=1}^{\infty} \frac{(a)_n (b)_n}{(a+b)_n^2} \\ &= 1 + \frac{1}{a+b-1}\left(A_0 - b^2 \sum_{n=1}^{\infty} \frac{(a)_{n-1}(b)_n}{(a+b)_n^2}\right) \quad \text{with } A_0 = b \\ &= \frac{a+2b-1}{a+b-1} - \frac{b^2}{a+b-1}\sum_{n=0}^{\infty} \frac{(a)_n (b)_{n+1}}{(a+b)_{n+1}^2} \end{aligned}$$

or, since $(b)_{n+1} = b \cdot (b+1)_n$ and $(a+b)_{n+1} = (a+b) \cdot (a+b+1)_n$, as:

$$v(a, b) = \frac{a+2b-1}{a+b-1} - \frac{b^3}{(a+b)^2(a+b-1)} v(a, b+1).$$

Through the repeated application of this recurrence, starting with $a = \frac{1}{2}$ and $b = \frac{3}{2}$ (Levrie and Nimbran 2018), we obtain the Bauer–Ramanujan formula as an immediate consequence of (27):

$$\frac{2}{\pi} = 1 - \frac{1}{4} v\left(\frac{1}{2}, \frac{3}{2}\right).$$

2.10. *Gosper's acceleration method*

A series acceleration method due to Gosper (1974) was applied by Ojanguren (2018) to obtain another proof of the Bauer–Ramanujan formula. Informally, Gosper's method is given by 'splitting' every term in a series into two new terms and by then joining the latter term derived from the nth term in the original series with the former term derived from the $(n+1)$th term from the original series.

Ojanguren obtained a proof of Dougall's generalization in (7) using a version of Gosper's series rearrangement method. A slight modification of the method used simplifies the proof. Adapting Ojanguren's work, we set $E_0 = 1$ and

$$E_n = \frac{s(1-s)(1+s)(2-s)\cdots(n-1+s)(n-s)(n+s)}{n!^2}$$

for positive integers n, so that

$$\lim_{n\to\infty} E_n = \frac{\sin \pi s}{\pi} \tag{29}$$

using Euler's product formula for sin. Gosper's method is as follows. We start with a series of the form

$$A_0^{(0)} + A_1^{(0)} + A_2^{(0)} + A_3^{(0)} + \cdots + A_n^{(0)} + \cdots.$$

Note that we can rewrite this series as a new series in the following manner:

$$\sum_{n=0}^{\infty} A_n^{(0)} = \underbrace{A_0 S_0^{(0)}}_{=B_0} + \underbrace{\left[(1 - S_0^{(0)})A_0^{(0)} + S_1^{(0)} A_1^{(0)}\right]}_{=A_0^{(1)}} + \underbrace{\left[(1 - S_1^{(0)})A_1^{(0)} + S_2^{(0)} A_2^{(0)}\right]}_{=A_1^{(1)}} + \cdots + \underbrace{\left[(1 - S_n^{(0)})A_n^{(0)} + S_{n+1}^{(0)} A_{n+1}^{(0)}\right]}_{=A_n^{(1)}} + \cdots$$

and this new series will converge faster than the original one if we choose the sequence of numbers $S_n^{(0)}$ in such a way that

$$\lim_{n\to\infty} \frac{A_n^{(1)}}{A_n^{(0)}} = \lim_{n\to\infty} \frac{(1 - S_n^{(0)})A_n^{(0)} + S_{n+1}^{(0)} A_{n+1}^{(0)}}{A_n^{(0)}} = 0.$$

We can then repeat the process, accelerating in a similar way the new series

$$A_0^{(1)} + A_1^{(1)} + A_2^{(1)} + A_3^{(1)} + \cdots + A_n^{(1)} + \cdots$$

using a sequence of well chosen numbers $S_n^{(1)}$ leading to

$$\sum_{n=0}^{\infty} A_n^{(0)} = B_0 + \sum_{n=0}^{\infty} A_n^{(1)} = B_0 + B_1 + \sum_{n=0}^{\infty} A_n^{(2)} = \cdots = \sum_{n=0}^{\infty} B_n.$$

