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# A Non-constructive Proof of the Four Colour Theorem.

D. M. Jackson and L. B. Richmond,
Dept. of Combinatorics and Optimization,
University of Waterloo, Waterloo, Ontario, Canada.

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**Dedicated to the Memory of W.T. Tutte.**

**Abstract**

The approach uses a singularity analysis of generating functions for particular sets of maps, and Tutte's enumerative and asymptotic work on planar maps and their chromatic polynomials.

## 1 Background and the approach.

A planar map or, more briefly, a *map*, is a 2-cell embedding of a connected graph in the plane. Determining the least number of colours required for colouring the vertices of a/ map so that no edge joins vertices of the same colour has a history extending over some 160 years [11]. The *Four Colour Conjecture* (in its dual form) asserted that every planar map is vertex 4-colourable. The first proof, by Appel and Haken [1, 2] in 1977, is heavily case-analytic, through the graph-theoretic operations of discharging and reducibility. A shorter such proof was given by Robertson *et al.* [9] in 1997. These proofs are *constructive*, giving a polynomial-time algorithm for a vertex 4-colouring.

Chromatic polynomials were introduced by Birkhoff and Lewis [3] who, with Tutte, believed they should assist in a proof. This insight and Tutte's work on map enumeration are shown to have been justified.

### 1.1 The chromatic polynomial.

The *chromatic polynomial* $P(G, \lambda)$, in $\lambda$, of a graph $G$ has the property that if $\lambda = k$, where $k$ is a positive integer, then $P(G, \lambda)$ is the number of colourings of $G$ with $k$ colours such that no edge of $G$ has vertices of the same colour. The colourings are called *proper colourings*, but since no other type of colouring will be considered, the usage here is unambiguous. Further material on the chromatic polynomial may be found in Bondy and Murty [4]. The Four Colour Theorem, restated in terms of $P(G, \lambda)$, is the following:

**Theorem 1.** *If $G$ is planar then $P(G, \lambda) > 0$ for $\lambda = 4$.*

Birkhoff and Lewis [3] proved that $P(G, \lambda) > 0$ for all real $\lambda \geq 5$. It is noted that this excludes the case $\lambda = 4$, namely, the Four Colour Problem. See also Appendix A. Royle's paper [10] explains why the condition $\lambda \geq 4$ is required in the inductive hypothesis in Section 4.2. He constructed a family $\mathcal{R}$ of 3-connected near-triangulations (all but one face is a triangle) with the property that for each real number $t > 0$ there is an element $R_t \in \mathcal{R}$ such that the chromatic polynomial $P(R_t, \lambda)$ has a chromatic zero $\lambda$ in the range $4 - t < \lambda < 4$. The number of edges in $R_t$ goes to infinity as $t \to 0$. However, from Royle [10], there is a planar map $M$ for each $t > 0$ with $n(t)$ edges such that $P(M, \lambda) = 0$ for some $\lambda$ in the above range. That is, for each $t$, there is a sequence of planar maps $M_i$, where $M_i$ has $n_t$ edges and $n_t \to \infty$ as $t \to 0$, such that $P(M_i, \lambda) = 0$ for $\lambda$ in the above range for each constant $t > 0$. Nevertheless, it will be proved that $P(M, 4) > 0$ for all $M$.

**Remark 2.** *The value $\lambda = 4$ is exceptional with respect to the plane in that for no other known value of $\lambda > 3$ does $P(M, \lambda) = 0$ hold for $\lambda$ approaching some fixed value $\lambda_0$, yet $P(M, \lambda_0) > 0$. Thomassen [12] showed that there are chromatic zeros approaching any $\lambda_0 \in (\frac{22}{7}, 3)$. He conjectured that this 3 may be replaced by 4. See also Appendix A.*

