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ON A SIMPLE OPTIMAL STOPPING PROBLEM

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Abstract. L.A. Shepp has posed and analyzed the problem of optimal random drawing without replacement from an urn containing predetermined numbers of plus and minus balls. Here Shepp's results are extended by improving the bounds on values of perturbed urns, deriving an exact algorithm for the urn values and computing the stopping boundary for urns of up to 200 balls.

1. Introduction

L.A. Shepp has posed the following problem [6, p. 999]: an urn contains m balls of value -1 and p balls of value $+1$ and one is allowed to draw balls randomly, without replacement, until one wants to stop. Which urns are favorable? That is, for what m and p is there a drawing strategy for which the expected total value of balls drawn is positive? More generally, what is the maximum expected value obtainable (the value $V(m, p)$ of the urn) and what strategy achieves it?

The problem was considered by Shepp because of the connections he found with the 'ESP problem' posed by Breiman [2] and studied by Chow and Robbins [3], Dvoretzky [4], Shepp [6], and others. It also serves as a prototype for a class of 'random urn' problems which have applications in financial modeling [1].

After stating the problem, Shepp observed that an optimal strategy was to stop drawing as soon as the depleted urn had value zero and that the value of any specific urn could be calculated by an easily-derived forward recursion. Shepp also proved that $V(m, p+1) \geq V(m, p)$ by a randomized-strategy argument and he pointed out that similarly $V(m+1, p) \leq V(m, p)$. Thus for each p there is a maximum m for

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which $V(m, p) > 0$, which Shepp denoted $\beta(p)$. From the main results of [6], he obtained the asymptotic formula

$$\beta(p) = p + \alpha\sqrt{2p} + \epsilon(p)\sqrt{2p}, \qquad \lim_{p\to\infty} \epsilon(p) = 0, \tag{1}$$

where $\alpha = 0.83992$ ... is the unique real root of the integral equation (1.3) of [6]. Although for smaller values of p, $\beta(p)$ can be computed numerically using the forward recursion, Shepp noted that for larger p round-off would cause numerical difficulties.

In this paper, we elaborate on Shepp's discussion of the problem and provide additional results. First we give a table of the values $V(m, p)$ for small m and p. Then we prove the three relations about adding balls referred to by Shepp, plus additional bounds; however, instead of a random device we use the recursion formula for a proof by induction. Next, we show that based on an interesting numerical property of the values $V(m, p)$, related to Pascal's triangle, we can construct an efficient integer algorithm for determining the values and boundary points $\beta(p)$. By using the algorithm to calculate the boundary for p up to 100, we show that with a small modification Shepp's asymptotic formula becomes very accurate in this range. Another finding is that for $p \leq 100$, only the urn with $m = 2$ and $p = 1$ has both 'draw' and 'don't draw' optimal strategies. Finally, we verify a conjecture of H.O. Pollak which gives an asymptotic expression for $V(m, p)$ for fixed m.

2. The problem and recursive solution

As described in [1, pp. 33–34], $V(m, p)$ denotes the value under optimal play of an urn with $n = m + p$ balls, m of value -1 and p of value $+1$. The player is told the composition (m, p) of the urn; then he can draw from zero to n of the balls, randomly, one at a time, without replacement. Thus he begins with all n in the urn and can stop any time, including before drawing even one ball. His objective is to draw in such a way as to maximize his expected score upon stopping.

Shepp's recursion formula for $V(m, p)$ goes as follows [1, p. 47]: for the (m, p) urn, we draw a minus ball with probability m/n, which gives us -1 and the opportunity to draw from the $(m-1, p)$ urn, with value $V(m-1, p)$; while drawing a plus ball with probability p/n gives us $+1$

and the reduced urn is the $(m, p-1)$ urn. Thus the total expected value $\mathrm{E}(m, p)$ of drawing from the (m, p) urn is

$$(2) \qquad \mathrm{E}(m,p) = (m/n)(-1 + V(m-1,p)) + (p/n)(+1 + V(m,p-1)).$$

Then, if $\mathrm{E}(m,p) > 0$ we draw, so $V(m,p) = \mathrm{E}(m,p) > 0$; and if $\mathrm{E}(m,p) < 0$ we don't draw, so $V(m,p) = 0$. If $\mathrm{E}(m,p) = 0$ (on which more later) we can either draw or not, and $V(m,p) = 0$. Thus,