Defining

$$R_n^{(k)} = \frac{A_{n+1}^{(k)}}{A_n^{(k)}} \quad \text{and} \quad U_n^{(k)} = 1 - S_n^{(k)} + S_{n+1}^{(k)} R_n^{(k)}$$

we have to choose $S_n^{(k)}$ in such a way that $\lim_{n\to\infty} U_n^{(k)} = 0$, if we want to accelerate the convergence of the series. Note that by the choices made we have that

$$U_n^{(k)} = \frac{A_n^{(k+1)}}{A_n^{(k)}} \quad \text{and} \quad R_n^{(k+1)} = \frac{U_{n+1}^{(k)}}{U_n^{(k)}} R_n^{(k)}.$$

Furthermore

$$B_0 = A_0^{(0)} S_0^{(0)}, \quad B_n = A_0^{(0)} U_0^{(0)} \cdots U_0^{(n-1)} S_0^{(n)} \ (n \geq 1).$$

Now, following Ojanguren, we start with (29) and make it into a series:

$$\frac{\sin \pi s}{\pi} = E_0 + \underbrace{(E_1 - E_0)}_{=A_0^{(0)}} + \underbrace{(E_2 - E_1)}_{=A_1^{(0)}} + \cdots + \underbrace{(E_{n+1} - E_n)}_{=A_n^{(0)}} + \cdots$$

with $E_0 = s$ and general term

$$E_{n+1} - E_n = -s^2 \cdot \frac{(s)_{n+1}(1-s)_n}{(2)_n^2}.$$

Note that

$$R_n^{(0)} = \frac{(n+1+s)(n+1-s)}{(n+2)^2}.$$

If we choose, as in the work of Ojanguren (2018), the value $S_n^{(k)} = an + b$, with a and b depending on k, and

$$R_n^{(k)} = \frac{(n+1+s+k)(n+1-s)}{(n+2+k)^2} \quad (k \geq 1),$$

we have to take $a = \frac{1}{k+1}$ and $b = \frac{2k+s+2}{k+1}$, and hence

$$S_k^{(n)} = \frac{n}{k+1} + \frac{2k+s+2}{k+1}.$$

This means that

$$U_n^{(k)} = -\frac{(k+s+1)^2(n+1+s+k)}{(k+1)(n+2+k)^2}$$

and

$$B_{n-1} = (-1)^n \frac{(s)_n^3(2n+s)}{n!^3} \quad (n \geq 1).$$

We conclude that Dougall's formula (7) holds true:

$$\frac{\sin \pi s}{\pi} = s + \sum_{n=1}^{\infty} B_{n-1} \Rightarrow \frac{\sin \pi s}{\pi} = \sum_{n=0}^{\infty} (-1)^n \frac{(s)_n^3(2n+s)}{n!^3}.$$

Taking $s = \frac{1}{2}$, we obtain the Bauer–Ramanujan series.

2.11. *An inverse series relation due to Mishev*

The first author has applied an inverse series-based identity due to Mishev (2018) so as to obtain a new proof of the Bauer–Ramanujan identity (Campbell to appear). A key to Mishev's inverse series relation identity is given by *Dixon's theorem*. More specifically, the following terminating version of Dixon's theorem (Bailey 1935, 13) is required to obtain Mishev's expansion for the operator L_a:

$${}_3F_2\left[\begin{matrix} a,\, b,\, -n \\ 1+a-b,\, 1+a+n \end{matrix} \,\middle|\, 1 \right] = \left[\begin{matrix} 1+a,\, 1+\frac{a}{2}-b \\ 1+\frac{a}{2},\, 1+a-b \end{matrix} \right]_n.$$

Mishev's inverse series relation may be formulated in the following way (Campbell to appear).

Theorem 2.1 (Mishev 2018) *Let* $(x_n{:}n \in \mathbb{N}_0)$ *be a sequence of complex numbers, and let* $a \in \mathbb{C} \setminus \{0, -1, -2, \dots\}$. *The equality*

$$x_n = \frac{1}{n!(a+1)_n} \sum_{k=0}^{n} \left[\begin{matrix} a, 1+\frac{a}{2}, -n \\ 1, \frac{a}{2}, 1+a+n \end{matrix} \right]_k \sum_{\ell=0}^{k} (-k)_\ell (k+a)_\ell x_\ell \tag{30}$$

holds true.