### 1.2 Rooted maps.

Several results of Tutte's on the enumeration of rooted planar maps will be required, especially those in [15] and an earlier paper [14], so an outline of these is included. Tutte [15] rooted a map $M$ in the plane by choosing an edge as the *root edge*, a direction on the root edge directed away from the *root vertex*, and a face incident with the root edge (on the left hand side of the root edge) as the *root face*. Let $E(M)$ denote the number of edges of $M$. If the map is asymmetrical with $E(M)$ edges it will have $4\,E(M)$ rootings. (If it is known that almost all of the maps with $E(M)$ edges in some family are asymmetrical then, given an asymptotic estimate for the number of rooted maps with $E(M)$ edges, an asymptotic formula for the number of unrooted maps with $E(M)$ edges may be obtained by dividing this asymptotic estimate by $4\,E(M)$). A rooted map satisfies the Four Colour Theorem if and only if the corresponding unrooted map does also, so the rooting is not important in the present context. However, the root is crucial in those enumerative results of Tutte's that are used here.

### 1.3 The approach.

The following assertion is to be proved. *If there is a set of maps $\mathcal{Q}$ that can be 4-coloured and if the number of these maps with $n$ edges is a positive fraction of the number of all maps with $n$ edges, then there is no map that cannot be 4-coloured.* ($\mathcal{Q}$ is constructed in Section 4.1.)

## 2 The map series $A(x)$ and $B(x)$.

### 2.1 Preliminaries.

Two map series, $A(x)$ and $B(x)$, that are fundamental to the approach may be expressed in terms of the hypergeometric series ${}_2F_1[a, b; c; x] := 1 + \frac{ab}{c}\frac{x}{1!} + \frac{a(a+1)b(b+1)}{c(c+1)}\frac{x^2}{2!} + \frac{a(a+1)(a+2)b(b+1)(b+2)}{c(c+1)(c+2)}\frac{x^3}{3!} + \cdots$. This satisfies the differential equation

$$x(1-x)\frac{d^2y}{dx^2} + (c - (1 + a + b)x)\,\frac{dy}{dx} - aby = 0. \tag{1}$$

Its evaluation at $x = 1$ is

$${}_2F_1[a, b; c; 1] = \tfrac{\Gamma(c)\,\Gamma(c-a-b)}{\Gamma(c-a)\,\Gamma(c-b)}. \tag{2}$$

The following are recalled $\Gamma(-\frac{1}{2}) = -2\sqrt{\pi}$, $\Gamma(-\frac{3}{2}) = \frac{4\sqrt{\pi}}{3}$, $\Gamma(1 + x) = x\,\Gamma(x)$, $\Gamma(1) = 1$ for later calculations. From Theorem VI.1 [5], ($[x^n]$ denotes the coefficient extraction operator)

$$[x^n](1-x)^{-\alpha} = \tfrac{n^{\alpha-1}}{\Gamma(\alpha)}(1 + \mathcal{O}(\tfrac{1}{n})). \tag{3}$$

### 2.2 $A(x)$ - the generating function for rooted maps.

Let $A(x) = \sum_{n=1}^{\infty} a_n x^n$ where $a_n$ denotes the number of rooted maps with $n$ edges. Tutte [15] showed that $a_n = g_n \frac{3^n}{n!}$, where $g_n = 2\frac{(2n)!}{(n+2)!}$. Thus $\frac{g_{n+1}}{g_n} = 4 \cdot \frac{(n+1)(n+\frac{1}{2})}{n+3}$, so

$$A(x) = {}_2F_1[\tfrac{1}{2}, 1; 3; 12x] - 1 = 2x + 9x^2 + \cdots. \tag{4}$$

Also, from Stirling's formula,

$$a_n \sim \tfrac{2}{\sqrt{\pi}} n^{-5/2} 12^n. \tag{5}$$

From the theory of linear differential equations, and (1), the only singularities of $A(x)$ in the finite complex plane are the zeros of $x(1-x)$, namely 0 or 1. Thus, singularities can occur only at roots of the leading coefficient of a linear ordinary differential equation ([5], *Regular Singularities, para.1, p. 519*). It follows from (5) that $A''(x)$ diverges at $x = \frac{1}{12}$, and therefore $A(x)$ has a unique singularity in the finite complex plane at $x = \frac{1}{12}$. *Equ. 17.6.24* of Hille [6] states that