$$(3) \qquad V(m,p) = \max\{0, \mathrm{E}(m,p)\} \geq \mathrm{E}(m,p).$$

For the values of urns with n balls, one needs only the values of urns with $n-1$ balls, so a table of values for small m and p is easily constructed; see Table 1 [1, p. 33]. We see that many urns with $m > p$ have a positive value. This initially surprising conclusion illustrates the advantage of the option to quit while ahead.

Table 1
$V(m, p)$

p (plus) \ m (minus)	0	1	2	3	4	5	6	7	8	9
9	9	8.10	7.20	6.31	5.43	4.58	3.75	2.95	2.21	1.53
8	8	7.11	6.22	5.35	4.49	3.66	2.86	2.11	1.43	0.84
7	7	6.13	5.25	4.39	3.56	2.76	2.01	1.34	0.75	0.30
6	6	5.14	4.29	3.45	2.66	1.91	1.23	0.66	0.23	0
5	5	4.17	3.33	2.54	1.79	1.12	0.55	0.15	0	0
4	4	3.20	2.40	1.66	1.00	0.44	0.07	0	0	0
3	3	2.25	1.50	0.85	0.34	0	0	0	0	0
2	2	1.33	0.67	0.20	0	0	0	0	0	0
1	1	0.50	0	0	0	0	0	0	0	0
0	0	0	0	0	0	0	0	0	0	0

Three conclusions about the local relationships of the values are clearly illustrated by Table 1. These are:

(A) adding a plus ball never hurts: $V(m, p+1) \geq V(m,p)$;

(B) adding a minus ball never helps: $V(m+1, p) \leq V(m,p)$;

(C) adding both a minus and a plus ball never hurts:

$$V(m+1, p+1) \geq V(m,p).$$

Of the three, (C) is the only surprise, since it is not at all intuitive that 'diluting' a (5,25) urn (5/6 plus) to a (10,30) urn (3/4 plus) should enhance the value. In [6], Shepp cited all these relationships, proved (A) by a randomized-strategy argument, and observed that (B) could be handled the same way. In the next section, we use (2) and (3) to obtain rigorous proofs of (C) and slightly stronger versions of (A) and (B), which can also be derived by randomizing strategies. (However, our method of proof generalizes more easily to the cases studied in [1].)

It has been conjectured that increasing the number of balls in an urn while maintaining or increasing the proportion of plus balls should increase the value of the urn. When the urn is at least half plus balls, this conclusion is a consequence of (A) and (C), since maintaining the proportion may be viewed as adding equal numbers of plus and minus balls, the adding some extra plus balls. But when $m > p$, it is possible to increase the number of balls and maintain proportions, but decrease the value of the urn. For example, in Table 1 we see that $V(3,2) = 0.20$, $V(6,4) = 0.07$ and $V(9,6) = 0.0$. The failure of the conjecture when $m > p$ may be considered as a corollary to the parabolic shape of the stopping boundary which is revealed by (1).

3. Local value relationships

We improve on inequalities (A), (B) and (C) in the following theorems.

Theorem 3.1 (a) *If* $\mathrm{E}(m, p) \geq 0$, *then* $V(m, p+1) \geq V(m, p) + 1/(n+1)$.
(b) $V(m, p+1) \leq V(m, p) + 1$.