We set $x_\ell = \frac{1}{(\ell!)^2}$ in Theorem 2.1, so that Gauss' ${}_2F_1(1)$-identity

$${}_2F_1\left[\begin{matrix} a, b \\ c \end{matrix} \,\middle|\, 1 \right] = \Gamma\left[\begin{matrix} c, c-a-b \\ c-a, c-b \end{matrix} \right]$$

gives us that

$${}_4F_3\left[\begin{matrix} a, a, -n, 1+\frac{a}{2} \\ 1, \frac{a}{2}, n+a+1 \end{matrix} \,\middle|\, -1 \right] = \frac{(a+1)_n}{n!} \tag{31}$$

for the cases whereby n is a nonnegative integer (Campbell to appear). Through an application of Carlson's theorem, we find that (31) holds for complex values including $n = -\frac{1}{2}$. So, by setting $a = \frac{1}{2}$ and $n = -\frac{1}{2}$, we obtain an equivalent form of the Bauer–Ramanujan formula.

3. Historical analyses and perspectives

One of the 'oldest tricks in the book' in the field of classical analysis is given by the interchange of limiting operations, and we argue, as below, that the history of the Bauer–Ramanujan formula is representative of this and of how modern and computer-based approaches toward summation problems differ in a paradigmatic way. We argue that the history of the Bauer–Ramanujan formula may be used to illustrate a paradigm shift in the history of mathematics given by the advent of computer proofs.

The study of Legendre polynomials and expansions in terms of Legendre polynomials may be regarded as having been a popular topic within areas of mathematical analysis in the 19th century, and Bauer's proof reviewed above is representative of this. Another popular topic in this time period and in the early 20th century is given by the development of generalized hypergeometric functions, and the proof chronologically following Bauer's original proof, as reviewed above, is representative of this. Another popular topic, again in the history of mathematical analysis in the time periods under consideration, was given by the acceleration of the convergence of series, with particular reference to the work of Kummer (1837) and of Markov in 1890 (Kondratieva and Sadov 2002). The advent of computers triggered Bill Gosper to write his paper about convergence acceleration (Gosper 1974). The Borwein brothers' text on Pi and the AGM (Borwein and Borwein 1987) brought Ramanujan-type series back to attention. The WZ proof by Zeilberger of the Bauer–Ramanujan formula then paved the way toward the modern era development of Ramanujan-type series, as in the work of Guillera considered below.

The proofs in our history can be divided into different types:

- Legendre-related: Sections 2.1/2.5;
- Generalized hypergeometric functions: Sections 2.2/2.3/2.6/2.7/2.8/2.11;
- Convergence acceleration of series: Sections 2.9/2.10; and
- WZ: Section 2.4.

Recall the key to Bauer's original 1859 proof, as given by the identity in (3). When dealing with identities involving infinite sums of orthogonal polynomials, the interchanging of limiting operations is typically involved in a meaningful way, even at an implicit or 'hidden' level. In this regard, an interchange of this form is implicit in the derivation of (3).

We see that (3) is equivalent to

$$\frac{P_m(x)}{\sqrt{1-x^2}} = \frac{\pi}{2}\sum_{n=0}^{\infty}(4n+1)\frac{\binom{2n}{n}^2}{16^n}P_{2n}(x)P_m(x)$$

for a given Legendre polynomial $P_m(x)$. So, as suggested in (6), the derivation of the key to Bauer's original 1859 proof relies on determining scalar coefficients a_n such that

$$\frac{P_m(x)}{\sqrt{1-x^2}} = \sum_{n=0}^{\infty} a_n P_n(x)P_m(x), \tag{32}$$

and thus an interchange of limiting operations naturally arises, in the following manner. To 'isolate' the coefficient a_n for the $m = n$ case, we apply the integral operator $\int_{-1}^{1} \cdot \, \mathrm{d}x$ to both sides of (32), yielding

$$\int_{-1}^{1} \frac{P_m(x)}{\sqrt{1-x^2}}\,\mathrm{d}x = \int_{-1}^{1}\left(\sum_{n=0}^{\infty} a_n P_n(x)P_m(x)\right)\mathrm{d}x, \tag{33}$$

and we would proceed to argue that the limiting operators $\int_{-1}^{1} \cdot \, \mathrm{d}x$ and $\sum_{n=0}^{\infty} \cdot$ may be reversed, so as to apply the orthogonality relations for Legendre polynomials to isolate a_m and to use known formulas for $P_m(x)$ to evaluate the integral on the left of (33). The interchange of integration and infinite summation operations may be seen as being of even central importance within classical analysis, real analysis, and related areas such as measure theory, and this is evidenced by the importance of the Dominated Convergence Theorem and the Monotone Convergence Theorem in these disciplines. If we compare this to much later proofs of the Bauer–Ramanujan formula as in Zeilberger's computer proof, then this supports our argument that the history of the Bauer–Ramanujan formula is representative of historical trends in the history of mathematics more broadly.