$$ {}_2F_1[a, b; c; 12z] = \frac{\Gamma(c)}{\Gamma(a) \cdot \Gamma(c-a)} \int_0^1 t^{a-1}(1-t)^{c-a-1}(1-12\,z\,t)^{-b}dt, \tag{6} $$

which is valid in the plane cut along $[1, \infty)$ if $\mathbb{R}(a) > 0$, $\mathbb{R}(c-a) > 0$. Following Hille [6], if these conditions are satisfied and $|z| < \frac{1}{12}$ then $(1-12\,z\,t)^{-b}$ may be expanded as a binomial series, and the integration in (6) then carried out termwise. The resulting integrals are of the form $\int_0^1 t^{\alpha-1}(1-t)^{\beta-1}dt = \frac{\Gamma(\alpha)\cdot\Gamma(\beta)}{\Gamma(\alpha+\beta)}$ for $\mathbb{R}(\alpha) > 0$ and $\mathbb{R}(\beta) > 0$. The integral exists in the real plane and gives the analytic continuation of the hypergeometric series outside the unit circle. This gives a representation of $A(x)$ in the form

$$ A(x) = A_1(x) + (1-12x)^{3/2}A_2(x) \tag{7} $$

where $A_1(x)$ and $A_2(x)$ are analytic in the complex plane. This explains why $A(x)$ has precisely one singularity in the complex plane, as observed earlier. Thus the singularity analysis of Flajolet *et al.* described in [5] is applicable, and gives a complete asymptotic expansion of $a_n$.

### 2.3 $B(x)$ - The generating function for non-separable rooted maps.

**Definition 3.** *(Tutte [15]) A rooted map $M$ is said to be* ***separable*** *if its edge-set can be partitioned into two disjoint non-null subsets $S$ and $T$ with just one vertex $v$ incident with both a member of $S$ and a member of $T$. The vertex $v$ is called a* ***cut vertex*** *of $M$. Note that $S$ and $T$ each must have at least one edge. $S$ and $T$ are called the* ***constituents*** *of $M$ under this partitioning. This is indicated through the notation $M = S \bullet T$ where $\bullet$ denotes a cut vertex.*

Let $B(x) = \sum_{n\geq 1} b_n x^n$, where $b_n$ denotes the number of non-separable rooted maps with $n$ edges. Tutte [15] showed that $b_n = \frac{h_n}{n!}$, where $h_n = 2\,\frac{(3n-3)!}{(2n-1)!}$. Now $\frac{h_{n+1}}{h_n} = \frac{27}{4}\frac{(n-\frac{1}{3})(n-\frac{2}{3})}{n+\frac{1}{2}}$, so

$$ B(x) = {}_2F_1\left[-\tfrac{2}{3}, -\tfrac{1}{3}; \tfrac{1}{2}; \tfrac{27}{4}x\right] - 1 = 2x + x^2 + \cdots \quad \text{and} \tag{8} $$

$$ b_n \sim \frac{\sqrt{3}}{27\sqrt{\pi}} n^{-5/2} \left(\frac{27}{4}\right)^n \tag{9} $$

from Stirling's formula. Thus $B(x)$ has only one singularity in the finite complex plane, and this is at $x = \frac{4}{27}$. In addition, the generating series for separable maps is