Proof is by induction on $n = m + p$. Clearly, the theorem is true for $n = 1$ and we assume that it is true for n. Then applying (2) and (3) to $\mathrm{E}(m, p+1)$ and multiplying by $n+1$ yields

$$(n+1)\, V(m, p+1) \geq m(-1 + V(m-1, p+1)) + (p+1)(1 + V(m, p)). \tag{4}$$

For part (a), if $\mathrm{E}(m, p) \geq 0$, then $V(m, p) = \mathrm{E}(m, p)$, so multiplying (2) by n and adding $V(m, p)$ to both sides yields

$$(n+1)\, V(m, p) = m(-1 + V(m-1, p)) + p(1 + V(m, p-1)) + V(m, p). \tag{5}$$

Subtracting (5) from (4) gives

$$(6) \qquad (n+1)\{V(m,p+1)-V(m,p)\} \geq m\{V(m-1,p+1)-V(m-1,p)\} + p\{V(m,p) - V(m,p-1)\} + 1.$$

By the induction hypothesis and the fact that values are never negative, the two expressions in braces on the right-hand side of (6) must be non-negative. Thus

$$(n+1)\{V(m,p+1) - V(m,p)\} \geq 1$$

when $E(m,p) \geq 0$, which establishes part (a). For part (b), consider that it is trivially true if $V(m,p+1) = 0$, so we may assume for the proof that $V(m,p+1) = E(m,p+1)$ and that (4) is an equality. However, we don't known if $V(m,p) = 0$, so (5) is replaced by an inequality. Then subtracting the modified (5) from the modified (4) gives (6) with the direction of the inequality reversed. Then the induction hypothesis is that expressions in braces on the right-hand side do not exceed 1, so

$$(n+1)\{V(m,p+1)-V(m,p)\} \leq m+p+1 = n+1,$$

as claimed in (b).

Theorem 3.2. (a) *If* $E(m+1,p) \geq 0$, *then* $V(m,p) \geq V(m+1,p)+1/(n+1)$. (b) $V(m,p) \leq V(m+1,p) + 1$.

Proof is similar to that of Theorem 1. Again, the theorem is clearly true for $n = 1$. For (a), subtracting

$$(n+1)\,V(m+1,p) = (n+1)\,E(m+1,p)$$

from

$$(n+1)\,V(m,p) \geq nE(m,p) + V(m,p)$$

gives

$$(7) \qquad (n+1)\{V(m,p)-V(m+1,p)\} \geq m\{V(m-1,p)-V(m,p)\} + p\{V(m,p-1)-V(m+1,p-1)\} + 1.$$

analogous to (6), and so (a) holds by induction. For part (b), we obtain (7) with the direction of the inequality reversed, as in Theorem 3.1, and (b) is proved by induction.

Corollary 3.3 (Shepp). $V(m+1,p) \leq V(m,p) \leq V(m,p+1)$.

Proof follows from 3.1 (a) and 3.2 (a) if the left-hand sides are positive, otherwise the nonnegativity of values is all that is required.

Corollary 3.4 [1, p. 47]. *Under optimal play, the last ball drawn is always a plus.*

Proof. If a minus ball is drawn from the (m,p) urn under optimal play, then we must have had $m > 0$ and $\mathrm{E}(m,p) \geq 0$. But the reduced urn is the $(m-1,p)$ urn, and by Theorem 3.2 $V(m-1,p) > V(m,p) \geq 0$, so another ball will be drawn. Thus a minus ball is never the last drawn.

With both Theorems 3.1 and 3.2 available, we can strengthen them so as to obtain limiting relationships for large p. Somewhat analogously to $\beta(p)$, we define $\gamma(m)$ to be the *least* p for which $\mathrm{E}(m,p) \geq 0$. Then it follows from Theorem 3.1 (a) that $V(m,p) > 0$ for $p > \gamma(m)$, while from Theorem 3.2 (a) we see that $\gamma(m)$ is a nondecreasing function of m.

Theorem 3.1. (c) *If $p \geq \gamma(m)$, then*

$$V(m,p+1) \geq V(m,p) + (p-\gamma(m)+1)/(n+1).$$

Proof is by double induction, first on m and then on p. When $m = 0$, then $\gamma(m) = 0$ and $p = n$, so the fraction equals 1 and equality holds for all p. Thus assume that the theorem is true for $m-1$. When $p = \gamma(m)$, the theorem reduces to Theorem 3.1 (a). If the theorem is true for $p-1 \geq \gamma(m)$, then in (6) we have

$$V(m-1,p+1) - V(m-1,p) \geq (p-\gamma(m-1)+1)/n > (p-\gamma(m))/n$$

and

$$V(m,p) - V(m,p-1) \geq ((p-1)-\gamma(m)+1)/n = (p-\gamma(m))/n,$$

so since $n = m + p$,

$$(n+1)\{V(m,p+1)-V(m,p)\} \geq (m+p)(p-\gamma(m))/(n+1) \\ = p-\gamma(m)+1$$

which yields the theorem.