We see that the interchange of limiting operations again plays a key role in the next classical proof of the Bauer–Ramanujan formula covered in our survey. In Hardy's proof via Dougall's ${}_5F_4$-identity, the interchange of $\lim_{z\to\infty} \cdot$ and an infinite summation operator corresponding to the ${}_5F_4$-series in (11) provides the key step.

Jumping ahead in history to the Borwein brothers' work on Ramanujan-type series, we again find that the interchange of limiting operations plays a key role, as in the term-by-term application of a differential operator to an infinite series indicated in (15). This leads us to the subsequent proof covered in Section 2.4, which, as considered below, is in stark contrast to the preceding proofs, due to its innovative use of telescoping and due to its computer-based nature.

The history surrounding the Bauer–Ramanujan formula and how the proofs of this formula relate to one another both mathematically and historically may be seen as being indicative of important aspects of and developments within both classical analysis and experimental mathematics. With regard to this former area, the interchange of limiting operations may be seen as being central within virtually every area of mathematical analysis. Our above account of the history of the earlier proofs of the Bauer–Ramanujan formula shows how interchanging limiting operations is of basic usefulness throughout the history of analysis. Indeed, it is difficult to produce an analysis-based proof of the Bauer–Ramanujan formula *without* the use of the interchange of limiting operations. This reflects the radically different nature of Zeilberger's computer proof relying on a discrete and telescoping-based approach, and this also reflects how the history of the Bauer–Ramanujan formula may be seen as 'capturing' broader aspects in the history of mathematics, as in a microcosm.

Observe the many different limiting processes being switched in the classical proofs of the Bauer–Ramanujan formula, reflecting the fundamentality and versatility of the interchange of limiting processes as a tool in the discipline of mathematical analysis and how the history of the Bauer–Ramanujan identity is representative of this. In our formulation of Bauer's 1859 proof, an operator of the form $\int_{-1}^{1} \cdot \,\mathrm{d}x$ is interchanged with an operator of the form $\sum_{n=0}^{\infty} \cdot$ with the use of orthogonal polynomials, so that the Bauer–Ramanujan formula may be seen as an instantiation of the historically important role of orthogonal polynomials within areas of mathematics such as special functions theory. As above, the exchange of the order of $\lim_{z\to\infty}$ and an infinite summation operator is of key importance in Hardy's proof, and the exchange of the differential operator $\frac{\mathrm{d}}{\mathrm{d}x} \cdot$ and $\sum_{n=0}^{\infty} \cdot$ is required in our formulation of the Borwein brothers' proof. So, for the three cases we have considered, an infinite summation operator is interchanged with three different limiting operators: an integration operator, a limiting operation given by setting a variable to approach ∞, and a differential operator. Again, this shows how the history of the Bauer–Ramanujan formula may be seen as a microcosm of the variety of techniques arising in classical analysis.

An historically significant aspect about the proof reviewed in Section 2.4 is given by its inherently 'finite' or discrete nature given by how this proof can be thought of as reducing the Bauer–Ramanujan formula to a finite sum proved via a computer-generated discrete difference equation. Although Carlson's theorem is required to generalize the discrete identity $\sum_{k=0}^{n} F(n, k)$ so as to allow for a non-integer argument n, this may be seen as something of a formality: Zeilberger's proof is non-analytic in the sense that interchanges of limiting operations are not required and in the sense that it ultimately relies on the use of telescoping via a discrete difference equation. We argue, as below, that Zeilberger's proof is representative of broader or 'bigger picture' changes when it comes to a paradigm shift in the nature of mathematical proofs over the centuries: Notably, this is representative of the importance of the advent of computer proofs and computer-based proofs and computer-assisted proofs in the history of mathematics.

If we consider the time span covering the history of proofs of the Bauer–Ramanujan formula, dating from 1859 to the present as reviewed in Sections 1 and 2, the historical 'gap' suggested above between the classical analysis-based proofs from 1859–1924 and Zeilberger's computer proof in 1994 may be seen as indicative of a paradigm shift in the history of mathematical proofs if we consider the given time periods in relation to what is usually considered as the first major result to have been initially proved via a computer-assisted proof, namely, the famous four colour theorem, as proved in 1976 by Kenneth Appel and Wolfgang Haken. The famous controversy resulting from the publication of this computer-assisted proof (Swart 1980) is in contrast to how this proof is now widely accepted by the mathematical, scientific, and philosophical communities (Gonthier 2008), and this is representative of its innovative and groundbreaking nature. The foregoing considerations illustrate how Zeilberger's computer proof may be seen as being of a similarly groundbreaking nature in the history of computer proofs.