$$ A(x) - B(x). \tag{10} $$

### 2.4 Maps with a submap $L$ that is not 4-colourable.

Tutte [15] showed that any rooted planar map $M$ has a uniquely determined rooted non-separable map $N$ from which it may be constructed. The following sketches a proof of this using *splitting* of edges in $N$ in the terminology of Section 6 of Tutte [15]. Given an edge $E = \{u, v\}$ in $N$, first replace it with two edges joining $u$ and $v$, one on each side of $E$, and then insert a rooted map attached to each new edge by identifying the root edge of each inserted map with one of the new edges. If the inserted maps have $n_1$ and $n_2$ edges, this may be done in $a_{n_1}a_{n_2}$ ways. To create a map with $n$ edges requires that $n_1 + n_2 = n - 1$ since the edge $E$ must also be counted precisely once. Thus

$$ A(x) = B(x(1 + A(x))^2). \tag{11} $$

The '1' in this is required since an insertion may not be made on one of the new edges.

Theorem IX.27 of Tutte [13] states that if $G$ is the union of two subgraphs $H$ and $K$ whose intersection is the complete graph on $n$ vertices, then

$$ (\lambda)_n \cdot P(G, \lambda) = P(H, \lambda) \cdot P(K, \lambda), \tag{12} $$

where $(\lambda)_n = \lambda(\lambda-1)\ldots(\lambda-n+1)$ if $n > 0$, and is 1 otherwise.

**Definition 4.** *If the root edge of a rooted map $L$ is attached to one of the new edges obtained by splitting an edge in $N$ then the map $M$ constructed from $N$ is said to* ***contain a copy*** *of $L$, a submap isomorphic to $L$.*

It is recalled from the above that every planar map may be derived (by edge splitting) from a unique non-separable map $N$. Here, $L$ and the rest of the map $M$ intersect in an edge (the complete graph with two vertices and one edge). The rest of the map is denoted by $M - L$. Then, from (12), $\lambda(\lambda-1) \cdot \lambda P(M,\lambda) = P(L,\lambda) \cdot P(M-L,\lambda)$. The next result follows immediately.

**Lemma 5.** *If $L$ denotes a map that is a counterexample to the 4-Colour Theorem then $P(M,4) = 0$ for any map $M$ containing a copy of $L$.*

**Remark 6.** *The strategy of this proof of the Four Colour Theorem is to show that such a map $L$ does not exist. This will require: (a) for any rooted map $L$, properties of the generating function $A_L(x)$ for rooted maps not containing a copy of $L$; (b) a construction of a set of maps $\mathcal{Q}$ (see Subsection 1.3).*

## 3 A singularity analysis for $A(x), B(x)$, and $A_L(x)$.

### 3.1 A relation between the radii of convergence $r_A, r_B$ and $r_L$ of $A(x), B(x), A_L(x)$.

Pringsheim's Theorem states that there is a smallest singularity, in absolute value, of an analytic function with non-negative coefficients on the positive real axis. Since $A(x)$ and $B(x)$ have only one singularity in the finite complex plane, this smallest singularity will be the smallest finite singularity of $A(x)$ and $B(x)$. Let $r_A$ and $r_B$ denote the radii of convergence of $A(x)$ and $B(x)$, respectively. From (5) and (9),

$$r_A = \tfrac{1}{12} \text{ and } r_B = \tfrac{4}{27}. \tag{13}$$

Next, for $r_B$: from (11), $B(x(1+A(x))^2) = A(x)$ is analytic for $|x| < r_A$. Thus $r_B \geq r_A(1+A(r_A))^2$, recalling that $A(r_A)$ is convergent. Suppose $B(x)$ is analytic at $x = r_A(1+A(r_A))^2$. This implies that, from (11), $B(x(1+A(x))^2) = A(x)$ so $A(x)$ is analytic at $x = r_A$. However, $A''(\frac{1}{12})$ diverges, from (5), giving a contradiction if $r_B > r_A(1+A(r_A))^2$. Thus

$$r_B = r_A(1+A(r_A))^2. \tag{14}$$

Then, from (2) and (4),

$$A(r_A) = A(\tfrac{1}{12}) = {}_2F_1[\tfrac{1}{2},1;3;1] - 1 = \frac{\Gamma(3)\,\Gamma(\frac{3}{2})}{\Gamma(\frac{5}{2})\,\Gamma(2)} - 1 = \tfrac{1}{3}. \tag{15}$$