A similar proof, which we omit, gives

Theorem 3.2. (c) *If* $p \geq \gamma(m+1)$, *then*

$$V(m,p) \geq V(m+1,p)+(p-\gamma(m+1)+1)/(n+1).$$

(If the randomized-strategy proof is used, 3.1 (c) and 3.2 (c) hold because if k draws are guaranteed, the difference in urn values is at least $k/(n+1)$.) Theorems 3.1 (c) and 3.2 (c) when combined with Theorems 3.1 (b) and 3.2 (b) yield the following corollary.

Corollary 3.5. $\lim_{p\to\infty} \{V(m,p+1)-V(m,p)\} = \lim_{p\to\infty} \{V(m,p)-V(m+1,p)\} = 1.$

The next lemma and theorem, although perhaps of some interest in themselves, are primarily directed toward the proof of (C) given in Theorem 3.8. We omit the proof of the lemma, which is a straightforward induction based on (2).

Lemma 3.6. $V(1,p) = p^2/(p+1) > p-1$.

Theorem 3.7. *If* $\mathrm{E}(m,p) \geq 0$, *then* $V(m+1,p)+V(m,p+1) > 2V(m,p)$.

Proof is again by induction, and again, the theorem holds for $n = 1$. The case $m = 0$ is special and is disposed of by the lemma, since $V(1,p) + V(0,p+1) > (p-1) + (p+1) = 2p = 2V(0,p)$. Thus for the rest of the proof we may assume that $m > 0$. For the induction step, we subtract twice $(n+1)\,V(m,p) = n\mathrm{E}(m,p)+V(m,p)$ from the sum of $(n+1)\,V(m+1,p) \geq (n+1)\mathrm{E}(m+1,p)$ and $(n+1)\,V(m,p+1) \geq (n+1)\mathrm{E}(m,p+1)$ to obtain

$$\begin{aligned}(n+1)\{V(m+1,p)+V(m,p+1)-2V(m,p)\}\\ \geq m\{V(m,p)+V(m-1,p+1)-2V(m-1,p)\}\\ +p\{V(m+1,p-1)+V(m,p)-2V(m,p-1)\}\ .\end{aligned}$$

By the induction hypothesis or the nonnegativity of values, the last expression in braces is nonnegative, and by Theorem 3.2, $V(m-1,p) > V(m,p) \geq 0$, so the middle braced term is positive by the induction hypothesis. Since $m > 0$, the left-hand side must be positive, which is the content of the theorem.

Theorem 3.8. *If* $\mathrm{E}(m,p) \geq 0$, *then* $V(m+1,p+1) > V(m,p)$.

Proof is by induction, as usual, and as usual the theorem is trivial for $n = 1$. For the induction step, we subtract $(n+2)\,V(m,p) = n\mathrm{E}(m,p) + 2V(m,p)$ from $(n+2)\,V(m+1,p+1) \geq (n+2)\mathrm{E}(m+1,p+1)$ to obtain

$$\begin{aligned}(n+2)\{V(m+1,p+1)-V(m,p)\} \geq\ & m\{V(m,p+1)-V(m-1,p)\}\\ & +p\{V(m+1,p)-V(m,p-1)\}\\ & +\{V(m+1,p)+V(m,p+1)\\ & \quad -2V(m,p)\}\ .\end{aligned}$$

By the induction hypothesis or nonnegativity of values, the two middle braced expressions are nonnegative, and by Theorem 3.7 the last expression is positive, so the left-hand side must be positive also, which proves the theorem.

In [1], we discuss the generalized problem of optimal drawing without replacement from a 'random urn', one for which the number of balls n is known, but the division of the n balls between plus and minus ones is specified by only a probability distribution. Analogs of (A), (B) and (C) can be stated in terms of the effect on the value of adding a plus ball, a minus ball, or both to such a random urn. It is curious that, for the analogs, the truth of (B) is virtually a corollary to Shepp's proof of (B) in the known-urn case, (A) is still true but is much more difficult to prove, and (C) is false in general.