Another way of illustrating how Zeilberger's computer proof is representative of a paradigm shift is given by how this computer proof is closely related to an important development in the legacy of Ramanujan's discoveries: The application of computer-based methods, especially Wilf–Zeilberger-based methods, in the determination of Ramanujan-type series. In this regard, the computer-based discoveries due to Guillera, via the Wilf–Zeilberger method, concerning Ramanujan-type series may be seen as especially notable in the history of Ramanujan-type series, and this is evidenced by numerous mainstream news items concerning Guillera's discoveries on π formulas (Ansede 2015; Cotera 2015; Población 2019; Sacristán 2016). Zeilberger's computer proof of the Bauer–Ramanujan formula appears to be the first 'purely' computer proof or computer-generated proof of a Ramanujan-type series expansion for $\frac{1}{\pi}$, and hence its groundbreaking nature, in view of the subsequent history of computer-based derivations of Ramanujan-type series formulas.

Computer-assisted discoveries and computer-assisted proofs may be seen as being at the core of what is meant by the discipline of *experimental mathematics*, with reference to a standard text on experimental mathematics (Borwein et al. 2004). The primary author of this textbook, Jonathan Borwein, has been described as a *Renaissance Mathematician* in a Special Issue of a journal of the Mathematical Association of America (Bailey 2021), with explicit reference to the pioneering nature of Jonathan Borwein's development of the field of experimental mathematics:

> …Borwein did notable research in a wide range of fields, ranging from experimental mathematics (in which he can rightly be regarded as a pioneer and leading exponent) and optimization to biomedical imaging, mathematical finance, and computer science (Bailey 2021, 773).

In a presentation on *Computer-assisted Discovery and Proof* by Jonathan Borwein, the 'Wilf–Zeilberger algorithm for proving summation identities' was highlighted as one of the main instances of an algorithm involved in experimental mathematics (Borwein 2006). This provides a further way of arguing based on the historical significance of Zeilberger's WZ proof of the Bauer–Ramanujan formula, since this 1994 proof is one of the first published WZ proofs in history, and since the WZ method is regarded as a main instance of a method involved in the burgeoning field of experimental mathematics. The renaissance-like nature associated with the period in the history of mathematics involving the development of experimental mathematics is representative of how computer-assisted proofs provide a foundation for a paradigm shift in mathematics and beyond (Zeilberger 2005). More generally, Artificial Intelligence is widely recognized as being a basis of a paradigm shift (Cristianini 2014).

4. Conclusion

The historical analyses given above concerning the Bauer–Ramanujan formula may be seen as forming something of a foundation for a future study based on variants and generalizations of the Bauer–Ramanujan formula. In this regard, our explorations based on Mishev's theorem, as reproduced in Theorem 2.1, have led us to discover the following Bauer–Ramanujan-type formula:

$$\frac{5}{6}\sqrt[3]{2} = \sum_{n=0}^{\infty} (-1)^n \begin{bmatrix} -\frac{1}{3}, \frac{1}{6}, \frac{1}{2} \\ 1, \frac{4}{3}, \frac{11}{6} \end{bmatrix}_n (4n+1). \tag{34}$$

The mysterious nature of the series in (34) is evidenced by how its partial sums are not evaluable in closed form. With regard to the classical proof of the Bauer–Ramanujan formula via Fourier–Legendre (FL) expansions reviewed in Section 2.1, it seems that it would not be possible to use a similar approach in the hope of proving (34), since there do not seem to be any known elementary functions that admit FL expansions with combinations of Pochhammer symbols as in (34). Remarkably, the Maple Computer Algebra System (CAS) actually is able to evaluate the series in (34), and this adds to the mystery of the evaluation in (34), since it is not at all clear how the specified CAS software was able to determine a closed form for the series (34), noting that the Mathematica CAS is not able to evaluate this same series. This motivates a full exploration of (34) and its relation to the history of the Bauer–Ramanujan formula. We leave this to a separate project.

Acknowledgments

The first author is grateful to acknowledge support from a Killam Postdoctoral Fellowship from the Killam Trusts, and thanks Karl Dilcher for a useful discussion. The authors are very grateful for the anonymous reviewer feedback that has been provided and that has led to many substantial improvements to this paper.

Disclosure statement

No potential conflict of interest was reported by the author(s).

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