The argument used to establish (11) may be applied, in turn, to $A_L(x)$ and $B(x)$ to show that $A_L(x) = B(x(1+A_L(x))^2)$ and, with $r_L$ denoting the radius of convergence of $A_L$, that

$$r_B = r_L(1+A_L(r_L))^2. \tag{16}$$

The coefficients of $A_L(x)$ are less than or equal to those of $A(x)$ since every map counted by $A_L(x)$ is also counted by $A(x)$, so $r_L \geq r_A$. Also $A_L(x)$ must converge at $x = r_L$ since, from (9), $B(x)$ converges at $x = r_B$. Hence $B(r_B)$ is defined. The *lim sup* definition of the radius of convergence, gives $r_L = \frac{1}{\alpha_L}$ where $\alpha_L = \limsup_{n\to\infty} |a_{n,L}|^{1/n}$. Suppose $r_L = r_A$. To show that this leads to a contradiction, consider $r_B = r_A(1+A(r_A))^2 = r_L(1+A(r_L))^2 = r_B$ from (14) and (16) so, if $r_A = r_L$, then $(1+A(r_A))^2 = (1+A_L(r_L))^2$ so $A(r_A) = A_L(r_A)$. If $L$ has $\ell$ edges then $[x^\ell]A_L(x) \leq a_L - 1$ so $A(r_A) > A_L(r_A)$, and it cannot be that $(1+A(r_A))^2 = (1+A_L(r_A))^2$. Thus to avoid a contradiction requires that

$$r_L > r_A. \tag{17}$$

### 3.2 Estimating the number of rooted maps with no copy of $L$.

Consider the maps which do not contain a copy of $L$ (it is assumed that these maps and $L$ are rooted). An estimate may be given for all $n$. Recall from Cauchy's coefficient formula ([5], Thm. IV.4, *p.* 237) that $[x^n]A_L(x) = \frac{1}{2\pi i}\int_{-\pi}^{\pi} \frac{A_L(re^{i\theta})}{(re^{i\theta})^{n+1}} d(re^{i\theta}) = \frac{1}{2\pi}\int_{-\pi}^{\pi} \frac{A_L(re^{i\theta})}{r^n} e^{-ni\theta} d\theta$, where $x = re^{i\theta}$. The coefficients of $A_L(x)$ are non-negative so $|A_L(re^{i\theta})| \leq A_L(r)$ for $-\pi \leq \theta \leq \pi$. Thus $|[x^n]A_L(x)| \leq \frac{1}{2\pi}\left|\int_{-\pi}^{\pi} r^{-n}A_L(r)d\theta\right| \leq A_L(r)r^{-n}$ for every $r < r_L$. Also $[x^n]A_L(x) \leq [x^n]A(x)$ since every map counted by $A_L(x)$ is counted by $A(x)$. Now $r_B = r_L(1 + A(r_L))^2 = r_A(1 + A(r_A))^2$, from (14) and (16), so taking the quotient of the two expressions for $r_B$ gives $1 = \frac{r_A(1+A(r_A))^2}{r_L(1+A_L(r_L))^2} < \frac{(1+A(r_A))^2}{(1+A_L(r_L))^2}$. Thus $A(r_A) > A_L(r_L)$. Then, from (15), $A_L(r_L) < \frac{1}{3}$. Choosing $r = (r_A + r_L)/2$ gives $r = r_A + \delta$ where $\delta > 0$. Since $r_A = \frac{1}{12}$ from (13) then $[x^n]A_L(x) \leq \frac{1}{3}\left(\frac{1}{12} + \delta\right)^{-n} = \frac{1}{3}12^n(1 + 12\delta)^{-n}$ (see (19) in Prop. IV.1, *p.* 246 [5]). Also $a_n \sim \frac{2}{\sqrt{\pi}}n^{-5/2}12^n$ (from (5)). Then dividing the left hand side of the above by $a_n$ and the right hand side by $\frac{2}{\sqrt{\pi}}n^{-5/2}12^n$ gives $[x^n]A_L(x) < \frac{\sqrt{\pi}}{6}n^{5/2}(1 + 12\delta)^{-n}a_n(1 + o(1))$ as $n \to \infty$. From Lemma 5, if there is a map $L$ which cannot be 4-coloured then no map containing $L$ as a copy can be 4-coloured. In other words, replacing $\delta$ by any $\delta_1$ where $0 < \delta_1 < \delta$ and dropping the $\sqrt{\pi}n^{5/2}$, gives the following.