4. An integer algorithm for $V(m, p)$

Consider now the problem of evaluating numerically the functions $V(m, p)$ and $\beta(p)$ for large values of m and p. The use of floating-point arithmetic not only leads to the accumulation of round-off error, but since the forward recursion involves subtraction there is great danger of losing accuracy through cancellation of significant digits. This is particularly serious in the case of $\beta(p)$ since it is precisely at $\beta(p)$ that the subtracted quantities will be nearly the same. To aggravate matters, the 'principle of smooth fit' [5, 6] assures us that $V(m, p)$ will be nearly zero and hence numerically sensitive in a wide area around the boundary as well as at boundary itself. Thus floating-point arithmetic is most inappropriate for investigating $\beta(p)$.

Other methods are available, however, since it is obvious that all the arithmetic is rational and that $V(m, p)$ is always rational. In fact, $(m+p)!\,V(m, p) = V'(m, p)$ is always an integer, since multiplying (2) and (3) by $n! = (m+p)!$ yields

$$V'(m,p) = \max\{0,(m+p-1)!\,(p-m)+m\,V'(m-1,p)+p\,V'(m,p-1)\}. \tag{8}$$

Thus we could compute $V'(m, p)$ using only integer arithmetic and divide by $(m+p)!$ if $V(m, p)$ itself were desired. For $\beta(p)$ we would only need $V'(m, p)$, since $V'(m, p)$ and $V(m, p)$ are zero simultaneously, and knowing where $V(m, p)$ is zero is all that is needed for $\beta(p)$. But using $V'(m, p)$ to calculate $\beta(p)$ for large p would still be very difficult due to the rapid growth of $n!$; for instance, $V'(5,5)$ already exceeds 4,000,000. The necessity of doing multiplication of large integers is another complication.

What we are leading up to is a much more efficient integer algorithm related to and using Pascal's triangle, or the binomial coefficients. To point up the parallelism we adopt the nonstandard notation

$$C(m, p) = \binom{m+p}{m} = \frac{(m+p)!}{m!\,p!} \;.$$

The 'Pascal's triangle' recurrence then becomes

$$C(m, p) = C(m-1, p) + C(m, p-1)\,. \tag{9}$$

The following observation is the key to the algorithm.

Theorem 4.1. *$B(m,p) = C(m,p)\, V(m,p)$ is an integer.*

Proof is by induction on $n = m + p$. The case $n = 1$ is trivial since $V(0,1)$ and $V(1,0)$ are integral. Assuming validity for $n-1$, we multiply (2) and (3) by $C(m,p)$ to obtain

$$B(m,p) = \max\{0, C(m,p)(m/n)(-1+V(m-1,p)) + C(m,p)(p/n)(1+V(m,p-1))\}\ .$$

Then, since $C(m,p)(m/n) = C(m-1,p)$ and $C(m,p)(p/n) = C(m,p-1)$, we have

$$(10)\quad B(m,p) = \max\{0,[C(m,p-1)-C(m-1,p)] + B(m-1,p) + B(m,p-1)\}\ .$$

Thus by induction $B(m,p)$ is integral.

Equations (9) and (10) form the basis of a multiplication-free integer algorithm for computing $V(m,p)$ and $\beta(p)$, and the size of the integers is considerably less than those appearing in (8); for example, $B(5,5)$ is only 282.

The discovery of Theorem 4.1 was much more round-about than is indicated by the brief proof. Logically, it is derived from Theorem 4 of [1], which described an algorithm for computing the value of the 'random urns' referred to earlier. In the algorithm, for each m between 0 and n the probability $\mathrm{P}(m)$ of m is divided by $\binom{n}{m} = C(m, n-m)$, and then the value is computed from the quotients by a succession of subtractions, additions and applications of the 'max $\{0,x\}$' operator. When $\mathrm{P}(m^*) = 1$ for one m^* and the rest are zero, it is immediate that the resulting value must be a fraction with denominator $C(m^*, n-m^*)$.