**Theorem 7.** *If there is a map $L$ which cannot be* 4*-coloured then only an exponentially small fraction of the maps with $n$ edges can be* 4*-coloured. ($L$ may be separable or non-separable.)*

Corollary 1 of Richmond *et al.* [7] states: if there is one 3-connected triangulation which cannot be 4-coloured then the radius of convergence of the generating function for 4-colourable 3-connected triangulations is strictly greater than that for 3-connected triangulations.

The above Theorem 7 is similar, the difference being that it concerns maps with at least one edge, whereas Corollary 1 of Richmond *et al.*[7] concerns 3-connected triangulations. It is this latter observation that allows the present approach, outlined in Section 1.3, to be carried out to completion.

## 4 Constructing a set, $\mathcal{Q}$, of maps.

### 4.1 Preliminaries

Let $G$ be a map with submaps $S$ and $T$ which have only a cut vertex in common. Moreover, let $S$ have constituents $S_1$ and $T_1$, and let $T$ have constituents $S_2$ and $T_2$ (so $S = S_1 \bullet T_1$ and $T = S_2 \bullet T_2$). These decompose $G$ into four maps, $S_1, T_1, S_2$ and $T_2$. Moreover, $S_1$ and $T_1$ have only one cut vertex in common, as do $S_2$ and $T_2$ . Moreover, the number of edges, $E(G)$, in $G$ satisfies $E(G) = E(S_1) + E(T_1) + E(S_2) + E(T_2)$. Since each constituent has at least one edge, the following inequalities hold: $E(S_1), E(T_1), E(S_2), E(T_2) < E(G)$. Then $\mathcal{Q}$ is defined as follows.

**Definition 8.** *$\mathcal{Q}$ is the set of all maps $(S_1 \bullet T_1) \bullet (S_2 \bullet T_2)$ with $E(S_1), E(S_2), E(T_1), E(T_2) \geq 1$ where '$\bullet$' indicates a cut vertex.*

It follows that $\sum_{n_1,n_2\geq 1, n_1+n_2=n}(a_{n_1} - b_{n_1})(a_{n_2} - b_{n_2}) = [x^n](A(x) - B(x))^2$ is the number of maps of the form $S_1 \bullet T_1$ with $n$ edges since, from (10), the generating function for separable maps is $A(x) - B(x)$. Thus the generating function for $\mathcal{Q}$ is

$$(A(x) - B(x))^4. \tag{18}$$

### 4.2 The 4-colourability of the maps of $\mathcal{Q}$.

In view of Theorem 7, to prove that the maps of $\mathcal{Q}$ it suffices to prove that the maps of a particular subset $\overline{\mathcal{Q}}$, defined below, are 4-colourable. Let $Q \in \mathcal{Q}$ (see Def. 8). Then, from (12), $\lambda P(Q, \lambda) = P(S_1 \bullet T_1, \lambda) \cdot P(S_2 \bullet T_2, \lambda)$ so

$$\lambda^3 P(Q, \lambda) = P(S_1, \lambda) \cdot P(T_1, \lambda) \cdot P(S_2, \lambda) \cdot P(T_2, \lambda). \tag{19}$$

Let $\overline{\mathcal{Q}}$ be the set of all maps of the form $\overline{Q} = (S_1 \bullet e_1) \bullet (e_2 \bullet e_3)$ where $e_1, e_2$ and $e_3$ are edges. The following Induction Hypothesis for $\lambda \geq 4$ will be used. *All maps in $\mathcal{Q}$ with at most $n-1$ edges are 4-colourable.* This induction hypothesis also applies to $\overline{\mathcal{Q}}$ since $\overline{\mathcal{Q}} \subset \mathcal{Q}$, and places no condition upon $S_1, T_1, S_2$ or $T_2$. From (19), $\lambda^3 P(\overline{Q}) = P(S_1, \lambda) \cdot P(e_1, \lambda) \cdot P(e_2, \lambda) \cdot P(e_3, \lambda) = \lambda^3(\lambda-1)^3 P(S_1, \lambda)$ so $P(\overline{Q}) = (\lambda-1)^3 P(S_1, \lambda)$. In addition, $E(Q) = E(S_1)+E(S_2)+E(T_1)+E(T_2)$ and $E(\overline{Q}) = E(S_1)+3$. Since $E(S_i) \geq 1$ and $E(T_i) \geq 1$ for $i = 1, 2$ then $E(S_1) = E(\overline{Q}) - E(S_2) - E(T_1) - E(T_2) < E(Q)$. It follows from the Induction Hypothesis that $S_1$ is 4-colourable, so $P(S_1, \lambda) > 0$. Similarly, $P(T_1, \lambda) > 0, P(S_2, \lambda) > 0$ and $P(T_2, \lambda) > 0$. Thus, from (19), so $Q$ is 4-colourable.

### 4.3 Non-existence of the map $L$.

From (5) and (9), the generating function $A(x) - B(x)$ for separable maps has radius of convergence $r_A = \frac{1}{12}$. Note that $a_n - b_n = a_n \left(1 - \frac{b_n}{a_n}\right) = a_n \left(1 + \mathcal{O}(\frac{9}{16})^n\right) = a_n + \mathcal{O}\left(n^{-5/2}(\frac{27}{4})^n\right)$. The fraction of the maps with $n$ edges which are non-separable is $\mathcal{O}\left((\frac{9}{16})^n\right)$. Hence, almost all maps are separable. From (4) and (8), $A(x) - B(x) = 8x^2 + \sum_{n\geq 3}(a_n - b_n)x^n$. Thus, since $a_n - b_n \geq 0$ for $n \geq 1$, then

$$A(\tfrac{1}{12}) - B(\tfrac{1}{12}) = \tfrac{8}{144} + \sum_{n\geq 3}(a_n - b_n)(\tfrac{1}{12})^n > \tfrac{1}{18} > 0. \tag{20}$$

Following Remark 6, it remains to show that a map $L$ does not exist. Let $Q(x)$ denote the generating series $(A(x) - B(x))^4$ for the number of maps in $\mathcal{Q}$ with respect to the number of edges. To estimate the number $[x^n]Q(x)$ of maps in $\mathcal{Q}$ with $n$ edges note that the only singularities of $A(x) - B(x)$ are at $x = \frac{1}{12}$ and $x = \frac{4}{27}$. Then, with (7), where $A_1(x)$ and $A_2(x)$ are analytic in the complex plane, and recalling that $B(x)$ is analytic in $|x| < \frac{4}{27}$ and has only one singularity in the complex plane, it follows from (18) that $Q(x) = \left(A_1(x) - B(x) + (1 - 12x)^{3/2}A_2(x)\right)^4$. Near $x = \frac{1}{12}$, the right hand side may be rewritten as $(A_1(x) - B(x))^4 + 4(A_1(x) - B(x))^3 A_2(x)(1 - 12x)^{3/2} + \cdots + \left((1 - 12x)^{3/2}A_2(x)\right)^4$. The smallest power of $1 - 12x$ in the above expression, *i.e.* the term containing $(1 - 12x)^{3/2}$, determines the asymptotic behaviour of $[x^n]Q(x)$. Thus, as $n \to \infty$, from (3) and singularity analysis,