To emphasize the connections with Theorem 4 of [1] and Pascal's triangle, we define

$$A(m,p) = C(m,p-1) - C(m-1,p)\ .$$

Then $A(m,p)$ inherits the Pascal recurrence (9),

$$A(m,p) = A(m-1,p) + A(m,p-1)$$

Table 2
$A(m, p)$

p (plus) \ m (minus)	0	1	2	3	4	5	6	7	8	9
9	1	8	35	110	275	572	1001	1430	1430	0
8	1	7	27	75	165	297	429	429	0	−1430
7	1	6	20	48	90	132	132	0	−429	−1430
6	1	5	14	28	42	42	0	−132	−429	−1001
5	1	4	9	14	14	0	−42	−132	−297	−572
4	1	3	5	5	0	−14	−42	−90	−165	−275
3	1	2	2	0	−5	−14	−28	−48	−75	−110
2	1	1	0	−2	−5	−9	−14	−20	−27	−35
1	1	0	−1	−2	−3	−4	−5	−6	−7	−8
0	0	−1	−1	−1	−1	−1	−1	−1	−1	−1

but has initial values $A(1,0) = -1$, $A(0,1) = 1$ instead of the binomial $C(1,0) = C(0,1) = 1$. Our integer algorithm is then described by the following theorem.

Theorem 4.2. *Set* $A(m,p) = B(m,p) = 0$ *whenever* m *or* p *is negative, and set* $A(0,0) = 0$, $A(1,0) = -1$ *and* $A(0,1) = 1$. *If* $A(m,p) = A(m-1,p) + A(m,p-1)$ *when* $m + p > 1$ *and* $B(m,p) = \max\{0, A(m,p) + B(m-1,p) + B(m,p-1)\}$, *then* $B(m,p) = C(m,p)\,V(m,p)$.

Table 3
$B(m, p)$

p (plus) \ m (minus)	0	1	2	3	4	5	6	7	8	9
9	9	81	396	1388	3885	9165	18760	33796	53683	74131
8	8	64	280	882	2222	4708	8594	13606	18457	20448
7	7	49	189	527	1175	2189	3457	4583	4851	3421
6	6	36	120	290	558	882	1136	1126	697	0
5	5	25	70	142	226	282	254	122	0	0
4	4	16	36	58	70	56	14	0	0	0
3	3	9	15	17	12	0	0	0	0	0
2	2	4	4	2	0	0	0	0	0	0
1	1	1	0	0	0	0	0	0	0	0
0	0	0	0	0	0	0	0	0	0	0

Proof follows from Theorem 4.1 and the fact that

$$A(m,p) = C(m,p-1) - C(m-1,p).$$

The values of $A(m,p)$ and $B(m,p)$ for $m, p < 10$ are given in Tables 2 and 3.

5. Numerical results on the boundary

As described in the introduction, Shepp's original question was, "Which urns are favorable?" That is, what is $\beta(p)$?

Ann R. Martin has calculated $\beta(p)$ for $p \leq 100$ on a Honeywell 635 computer, using the algorithm of Theorem 4.2 and extended-precision integer operations. We find that for $2 \leq p \leq 100$, an urn is favorable if and only if

(11) $$m < p + 0.83992\sqrt{2p} - 0.1427.$$

A less 'digital' form of the right-hand side of (11) is

(12) $$p + \alpha\sqrt{2p} - c\,,$$

where $\alpha = 0.83992\ ...$ is the same as in (1), and c must satisfy

$$\alpha\sqrt{2 \cdot 7} - 3 < c < \alpha\sqrt{2 \cdot 88} - 11$$

to insure the correctness of (11) and (12) for $p = 7$ and $p = 88$. The constant c exists by only a narrow margin since $x\sqrt{2 \cdot 88} - 11$ exceeds $x\sqrt{2 \cdot 7} - 3$ only for

$$x > 0.839909,$$

and the allowable range of c is but 10^{-4}.

We may put (11) and (12) in terms of $\beta(p)$ to obtain a remarkable sequel to Shepp's formula (1). The complication is that $\beta(p)$ is integral while (12) is not. Thus our numerical finding is that for $2 \leq p \leq 100$,

(13) $$\beta(p) = [p + \alpha\sqrt{2p} - c]\,,$$

where $[x]$ denotes the integral part of x. If the fractional part of (12) is denoted by $f(p)$, then setting $\epsilon(p) = -(c+f(p))/\sqrt{2p}$ gives a perfect match with (1).

The only exception found for (13), namely $p = 1$, may really not be an exception at all, since (13) 'fails' there in a noteworthy way. The validity of (13) for $p = 1$ would indicate that $(m, p) = (2,1)$ is a favorable urn, whereas no strategy for this urn gives a positive pay-off; that is, $V(2,1) = 0$. Thus not drawing at all is an optimal strategy for the (2,1) urn, as it is for all urns with value zero. But since $E(2,1) = 0$, an expected value of zero may be obtained by drawing as well as not drawing; in fact, 'draw until you get a plus' is also an optimal strategy for the (2,1) urn. We will call an urn with $E(m, p) = 0$ a *neutral* urn, since drawing from it may included or omitted from an optimal strategy as one chooses. In the integer algorithm, it is easy to check for $E(m, p) = 0$, so our calculation of $\beta(p)$ yielded another finding: *for* $p \leq 100$, *there is no neutral urn other than the* (2,1) *urn.* (In fact, we conjecture, but are unable to prove, that the (2,1) urn is the only one.) Thus if a 'favorable' urn is redefined as one from which it is optimal to draw, or equivalently one for which $E(m, p) \geq 0$, then (13) describes favorable urns for all $p \leq 100$.

Our calculations extended only to $p = 100$ because that was a convenient goal, not because of any problems with the algorithm. The largest integer encountered at that point was only about 70 decimal digits, and the cost of the computer run was less than 10 dollars.

6. $B(m, p)$ as a polynomial

In Table 3, H.O. Pollak noted that for each m the positive values of $B(m, p)$ match an $(m+1)^{st}$ degree polynomial in p which differs from

$$P(m, p) = p(p-m+1)\, C(m, p)/(p+1),$$

only in the terms of lowest degree. (In Table 3, $B(m, p) = P(m, p)$ for $m = 0,1,2$ and for $m = 3$ is off by only 2.) If true in general, as conjectured by Pollak, $V(m, p)$ would asymptotically equal

$$P(m, p)/C(m, p) = (p-m) + m/(p+1)\,.$$

The following theorems verify these conjectures.

First, we note that $C(m, p)$ is an m^{th} degree polynomial in p, thus $A(m, p) = C(m, p-1) - C(m-1, p)$ is also. Furthermore, if $p \geq \gamma(m)$, then

$$(14) \qquad B(m, p) - B(m, p-1) = A(m, p) + B(m-1, p) .$$

Theorem 6.1. *$B(m, p)$ is a polynomial of $(m+1)^{\text{st}}$ degree in p for $p \geq \gamma(m) - 1$.*

Proof. Obviously, $B(0, p) = p$ and by Lemma 3.6 $B(1, p) = p^2$. Thus we assume that $B(m-1, p)$ is a polynomial of degree m in p for $p \geq \gamma(m-1) - 1$. Then (14) holds when $p \geq \gamma(m) \geq \gamma(m-1)$, so for $p \geq \gamma(m)$ the right-hand side of (14) is a polynomial of degree m in p. Thus the first differences of $B(m, p)$ follow an m^{th} degree polynomial, hence $B(m, p)$ itself must be an $(m+1)^{\text{st}}$ degree polynomial, determined by its first differences up to the constant term. The constant we establish by setting $B(m, \gamma(m) - 1) = 0$. Thus $B(m, p)$ is given by the polynomial when $p \geq \gamma(m) - 1$.

It looks like we might need to know $\gamma(m)$ beforehand in order to determine $B(m, p)$, but we do not, for the next theorem shows that $\gamma(m)$ is simply the least p for which the right-hand side of (14) is nonnegative.