$$\begin{aligned} [x^n]Q(x) &= [x^n]\, 4\, A_2(\tfrac{1}{12}) \left(A_1(\tfrac{1}{12}) - B(\tfrac{1}{12})\right)^3 (1 - 12x)^{3/2}(1 + \mathcal{O}(\tfrac{1}{n})) \\ &= 4\, A_2(\tfrac{1}{12}) \left(A_1(\tfrac{1}{12}) - B(\tfrac{1}{12})\right)^3 12^n \left(\Gamma(-\tfrac{3}{2})\right)^{-1} n^{-5/2}(1 + \mathcal{O}(\tfrac{1}{n})). \end{aligned}$$

Now $A(\frac{1}{12}) - B(\frac{1}{12}) > \frac{1}{18}$ from (20). Also, from (5) and (7), it follows from singularity analysis that $a_n = \frac{2}{\sqrt{\pi}} n^{-5/2} 12^n (1 + \mathcal{O}(\frac{1}{n})) = A_2(\frac{1}{12}) \cdot [x^n](1 - 12x)^{3/2} = A_2(\frac{1}{12})\, n^{-5/2} \left(\Gamma(-\frac{3}{2})\right)^{-1} 12^n (1 + \mathcal{O}(\frac{1}{n}))$. Also, from (7), $A_1(\frac{1}{12}) = A(\frac{1}{12})$. Thus, $[x^n]Q(x) = \frac{8}{\sqrt{\pi}} \left(A(\frac{1}{12}) - B(\frac{1}{12})\right)^3 n^{-5/2} 12^n (1 + \mathcal{O}(\frac{1}{n})) = 4\left(A(\frac{1}{12}) - B(\frac{1}{12})\right)^3 a_n (1 + \mathcal{O}(\frac{1}{n})) > \frac{4}{18^3}\, a_n (1 + \mathcal{O}(\frac{1}{n}))$ from (5) and (20).

Thus the number of maps in $\mathcal{Q}$ with $n$ edges is greater than the positive fraction $\frac{1}{1458}$ times the number, $a_n$, of maps with $n$ edges. So a map $L$ posited in Theorem 7 does not exist. It therefore follows that all planar maps are 4-colourable, completing a proof of Theorem 1.

# Appendices

## A Observations

Two tantalising results on the chromatic polynomial: (a) Thomassen [12] showed that the zeros of chromatic polynomials of planar maps contain a dense subset of $(\frac{32}{27}, 3)$ and conjectured that 3 can be replaced by 4; (b) Royle [10] described families of planar near-triangulations with real chromatic roots arbitrarily close to 4, approached from below. This was mentioned in Section 1.

## B Tutte's rooting convention for maps in the plane.

The reason this convention is not necessary (*i.e.* the result is independent of this convention) is that every planar map may be embedded on the sphere. The stereographic projection projects any spherical map onto the plane (or any planar map onto the sphere). A spherical map may be rotated so that the north pole of the sphere is inside a face of the spherical map. Every point of a spherical map may be projected from the north pole of the sphere onto a plane tangent to the south pole of the sphere. The face of the spherical map containing the north pole is projected onto the external face of the planar map in this way. Any face of a planar map may also be projected onto a face of the spherical map. This face may be rotated to contain the north pole and projected back onto a plane tangential to the south pole where it will be an external face. Since any edge of the external face may be chosen as the root edge, it follows that any edge of a planar map may be chosen as the root edge.

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