Theorem 6.2. *$p \geq \gamma(m)$ if and only if $A(m, p) + B(m-1, p) \geq 0$.*

Proof. If $p > \gamma(m)$, then $\mathrm{E}(m, p-1) \geq 0$ and so $V(m, p) > V(m, p-1)$ by Theorem 3.1. But $B(m, p) = C(m, p)\, V(m, p)$ and $C(m, p) > C(m, p-1) > 0$, hence $B(m, p) > B(m, p-1)$. Since (14) applies, both sides must be positive. If $p = \gamma(m)$ then $\mathrm{E}(m, p-1) < 0$, and so $B(m, p-1) = V(m, p-1) = 0$. But (14) holds, and $B(m, p) \geq 0$, hence both sides of (14) are nonnegative.

If $A(m, p) + B(m-1, p) \geq 0$ then since $B(m, p-1) \geq 0$ we have

$$C(m, p)\, \mathrm{E}(m, p) = A(m, p) + B(m-1, p) + B(m, p-1) \geq 0.$$

Since $C(m, p) > 0$, we have $\mathrm{E}(m, p) \geq 0$ and thus $p \geq \gamma(m)$.

Next we verify Pollak's conjectured asymptotic form for $B(m, p)$. $\mathrm{P}(m, p)$ is an $(m+1)^{\text{st}}$ degree polynomial in p since $p+1$ is always a factor of $C(m, p)$.

Theorem 6.3. *For $p \geq \gamma(m)-1$, $B(m,p) = \mathrm{P}(m,p) + Q(m,p)$, where $Q(m,p) = 0$ for $m = 0,1,2$ and $Q(m,p)$ is an $(m-3)^{\mathrm{rd}}$ degree polynomial in p for $m \geq 3$.*

Proof. We use the identity

$$\mathrm{P}(m,p) = A(m,p) + \mathrm{P}(m-1,p) + \mathrm{P}(m,p-1).$$

Substituting the formula for $A(m,p)$ and the hypothesized formula for $B(m,p)$ into (14) yields the condition

$$Q(m,p) - Q(m,p-1) = Q(m-1,p)\,. \tag{15}$$

The proof that $Q(m,p) = 0$ for $m = 0,1,2$ is obtained from a comparison of Table 3 with $P(m,p)$, since k^{th} degree polynomials which agree at $k+1$ places are identical. We find that $Q(3,p) = 2$ and $Q(4,p) = 2p+6$, so $Q(m,p)$ is an $(m-3)^{\mathrm{rd}}$ degree polynomial for $m = 3,4$. The proof for higher values of m follows by induction from (15) as in the proof of Theorem 6.1.

Corollary 6.4. *For fixed m, $V(m,p) = (p-m) + m/(p+1) + \mathrm{O}(p^{-3})$.*

Finally we note that for each i

$$Q_i(m,p) = C(m-i,p)$$

is an $(m-i)^{\mathrm{th}}$ degree polynomial in p which satisfies (15). Thus, if we define $C(l,p) = 0$ for $l < 0$, we may expand Q in a Q_i series with the i^{th} coefficient set at $m = i$ as follows:

$$Q = 2Q_3 + 4Q_4 + 6Q_5 + 22Q_6 + \dots$$

The 2-4-6 sequence is merely fortuitous, as no overall pattern emerges from the coefficients.

References

[1] W.M. Boyce, Stopping rules for selling bonds, Bell J. Econ. Mgmt. Sci. 1 (1970) 27–53.

[2] L. Breiman, Stopping rule problems, in: E.F. Beckenbach, ed., Applied combinatorial mathematics (Wiley, New York, 1964).

[3] Y.S. Chow and H. Robbins, On optimal stopping rules for S_n/n, Illinois J. Math. 9 (1965) 444–454.
[4] A. Dvoretzky, Existence and properties of certain optimal stopping rules, Proc. Fifth Berkeley Symp. Math. Statist. Prob. 1 (1967) (Univ. of California Press) 441–452.
[5] B.I. Grigelionis and A.N. Shiryaev, On Stefan's problem and optimal stopping rules for Markov processes, Teor. Verojatnost. i Primemen 11 (1966) 611–631.
[6] L.A. Shepp, Explicit solutions to some problems of optimal stopping, Ann. Math. Statist. 40 (1969) 993–1010.